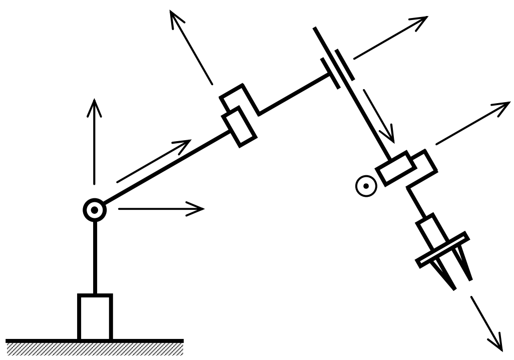
SOLVED PROBLEMS
SET 2

PROBLEM 1



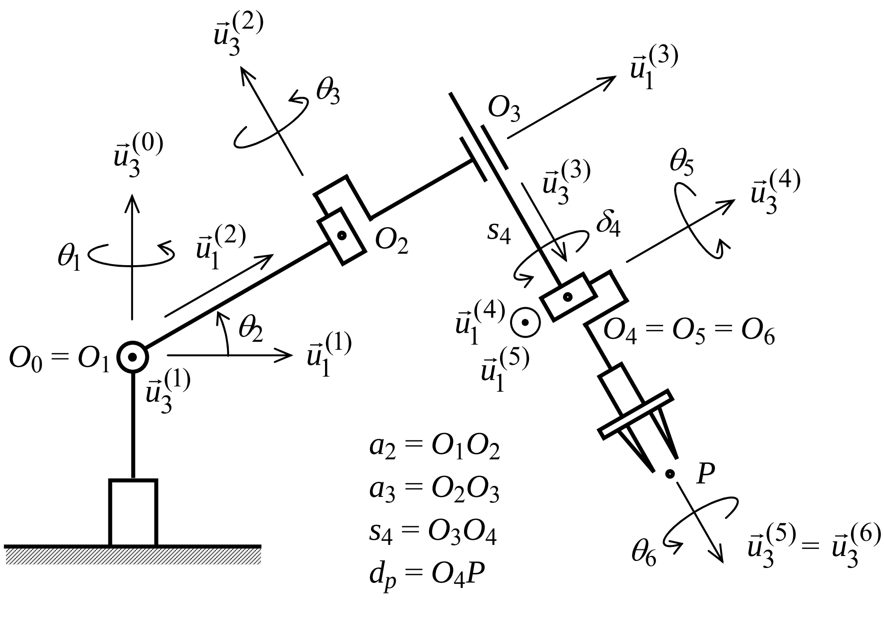
a) The R^3PR^2 manipulator sketched above does not have any shoulder offset. Label the indicated unit vectors. Then, accordingly, identify the joint variables, the link parameters, and indicate the link frame origins. Show the joint variables clearly by drawing auxiliary diagrams, if necessary.

b) Verify the following equations for the hand orientation matrix and the wrist point position vector:

$$\hat{C} = \hat{C}^{(0,6)} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\theta_5} e^{-\tilde{u}_3\theta_6'} e^{\tilde{u}_1\pi}, \quad \theta_6' = \theta_6 + \pi/2\,;$$

$$\bar{r} = \bar{r}^{(0)} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_2} [\bar{u}_1(a_2 + a_3\cos\theta_3) + \bar{u}_2(a_3\sin\theta_3) - \bar{u}_3 s_4]\,.$$

SOLUTION



a) The labels are inserted into the picture as shown above. By inspection, the link-to-link rotation matrices can be written as follows:

$$\hat{C}^{(0,1)} = e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\pi/2}, \quad \hat{C}^{(1,2)} = e^{\tilde{u}_3\theta_2} e^{-\tilde{u}_1\pi/2}, \quad \hat{C}^{(2,3)} = e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\pi},$$

$$\hat{C}^{(3,4)} = e^{\tilde{u}_3\pi/2} e^{\tilde{u}_1\pi/2}, \quad \hat{C}^{(4,5)} = e^{\tilde{u}_3\theta_5} e^{-\tilde{u}_1\pi/2}, \quad \hat{C}^{(5,6)} = e^{\tilde{u}_3\theta_6}.$$

The product of these matrices can be simplified as shown below:

$$\hat{C} = e^{\tilde{u}_3\theta_1} (e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{-\tilde{u}_1\pi/2}) e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\pi} e^{\tilde{u}_3\pi/2} (e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_5} e^{-\tilde{u}_1\pi/2}) e^{\tilde{u}_3\theta_6},$$

$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\pi} e^{\tilde{u}_3\pi/2} e^{-\tilde{u}_2\theta_5} e^{\tilde{u}_3\theta_6},$$

$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} e^{-\tilde{u}_3\pi/2} e^{\tilde{u}_2\theta_5} e^{-\tilde{u}_3\theta_6} e^{\tilde{u}_1\pi},$$

$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} \underline{e^{-\tilde{u}_3\pi/2} e^{\tilde{u}_2\theta_5} (e^{\tilde{u}_3\pi/2}} \overline{e^{-\tilde{u}_3\pi/2}) e^{-\tilde{u}_3\theta_6}} e^{\tilde{u}_1\pi},$$

$$\hat{C} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_2} e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\theta_5} e^{-\tilde{u}_3(\theta_6+\pi/2)} e^{\tilde{u}_1\pi}. \quad \text{Q.E.D.}$$

Referring to the figure, the wrist point position vector can be expressed as

$$\vec{r} = a_2\vec{u}_1^{(2)} + a_3\vec{u}_1^{(3)} + s_4\vec{u}_3^{(3)},$$

This vector equation can be written as the following matrix equation in the base frame:

$$\bar{r} = a_2\hat{C}^{(0,2)}\bar{u}_1 + a_3\hat{C}^{(0,3)}\bar{u}_1 + s_4\hat{C}^{(0,3)}\bar{u}_3 = \hat{C}^{(0,2)}[a_2\bar{u}_1 + \hat{C}^{(2,3)}(a_3\bar{u}_1 + s_4\bar{u}_3)],$$

$$\bar{r} = (e^{\tilde{u}_3\theta_1} e^{\tilde{u}_1\pi/2} e^{\tilde{u}_3\theta_2} e^{-\tilde{u}_1\pi/2})[a_2\bar{u}_1 + (e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\pi})(a_3\bar{u}_1 + s_4\bar{u}_3)].$$

Note that $e^{\tilde{u}_1\pi}\bar{u}_1 = \bar{u}_1$ but $e^{\tilde{u}_1\pi}\bar{u}_3 = -\bar{u}_3$. Therefore,

$$\bar{r} = (e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_2})[a_2\bar{u}_1 + e^{\tilde{u}_3\theta_3}(a_3\bar{u}_1 - s_4\bar{u}_3)],$$

$$\bar{r} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_2}[\bar{u}_1(a_2 + a_3\cos\theta_3) + \bar{u}_2(a_3\sin\theta_3) - s_4\bar{u}_3]. \quad \text{Q.E.D.}$$

PROBLEM 2

a) Note that the inverse kinematic solution for this manipulator can not be obtained analytically if $a_3 \neq 0$. So, take $a_3 = 0$ and obtain the inverse kinematic solution following the guideline described below:

- First, determine s_4 using the fact that an equation such as $\bar{y} = \hat{R}\bar{x}$ implies that $\bar{y}^t\bar{y} = \bar{x}^t\bar{x}$ if $\hat{R}$ is an orthonormal matrix.
- Next, taking the relevant matrix to the other side of the equation, determine θ_1.
- Then, determine θ_2.
- Finally, determine the remining angles θ_5, θ_3, and θ_6.

b) Noting that s_4 can not be negative, show that there are only two multiplicities, which are associated with θ_1 and θ_5. Illustrate these multiplicities by drawing simple sketches.

c) Identify the singular configurations. Discuss what happens in these configurations. Illustrate these configurations also by drawing simple sketches.

SOLUTION

a) With $a_3 = 0$, the wrist point location equation simplifies further to

$$\bar{r} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_2} (\bar{u}_1 a_2 - s_4 \bar{u}_3).$$

Multiplying each side by its transpose leads to

$$\bar{r}^{\mathrm{t}}\bar{r} = (\bar{u}_1 a_2 - s_4 \bar{u}_3)^{\mathrm{t}} e^{\tilde{u}_2\theta_2} e^{-\tilde{u}_3\theta_1} e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_2} (\bar{u}_1 a_2 - s_4 \bar{u}_3),$$

$$\bar{r}^{\mathrm{t}}\bar{r} = (\bar{u}_1 a_2 - s_4 \bar{u}_3)^{\mathrm{t}} (\bar{u}_1 a_2 - s_4 \bar{u}_3) = a_2^2 + s_4^2.$$

Note that s_4 can not be negative. So,

$$s_4 = \sqrt{\bar{r}^{\mathrm{t}}\bar{r} - a_2^2} = \sqrt{r_1^2 + r_2^2 + r_3^2 - a_2^2}.$$

Having s_4 available, the wrist equation can be written as

$$e^{-\tilde{u}_3\theta_1}\bar{r} = e^{-\tilde{u}_2\theta_2} (\bar{u}_1 a_2 - s_4 \bar{u}_3).$$

Pre-multiplication by $\bar{u}_2^{\mathrm{t}}$ results in

$$\bar{u}_2^{\mathrm{t}} e^{-\tilde{u}_3\theta_1}\bar{r} = \bar{u}_2^{\mathrm{t}} e^{-\tilde{u}_2\theta_2} (\bar{u}_1 a_2 - s_4 \bar{u}_3) = \bar{u}_2^{\mathrm{t}} (\bar{u}_1 a_2 - s_4 \bar{u}_3) = 0,$$

$$\bar{u}_2^{\mathrm{t}} e^{-\tilde{u}_3\theta_1}\bar{r} = (\bar{u}_2^{\mathrm{t}} \cos\theta_1 - \bar{u}_1^{\mathrm{t}} \sin\theta_1)\bar{r} = r_2\cos\theta_1 - r_1\sin\theta_1 = 0,$$

$$\sigma_1 r_2 \cos\theta_1 = \sigma_1 r_1 \sin\theta_1, \quad \sigma_1 = \pm 1.$$

If $r_1^2 + r_2^2 > 0$, θ_1 is determined with two possible values as

$$\theta_1 = \mathrm{atan}_2(\sigma_1 r_2, \sigma_1 r_1).$$

With the selected value of θ_1, the wrist equation now leads to the following equation pair to determine θ_2 without any further ambiguity:

$$\bar{u}_1^{\mathrm{t}} e^{-\tilde{u}_2\theta_2} (\bar{u}_1 a_2 - s_4 \bar{u}_3) = \bar{u}_1^{\mathrm{t}} e^{-\tilde{u}_3\theta_1}\bar{r} \;\rightarrow\; a_2\cos\theta_2 + s_4\sin\theta_2 = r_1\cos\theta_1 + r_2\sin\theta_1,$$

$$\bar{u}_3^{\mathrm{t}} e^{-\tilde{u}_2\theta_2} (\bar{u}_1 a_2 - s_4 \bar{u}_3) = \bar{u}_3^{\mathrm{t}} e^{-\tilde{u}_3\theta_1}\bar{r} \;\rightarrow\; a_2\sin\theta_2 - s_4\cos\theta_2 = r_3.$$

For convenience, let $r_{12} = r_1\cos\theta_1 + r_2\sin\theta_1$. Substituting $\theta_1 = \mathrm{atan}_2(\sigma_1 r_2, \sigma_1 r_1)$, it can be shown that $r_{12} = \sigma_1\sqrt{r_1^2 + r_2^2} = \sigma_1 |r_{12}|$. Then, $\sin\theta_2$ and $\cos\theta_2$ can be found separately as

$$\sin\theta_2 = (\sigma_1 s_4 |r_{12}| + a_2 r_3)/(a_2^2 + s_4^2), \quad \cos\theta_2 = (\sigma_1 a_2 |r_{12}| - s_4 r_3)/(a_2^2 + s_4^2).$$

Hence,

$$\theta_2 = \mathrm{atan}_2(\sigma_1 s_4 |r_{12}| + a_2 r_3, \sigma_1 a_2 |r_{12}| - s_4 r_3).$$

To determine the remaining joint variables, the orientation equation can be written as

$$e^{\tilde{u}_3\theta_3} e^{\tilde{u}_1\theta_5} e^{-\tilde{u}_3\theta_6'} = e^{\tilde{u}_2\theta_2} e^{-\tilde{u}_3\theta_1} \hat{C} e^{-\tilde{u}_1\pi} = \hat{C}^* \text{ (known).}$$

From this matrix equation, the following scalar equations can be generated:

$$\cos\theta_5 = c_{33}^* \quad \rightarrow \quad \sin\theta_5 = \sigma_5\sqrt{1-c_{33}^{*2}}, \quad \sigma_5 = \pm 1;$$

$$\sin\theta_3 \sin\theta_5 = c_{13}^*, \quad \cos\theta_3 \sin\theta_5 = -c_{23}^*;$$

$$\sin\theta_6' \sin\theta_5 = -c_{31}^*, \quad \cos\theta_6' \sin\theta_5 = c_{32}^*.$$

The first equation pair gives θ_5 with two possible values as

$$\theta_5 = \operatorname{atan}_2(\sigma_5\sqrt{1-c_{33}^{*2}}, c_{33}^*).$$

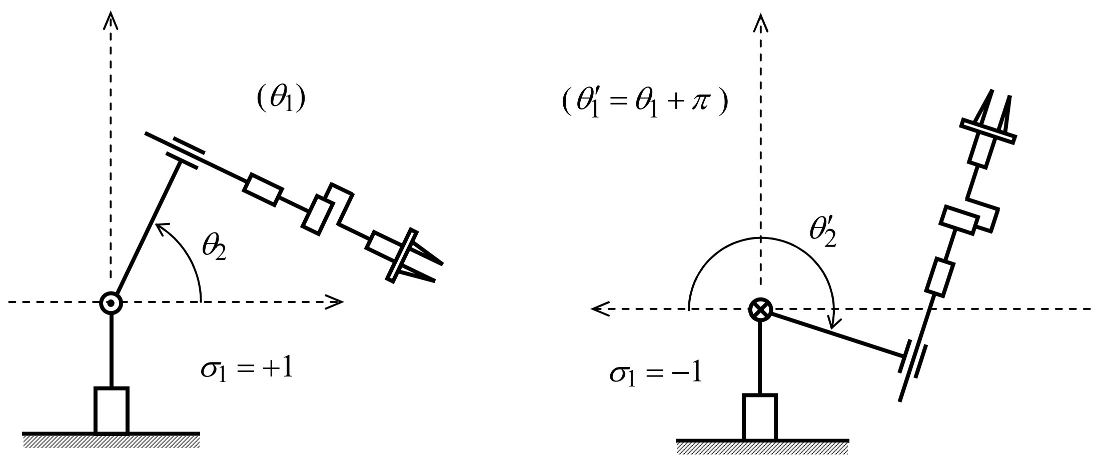
If $\sin\theta_5 \neq 0$, then the other angles are determined as

$$\theta_3 = \operatorname{atan}_2(\sigma_5 c_{13}^*, -\sigma_5 c_{23}^*);$$

$$\theta_6' = \operatorname{atan}_2(-\sigma_5 c_{31}^*, \sigma_5 c_{32}^*), \qquad \theta_6 = \theta_6' - \pi/2.$$

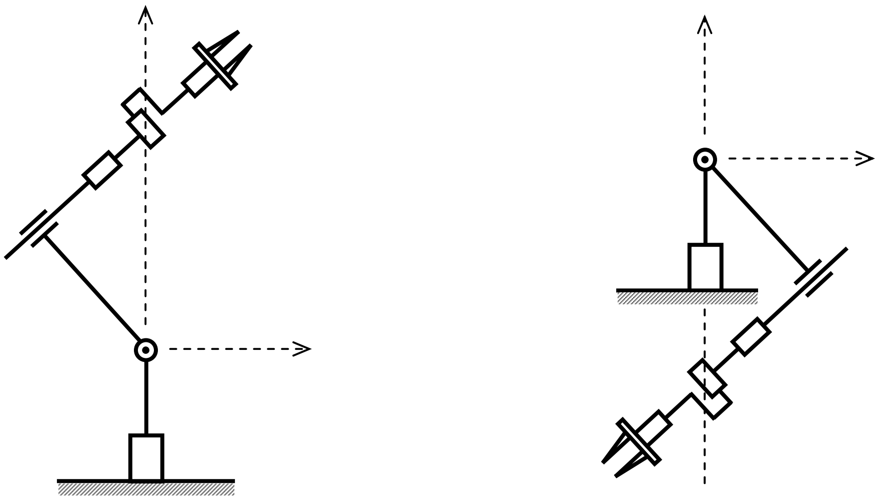
b) Optional Configurations Defined by σ_1 (with $a_3 = 0$):

Two possible configurations with the same wrist point location are illustrated below.



c) Singularity Analysis (with $a_3 = 0$):

(*i*) If $r_1 = r_2 = 0$, the first positional singularity occurs. Then, θ_1 can not be determined and therefore it can be selected arbitrarily. At this singularity, the wrist point happens to be located on the axis of the first joint as illustrated below:



(*ii*) If $\sin\theta_5 = 0$, i.e. if $\theta_5 = 0$ or $\theta_5 = \sigma_5\pi$ (which is not physically possible), the second positional singularity occurs. Then, θ_3 and θ_6 can not separately be determined because their axes become coincident. However, their difference can still be determined as follows, when $\theta_5 = 0$:

$$e^{\tilde{u}_3(\theta_3-\theta_6')} = \hat{C}^* \quad \rightarrow \quad \cos(\theta_3-\theta_6') = c_{11}^* = c_{22}^*, \quad \sin(\theta_3-\theta_6') = c_{21}^* = -c_{12}^*;$$

$$(\theta_3-\theta_6') = \mathrm{atan}_2(c_{21}^*, c_{11}^*) \quad \text{or} \quad (\theta_3-\theta_6) = \mathrm{atan}_2(c_{21}^*, c_{11}^*) + \pi/2.$$

Thus, one of θ_3 and θ_6 can be selected arbitrarily and the other can be determined from the preceding relationship.

PROBLEM 3

The figure on the next page shows the sketch of a Unimate Robot. Its joint arrangement is R^2PR^3. An additional auxiliary sketch is also shown.

a) Identify the unit vectors shown on the sketches and indicate all the joint variables clearly. Identify the constant link and joint parameters together with the link frame origins as well.

b) Show that the orientation of the hand and the location of the wrist point can be expressed as follows with respect to the base frame.

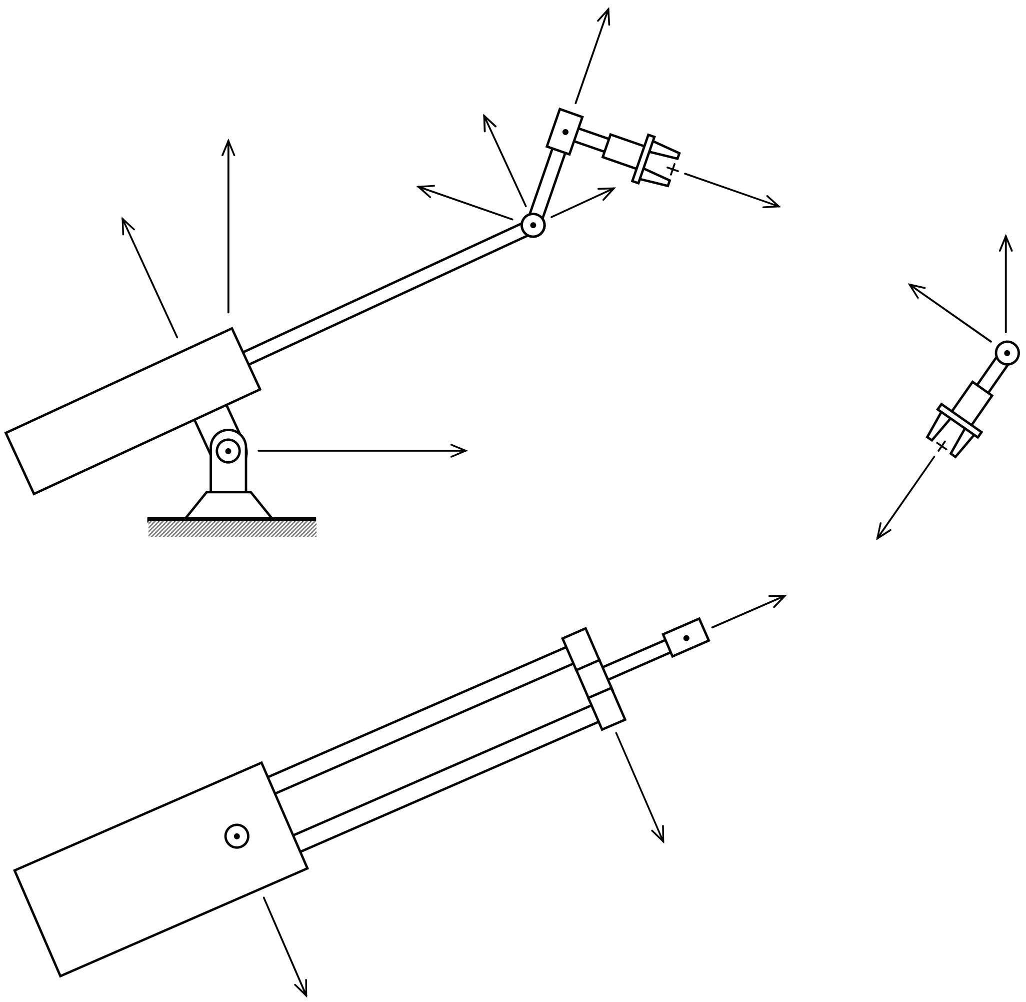
$$\hat{C} = \hat{C}^{(0,6)} = e^{\tilde{u}_3\theta_1} e^{-\tilde{u}_2\theta_{24}'} e^{\tilde{u}_1\theta_5} e^{-\tilde{u}_2\theta_6'} e^{\tilde{u}_1\pi/2},$$

$$\overline{r} = \overline{r}^{(0)} = e^{\tilde{u}_3\theta_1}[a_2 e^{-\tilde{u}_2\theta_2'}\overline{u}_3 + s_3 e^{-\tilde{u}_2\theta_2'}\overline{u}_1 + d_5 e^{-\tilde{u}_2\theta_{24}'}\overline{u}_1];$$

$$\theta_2' = \theta_2 - \pi/2, \quad \theta_{24}' = \theta_2' + \theta_4, \quad \theta_6' = \theta_6 + \pi/2.$$

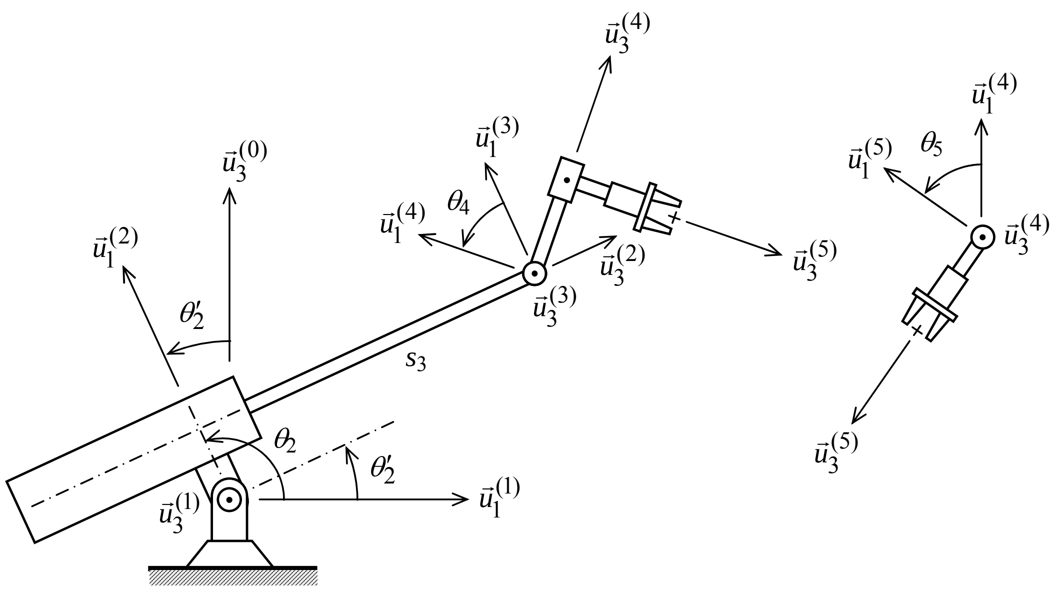
Note that θ_2' and θ_{24}' are more convenient angles than θ_2 and $\theta_{24} = \theta_2 + \theta_4$. Show θ_2' and θ_{24}' also on the sketches.

Sketches of a Unimate Robot



SOLUTION

a)



b) $\hat{C}=(e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\pi/2})(e^{\tilde{u}_3\theta_2}e^{\tilde{u}_1\pi/2})(e^{\tilde{u}_3 0}e^{-\tilde{u}_1\pi/2})(e^{\tilde{u}_3\theta_4}e^{\tilde{u}_1\pi/2})(e^{\tilde{u}_3\theta_5}e^{-\tilde{u}_1\pi/2})(e^{\tilde{u}_3\theta_6})$,

$$\hat{C}=e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\pi/2}e^{\tilde{u}_3\theta_{24}}e^{\tilde{u}_1\pi/2}e^{\tilde{u}_3\theta_5}e^{-\tilde{u}_1\pi/2}e^{\tilde{u}_3\theta_6},$$

$$\hat{C}=e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta_{24}}e^{\tilde{u}_1\pi}e^{\tilde{u}_3\theta_5}e^{-\tilde{u}_1\pi/2}e^{\tilde{u}_3\theta_6}=e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta_{24}}e^{-\tilde{u}_3\theta_5}e^{\tilde{u}_1\pi/2}e^{\tilde{u}_3\theta_6},$$

$$\hat{C}=e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta_{24}}e^{-\tilde{u}_3\theta_5}e^{-\tilde{u}_2\theta_6}e^{\tilde{u}_1\pi/2};$$

$$\hat{C}=e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta'_{24}}e^{-\tilde{u}_2\pi/2}e^{-\tilde{u}_3\theta_5}e^{-\tilde{u}_2\theta_6}e^{\tilde{u}_1\pi/2}=e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta'_{24}}e^{\tilde{u}_1\theta_5}e^{-\tilde{u}_2\pi/2}e^{-\tilde{u}_2\theta_6}e^{\tilde{u}_1\pi/2},$$

$$\hat{C}=e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta'_{24}}e^{\tilde{u}_1\theta_5}e^{-\tilde{u}_2\theta'_6}e^{\tilde{u}_1\pi/2}.$$

$$\vec{r}=a_2\vec{u}_1^{(2)}+s_3\vec{u}_3^{(2)}+d_5\vec{u}_3^{(4)}\quad\rightarrow\quad\overline{r}=\overline{r}^{(0)}=a_2\overline{u}_1^{(2/0)}+s_3\overline{u}_3^{(2/0)}+d_5\overline{u}_3^{(4/0)},$$

$$\overline{r}=a_2\hat{C}^{(0,2)}\overline{u}_1+s_3\hat{C}^{(0,2)}\overline{u}_3+d_5\hat{C}^{(0,4)}\overline{u}_3,$$

$$\overline{r}=a_2(e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\pi/2})(e^{\tilde{u}_3\theta_2}e^{\tilde{u}_1\pi/2})\overline{u}_1+s_3(e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\pi/2})(e^{\tilde{u}_3\theta_2}e^{\tilde{u}_1\pi/2})\overline{u}_3$$
$$+d_5(e^{\tilde{u}_3\theta_1}e^{\tilde{u}_1\pi/2})(e^{\tilde{u}_3\theta_2}e^{\tilde{u}_1\pi/2})(e^{\tilde{u}_3 0}e^{-\tilde{u}_1\pi/2})(e^{\tilde{u}_3\theta_4}e^{\tilde{u}_1\pi/2})\overline{u}_3,$$

$$\overline{r}=a_2e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta_2}\overline{u}_1-s_3e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta_2}\overline{u}_3-d_5e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta_{24}}\overline{u}_3,$$

$$\overline{r}=a_2e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta'_2}e^{-\tilde{u}_2\pi/2}\overline{u}_1-s_3e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta'_2}e^{-\tilde{u}_2\pi/2}\overline{u}_3-d_5e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta'_{24}}e^{-\tilde{u}_2\pi/2}\overline{u}_3,$$

$$\overline{r}=a_2e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta'_2}\overline{u}_3+s_3e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta'_2}\overline{u}_1+d_5e^{\tilde{u}_3\theta_1}e^{-\tilde{u}_2\theta'_{24}}\overline{u}_1,$$

$$\overline{r}=e^{\tilde{u}_3\theta_1}(a_2e^{-\tilde{u}_2\theta'_2}\overline{u}_3+s_3e^{-\tilde{u}_2\theta'_2}\overline{u}_1+d_5e^{-\tilde{u}_2\theta'_{24}}\overline{u}_1).$$

PROBLEM 4

Derive expressions to determine the joint variables corresponding to a specified orientation of the hand and a specified location of the tip point.

Indicate and discuss multiple solutions and postural singularities.

SOLUTION

Re-write the wrist location equation as

$$a_2e^{-\tilde{u}_2\theta'_2}\overline{u}_3+s_3e^{-\tilde{u}_2\theta'_2}\overline{u}_1+d_5e^{-\tilde{u}_2\theta'_{24}}\overline{u}_1=e^{-\tilde{u}_3\theta_1}\overline{r}.$$

Pre-multiply by $\overline{u}_2^t$:

$$0=\overline{u}_2^te^{-\tilde{u}_3\theta_1}\overline{r}=(\overline{u}_2^t\cos\theta_1-\overline{u}_1^t\sin\theta_1)\overline{r},$$

$$r_2\cos\theta_1-r_1\sin\theta_1=0\quad\rightarrow\quad\sigma_1r_2\cos\theta_1=\sigma_1r_1\sin\theta_1;\quad\sigma_1=\pm1.$$

If $r_1^2+r_2^2\neq0$,

$$\theta_1=\text{atan}_2(\sigma_1r_2,\sigma_1r_1).$$

If $r_1^2 + r_2^2 = 0$ or $r_1 = r_2 = 0$, the first singularity occurs and θ_1 becomes arbitrary.

With known θ_1, re-write the hand orientation equation as

$$e^{-\tilde{u}_2\theta'_{24}} e^{\tilde{u}_1\theta_5} e^{-\tilde{u}_2\theta'_6} = e^{-\tilde{u}_3\theta_1}\hat{C}e^{-\tilde{u}_1\pi/2} = \hat{C}^* .$$

Hence,

$$\bar{u}_2^t e^{\tilde{u}_1\theta_5}\bar{u}_2 = \cos\theta_5 = c_{22}^* \quad \to \quad \sin\theta_5 = \sigma_5\sqrt{1-c_{22}^{*2}}\,, \quad \sigma_5 = \pm 1\,;$$

$$\theta_5 = \operatorname{atan}_2(\sigma_5\sqrt{1-c_{22}^{*2}}\,, c_{22}^*)\,.$$

$$\bar{u}_3^t e^{-\tilde{u}_2\theta'_{24}} e^{\tilde{u}_1\theta_5}\bar{u}_2 = \cos\theta'_{24}\sin\theta_5 = c_{32}^*\,, \quad \bar{u}_1^t e^{-\tilde{u}_2\theta'_{24}} e^{\tilde{u}_1\theta_5}\bar{u}_2 = -\sin\theta'_{24}\sin\theta_5 = c_{12}^*\,;$$

$$\bar{u}_2^t e^{\tilde{u}_1\theta_5} e^{-\tilde{u}_2\theta'_6}\bar{u}_3 = -\sin\theta_5\cos\theta'_6 = c_{23}^*\,, \quad \bar{u}_2^t e^{\tilde{u}_1\theta_5} e^{-\tilde{u}_2\theta'_6}\bar{u}_1 = -\sin\theta_5\sin\theta'_6 = c_{21}^*\,.$$

If $\sin\theta_5 \neq 0$,

$$\theta'_{24} = \operatorname{atan}_2(-\sigma_5 c_{12}^*, \sigma_5 c_{32}^*)\,, \quad \theta'_6 = \operatorname{atan}_2(-\sigma_5 c_{21}^*, -\sigma_5 c_{23}^*)\,.$$

If $\sin\theta_5 = 0$, i.e. if $\theta_5 = 0$ or $\theta_5 = \sigma'_5\pi$ with $\sigma'_5 = \pm 1$, the second singularity occurs. In such a case, θ'_{24} and θ'_6 can't be determined separately. However, their combinations can be found as follows:

$$\theta_5 = 0 \ \to \ e^{-\tilde{u}_2(\theta'_{24}+\theta'_6)} = \hat{C}^* \ \to \ (\theta'_{24} + \theta'_6) = \operatorname{atan}_2(c_{31}^*, c_{11}^*)\,.$$

$$\theta_5 = \sigma'_5\pi \ \to \ e^{-\tilde{u}_2\theta'_{24}} e^{\tilde{u}_1\sigma_5\pi} e^{-\tilde{u}_2\theta'_6} = e^{\tilde{u}_2(\theta'_6 - \tilde{u}_2\theta'_{24})} e^{\tilde{u}_1\sigma_5\pi} = \hat{C}^*\,;$$

$$e^{\tilde{u}_2(\theta'_6 - \tilde{u}_2\theta'_{24})} = \hat{C}^* e^{-\tilde{u}_1\sigma_5\pi} = \hat{C}^{\#} \ \to \ (\theta'_6 - \theta'_{24}) = \operatorname{atan}_2(c_{13}^{\#}, c_{11}^{\#})\,.$$

With known θ_1 and θ'_{24}, re-write the wrist point equation as

$$a_2\bar{u}_3 + s_3\bar{u}_1 = e^{\tilde{u}_2\theta'_2}\bar{x}\,,$$

where

$\bar{x} = e^{-\tilde{u}_3\theta_1}\bar{r} - d_5 e^{-\tilde{u}_2\theta'_{24}}\bar{u}_1$, which is known.

Hence,

$$a_2 = \bar{u}_3^t e^{\tilde{u}_2\theta'_2}\bar{x} \quad \to \quad x_3\cos\theta'_2 - x_1\sin\theta'_2 = a_2\,.$$

$$s_3 = \bar{u}_1^t e^{\tilde{u}_2\theta'_2}\bar{x} \quad \to \quad s_3 = x_1\cos\theta'_2 + x_3\sin\theta'_2\,.$$

As noticed, once θ'_2 is found from the first equation, s_3 is obtained readily from the second equation. Find θ'_2 as follows:

$$x_3 = x_{31}\cos\gamma_2\,, \quad x_1 = x_{31}\sin\gamma_2 \quad \to \quad x_{31} = \sqrt{x_3^2 + x_1^2}\,, \quad \gamma_2 = \operatorname{atan}_2(x_1, x_3)\,;$$

$$\cos(\theta'_2 + \gamma_2) = \xi_2 = a_2 / x_{31} \quad \to \quad \sin(\theta'_2 + \gamma_2) = \sigma_2\sqrt{1-\xi_2^2}\,, \quad \sigma_2 = \pm 1\,;$$

$$\theta_2' = \mathrm{atan}_2(\sigma_2\sqrt{1-\xi_2^2}\,,\,\xi_2) - \gamma_2\,.$$

Then,

$$s_3 = x_{31}\sin\gamma_2\cos\theta_2' + x_{31}\cos\gamma_2\sin\theta_2' = x_{31}\sin(\theta_2'+\gamma_2) = \sigma_2 x_{31}\sqrt{1-\xi_2^2}\,,$$

$$s_3 = \sigma_2\sqrt{x_1^2+x_3^2-a_2^2} \quad\rightarrow\quad s_3 = \sqrt{x_1^2+x_3^2-a_2^2}\,.$$

Note that $s_3 > 0$. Therefore, $\sigma_2 = +1$ and this also removes the ambiguity in θ_2'. That is,

$$\theta_2' = \mathrm{atan}_2(\sqrt{1-\xi_2^2}\,,\,\xi_2) - \gamma_2\,.$$

Here is an alternative solution for the last two joint variables:

$$a_2\bar{u}_3 + s_3\bar{u}_1 = e^{\tilde{u}_2\theta_2'}\bar{x}\,.$$

Multiply both sides by their transposes:

$$(a_2\bar{u}_3 + s_3\bar{u}_1)^t(a_2\bar{u}_3 + s_3\bar{u}_1) = \bar{x}^t e^{-\tilde{u}_2\theta_2'}e^{\tilde{u}_2\theta_2'}\bar{x} = \bar{x}^t\bar{x} \quad\rightarrow\quad a_2^2 + s_3^2 = x_1^2 + x_2^2 + x_3^2\,.$$

Note that $\bar{u}_2^t(a_2\bar{u}_3 + s_3\bar{u}_1) = \bar{u}_2^t e^{\tilde{u}_2\theta_2'}\bar{x}$ or $0 = \bar{u}_2^t\bar{x} = x_2$. Therefore,

$$s_3 = \sqrt{x_1^2+x_3^2-a_2^2}\,,\quad s_3 > 0\,.$$

Then,

$$a_2 = \bar{u}_3^t e^{\tilde{u}_2\theta_2'}\bar{x} = x_3\cos\theta_2' - x_1\sin\theta_2',\quad s_3 = \bar{u}_1^t e^{\tilde{u}_2\theta_2'}\bar{x} = x_1\cos\theta_2' + x_3\sin\theta_2'\,.$$

Write these equations in matrix form and obtain the solution as follows:

$$\begin{bmatrix} x_3 & -x_1 \\ x_1 & x_3 \end{bmatrix}\begin{bmatrix} \cos\theta_2' \\ \sin\theta_2' \end{bmatrix} = \begin{bmatrix} a_2 \\ s_3 \end{bmatrix} \quad\rightarrow\quad \begin{bmatrix} \cos\theta_2' \\ \sin\theta_2' \end{bmatrix} = \frac{1}{x_3^2+x_1^2}\begin{bmatrix} x_3 & x_1 \\ -x_1 & x_3 \end{bmatrix}\begin{bmatrix} a_2 \\ s_3 \end{bmatrix}.$$

Note that $x_3^2 + x_1^2 > a_2^2 > 0$. So,

$$\cos\theta_2' = (x_3a_2 + x_1s_3)/(x_3^2+x_1^2),\quad \sin\theta_2' = (x_3s_3 - x_1a_2)/(x_3^2+x_1^2)\,;$$

$$\theta_2' = \mathrm{atan}_2[(x_3s_3 - x_1a_2),\,(x_3a_2 + x_1s_3)]\,.$$