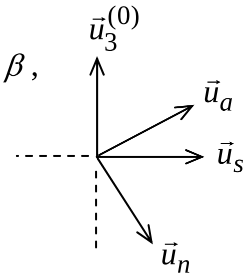
SOLVED PROBLEMS
SET 3

PROBLEM 1

During a task performed by a robotic manipulator, its end-effector is required to be oriented with respect to the base frame as described below.

The approach vector is oriented by two angles as



$$\vec{u}_a = \vec{u}_1^{(0)} \cos\alpha \cos\beta + \vec{u}_2^{(0)} \sin\alpha \cos\beta + \vec{u}_3^{(0)} \sin\beta ,$$

where α is the azimuth angle and β is the elevation angle.

The side vector is required to remain horizontal so that

$$\vec{u}_s = -\vec{u}_1^{(0)} \sin\alpha + \vec{u}_2^{(0)} \cos\alpha .$$

a) Show that the normal vector ($\vec{u}_n$) is expressed as follows in this task.

$$\vec{u}_n = \vec{u}_1^{(0)} \cos\alpha \sin\beta + \vec{u}_2^{(0)} \sin\alpha \sin\beta - \vec{u}_3^{(0)} \cos\beta .$$

Solution

It is straightforward.

b) Express the angular velocity $\vec{\omega}$ of the end-effector in the base frame in terms of α, β, and their rates.

Solution

$$d_0\vec{u}_s / dt = -\vec{u}_1^{(0)} \dot{\alpha} \cos\alpha - \vec{u}_2^{(0)} \dot{\alpha} \sin\alpha ,$$

$$d_0\vec{u}_s / dt = \vec{\omega} \times \vec{u}_s = [\omega_1 \vec{u}_1^{(0)} + \omega_2 \vec{u}_2^{(0)} + \omega_3 \vec{u}_3^{(0)}] \times [-\vec{u}_1^{(0)} \sin\alpha + \vec{u}_2^{(0)} \cos\alpha],$$

$$d_0\vec{u}_s / dt = -\vec{u}_1^{(0)} (\omega_3 \cos\alpha) - \vec{u}_2^{(0)} (\omega_3 \sin\alpha) + \vec{u}_3^{(0)} (\omega_2 \sin\alpha + \omega_1 \cos\alpha).$$

These equations imply that

$$\omega_3 = \dot{\alpha} ,$$

$$\omega_2 \sin\alpha + \omega_1 \cos\alpha = 0 .$$

See below that the second equation is automatically satisfied by ω_1 and ω_2.

Similarly,

$$d_0\vec{u}_a / dt = \vec{u}_1^{(0)} (-\dot{\alpha} s\alpha c\beta - \dot{\beta} c\alpha s\beta) + \vec{u}_2^{(0)} (\dot{\alpha} c\alpha c\beta - \dot{\beta} s\alpha s\beta) + \vec{u}_3^{(0)} (\dot{\beta} c\beta),$$

$$d_0\vec{u}_a / dt = \vec{\omega} \times \vec{u}_a = [\omega_1 \vec{u}_1^{(0)} + \omega_2 \vec{u}_2^{(0)} + \omega_3 \vec{u}_3^{(0)}] \times [\vec{u}_1^{(0)} c\alpha c\beta + \vec{u}_2^{(0)} s\alpha c\beta + \vec{u}_3^{(0)} s\beta],$$

$$d_0\vec{u}_a / dt = \vec{u}_1^{(0)} (\omega_2 s\beta - \omega_3 s\alpha c\beta) + \vec{u}_2^{(0)} (\omega_3 c\alpha c\beta - \omega_1 s\beta) + \vec{u}_3^{(0)} (\omega_1 s\alpha - \omega_2 c\alpha) c\beta .$$

Recalling that $\omega_3 = \dot{\alpha}$, these equations imply that

$$\omega_1 = \dot{\beta} \sin\alpha , \quad \omega_2 = -\dot{\beta} \cos\alpha ;$$

$$\omega_1 s\alpha - \omega_2 c\alpha = \dot{\beta} .$$

This third equation is also automatically satisfied by ω_1 and ω_2.

Hence, the angular velocity is obtained as

$$\vec{\omega} = \vec{u}_1^{(0)}\dot{\beta}\sin\alpha - \vec{u}_2^{(0)}\dot{\beta}\cos\alpha + \vec{u}_3^{(0)}\dot{\alpha} .$$

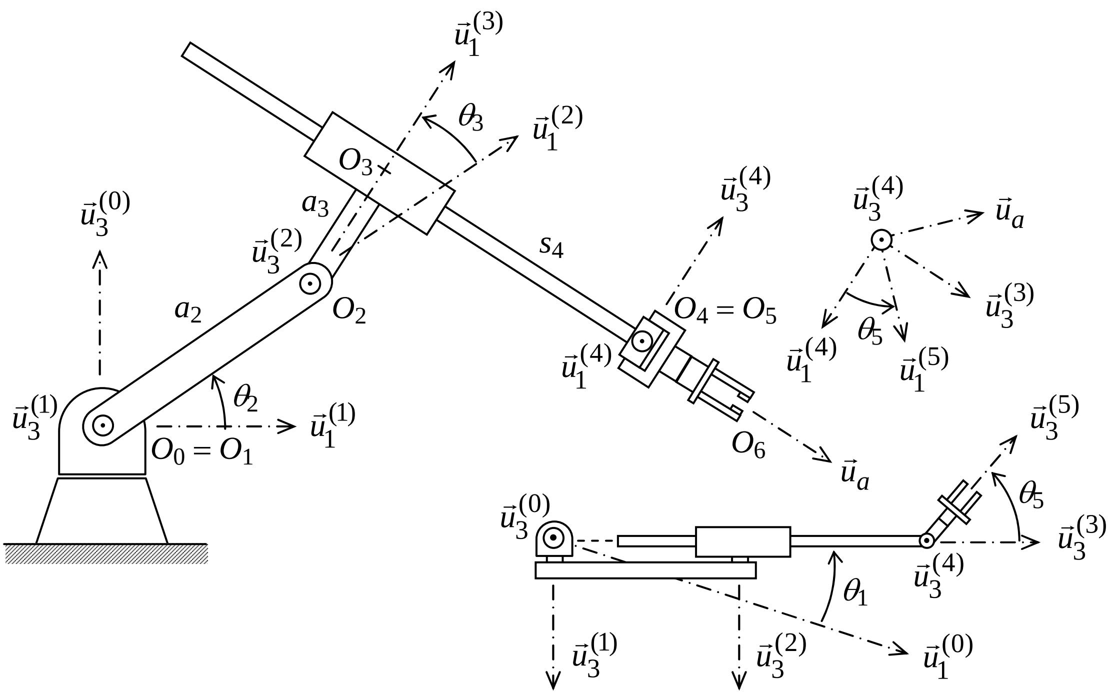
c) Using the results of part (b), express also the angular acceleration $\vec{\alpha}$ of the end-effector in the base frame in terms of α, β, and their first and second rates.

Solution

$$\vec{\alpha} = d_0\vec{\omega} / dt = d_0[\vec{u}_1^{(0)}\dot{\beta}\sin\alpha - \vec{u}_2^{(0)}\dot{\beta}\cos\alpha + \vec{u}_3^{(0)}\dot{\alpha}] / dt ,$$

$$\vec{\alpha} = \vec{u}_1^{(0)}(\ddot{\beta}\sin\alpha + \dot{\alpha}\dot{\beta}\cos\alpha) - \vec{u}_2^{(0)}(\ddot{\beta}\cos\alpha - \dot{\alpha}\dot{\beta}\sin\alpha) + \vec{u}_3^{(0)}\ddot{\alpha} .$$

PROBLEM 2



For the R^3PR^2 manipulator shown in the figure, it has already been shown that the orientation of its hand and the location of its wrist point can be expressed as follows with respect to the base frame:

$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_{23}} e^{\tilde{u}_1\theta'_5} e^{\tilde{u}_3\theta'_6} ;$$

$$\theta_{23} = \theta_2 + \theta_3 , \quad \theta'_5 = \theta_5 + \pi , \quad \theta'_6 = \theta_6 + \pi / 2 .$$

$$\overline{r} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_{23}} [\overline{u}_1(a_3 + a_2\cos\theta_3) - \overline{u}_3(s_4 + a_2\sin\theta_3)] .$$

Show that the angular velocity of the hand ($\overline{\omega}$) and the linear velocity of the wrist point ($\overline{w}$) can be expressed by the following formulas:

$$\overline{\omega} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_{23}} \overline{\omega}^* ;$$

$$\bar{\varpi}^* = \bar{u}_1(\dot{\theta}_1 \sin\theta_{23} + \dot{\theta}_5) - \bar{u}_2(\dot{\theta}_{23} + \dot{\theta}_6 \sin\theta_5') + \bar{u}_3(\dot{\theta}_1 \cos\theta_{23} + \dot{\theta}_6 \cos\theta_5') .$$

$$\bar{w} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_{23}} \bar{w}^* ;$$

$$\bar{w}^* = \bar{u}_2(a_2 \cos\theta_2 + a_3 \cos\theta_{23} + s_4 \sin\theta_{23})\dot{\theta}_1$$
$$+ \bar{u}_3(a_2\dot{\theta}_2 \cos\theta_3 + a_3\dot{\theta}_{23} - \dot{s}_4) + \bar{u}_1(a_2\dot{\theta}_2 \sin\theta_3 + s_4\dot{\theta}_{23}) .$$

SOLUTION

Starting from the given $\hat{C}$, $\bar{\varpi}$ can be written directly as

$$\bar{\varpi} = \dot{\theta}_1\bar{u}_3 - \dot{\theta}_{23} e^{\tilde{u}_3\theta_1}\bar{u}_2 + \dot{\theta}_5 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_{23}}\bar{u}_1 + \dot{\theta}_6 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_{23}} e^{\tilde{u}_1\theta_5'}\bar{u}_3$$

Let $\bar{\varpi}^* = e^{\tilde{u}_2\theta_{23}} e^{-\tilde{u}_3\theta_1}\bar{\varpi}$. Then,

$$\bar{\varpi}^* = \dot{\theta}_1 e^{\tilde{u}_2\theta_{23}}\bar{u}_3 - \dot{\theta}_{23}\bar{u}_2 + \dot{\theta}_5\bar{u}_1 + \dot{\theta}_6 e^{\tilde{u}_1\theta_5'}\bar{u}_3$$

$$\bar{\varpi}^* = \dot{\theta}_1(\bar{u}_3 \cos\theta_{23} + \bar{u}_1 \sin\theta_{23}) - \dot{\theta}_{23}\bar{u}_2 + \dot{\theta}_5\bar{u}_1 + \dot{\theta}_6(\bar{u}_3 \cos\theta_5' - \bar{u}_2 \sin\theta_5')$$

$$\bar{\varpi}^* = \bar{u}_1(\dot{\theta}_1 \sin\theta_{23} + \dot{\theta}_5) - \bar{u}_2(\dot{\theta}_{23} + \dot{\theta}_6 \sin\theta_5') + \bar{u}_3(\dot{\theta}_1 \cos\theta_{23} + \dot{\theta}_6 \cos\theta_5')$$

Differentiation of the given $\bar{r}$ leads to

$$\bar{w} = \dot{\theta}_1 e^{\tilde{u}_3\theta_1}\tilde{u}_3 e^{-\tilde{u}_2\theta_{23}}[\bar{u}_1(a_3 + a_2 \cos\theta_3) - \bar{u}_3(s_4 + a_2 \sin\theta_3)]$$
$$- \dot{\theta}_{23} e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_{23}}\tilde{u}_2[\bar{u}_1(a_3 + a_2 \cos\theta_3) - \bar{u}_3(s_4 + a_2 \sin\theta_3)]$$
$$+ e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_{23}}[\bar{u}_1(-a_2\dot{\theta}_3 \sin\theta_3) - \bar{u}_3(\dot{s}_4 + a_2\dot{\theta}_3 \cos\theta_3)]$$

$$\bar{w} = \dot{\theta}_1 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_{23}} e^{\tilde{u}_2\theta_{23}}\tilde{u}_3 e^{-\tilde{u}_2\theta_{23}}[\bar{u}_1(a_3 + a_2 \cos\theta_3) - \bar{u}_3(s_4 + a_2 \sin\theta_3)]$$
$$+ \dot{\theta}_{23} e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_{23}}[\bar{u}_3(a_3 + a_2 \cos\theta_3) + \bar{u}_1(s_4 + a_2 \sin\theta_3)]$$
$$- e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_{23}}[\bar{u}_1(a_2\dot{\theta}_3 \sin\theta_3) + \bar{u}_3(\dot{s}_4 + a_2\dot{\theta}_3 \cos\theta_3)]$$

Again, let $\bar{w}^* = e^{\tilde{u}_2\theta_{23}} e^{-\tilde{u}_3\theta_1}\bar{w}$. Then,

$$\bar{w}^* = \dot{\theta}_1(\tilde{u}_3 \cos\theta_{23} + \tilde{u}_1 \sin\theta_{23})[\bar{u}_1(a_3 + a_2 \cos\theta_3) - \bar{u}_3(s_4 + a_2 \sin\theta_3)]$$
$$+ \bar{u}_3[(a_3 + a_2 \cos\theta_3)\dot{\theta}_{23} - \dot{s}_4 - a_2\dot{\theta}_3 \cos\theta_3] + \bar{u}_1[(s_4 + a_2 \sin\theta_3)\dot{\theta}_{23} - a_2\dot{\theta}_3 \sin\theta_3]$$

$$\bar{w}^* = \bar{u}_2\dot{\theta}_1(a_3 + a_2 \cos\theta_3)\cos\theta_{23} + \bar{u}_2\dot{\theta}_1(s_4 + a_2 \sin\theta_3)\sin\theta_{23}$$
$$+ \bar{u}_3[(a_3 + a_2 \cos\theta_3)\dot{\theta}_{23} - \dot{s}_4 - a_2\dot{\theta}_3 \cos\theta_3] + \bar{u}_1[(s_4 + a_2 \sin\theta_3)\dot{\theta}_{23} - a_2\dot{\theta}_3 \sin\theta_3]$$

$$\bar{w}^* = \bar{u}_2(a_2 \cos\theta_2 + a_3 \cos\theta_{23} + s_4 \sin\theta_{23})\dot{\theta}_1$$
$$+ \bar{u}_3(a_2\dot{\theta}_2 \cos\theta_3 + a_3\dot{\theta}_{23} - \dot{s}_4) + \bar{u}_1(a_2\dot{\theta}_2 \sin\theta_3 + s_4\dot{\theta}_{23})$$

PROBLEM 3

Consider the same manipulator introduced in Problem 1.

a) Determine the *joint velocities* (the first derivatives of the joint variables) corresponding to specified $\overline{w}$ and $\overline{\omega}$ at a certain position of the manipulator.

b) Identify the *motion singularities*. Illustrate them and discuss their consequences.

Hint: As the first step, obtain $\overline{\omega}^*$ and $\overline{w}^*$ in terms of $\overline{\omega}$, $\overline{w}$, and the current position. Then, proceed with $\overline{\omega}^*$ and $\overline{w}^*$.

SOLUTION

a) The given equations for $\overline{w}^*$ and $\overline{\omega}^*$ lead to the following scalar equations:

$$(a_2\cos\theta_2 + a_3\cos\theta_{23} + s_4\sin\theta_{23})\dot{\theta}_1 = w_2^*, \tag{1}$$

$$a_2\dot{\theta}_2\cos\theta_3 + a_3\dot{\theta}_{23} - \dot{s}_4 = w_3^*, \tag{2}$$

$$a_2\dot{\theta}_2\sin\theta_3 + s_4\dot{\theta}_{23} = w_1^*; \tag{3}$$

$$\dot{\theta}_1\sin\theta_{23} + \dot{\theta}_5 = \omega_1^*, \tag{4}$$

$$\dot{\theta}_{23} + \dot{\theta}_6\sin\theta_5' = -\omega_2^*, \tag{5}$$

$$\dot{\theta}_1\cos\theta_{23} + \dot{\theta}_6\cos\theta_5' = \omega_3^*. \tag{6}$$

If $a_2\cos\theta_2 + a_3\cos\theta_{23} + s_4\sin\theta_{23} \neq 0$, Eq. (1) gives

$$\dot{\theta}_1 = w_2^* / (a_2\cos\theta_2 + a_3\cos\theta_{23} + s_4\sin\theta_{23}).$$

With the knowledge of $\dot{\theta}_1$, and if $\cos\theta_5' \neq 0$, Eqs. (4), (6), and (5) give

$$\dot{\theta}_5 = \omega_1^* - \dot{\theta}_1\sin\theta_{23},$$

$$\dot{\theta}_6 = (\omega_3^* - \dot{\theta}_1\cos\theta_{23}) / \cos\theta_5',$$

$$\dot{\theta}_{23} = -(\omega_2^* + \dot{\theta}_6\sin\theta_5').$$

With the knowledge of $\dot{\theta}_{23}$, and if $\sin\theta_3 \neq 0$, Eqs. (3) and (2) give

$$\dot{\theta}_2 = (w_1^* - s_4\dot{\theta}_{23}) / (a_2\sin\theta_3),$$

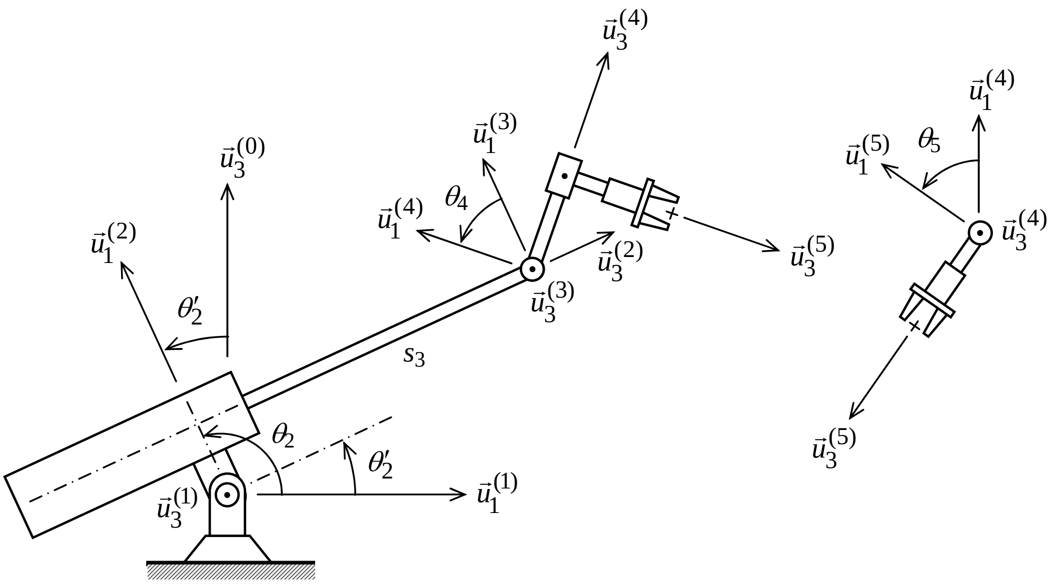
$$\dot{s}_4 = a_2\dot{\theta}_2\cos\theta_3 + a_3\dot{\theta}_{23} - w_3^*.$$

b) As noted above, this manipulator has three kinds of motion singularities:

* The first kind of singularity occurs if $a_2\cos\theta_2 + a_3\cos\theta_{23} + s_4\sin\theta_{23} = 0$. This equation implies that the wrist point lies on the axis of the first joint. In this kind of singularity, $\dot{\theta}_1$ becomes arbitrary as an extra joint space freedom but w_2^* vanishes as a task space restriction.

* The second kind of singularity occurs if $\cos\theta_5' = \cos(\theta_5 + \pi) = 0$, i.e. if $\theta_5 = \pm\pi/2$. This implies that the approach vector becomes parallel to the parallel axes of the second and third joints. Consequently, $\dot{\theta}_{23}$ and $\dot{\theta}_6$ become arbitrary within the combination $\dot{\theta}_{23} \pm \dot{\theta}_6 = -\omega_2^*$ as an extra joint space freedom but ω_3^*, becoming $\omega_3^* = \dot{\theta}_1 \cos\theta_{23}$, loses its independence as a task space restriction.
* The third kind of singularity occurs if $\sin\theta_3 = 0$, i.e. if $\theta_3 = 0$ or $\theta_3 = \pm\pi$. This implies that O_3O_4 becomes orthogonal to aligned O_1O_2 and O_2O_3. Thus, both $\dot{\theta}_2$ and $\dot{s}_4$ drive the wrist point in the same direction and therefore they become undistinguishable. In this kind of singularity, Eq. (3) implies that $w_1^* = s_4\dot{\theta}_{23}$, i.e. w_1^* loses its independence as a task space restriction. On the other hand, Eq. (2) implies that $\dot{s}_4 \pm a_2\dot{\theta}_2 = a_3\dot{\theta}_{23} - w_3^*$, i.e. $\dot{s}_4$ and $\dot{\theta}_2$ become arbitrary within the indicated combination as an extra joint space freedom.

PROBLEM 4



The figure shows the sketch of a Unimate Robot. The orientation of its hand and the location of its wrist point has already been expressed as follows with respect to the base frame.

$$\hat{C} = \hat{C}^{(0,6)} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_{24}'} e^{\tilde{u}_1\theta_5} e^{-\tilde{u}_2\theta_6'} e^{\tilde{u}_1\pi/2},$$

$$\overline{r} = \overline{r}^{(0)} = e^{\tilde{u}_3\theta_1}[e^{-\tilde{u}_2\theta_2'}(s_3\overline{u}_1 + a_2\overline{u}_3) + d_5 e^{-\tilde{u}_2\theta_{24}'}\overline{u}_1];$$

$$\theta_2' = \theta_2 - \pi/2, \quad \theta_{24}' = \theta_2' + \theta_4, \quad \theta_6' = \theta_6 + \pi/2.$$

a) Show that the angular velocity of its hand and the linear velocity of its wrist point can be expressed in the base frame as follows:

$$\overline{\omega} = \dot{\theta}_1\overline{u}_3 - \dot{\theta}_{24} e^{\tilde{u}_3\theta_1}\overline{u}_2 + \dot{\theta}_5 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_{24}'}\overline{u}_1 - \dot{\theta}_6 e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_{24}'} e^{\tilde{u}_1\theta_5}\overline{u}_2,$$

$$\bar{w} = e^{\tilde{u}_3\theta_1}[\bar{u}_2\dot{\theta}_1(d_5c\theta'_{24} + s_3c\theta'_2 - a_2s\theta'_2) + \dot{\theta}_2 e^{-\tilde{u}_2\theta'_2}(s_3\bar{u}_3 - a_2\bar{u}_1) + d_5\dot{\theta}_{24}e^{-\tilde{u}_2\theta'_{24}}\bar{u}_3] + \dot{s}_3 e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta'_2}\bar{u}_1.$$

Solution

It is straightforward.

b) Referring to the equations given above, identify the *velocity influence coefficients* that constitute the wrist point Jacobian matrix of the manipulator.

Solution

$$\bar{J}_{a1} = \bar{u}_3,\ \bar{J}_{a2} = \bar{J}_{a4} = -e^{\tilde{u}_3\theta_1}\bar{u}_2,\ \bar{J}_{a5} = e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta'_{24}}\bar{u}_1,\ \bar{J}_{a6} = -e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta'_{24}}e^{\tilde{u}_1\theta_5}\bar{u}_2.$$

$$\bar{J}_{r1} = e^{\tilde{u}_3\theta_1}\bar{u}_2(d_5c\theta'_{24} + s_3c\theta'_2 - a_2s\theta'_2),\ \bar{J}_{r2} = e^{\tilde{u}_3\theta_1}[e^{-\tilde{u}_2\theta'_2}(s_3\bar{u}_3 - a_2\bar{u}_1) + d_5e^{-\tilde{u}_2\theta'_{24}}\bar{u}_3],$$
$$\bar{J}_{r3} = e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta'_2}\bar{u}_1,\ \bar{J}_{r4} = d_5e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta'_{24}}\bar{u}_3,\ \bar{J}_{r5} = \bar{J}_{r6} = \bar{0}.$$

PROBLEM 5

Consider the same manipulator introduced in Problem 4.

a) Derive expressions to determine the rates of the joint variables corresponding to a specified motion of the hand.

b) Indicate and discuss the motion singularities as well.

a) **Inverse Kinematics Solution:**

Let's re-write the wrist point velocity equation as

$$e^{-\tilde{u}_3\theta_1}\bar{w} = \bar{u}_2\dot{\theta}_1(d_5c\theta'_{24} + s_3c\theta'_2 - a_2s\theta'_2) + \dot{\theta}_2e^{-\tilde{u}_2\theta'_2}(s_3\bar{u}_3 - a_2\bar{u}_1) + d_5\dot{\theta}_{24}e^{-\tilde{u}_2\theta'_{24}}\bar{u}_3 + \dot{s}_3e^{-\tilde{u}_2\theta'_2}\bar{u}_1.$$

Let's pre-multiply both sides by $\bar{u}_2^t$:

$$\bar{u}_2^t e^{-\tilde{u}_3\theta_1}\bar{w} = w_2c\theta_1 - w_1s\theta_1 = \dot{\theta}_1(d_5c\theta'_{24} + s_3c\theta'_2 - a_2s\theta'_2).$$

Hence, if $d_5c\theta'_{24} + s_3c\theta'_2 - a_2s\theta'_2 \neq 0$,

$$\dot{\theta}_1 = \frac{w_2c\theta_1 - w_1s\theta_1}{d_5c\theta'_{24} + s_3c\theta'_2 - a_2s\theta'_2}.$$

With known $\dot{\theta}_1$, let's write the angular velocity equation as

$$-\dot{\theta}_{24}\bar{u}_2 + \dot{\theta}_5\bar{u}_1 - \dot{\theta}_6e^{\tilde{u}_1\theta_5}\bar{u}_2 = \bar{\varpi}^*,$$

where $\bar{\varpi}^*$ is known as

$$\bar{\varpi}^* = e^{\tilde{u}_2\theta'_{24}}e^{-\tilde{u}_3\theta_1}(\bar{\varpi} - \dot{\theta}_1\bar{u}_3).$$

Let's write this equation further as

$$-(\dot{\theta}_{24} + \dot{\theta}_6c\theta_5)\bar{u}_2 + \dot{\theta}_5\bar{u}_1 - \bar{u}_3\dot{\theta}_6s\theta_5 = \bar{\varpi}^*.$$

Hence, if $\sin\theta_5 \neq 0$,

$$\dot\theta_5 = \omega_1^*, \quad \dot\theta_6 = -\omega_3^* / \sin\theta_5, \quad \dot\theta_{24} = -(\omega_2^* + \dot\theta_6 \cos\theta_5) = \omega_3^* / \tan\theta_5 - \omega_2^* .$$

With known $\dot\theta_1$ and $\dot\theta_{24}$, let's write the wrist point velocity equation this time as

$$\dot\theta_2 (s_3 \bar{u}_3 - a_2 \bar{u}_1) + \dot{s}_3 \bar{u}_1 = (s_3 \dot\theta_2) \bar{u}_3 + (\dot{s}_3 - a_2 \dot\theta_2) \bar{u}_1 = w^* .$$

Here, w^* is known as

$$w^* = e^{\tilde{u}_2 \theta_2'} e^{-\tilde{u}_3 \theta_1} \bar{w} - \bar{u}_2 \dot\theta_1 (d_5 c\theta_{24}' + s_3 c\theta_2' - a_2 s\theta_2') - d_5 \dot\theta_{24} e^{-\tilde{u}_2 \theta_4} \bar{u}_3 .$$

Hence, noting that $s_3 > 0$ always,

$$\dot\theta_2 = w_3^* / s_3, \quad \dot{s}_3 = w_1^* + a_2 \dot\theta_2 .$$

Finally,

$$\dot\theta_4 = \dot\theta_{24} - \dot\theta_2 .$$

b) **Motion Singularities:**

First kind of motion singularity occurs if

$$d_5 c\theta_{24}' + s_3 c\theta_2' - a_2 s\theta_2' = 0 .$$

This equation implies that the wrist point becomes located on the axis of the first joint. In this kind of singularity, $\dot\theta_1$ becomes arbitrary while the wrist point motion becomes restricted so that

$$w_2 c\theta_1 - w_1 s\theta_1 = 0 \quad \text{or} \quad w_2 = w_1 \tan\theta_1 .$$

Second kind of motion singularity occurs if

$$\sin\theta_5 = 0 .$$

In such a configuration, the axis of joint 6 becomes parallel to the axes of joints 2 and 4. Then, $\dot\theta_6$ and $\dot\theta_{24}$ become arbitrary. However, their sum or difference can still be determined as

$$\dot\theta_{24} + \sigma_5' \dot\theta_6 = -\omega_2^* ; \quad \sigma_5' = \cos\theta_5 = \pm 1.$$

The corresponding restriction in the hand's angular motion is $\omega_3^* = 0$.