1. Find the solution of the partial differential equation

\[ U_x = x^2 + y^2 \]

that satisfies

\[ U(1, y) = e^y. \]

2. Show that, for an appropriate choice of the constants \( a, b \in \mathbb{R} \) in each case,

\[ U(x, y) = f(ax + by) \]

where \( f \) is an arbitrary function, is a solution of the following equations:

(A) \( U_x + 7U_y = 0 \)

(B) \( 2U_x - 5U_y = 0 \).

3. Show that, for an appropriate choice of the constants \( a, b \in \mathbb{R} \) in each case,

\[ U(x, y) = e^{ax+by} \]

is a solution of the following equations:

(A) \( U_x + 4U_y + U = 0 \)

(B) \( U_{xx} + U_{yy} = 5e^{x-2y} \)

(C) \( U_{xxxx} + U_{yyyy} - 3U_{xxyy} = 0 \).

4. Find a solution of the equation

\[ yU_x + U = x \]

subject to the condition

\[ U(x, x) = x^2 + 1. \]

5. Find the solution of the partial differential equation

\[ U_{xx} + 2U_{xy} + U_{yy} = 0 \]

that satisfies the conditions

\[ U(0, y) = \sin y \quad \text{and} \quad U_x(0, y) = 0 \]

for all \( x, y \in \mathbb{R} \), by introducing new coordinates

\[ \xi = x - y \quad \text{and} \quad \eta = y. \]
6. Consider the differential equation
\[ y^5 U_{xx} - y U_{yy} + 2U_y = 0 . \]
(A) Simplify this equation by introducing new coordinates
\[ \xi = x + \frac{y^3}{3} \quad \text{and} \quad \eta = x - \frac{y^3}{3} . \]
(B) Find a solution \( U : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R} \) of this equation that satisfies the conditions
\[ U(0, y) = 2y^6/9 \quad \text{and} \quad U_x(0, y) = 0 \]
for all \( x \in \mathbb{R}, \ y \in (0, \infty) \).

7. (A) Prove that there exists a unique smooth function \( U : \mathbb{R}^2 \rightarrow \mathbb{R} \) such that \( U(1, 1) = 4 \) and
\[ U_x = 6x^5y^2 - y \sin(xy) \quad \text{and} \quad U_y = 2x^6y - x \sin(xy) . \]
(B) Prove that there exists \( \text{no} \) smooth function \( U : \mathbb{R}^2 \rightarrow \mathbb{R} \) such that
\[ U_x = 5x^5y^2 - y \sin(xy) \quad \text{and} \quad U_y = 2x^6y - x \sin(xy) . \]
(C) Find a smooth function \( U : \mathbb{R}^2 \rightarrow \mathbb{R} \) with \( U(0, 1) = 5 \) such that
\[ U_x = U_y + 3 = 3y + U . \]

8. If \( U = U(x, t) \) is a solution of the Korteweg-de Vries equation
\[ U_{xxx} + 6UU_x + U_t = 0 \]
with \( \lim_{x \to \pm\infty} xU(x, t) = \lim_{x \to \pm\infty} xU_x(x, t) = \lim_{x \to \pm\infty} xU_{xx}(x, t) = 0 \) prove that the following functions are constant:
\[ M(t) = \int_{-\infty}^{\infty} U(x, t) \, dx \]
\[ P(t) = \int_{-\infty}^{\infty} \left( U(x, t) \right)^2 \, dx \]
\[ E(t) = \int_{-\infty}^{\infty} \left[ \frac{1}{2} \left( U_x(x, t) \right)^2 - \left( U(x, t) \right)^3 \right] \, dx \]
\[ I(t) = \int_{-\infty}^{\infty} \left[ xU(x, t) + 3t \left( U(x, t) \right)^2 \right] \, dx \]
\[ ^1\text{Of the "tsunami" fame.} \]
\[ ^2\text{Assume that the differentiation under the improper integral is allowed.} \]