# MATH 358 - PARTIAL DIFFERENTIAL EQUATIONS

## FIRST MIDTERM

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<th>FAMILY NAME</th>
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28th March 2012. Duration: 90 minutes. Three questions: 10 + 20, 35, (3 + 2) + 15 + 5 + 10

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Solutions
1. (A) Find all \( a \in \mathbb{R} \) for which \( U(x, y) = \exp \left( a x - (a^2 - a - 1) y \right) \) constitutes a solution of
\[
U_{xxx} - U_y = 0.
\]

(B) Find the solution of the equation
\[
V_{xx} = x + y^2
\]
which satisfies
\[
V(0, y) = y \quad \text{and} \quad V_x(0, y) = \sin(y).
\]

(A) \[
U_{xxx} - U_y = \left[ \frac{a^3}{a^2 - a - 1} \right] e^{ax - (a^2 - a - 1)y} = 0
\]
iff
\[
a^3 + a^2 - a - 1 = \frac{(a^2 - 1)(a+1)}{(a-1)(a+1)^2} = 0
\]
which happens iff \[
a = \pm 1
\]

(B) \[
V_{xx} = x + y^2 \rightarrow V_x = \frac{x^2}{2} + y^x + F(y),
\]
\[
\rightarrow V = \frac{x^3}{6} + \frac{y^x}{2} + F(y) + G(y)
\]
Since \[
V(0, y) = y \rightarrow G(y) = y
\]
\[
V_x(0, y) = \sin y \rightarrow F(y) = \sin(y)
\]
\[
\therefore V = \frac{x^3}{6} + \frac{x^3 y^x}{4} + x \sin(y) + y.
\]
It can be readily checked that this is a solution of the above PDE and satisfies the conditions.
2. Find a smooth function $U : (0, \infty) \times [0, \infty) \rightarrow (0, \infty)$ which satisfies

$$xU_x + U_y - \sqrt{U} = 0$$

for all $x, y > 0$ such that $U(x, 0) = x^2$ for all $x > 0$.

Since the equation can be written in the form $V = \begin{bmatrix} x \\ 1 \\ \sqrt{U} \end{bmatrix}$, the "surface" $U = U(x, y)$ is traced out by the integral curves of $V$ starting from the points on the curve $(0, 0, s^2)$. These curves can be obtained by solving the system of ODEs:

$$\begin{align*}
\dot{x} &= x \\
\dot{y} &= 1 \\
\dot{U} &= \sqrt{U}
\end{align*}$$

subject to $U(0) = s^2$.

Clearly, the parametric representation

$$\begin{bmatrix} x \\ y \\ U \end{bmatrix} = \begin{bmatrix} se^t \\ t \\ \left(\frac{t}{2} + s\right)^2 \end{bmatrix}$$

is a parametric representation of the surface $U = U(x, y)$.

Clearly, $t = y$ and $s = xe^{-y}$, hence

$$U = \left(\frac{y}{2} + xe^{-y}\right)^2$$

which can be easily checked to be the solution of the above PDE, and to satisfy the conditions.
3. Consider \( L[U] = y \, U_{xx} + x \, U_{xy} - \left( \frac{y}{x} \right) \, U_x \) for \( x > 0 \).

(A) Determine the type of \( L[U] \) and verify that \( U(x, y) = x^3 \) is a solution of the equation \( L[U] = 3xy \).

(B) Find the general solution of the equation \( L[U] = 0 \).

(C) Find the general solution of the equation \( L[U] = 3xy \).

(D) Find the solution of the equation \( L[U] = 3xy \) which satisfies

\[
U(0, y) = y + 1 - \frac{y^3}{2} \quad \text{and} \quad U_x(x, 0) = 3x^2 + x.
\]

\[\begin{align*}
\text{(A)} & \quad A = y, \quad B = \frac{x}{2}, \quad C = 0 \quad \implies \quad \Delta = B^2 - 4AC = \frac{x^2}{4} > 0 \quad \text{for} \ x \neq 0 \\
& \quad \text{This is a hyperbolic equation. Clearly,} \\
& \quad y(x^3)_{xx} + x(x^3)_{xy} - \frac{y}{x} (x^3)_x = 6xy + 0 \implies 3xy = 3xy. \\
\text{(B)} & \quad \text{Introduce new variables} \ \bar{\xi}, \ \bar{\eta} \ \text{where} \\
& \quad A \bar{\xi} + (B - \sqrt{A}) \bar{\eta} = 0 \quad \implies \quad \bar{\xi}_x = 0 \quad \implies \quad \bar{\xi} = \eta \\
& \quad A \bar{\eta}_x + (B + \sqrt{A}) \bar{\eta} = 0 \quad \implies \quad \eta_{xx} + x_\eta = 0 \quad \implies \quad \eta = x^2 - y^2 \\
& \quad \text{and notice that in view of} \\
& \quad \frac{U \gamma}{\gamma} = 2x \, U_\gamma \\
& \quad U_{xx} = 2U_\gamma + (4x^2) \, U_\eta \gamma \\
& \quad U_{xy} = 2 \left( U_{x\gamma} - 2y \, U_{\gamma \gamma} \right) \\
& \quad \text{the equation} \ L[U] = 0 \ \text{reduces to} \ U_{x\gamma} = 0. \ \text{Consequently,} \\
& \quad U(x, y) = F(y) + G(x^2 - y^2) \quad \text{is the general solution of} \ L[U] = 0. \\
\end{align*}\]

\[\begin{align*}
\text{(C)} & \quad \text{In view of the special solution obtained in (A)} \\
& \quad U(x, y) = F(y) + G(x^2 - y^2) + x^3 \quad \text{is the general solution of} \ L[U] = xy \\
\end{align*}\]

\[\begin{align*}
\text{(D)} & \quad U(x, y) = F(y) + G(-y^2) = y + 1 - \frac{y^3}{2} \quad \implies \quad F(y) = y + 1 - \frac{y^3}{2} - \left( \frac{1}{2} y^2 \right) + C \\
& \quad U_x(x, y) = G'(x^2) \, 2x + 3x^2 = 3x^2 + x \quad \implies \quad G'(x^2) = \frac{3}{2} \\
& \quad G(x) = \frac{1}{2} x + C \\
& \quad \text{Consequently} \\
& \quad U(x, y) = y + 1 + \frac{1}{2} (x^2 - y^2) + x^3 \\
\end{align*}\]

\[\begin{align*}
1 \text{Notice that} \ M = x^2 - y^2 \ \text{is a solution of the equation} \ yM_x + x M_y = 0. \quad F(y) = y + 1 - C
\end{align*}\]