SOLVED PROBLEMS
SET 1

PROBLEM 1

During a certain task, the approach vector of the gripper of the manipulator used is required to be oriented with respect to the base frame by the yaw (ϕ_1) and pitch (ϕ_2) angles as follows:

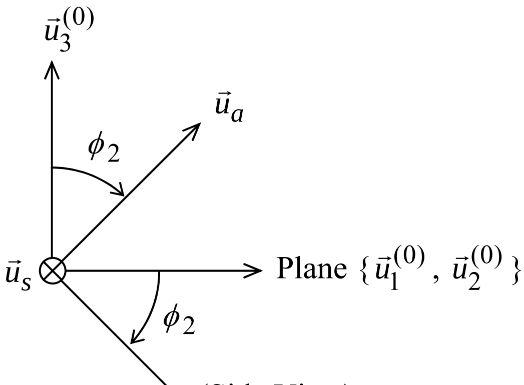
$$\vec{u}_a = \vec{u}_1^{(0)} \cos\phi_1 \sin\phi_2 + \vec{u}_2^{(0)} \sin\phi_1 \sin\phi_2 + \vec{u}_3^{(0)} \cos\phi_2 .$$

Meanwhile, the normal vector ($\vec{u}_n$) of the gripper is required to remain always in the vertical plane pointing downward.

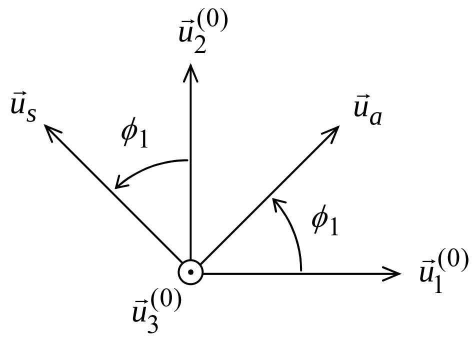
Determine the required orientation matrix $\hat{C} = \hat{C}^{(0,g)}$ of the gripper.

SOLUTION

The easiest way to illustrate the gripper frame unit basis vectors is to draw the side and top views as shown below:



(Side View)



(Top View)

Hence, it is seen that

$$\vec{u}_a = \vec{u}_1^{(0)} \cos\phi_1 \sin\phi_2 + \vec{u}_2^{(0)} \sin\phi_1 \sin\phi_2 + \vec{u}_3^{(0)} \cos\phi_2 \quad \text{(as given)},$$

$$\vec{u}_n = \vec{u}_a\big|_{\phi_2 \to \phi_2 + \pi/2} = \vec{u}_1^{(0)} \cos\phi_1 \cos\phi_2 + \vec{u}_2^{(0)} \sin\phi_1 \cos\phi_2 - \vec{u}_3^{(0)} \sin\phi_2 ,$$

$$\vec{u}_s = -\vec{u}_1^{(0)} \sin\phi_1 + \vec{u}_2^{(0)} \cos\phi_1 .$$

The column matrix representations of these vectors in the base frame are

$$\bar{u}_a = \bar{u}_a^{(0)} = \bar{u}_3^{(g/0)} = \bar{u}_1 \cos\phi_1 \sin\phi_2 + \bar{u}_2 \sin\phi_1 \sin\phi_2 + \bar{u}_3 \cos\phi_2 ,$$

$$\bar{u}_n = \bar{u}_n^{(0)} = \bar{u}_1^{(g/0)} = \bar{u}_1 \cos\phi_1 \cos\phi_2 + \bar{u}_2 \sin\phi_1 \cos\phi_2 - \bar{u}_3 \sin\phi_2 ,$$

$$\bar{u}_s = \bar{u}_s^{(0)} = \bar{u}_2^{(g/0)} = -\bar{u}_1 \sin\phi_1 + \bar{u}_2 \cos\phi_1 .$$

On the other hand,

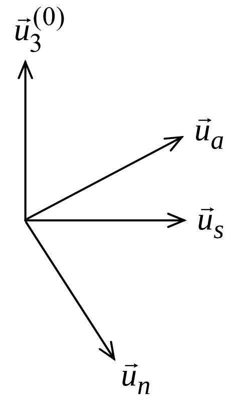
$$\hat{C} = \hat{C}^{(0,g)} = [\bar{u}_1^{(g/0)} \,\vdots\, \bar{u}_2^{(g/0)} \,\vdots\, \bar{u}_3^{(g/0)}] = [\bar{u}_n \,\vdots\, \bar{u}_s \,\vdots\, \bar{u}_a] .$$

Thus,

$$\hat{C} = \left[\begin{array}{c:c:c} \cos\phi_1 \cos\phi_2 & -\sin\phi_1 & \cos\phi_1 \sin\phi_2 \\ \hdashline \sin\phi_1 \cos\phi_2 & \cos\phi_1 & \sin\phi_1 \sin\phi_2 \\ \hdashline -\sin\phi_2 & 0 & \cos\phi_2 \end{array}\right] .$$

PROBLEM 2

During a task performed by a robotic manipulator, its end-effector is required to be oriented with respect to the base frame as described below.



The approach vector is oriented by two angles as

$$\vec{u}_a = \vec{u}_1^{(0)} \cos\alpha \cos\beta + \vec{u}_2^{(0)} \sin\alpha \cos\beta + \vec{u}_3^{(0)} \sin\beta ,$$

where α is the azimuth angle and β is the elevation angle.

The side vector is required to remain horizontal so that

$$\vec{u}_s = \vec{u}_1^{(0)} \cos\gamma + \vec{u}_2^{(0)} \sin\gamma .$$

The normal vector ($\vec{u}_n$) is required to be pointing downwardly.

a) For specified values of α and β, determine the vectors $\vec{u}_s$ (i.e. γ) and $\vec{u}_n$ without any sign ambiguity.

b) At an instant, the angles of the approach vector are specified as $\alpha = 50^\circ$ and $\beta = 120^\circ$. Determine the 1-2-3 Euler angles for the end-effector with ϕ_2 being an *acute* angle.

SOLUTION

a) Determination of $\vec{u}_s$ and $\vec{u}_n$:

$$\vec{u}_a = \vec{u}_1^{(0)} \cos\alpha \cos\beta + \vec{u}_2^{(0)} \sin\alpha \cos\beta + \vec{u}_3^{(0)} \sin\beta ,$$

$$\vec{u}_s = \vec{u}_1^{(0)} \cos\gamma + \vec{u}_2^{(0)} \sin\gamma .$$

$$\vec{u}_a \cdot \vec{u}_s = 0 \quad \rightarrow \quad \cos\alpha \cos\beta \cos\gamma + \sin\alpha \cos\beta \sin\gamma = 0 .$$

If $\cos\beta \neq 0$,

$$\cos\alpha \cos\gamma + \sin\alpha \sin\gamma = \cos(\gamma - \alpha) = 0 \quad \rightarrow \quad \gamma = \alpha + \sigma\pi/2 , \quad \sigma = \pm 1 .$$

As for σ, it can be determined using the condition on the normal vector:

$$\vec{u}_n = \vec{u}_s \times \vec{u}_a ,$$

$$\vec{u}_n = \vec{u}_3^{(0)} (\sin\alpha \cos\beta \cos\gamma - \cos\alpha \cos\beta \sin\gamma) - \vec{u}_2^{(0)} (\sin\beta \cos\gamma) + \vec{u}_1^{(0)} (\sin\beta \sin\gamma) .$$

This vector will be pointing downwardly if $\vec{u}_n \cdot \vec{u}_3^{(0)} < 0$. That is,

$$\sin\alpha\cos\beta\cos\gamma - \cos\alpha\cos\beta\sin\gamma < 0 \quad \rightarrow \quad \sin(\gamma-\alpha)\cos\beta > 0;$$

$$\sin(\sigma\pi/2)\cos\beta > 0 \quad \rightarrow \quad \sigma\cos\beta > 0 \quad \rightarrow \quad \sigma = \operatorname{sgn}(\cos\beta).$$

b) Determination of the 1-2-3 Euler angles:

For $\alpha = 50^\circ$ and $\beta = 120^\circ$, as found above, $\sigma = -1$ and $\gamma = 50^\circ - 90^\circ = -40^\circ$. Then,

$$\vec{u}_a = \vec{u}_1^{(0)}(-0.3214) + \vec{u}_2^{(0)}(-0.383) + \vec{u}_3^{(0)}(0.866),$$

$$\vec{u}_s = \vec{u}_1^{(0)}(0.766) + \vec{u}_2^{(0)}(-0.6428),$$

$$\vec{u}_n = \vec{u}_1^{(0)}(-0.5567) - \vec{u}_2^{(0)}(0.6634) - \vec{u}_3^{(0)}(0.5).$$

Hence,

$$\hat{C} = \begin{bmatrix} \bar{u}_n^{(0)} & \bar{u}_s^{(0)} & \bar{u}_a^{(0)} \end{bmatrix} = \begin{bmatrix} -0.5567 & 0.766 & -0.3214 \\ -0.6634 & -0.6428 & -0.383 \\ -0.5 & 0 & 0.866 \end{bmatrix} = e^{\tilde{u}_1\phi_1} e^{\tilde{u}_2\phi_2} e^{\tilde{u}_3\phi_3}.$$

$$c_{13} = -0.3214 = \bar{u}_1^t e^{\tilde{u}_2\phi_2}\bar{u}_3 = s\phi_2 \quad \rightarrow \quad \phi_2 = -18.75^\circ \text{ (an acute angle)}, \quad c\phi_2 > 0.$$

$$c_{23} = -0.383 = \bar{u}_2^t e^{\tilde{u}_1\phi_1} e^{\tilde{u}_2\phi_2}\bar{u}_3 = -s\phi_1 c\phi_2, \quad c_{33} = 0.866 = \bar{u}_3^t e^{\tilde{u}_1\phi_1} e^{\tilde{u}_2\phi_2}\bar{u}_3 = c\phi_1 c\phi_2;$$

$$\phi_1 = \operatorname{atan}_2(0.383,\ 0.866) = 23.86^\circ.$$

$$c_{12} = 0.766 = \bar{u}_1^t e^{\tilde{u}_2\phi_2} e^{\tilde{u}_3\phi_3}\bar{u}_2 = -c\phi_2 s\phi_3, \quad c_{11} = -0.5567 = \bar{u}_1^t e^{\tilde{u}_2\phi_2} e^{\tilde{u}_3\phi_3}\bar{u}_1 = c\phi_2 c\phi_3;$$

$$\phi_3 = \operatorname{atan}_2(-0.766,\ -0.5567) = -126.01^\circ.$$