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| **METU Department of Physics** | **TIME:** | **Instructor:** |
| **PHYS 503 Midterm Examination I** | 12:00-14:30 | İsmail Rafatov |
| November 13, 2016 | **150 minutes** | |

| First Name | Last Name | Signature |
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| Problem No | 1 | 2 | 3 | 4 | 5 | 6 | Total |
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| Grade | | | | | | | |

**Cauchy integral formula** $\oint_C \frac{f(z)dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0),\ n = 0,1,2,\ldots,$

**Residue at the pole of the order $m$** $\text{Res}\{f(z), z_0\} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\Big|_{z=z_0}$

**ASSIGNMENT:**

**Problem 1 (10 pts):** Let

$$f(z) = \begin{cases} \dfrac{xy^2(x+iy)}{x^2+y^2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Determine where, if anywhere, this function is
(a) differentiable,
(b) analytic.

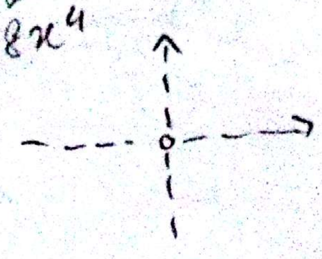
$f(z) = u + iv$ with $u = \frac{x^2y^2}{x^2+y^2}$ and $v = \frac{xy^3}{x^2+y^2}$, consider $(x,y) \neq (0,0)$.

$$\frac{\partial u}{\partial x} = \frac{2xy^2(x^2+y^2) - 2x^3y^2}{(x^2+y^2)^2} = \frac{2xy^4}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial x} = \frac{y^3(x^2+y^2) - 2x^2y^3}{(x^2+y^2)^2} = \frac{y^5 - x^2y^3}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{2x^2y(x^2+y^2) - 2x^2y^3}{(x^2+y^2)^2} = \frac{2x^4y}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = \frac{3xy^2(x^2+y^2) - 2xy^4}{(x^2+y^2)^2} = \frac{3x^3y^2 + xy^4}{(x^2+y^2)^2}$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} &\Rightarrow 2xy^4 = 3x^3y^2 + xy^4 \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} &\Rightarrow 2x^4y = -y^5 + x^2y^3 \end{aligned}\right\} \qquad \left.\begin{aligned} xy^2(y^2 - 3x^2) &= 0 \\ y(2x^4 + y^4 - x^2y^2) &= 0 \end{aligned}\right\}$$

$$\begin{cases} xy^2(y^2 - 3x^2) = 0 \\ y(\underbrace{2x^4 + 9x^4 - x^2 \cdot 3x^2}_{8x^4}) = 0 \end{cases}$$



$\Rightarrow$ C.-R. conditions are satisfied for $y = 0$, $x$ is any and $x = 0$, $y$ is any, except at $(0,0)$.

$\Rightarrow f(z)$ differentiable along these lines.

However, there is no region, in which $f(z)$ is differentiable

$\Rightarrow f(z)$ is analytic nowhere.

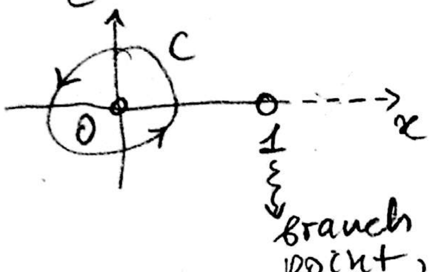
**Problem 2** (20 pts): Given the function $f(z) = \csc^2 z \, \ln(1-z)$, find
(a) the order of the pole at the origin,
(b) the residue there, and
(c) the integral around a (small) path $C$ enclosing the origin, but no other singularities.

$\ln(1-z)$ is multiple-valued. Let $\ln(1-z) = \ln|1-z| + i \cdot \arg(1-z)$, with $0 < \arg(1-z) < 2\pi$.

a) $z=0$ is a simple pole, because $\lim\limits_{z\to 0} z^1 \cdot f(z) = \lim\limits_{z\to 0} \frac{z}{\sin^2 z} \ln(1-z) =$

$= \underbrace{\lim\limits_{z\to 0} \frac{z}{\sin z}}_{=1} \cdot \lim\limits_{z\to 0} \frac{\ln(1-z)}{\sin z} = \left(\frac{0}{0}\right) = \lim\limits_{z\to 0} \frac{\frac{-1}{1-z}}{\cos z} = -1$ is finite.

b) $\operatorname{Res}\{\csc^2 z \ln(1-z); z=0\} = \lim\limits_{z\to 0}\left(z \cdot \csc^2 z \cdot \ln(1-z)\right) = -1$

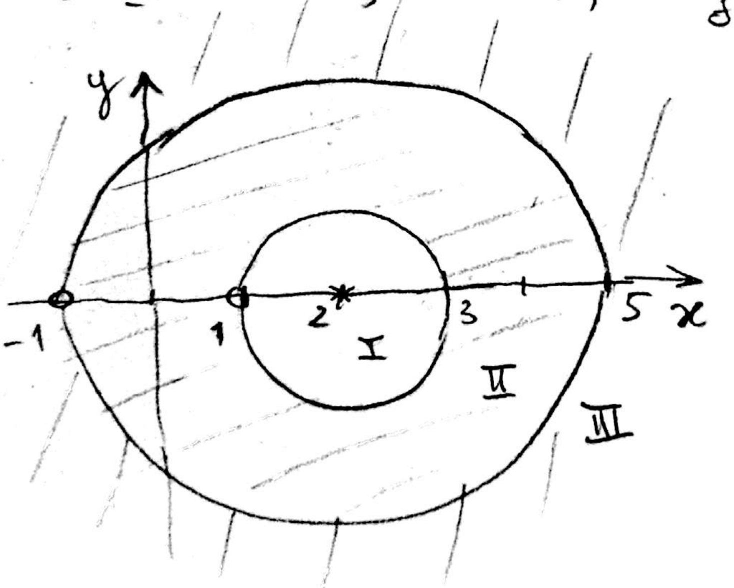
c) 


$\oint_C \csc^2 z \ln(1-z) dz = 2\pi i \cdot \underbrace{\operatorname{Res}\{\ldots, 0\}}_{= -1} = -2\pi i$

**Problem 3** (15 pts): Find all Laurent (or Taylor) expansions of the function $f(z) = \frac{z}{z^2-1}$ about $z = 2$.

$z = \pm 1$ are singularities;

$f(z) = \frac{z}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} + \frac{1}{z+1}\right)$




I: $|z-2| < 1$

$f(z) = \frac{1}{2}\left(\frac{1}{1+(z-2)} + \frac{1}{3+(z-2)}\right) =$

$= \frac{1}{2}\left(\frac{1}{1+(z-2)} + \frac{1}{3} \cdot \frac{1}{1+\frac{z-2}{3}}\right) =$

$= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (z-2)^n + \frac{1}{6} \sum_{n=0}^{\infty} (-1)^n \frac{(z-2)^n}{3^n} = \ldots$

II: $1 < |z-2| < 3$

$f(z) = \frac{1}{2} \frac{1}{z-2} \cdot \frac{1}{1+\frac{1}{z-2}} + \frac{1}{6} \frac{1}{1+\frac{z-2}{3}}$

$= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(z-2)^{n+1}} + \frac{1}{6} \sum_{n=0}^{\infty} (-1)^n \frac{(z-2)^n}{3^n}$

III: $|z-2| > 3$

$f(z) = \frac{1}{2} \frac{1}{z-2} \frac{1}{1+\frac{1}{z-2}} + \frac{1}{2} \frac{1}{z-2} \frac{1}{1+\frac{3}{z-2}} =$

$= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(z-2)^{n+1}} + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{(z-2)^{n+1}} = \ldots$

**Problem 4** (10 pts): Find $J = \oint_C \frac{z^4+2z+1}{(z-z_0)^4}\,dz$ using Cauchy's integral formula, where $C$ is any closed contour containing $z_0$.

$$J = \frac{2\pi i}{3!}\frac{d^3}{dz^3}\left(z^4+2z+1\right)\Big|_{z_0} = \frac{2\pi i}{6}\cdot\left(4z^3+2\right)''_{z_0} = \frac{\pi i}{3}\left(12z^2\right)'_{z_0} =$$

$$= \frac{\pi i}{3}\cdot 12\cdot 2z_0 = 8\pi i z_0.$$

**Problem 5** (20 pts): Use the residue theorem in evaluation of the following real integral: $\int_0^{2\pi}\frac{d\theta}{1+\epsilon\cos\theta} = \frac{2\pi}{\sqrt{1-\epsilon^2}}$ $(|\epsilon|<1)$.

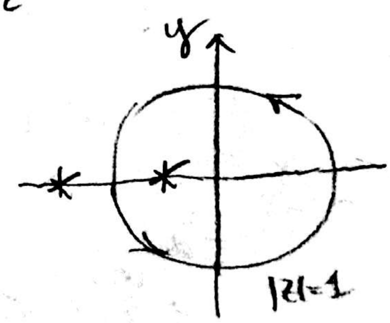
Let $z = e^{i\theta}$; $0 \le \theta \le 2\pi$

$$\Rightarrow \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z};\quad dz = ie^{i\theta}d\theta = iz\,d\theta;\quad d\theta = \frac{1}{iz}dz$$

$$\int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta} = \oint_{|z|=1}\frac{dz}{iz\cdot\left(1+\varepsilon\frac{z^2+1}{2z}\right)} = \frac{1}{i}\oint_{|z|=1}\frac{dz}{z+\frac{\varepsilon}{2}(z^2+1)} =$$

$$= \frac{2}{i}\oint_{|z|=1}\frac{dz}{\varepsilon z^2+2z+\varepsilon} = \frac{2}{\cancel{i}}\cdot 2\pi\cancel{i}\cdot\frac{1}{2\varepsilon z+2}\Bigg|_{\frac{1}{\varepsilon}(-1+\sqrt{1-\varepsilon^2})} = 2\pi\frac{1}{-\cancel{1}+\sqrt{1-\varepsilon^2}+\cancel{1}}$$

$$= \frac{2\pi}{\sqrt{1-\varepsilon^2}}.$$




$$\varepsilon z^2+2z+\varepsilon = 0$$

$$\frac{D}{4} = 1-\varepsilon^2$$

$$z = \frac{1}{\varepsilon}\left(-1+(1-\varepsilon^2)^{\frac{1}{2}}\right) =$$

$$= \frac{1}{\varepsilon}\left(-1+\sqrt{1-\varepsilon^2}\,e^{i\frac{2\pi k}{2}}\right)\quad (k=0,1)$$

$$= \begin{cases}\frac{1}{\varepsilon}\left(-1+\sqrt{1-\varepsilon^2}\right) \approx \frac{1}{\varepsilon}\left[-1+\left(1-\frac{\varepsilon^2}{2}\right)\right] = -\frac{\varepsilon}{2} \text{ is inside } |z|=1\\ \frac{1}{\varepsilon}\left(-1-\sqrt{1-\varepsilon^2}\right) \approx \frac{1}{\varepsilon}\cdot\left[-1-\left(1-\frac{\varepsilon^2}{2}\right)\right] = \frac{\varepsilon}{2}-\frac{2}{\varepsilon} \text{ is outside } |z|=1.\end{cases}$$

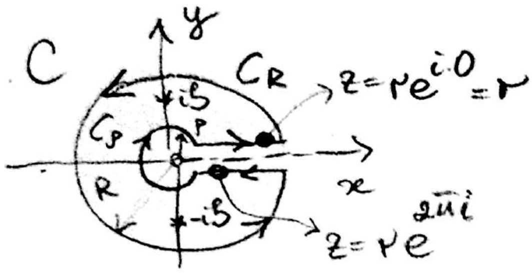
<u>simple poles</u>

**Problem 6** (25 pts): By integration through a branch cut, show that $\overbrace{\int_0^\infty \frac{x^{2a-1}dx}{b^2+x^2}}^{\mathcal{I}} = \frac{\pi b^{2(a-1)}}{2}\csc \pi a$, $0 < a < 1$.

Let $f(z) = \dfrac{z^{2a-1}}{b^2+z^2} = \dfrac{z^{2a-1}}{(z+ib)(z-ib)} = \dfrac{z^{2a}}{z(z+ib)(z-ib)}$, where

$z = \pm ib$ are simple poles $\qquad z^{2a} = e^{2a \ln z} = e^{2a(\ln|r| + i\theta)}$,

$z=0$ is a branch point $\qquad$ consider the branch with $0<\theta<2\pi$.




$$\oint_C f(z)dz = 2\pi i\left(\text{Res}\{f(z), ib\} + \text{Res}\{f(z), -ib\}\right)$$

$$\underset{①}{\int_{C_R} f(z)dz} + \underset{②}{\int_{C_\rho} f(z)dz} + \underset{③}{\int_\rho^R \frac{r^{2a-1}}{b^2+r^2}dr} + \underset{④}{\int_R^\rho \frac{(re^{2\pi i})^{2a-1}}{b^2+(re^{2\pi i})^2} d(re^{2\pi i})} =$$

$$= 2\pi i(\dots)$$

In the limit as $R\to\infty$ and $r\to 0$, $\qquad (0<a<1)$

① $\to 0$ because $z \cdot \dfrac{z^{2a-1}}{b^2+z^2} = \dfrac{z^{2a}}{b^2+z^2} \xrightarrow[|z|=R\to\infty]{} 0$

② $\to 0$ because $z \cdot \dfrac{z^{2a-1}}{b^2+z^2} = \dfrac{z^{2a}}{b^2+z^2} \xrightarrow[|z|=\rho\to 0]{} 0$

③ $\to \mathcal{I}$

④ $\to e^{2\pi i(2a-1)} \cdot (-\mathcal{I}) = -e^{4\pi a i} \cdot \mathcal{I}$

$$\Rightarrow (1-e^{4\pi a i}) \cdot \mathcal{I} = 2\pi i \cdot \left( \frac{z^{2a-1}}{2z}\Big|_{z=ib} + \frac{z^{2a-1}}{2z}\Big|_{z=-ib} \right) =$$

$$= \frac{2\pi i}{\cancel{2}}\left( z^{2a-2}\Big|_{be^{i\pi/2}} + z^{2a-2}\Big|_{be^{3\pi i/2}} \right) = \pi i b^{2a-2} \cdot \left(e^{-i\pi} e^{i\pi a} + e^{-3\pi i} \cdot e^{3\pi i a}\right)$$

$$= \pi i b^{2a-2} \cdot \left( e^{i\frac{\pi}{2}(2a-2)} + e^{\frac{3\pi i}{2}(2a-2)} \right) =$$

$$= -\pi i b^{2a-2} \cdot \left( e^{i\pi a} + e^{3i\pi a} \right)$$

$$\Rightarrow \mathcal{I} = -\pi i b \cdot \frac{{}^{2a-2}e^{i\pi a} + e^{3i\pi a}}{1 - e^{4i\pi a}} = \pi i b \cdot \frac{{}^{2a-2}\cancel{e^{2i\pi a}}(e^{i\pi a} + e^{-i\pi a})}{\cancel{e^{2i\pi a}}(e^{2i\pi a} - e^{-2i\pi a})} = \pi b^{2a-2}\frac{\cos \pi a}{\sin 2\pi a} =$$

$$= \pi b^{2a-2} \frac{\cancel{\cos \pi a}}{2\sin \pi a \cdot \cancel{\cos \pi a}} = \frac{\pi b^{2(a-1)}}{2} \cdot \frac{1}{\sin \pi a} = \frac{\pi b^{2(a-1)}}{2} \cdot \csc \pi a.$$