Solutions

(C) of question 2:

The curve reaches from the north pole to the south pole - never really attaining either of them - impinging on each circle of latitude (as well as meridians!) at 45°.

\[ \text{Length} = \int_{0}^{\infty} \left( \frac{1}{t} + \frac{2}{1+t^2} \right) \, dt \]

\[ = \int_{0}^{\infty} \left( \frac{8}{(1+t^2)^2} \right) \, dt = \left[ \frac{2 \sqrt{2}}{1+t^2} \right]_{0}^{\infty} = \frac{2 \sqrt{2}}{2} = \frac{\pi}{2} \]
1. Consider the surface \( q : \mathbb{R} \rightarrow \mathbb{R}^3 \) defined by

\[
q(u, v) = \left( u, v, \frac{1}{uv} \right)
\]

Prove that the volume of the tetrahedron formed by a tangent plane at a point on the surface and the three coordinate planes is a constant (that is, independent of \( u, v \)).

Since \( \frac{\partial q}{\partial u} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/uv^2 & 1/uv \\ -1 & 0 & 0 \end{bmatrix} \)

the normal direction is given by

\[
\frac{\partial q}{\partial u} \times \frac{\partial q}{\partial v} = \begin{bmatrix} 1/uv \\ 1/uv^2 \\ 1 \end{bmatrix}
\approx \begin{bmatrix} v \\ u \end{bmatrix}
\]

and the equation of the tangent plane for the parameter value \( (u, v) \) is

\[
v(x - u) + u(y - v) + u^2v^2(z - \frac{1}{uv}) = 0
\]

Let the tangent plane intersect the \( x, y, z \)-axes in \((x, 0, 0), (0, y, 0), (0, 0, z)\). Clearly,

\[
y = 0 \quad \Rightarrow \quad v(x - u) + u(-v) + u^2v^2(-\frac{1}{uv}) \quad \Rightarrow \quad x = 3u
\]

\[
z = 0 \quad \Rightarrow \quad v(-u) + u(y - v) + u^2v^2(-\frac{1}{uv}) \quad \Rightarrow \quad y = 3v
\]

\[
x = y = 0 \quad \Rightarrow \quad v(-u) + u(-v) + u^2v^2(\frac{1}{uv}) \quad \Rightarrow \quad z = \frac{3}{uv}
\]

\[
\text{Volume of the tetrahedron} = \frac{1}{6}xyz = \frac{1}{6} \cdot 3u \cdot 3v \cdot \frac{3}{uv} = \frac{9}{2}
\]
2. (A) Evaluate the first fundamental form of the sphere

\[ q(\theta, \varphi) = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi). \]

(B) Prove that the curve \( \tau(t) = q(\theta(t), \varphi(t)) \) with \( t \in (0, \infty) \) and

\[ \theta(t) = \log t, \varphi(t) = 2 \arctan t \]

intersects all meridians under a constant angle. What is this angle?

(C) Give a rough sketch of the curve, supporting it by brief verbal descriptions. Evaluate the length of the curve.

(A) With

\[ \frac{\partial}{\partial \theta} \mathbf{q}_\theta = \begin{bmatrix} -\sin \varphi \sin \theta \\ \sin \varphi \cos \theta \\ 0 \end{bmatrix}, \quad \frac{\partial}{\partial \varphi} \mathbf{q}_\varphi = \begin{bmatrix} \cos \varphi \cos \theta \\ \cos \varphi \sin \theta \\ -\sin \varphi \end{bmatrix} \]

it is easily computed that

\[ E = \mathbf{q}_\theta \cdot \mathbf{q}_\theta = \sin^2 \varphi, \quad F = \mathbf{q}_\theta \cdot \mathbf{q}_\varphi = 0, \quad G = \mathbf{q}_\varphi \cdot \mathbf{q}_\varphi = 1 \]

(B) A tangent vector \( \mathbf{a} = D \mathbf{q}(\theta, \varphi) \) represents the direction of a meridian if \( \mathbf{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). Consequently,

\[ \cos \left( \chi \text{ (curve, meridian)} \right) = \frac{\mathbf{a}_\theta \cdot \mathbf{a}}{\sqrt{E}} \]

\[ \mathbf{a}_\theta = \begin{bmatrix} \frac{2}{1+t^2} \\ 0 \end{bmatrix} \]

\[ \mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ \chi \text{ (curve, meridian)} = \frac{\pi}{4} \]

Notice:

\[ \sin^2 \varphi = \frac{4\left(\frac{t}{1+t^2}\right)^2 \left(\frac{1}{1+t^2}\right)^2}{1+4t^2} = \frac{4t^2}{(1+t^2)^2} \]
3. Given the surface $q : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by

$$q(z, \theta) = (\cosh(z) \cos \theta, \cosh(z) \sin \theta, z)$$

evaluate its mean and Gaussian curvature.