METU Department of Physics | **TIME:** | **Instructor:**

PHYS 504 Midterm Examination I | 17:40-19:40 | İsmail Rafatov

November 14, 2016 | **120 minutes**

First Name	Last Name	Signature

Problem No	1	2	3	4	5	6	Total
Grade							

Transversality condition:	$\left[f + (g_x - y_x)\frac{\partial f}{\partial y_x}\right]_{x=x_0} = 0$
Fourier transformation:	$F(\omega) = \mathcal{F}\{f(t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$
Inverse transform	$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega$
Fourier transform of derivative	$\mathcal{F}\{f^{(n)}(t)\} = (-i\omega)^n\mathcal{F}\{f(t)\}$
Fourier convolution	$F(\omega)G(\omega) = \mathcal{F}\{\int_{-\infty}^{\infty} f(t-y)g(y)dy\}$
Laplace transform	$f(s) = \mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st}F(t)dt$
Laplace convolution	$f(s)g(s) = \mathcal{L}\left\{\int_0^t F(t-z)G(z)dz\right\}$

ASSIGNMENT:

Problem 1 (10 pts): The ground-state energy of a quantum particle of mass m in a pillbox (right-circular cylinder) is given by

$$E = \frac{\hbar}{2m}\left(\frac{(2.4048)^2}{R^2} + \frac{\pi^2}{H^2}\right),$$

in which R is the radius and H is the height of the pillbox. Use Lagrange multipliers method to find the ratio of R to H that will minimize the energy for a fixed volume.

Volume $= \pi R^2 H = V_0$.

Let $\varphi(H,R) = \pi R^2 H - V_0 = 0$ and $f = E(R,H) + \lambda\varphi(R,H)$

$$\begin{cases}\frac{\partial f}{\partial R} = 0\\ \frac{\partial f}{\partial H} = 0\end{cases} \Rightarrow \begin{cases}\frac{\partial E}{\partial R} + \lambda\frac{\partial \varphi}{\partial R} = 0\\ \frac{\partial E}{\partial H} + \lambda\frac{\partial \varphi}{\partial H} = 0\end{cases} \quad \begin{cases}\frac{\hbar}{2m}(2,4048)^2\cdot\frac{-2}{R^3} + \lambda\cdot 2\cdot\pi R H = 0\\ \frac{\hbar}{2m}\pi^2\cdot\frac{-2}{H^3} + \lambda\cdot\pi R^2 = 0\end{cases}$$

$$\frac{(2,4048)^2}{\pi^2}\frac{H^3}{R^3} = \frac{\cancel{\lambda}\cdot 2\cancel{\pi}\cancel{R}\cdot H}{\cancel{\lambda}\cancel{\pi} R^{\cancel{2}}} = 2\cdot\frac{H}{R}$$

$$\frac{(2,4048)^2}{\pi^2}\cdot\frac{H^2}{R^2} = 2 \Rightarrow \frac{R}{H} = \frac{1}{\sqrt{2}}\frac{2,4048}{\pi}.$$

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Problem 2 (20 pts): (i) Find the extremal of the functional $J[y] = \int_0^2 (y'^2 - 16y)dx$, subject to the conditions $y(0) = 0, y(2) = -16$.

$f = y_x^2 - 16y$

$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} = 0$

$-16 - \frac{d}{dx}(2y_x) = 0$

$y_{xx} = -8$

$y_x = -8x + C_1$

$y = -4x^2 + C_1 x + C_2$

$y(0) = C_2 = 0$

$y(2) = -4 \cdot 4 + C_1 \cdot 2 = -16 \Rightarrow C_1 = 0$

$\Rightarrow y(x) = -4x^2$

(ii) Find the extremal when one of the end point is not fixed: $J[y] = \int_b^2 (y'^2 - 16y)dx$. End point conditions are $y(b) = 1, y(2) = -16$.

$y = -4x^2 + C_1 x + C_2$

$y(2) = -4 \cdot 4 + 2C_1 + C_2 = -16$

$\Rightarrow 2C_1 + C_2 = 0$

$y(b) = -4b^2 + C_1 \cdot (b-2) = 1$

$f + (g_x - y_x)\frac{\partial f}{\partial y_x}\Big|_b = 0$

$g(x) = 1 \Rightarrow g_x = 0.$

$y_x^2 - 16y - y_x \cdot 2y_x \Big|_b = 0$

$y_x^2 + 16y \Big|_b = 0$

$(-8b + C_1)^2 + 16 \cdot (-4b^2 + C_1 b - 2C_1) = 0$

$C_1^2 + \cancel{64b^2} - \cancel{16bC_1} - \cancel{64b^2} + \cancel{16b \cdot C_1} - 32C_1 = 0$

$C_1 \cdot (C_1 - 32) = 0$

1) $C_1 = 0 \Rightarrow -4b^2 = 1 \Rightarrow b \in \emptyset$

2) $C_1 = 32 \Rightarrow +4b^2 - 32b + 65 = 0$

$\frac{D}{4} = 16^2 - 4 \cdot 65 = 256 - 260 < 0$

$\Rightarrow b \in \emptyset$?!

Problem 3 (15 pts): Find the inverse Laplace transform $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2)}\right\}$ by using

(i) partial fraction expansion

$\frac{1}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} = \frac{As + B(s+2) + Cs^2}{s^2(s+2)}$

$\Rightarrow \begin{cases} 2B = 1 \\ 4C = 1 \\ A + C = 0 \end{cases} \quad \begin{cases} B = 1/2 \\ C = 1/4 \\ A = -1/4 \end{cases}$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2)}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{-\frac{1}{2s} + \frac{1}{s^2} + \frac{1}{2}\frac{1}{s+2}\right\} =$

$= \frac{1}{2} \cdot \left(-\frac{1}{2} + t + \frac{e^{-2t}}{2}\right) = -\frac{1}{4} + \frac{t}{2} + \frac{1}{4}e^{-2t}.$

(ii) convolution theorem.

$\mathcal{L}^{-1}\left\{\frac{1}{s^2} \cdot \frac{1}{s+2}\right\} = \int_0^t (t-z) e^{-2z} dz = \frac{(t-z)e^{-2z}}{-2}\Big|_0^t + \frac{1}{2}\int_0^t e^{-2z} \cdot (-1) dz =$

$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t; \ \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}}$

$= \frac{1}{2} \cdot t + \frac{1}{4} e^{-2z}\Big|_0^t = \frac{1}{2}t + \frac{1}{4}e^{-2t} - \frac{1}{4}.$

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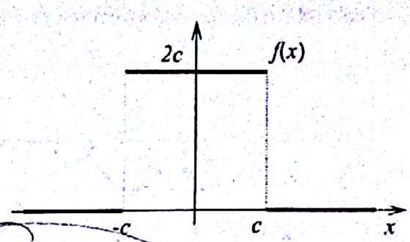
Problem 4 (10 pts): Find the Fourier transform of the function $f(x)$.



$$f(x) = \begin{cases} 2c, & \text{if } |x| < c \\ 0, & \text{if } |x| > c \end{cases}$$

$$\mathcal{F}\{f\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-c}^{c} 2 \cdot c\, e^{i\omega x} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot 2c \cdot \frac{1}{i\omega} e^{i\omega x} \Big|_{-c}^{c} = \frac{1}{\sqrt{2\pi}} \frac{2c}{i\omega} \left(e^{i\omega c} - e^{-i\omega c}\right) = \frac{4c}{\sqrt{2\pi}} \cdot \frac{\sin \omega c}{\omega}$$

Problem 5 (25 pts): Use Fourier transform method to solve the Cauchy problem for the heat equation:

$$u_t = \kappa^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = f(x), \quad -\infty < x < \infty$$

$\Rightarrow \mathcal{F}\{u_t\} = \mathcal{F}\{\kappa^2 u_{xx}\}$

$\mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\}.$

where function $f(x)$ is given in Problem 4.

Hint: 1. Use convolution theorem to express the final solution; 2. $\mathcal{F}\{e^{-a^2t^2}\} = \frac{1}{a\sqrt{2}} e^{-\frac{\omega^2}{4a^2}}$

$$\frac{\partial}{\partial t} \underbrace{\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{i\omega x} dx \right)}_{U(\omega,t)} = \kappa^2 \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u(x,t)}{\partial x^2} e^{i\omega x} dx}_{(-i\omega)^2 \cdot U(\omega,t)}$$

$$\begin{cases} \dfrac{\partial U(\omega,t)}{\partial t} = -\kappa^2 \omega^2 \cdot U(\omega,t) \\ U(\omega,0) = \mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\} \equiv F(\omega) \end{cases}$$

$$\ln |U(\omega,t)| = -\kappa^2\omega^2 t + \ln C$$

$$U(\omega,t) = C \cdot e^{-\kappa^2\omega^2 t}$$

$$U(\omega,0) = C \cdot e^0 = C = F(\omega)$$

$$\Rightarrow U(\omega,t) = F(\omega) \cdot e^{-\kappa^2\omega^2 t};$$

$$\mathcal{F}^{-1}\{e^{-\kappa^2\omega^2 t}\} = \mathcal{F}^{-1}\left\{e^{-\frac{\omega^2}{4 \cdot \left(\frac{1}{2\kappa\sqrt{t}}\right)^2}}\right\} = \frac{\sqrt{2}}{2\kappa\sqrt{t}} e^{-\frac{x^2}{4\kappa^2 t}}$$

$$u(x,t) = \mathcal{F}^{-1}\{F(\omega) \cdot e^{-\kappa^2\omega^2 t}\} =$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\kappa^2 t}} \cdot e^{-\frac{(x-y)}{4\kappa^2 t}} \cdot f(y)\, dy$$

$$= \frac{1}{\sqrt{2\kappa^2 t}} \cdot 2c \cdot \int_{-c}^{c} e^{-\frac{(x-y)^2}{4\kappa^2 t}} dy.$$

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Problem 6 (20 pts): Find expansion of the delta function $\delta(x - \pi/2)$ in a series of eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$$

1) Consider $\lambda < 0$. Let $\lambda = -n^2$.
Looking for $y(x) = e^{kx} \Rightarrow k^2 - n^2 = 0$
$k = \pm n \Rightarrow y(x) = C_1 e^{-nx} + C_2 e^{nx}$
$y(0) = C_1 + C_2 = 0 \Rightarrow C_1 = -C_2$
$y(\pi) = C_2(e^{n\pi} - e^{-n\pi}) = 0 \Rightarrow C_2 = 0$
$\Rightarrow y(x) \equiv 0 \Rightarrow \lambda$ cannot be < 0

2) Let $\lambda = n^2 > 0$.
$k^2 + n^2 = 0 \Rightarrow k = \pm in$
$\Rightarrow y(x) = C_1 \sin nx + C_2 \cos nx$
$y(0) = C_2 = 0$
$y(\pi) = C_1 \sin n\pi = 0$
$\Rightarrow n = 1, 2, 3, \ldots$
$\Rightarrow \lambda = \lambda_n = 1, 4, 9, \ldots$
$y(x) = y_n(x) = C_{1,n} \cdot \sin nx.$
$(n = 1, 2, 3, \ldots)$

3) Let $\lambda = 0$
$y'' = 0 \Rightarrow y(x) = C_1 x + C_2$
$y(0) = y(\pi) = 0 \Rightarrow y(x) \equiv 0$
$\Rightarrow \lambda$ cannot be $= 0$.

$\Rightarrow \{\sin x, \sin 2x, \ldots, \sin nx, \ldots\}$

$$\|\sin nx\|^2 = \int_0^\pi \sin^2 nx \, dx = \frac{1}{2}\int_0^\pi (1 - \cos 2nx)\, dx = \frac{\pi}{2}$$

$\Rightarrow \left\{\sqrt{\frac{2}{\pi}} \sin x, \sqrt{\frac{2}{\pi}} \sin 2x, \ldots, \sqrt{\frac{2}{\pi}} \sin nx, \ldots\right\}$
(Fourier sine series)

$$f(x) \sim \sum_{n=1}^{\infty} \left\langle f(x) \middle| \sqrt{\tfrac{2}{\pi}} \sin nx \right\rangle \sqrt{\tfrac{2}{\pi}} \sin nx$$
$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \langle f(x) | \sin nx \rangle \sin nx$$

$$\left\langle \delta\left(x - \tfrac{\pi}{2}\right) \middle| \sin nx \right\rangle = \int_0^\pi \delta\left(x - \tfrac{\pi}{2}\right) \cdot \sin nx \, dx = \sin \frac{n\pi}{2}$$

$$\Rightarrow \delta\left(x - \frac{\pi}{2}\right) = \frac{2}{\pi} \sum_{n=1}^{\infty} \sin \frac{n\pi}{2} \sin nx$$
$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \sin (2n-1)\frac{\pi}{2} \sin (2n-1)x$$
$$= \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \sin (2n-1)x.$$

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