TKMD Summer School on Algebraic Topology / METU / Math. Dept. / June 20-24, 2016

(Based on the book Algebraic Topology by Allen Hatcher.)

Introduction:

1) Definition of topology: Open closed, closure, interior.

2) Examples

- Some finite topologies
- $\mathbb{R}_{std}$, $\mathbb{R}^n_{std}$
- Real line with double origin

3) Equivalence of topologies:

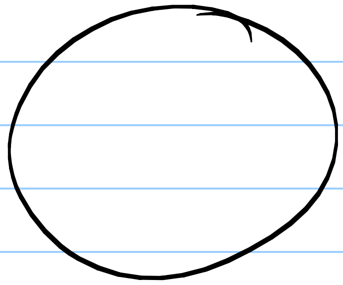
- Continuity
- Homeomorphism
- Homeomorphic spaces will be regarded the same

4) Basis, subbasis of a topology. Example $\mathbb{R}^n_{std}$.

5) Strong / Weak Topology

6) New topologies from old ones.

- Subspace topology



$S^1 \subseteq \mathbb{R}^2$

- Topological Embedding

<u>Example</u>: $X = [0, \infty)$

$$\mathcal{B} = \{(a,b) \mid 0 < a < b\}$$
$$\cup \{[0,a) \cup (b,\infty) \mid a, b > 0\}.$$

Let τ be the topology on X generated by $\mathcal{B}$. Then

$f : [0,\infty) \longrightarrow S^1$, $f(t) = e^{2\pi i t/1+t}$

is a homeomorphism.

<u>Example</u> $X = \mathbb{R}$

$\mathcal{B}' = \{(a,b) \mid ab > 0\}$
$\cup \{(a,b) \cup (c_1, \infty) \cup (-\infty, c_2) \mid$
$a < 0 < b\}$

If τ' is the topology on $\mathbb{R}$ generated by $\mathcal{B}'$ then (X, τ') is homeomorphic to the figure 8 in the plane.

- <u>Product Topology</u>
  - (X_α, τ_α) $\alpha \in \Lambda$ a family of spaces

$$\prod_\alpha X_\alpha \doteq \{f : \Lambda \to \dot{\bigcup} X_\alpha \mid f(\alpha) \in X_\alpha\}.$$

(Axiom of Choice !!)

- Box Topology
- Comparison of the two topologies.

<u>Proposition</u>: A function

$f: X \to \prod_\alpha X_\alpha$ is continuous

if and only if each

$f_\alpha : X \to X_\alpha$ is continuous.

<u>Remark</u> This is <u>not</u> true

if we put the box topology on $\prod_\alpha X_\alpha$.

<u>Example</u> $f: \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$

$t \mapsto (t, t, t, \ldots)$

- <u>Quotient Topology</u>

- Equivalence relations on a set X

$\updownarrow$ 1-1 correspondence

Surjective functions $\pi : X \to Y$.

- If $\sim$ is an equivalence relation on X and $\pi : X \to X/\sim$ $f(x) = [x]$ then π is called a quotient map.

Let $f : X \to Z$ be another map.

<u>Question:</u> Is there a map $g : X/\sim \to Z$ s.t. the diagram below is commutative

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ X/\sim \arrow[r, "g"'] & Z \end{tikzcd}$$

?

<u>Answer</u>: There is such g if and only if f is constant on the fibers of π: If $[x]=[y]$ for some $x, y \in X$ then $f(x)=f(y)$.

Moreover, in this case g is unique!

- Now let τ be a topology on X. Then there is a unique topology on $X/\sim$ so that whenever $f: X \to Z$ is a continuous map which is constant on the fibers of $\pi : X \to X/\sim$ then g is continuous. This topology is given by:

$U \subseteq X/\sim$ is open if and only if $\pi^{-1}(U)$ is open.

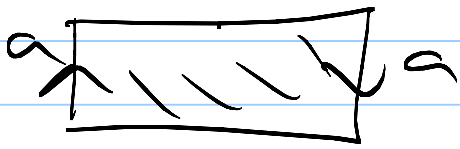
- This is the weakest topology on $X/\sim$ making $\pi : X \to X/\sim$ continuous.

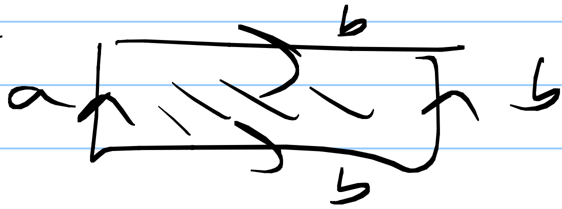
<u>Example</u>: $\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / x \sim \lambda x$

$x \in \mathbb{R}^{n+1} \setminus \{0\}$, $\lambda \in \mathbb{R} \setminus \{0\}$.

$\mathbb{CP}^n = \mathbb{C}^{n+1} \setminus \{0\} / x \sim \lambda x$

$x \in \mathbb{C}^{n+1} \setminus \{0\}$, $\lambda \in \mathbb{C} \setminus \{0\}$.

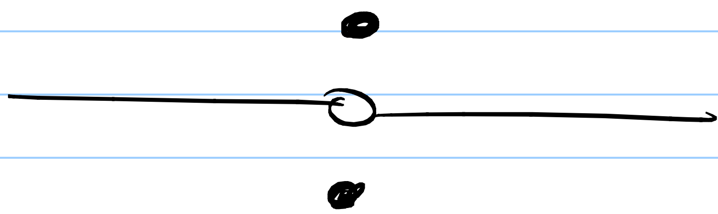
Example 1) MB: 

2) T^2: 

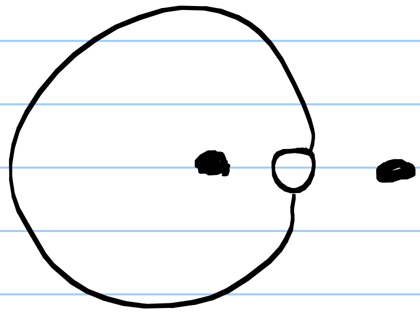
3) $\mathbb{R}P^2$: 

7) Separation Axioms:
T_0, T_1, T_2, T_3 and T_4.

Examples and counter examples.

Example 

and

X: 

$f: S^1 \longrightarrow X$ degree two map
(local homeomorphism!)

8) Compactness:

- General definition
- Sequential compactness for metric spaces.
- $\mathbb{R}, \mathbb{R}^n$

Theorem (Heine–Borel)
A subspace X of $\mathbb{R}^n$ is compact if and only if A is closed and bounded.

- Product of compact spaces are compact: Tychonoff Theorem

Theorem (Homeomorphism)
A continuous bijection $f: X \to Y$ is a homeomorphism provided that the spaces are Hausdorff and X is compact.

9) Connectedness:

- Definition
- Arc connectedness

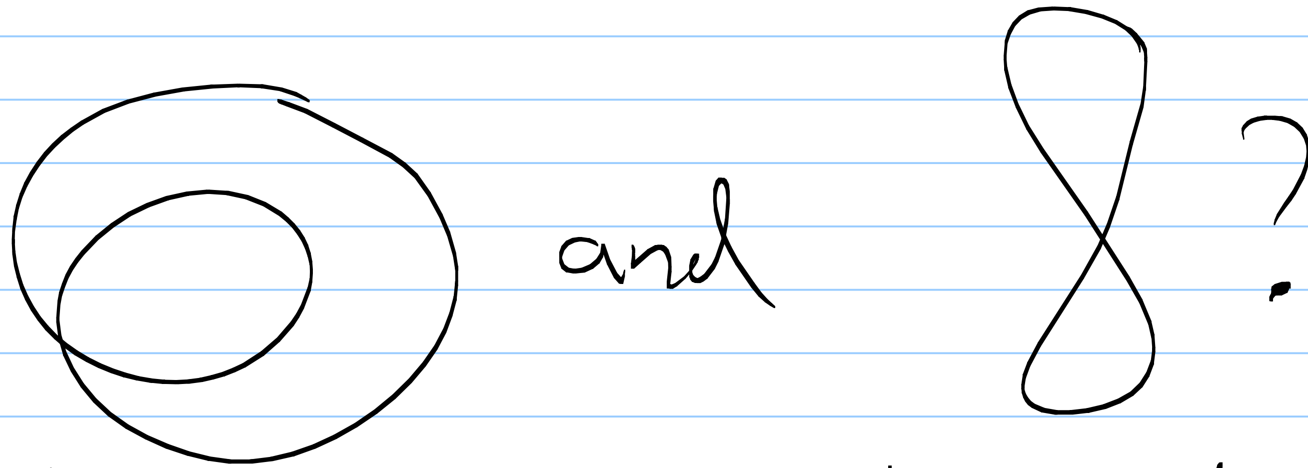
Example Topologist Sine curve

Example: An embedding of $\mathbb{RP}^2$ into $\mathbb{R}^5 / \mathbb{R}^4$.

(10) Why algebraic Topology?

- Comparing topological objects is just to difficult.

Example How to distinguish



They are different embeddings of $S^1 \vee S^1$ = figure eight.

- Algebraic objects are easier to compare, like numbers. The first has rotation number 2 and the second has rotation number zero.

In general topological classification is too difficult, so we use a coarser classification: Homotopy equivalence:

11) • Definition of Homotopy of maps / Relative homotopy.

- Homotopy equivalence
- Retraction $r: X \Rightarrow A$
- Deformation Retraction of a space X onto a subspace A

$f: X \times I \Rightarrow X$ cont. map s.t.

1) $f(x,0) = x, \forall x \in X$
2) $f(a,t) = a, \forall a \in A, \forall t \in [0,1]$.
3) $f(x,1) \in A, \forall x \in X$.

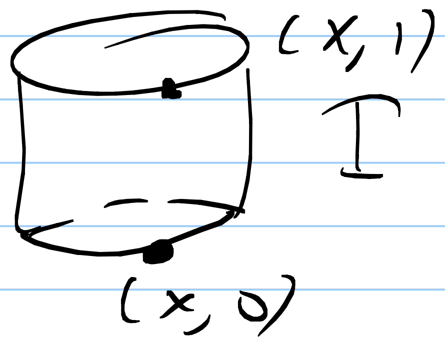
- Contractibility: $1_X \simeq *$.

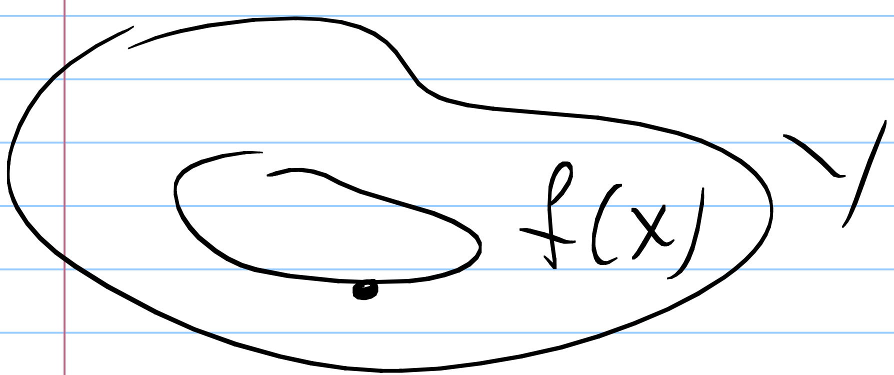
Remark: X deformation retracts onto a subspace A then X and A are homotopy equivalent.

12) Mapping Cylinder

$f: X \to Y$ any map

$$M_f = X \times I \,\dot{\cup}\, Y \Big/ _{(x,0) \sim f(x)}$$





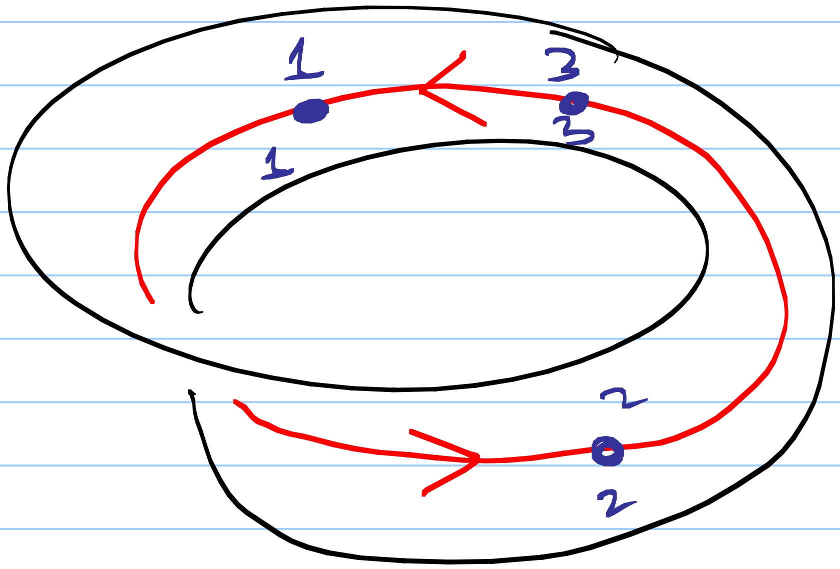
Remark

M_f deformation retracts onto Y.

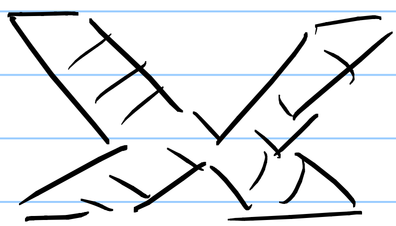
Example $f : S^1 \to S^1$
$z \mapsto z^2$

$M_f = MB$.

Proof: Cut through a MB along the center circle.

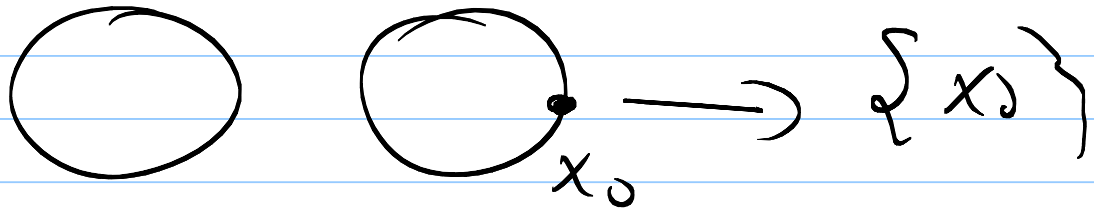


Remarks 1) Not all deformation retractions are obtained from mapping cylinders.

 $\to X \to \bullet = A$

Certain pairs of points follow paths that merge before they reach final destination. This does not happen in a mapping cylinder.

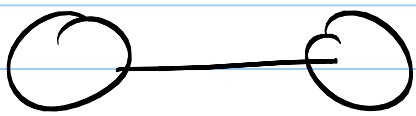
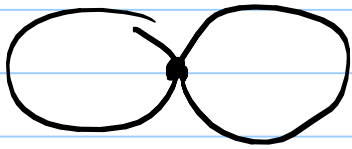
2) Not all retractions come from deformation retractions.

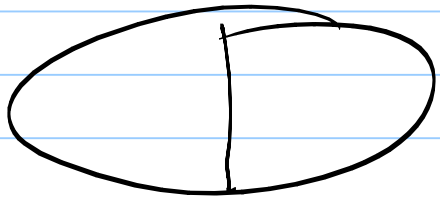
X: 

is a retraction. However, if $f: X \times I \to X$ is a deformation retraction onto $A = \{x_0\}$ then X must be path connected!

3) Homotopy equivalence is

an equivalence relation but deformation retraction is not:

,  and

 are all deformation retractions of the same space $\mathbb{R}^2 \setminus \{(\pm 1, 0)\}$ but they are <u>not</u> deformation retractions of each other. (Exercise!)

<u>Corollary 0.21</u> Two spaces X and Y are homotopy equivalent if and only if there is a third space Z which deformation retracts onto X and Y.

13) Cell complexes: We build a space inductively as follows:

1) Start with a discrete set of points X^0, whose elements are called 0-cells.

2) Form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps

$$\varphi_\alpha : S^{n-1} = \partial D^n \to X^{n-1}.$$

Hence, $X^n = X^{n-1} \coprod_\alpha D^n_\alpha \Big/ \underset{x \in \partial D^n}{x \sim \varphi_\alpha(x)}$

where e^n_α is $\mathrm{Int}(D^n_\alpha)$.

3) One can stop at some stage n and let $X = X^n$ or continue indefinitely, setting

$X = \bigcup_n X^n$. In this case, a subset $A \subset X$ is called open (closed) if and only if $X^n \cap A$ is open (resp. closed in X^n).

Such a space is called a CW-complex.

C: closure finite

W: weak topology

If $X = X^n$ then we say that X has dimension n.

<u>Examples</u> S^n, T^2, KB, S^∞,

1-dim'l CW complex = Graph

For any n-cell e^n_α the map $\varphi_\alpha : D^n_\alpha \to X^n \to X$ is called the characteristic map. The map $X^n \to X$ is continuous since it has the weak topology. The map $D^n_\alpha \to X^n$ is continuous since it is the union of an inclusion via a quotient map.

<u>Remark</u> Any CW-complex is the disjoint union of its cells.

<u>Definition</u> 1) If each characteristic map $\varphi_\alpha : D^n_\alpha \to X$ is an embedding then the

complex D called a regular complex.

2) Simplicial complex when each D^n_α is Δ^n and gluing maps are linear homeomorphisms.

14) Euler characteristic of a finite cell complex.

15) <u>Operation on Spaces</u>

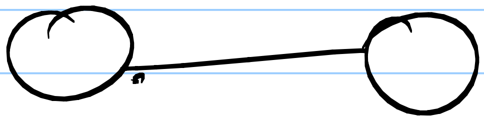
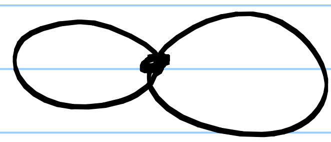
- Product of cell complexes
- Quotients $A \subseteq X$ subcomplex. then X/A has a complex structure.
- Suspension
- Cone on X
- Join of X and Y : $X * Y$

- Wedge $X \vee Y$
- Smash product: $X \times Y / X \vee Y$

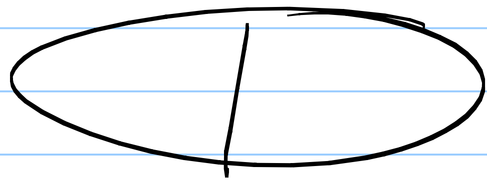
$S^1 \times S^1 / S^1 \vee S^1 \simeq S^2$

$S^n \times S^m / S^n \vee S^m \simeq S^{n+m}$

16) <u>Theorem</u> If (X,A) is a CW pair where A is contractible that $X \longrightarrow X/A$ is a homotopy equivalence.

<u>Example</u>  $\underset{h.e.}{\simeq}$ 

is h.e.



<u>Example</u> Any connected finite graph is homotopy

equivalent to $\bigvee_n S^1$ for some n.

<u>Theorem</u> If (X_1, A) is a CW pair and two attaching maps $f, g : A \to X_0$ are homotopic then $X_0 \coprod_f X_1$ homotopy equivalent to $X_0 \coprod_g X_1$.

<u>Exercises</u>

1) $S^\infty = \lim_n S^n$, $i_n : S^n \to S^{n+1}$
$x \mapsto (0, x)$

$= \{ (x_1, x_2, \dots, x_n, \dots) \mid x_i \in \mathbb{R}$
$x_i = 0$ for all but finitely many i
and $\sum_i x_i^2 = 1 \}$

Show that S^∞ is contractible.

Solution: Let $T: S^\infty \to S^\infty$

$T((x_1, x_2, \dots, x_n, \dots)) = (0, x_1, x_2, \dots, x_n, \dots)$

Let $F_t(x) = \dfrac{(1-t)x + tT(x)}{\|(1-t)x + tT(x)\|}$, $t \in [0,1]$.

$F_0 = \mathrm{Id}_{S^\infty}$ and $F_1(x) = T(x)$.

$F: S^\infty \times [0,1] \to S^\infty$, $F(x,t) = F_t(x)$.

Also, let $S_0^\infty = \{(x_i) \in S^n \mid x_1 = 0\}$

and $G: S_0^\infty \times [0,1] \to S^\infty$ by

$$G(x,t) = \frac{t(1,0,0,\dots) + (1-t)x}{\|t(1,0,0,\dots) + (1-t)x\|}.$$

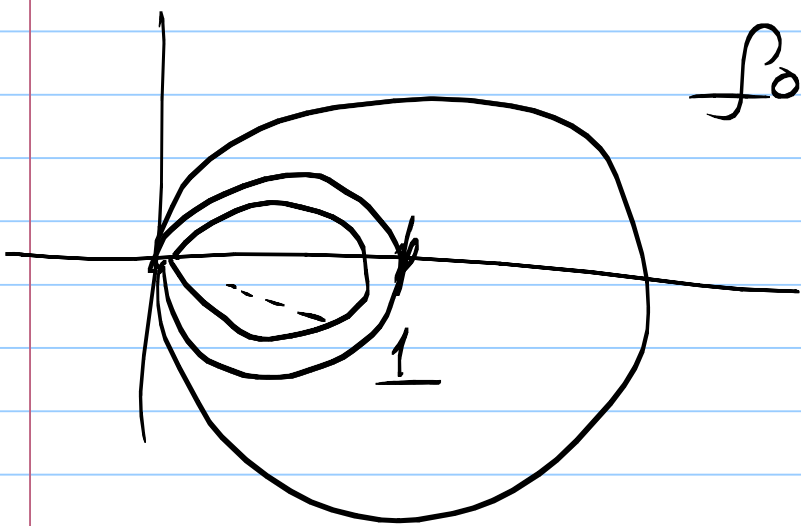
$G(x,0) = x$ and $G(x,1) = (1,0,0,\dots)$.

Finally, let $H: S^\infty \times [0,1] \to S^\infty$

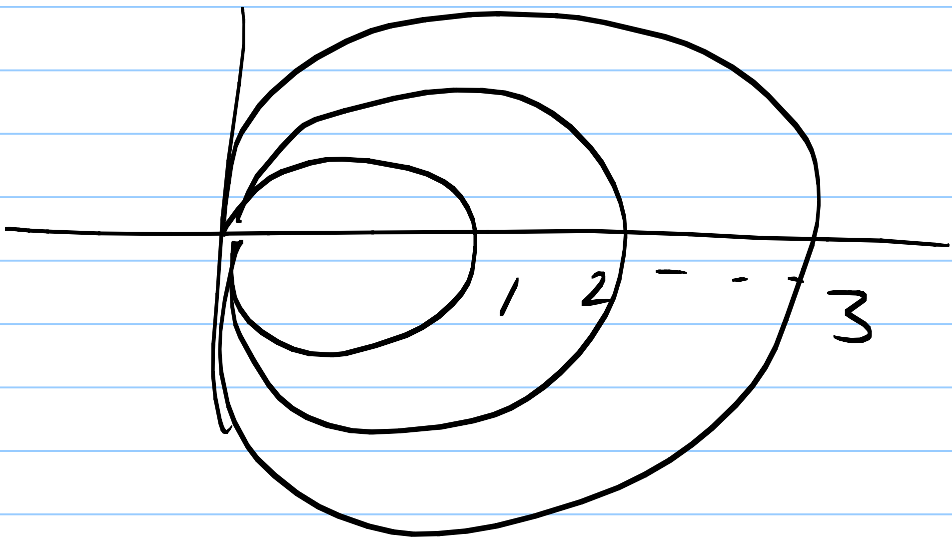
$$H(x,t) = \begin{cases} F(x,2t), & 0 \le t \le 1/2 \\ G(x, 2t-1), & 1/2 \le t \le 1. \end{cases}$$

2) $X_1 = \bigvee_{i=1}^{\infty} S^1$

$X_2 = \{(x,y) \in \mathbb{R}^2 \mid (x-\frac{1}{n})^2 + y^2 = \frac{1}{n^2}$ for some $n \in \mathbb{Z}^+\}$

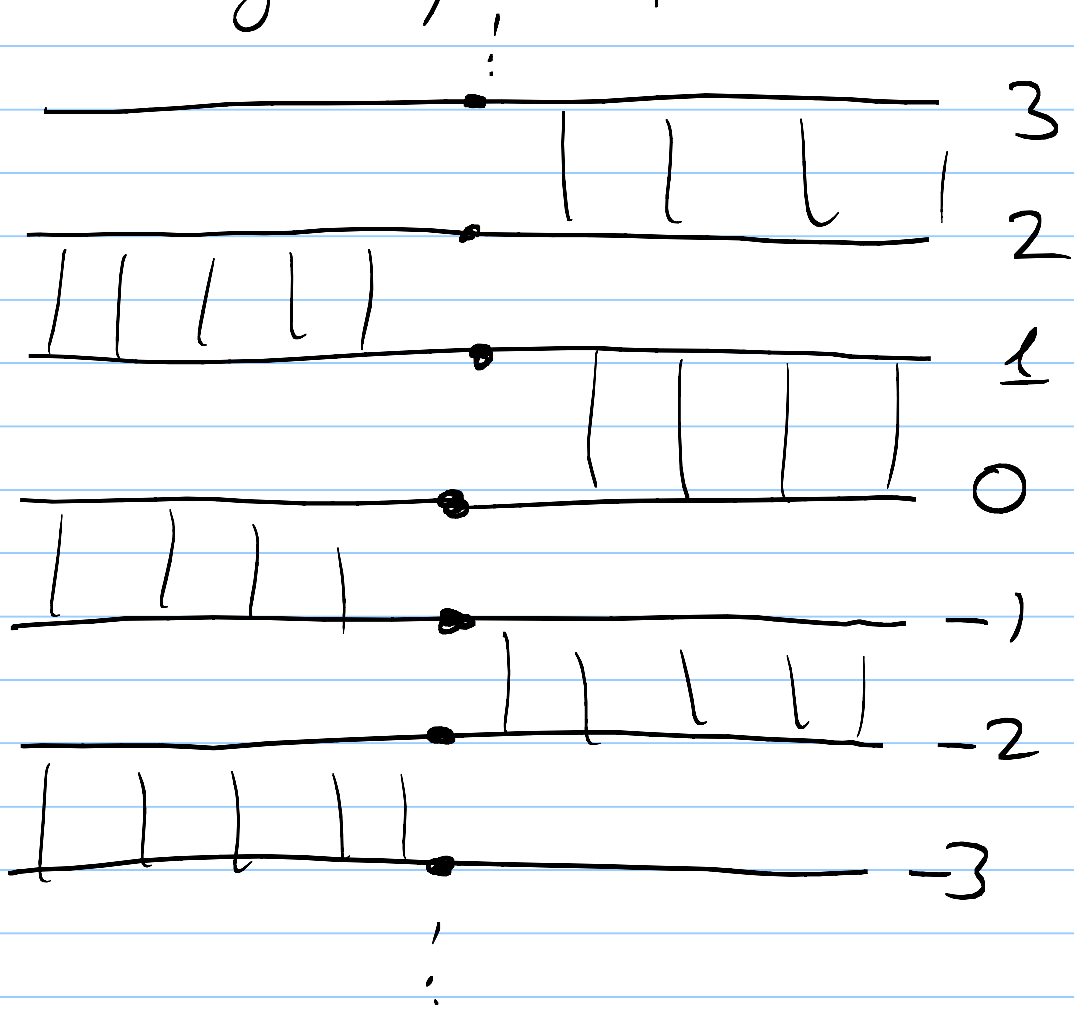


$X_3 = \{(x,y) \in \mathbb{R}^2 \mid (x-n)^2 + y^2 = n^2$, for some $n \in \mathbb{Z}^+\}$.



Is X_i homeomorphic to X_j?

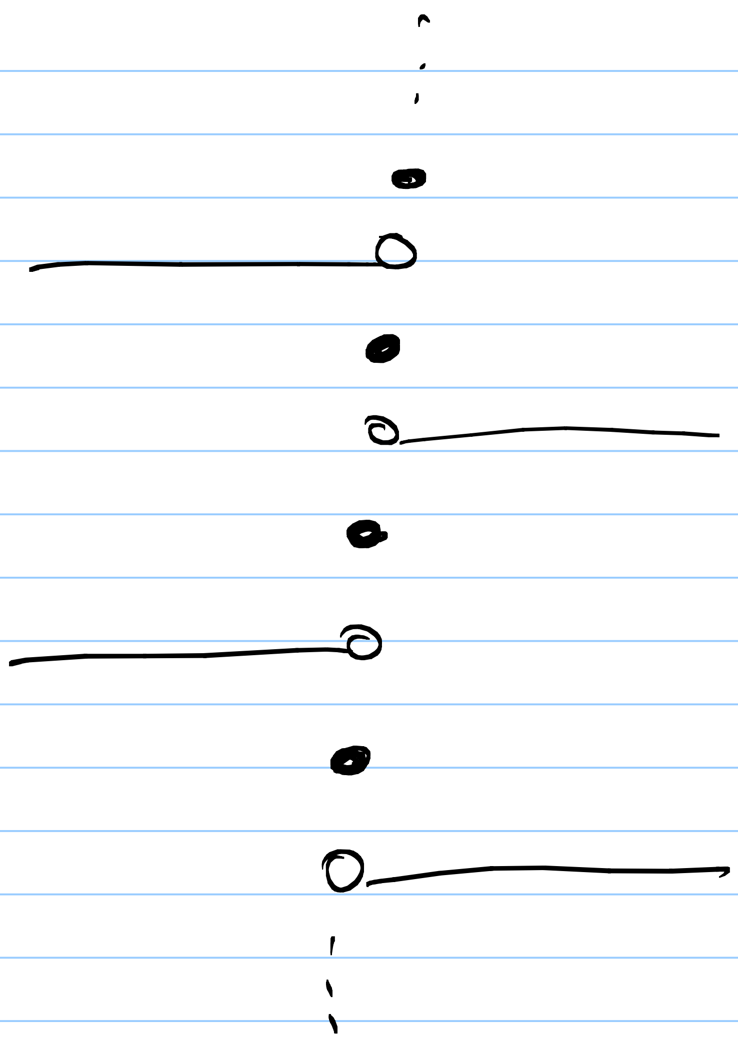
3) Let $X = \mathbb{R} \times \mathbb{Z} / \sim$ be the following space:



$(x,n) \sim (x,n+1)$ for $x > 0$ and n even

$(x,n) \sim (x,n+1)$ for $x < 0$ and n odd.

X:



Show that if K is a compact space and $f: K \to X$ is a continuous map then f is homotopic to a constant map.

<u>Hint</u>: Note that each $\mathbb{R} \times \{n\}$ is open in $\mathbb{R} \times \mathbb{Z}$ and thus its image is open in X.

The Fundamental Group

1) It is a functor from the category of topological spaces with based points to the category of groups:

$(X, x_0) \longmapsto \pi_1(X, x_0)$.

$x_0 \in X$

<u>Definition</u>: Given a based space (X, x_0) let $\mathcal{L}$ be the set of based loops at x_0:

$\mathcal{L} = \{ \gamma : [0,1] \to X \text{ cont}, \gamma(0) = x_0, \gamma(1) = x_0 \}$

Define the homotopy relation on $\mathcal{L}$: $\alpha, \beta \in \mathcal{L}$, $\alpha \sim \beta$ if and only if there is an homotopy $F : [0,1] \times [0,1] \to X$

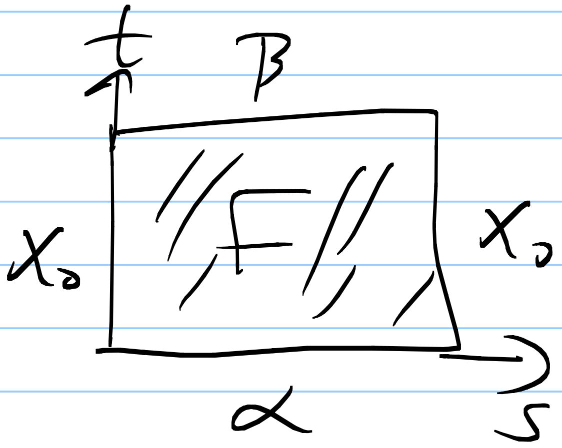
such that for all $0 \leq s, t \leq 1$,

i) $F(s, 0) = \alpha(s)$

ii) $F(s, 1) = \beta(s)$

iii) $F(0, t) = x_0$

iv) $F(1, t) = x_0$



Define $\pi_1(X, x_0)$ as the set of equivalence classes $\Omega/\sim$.

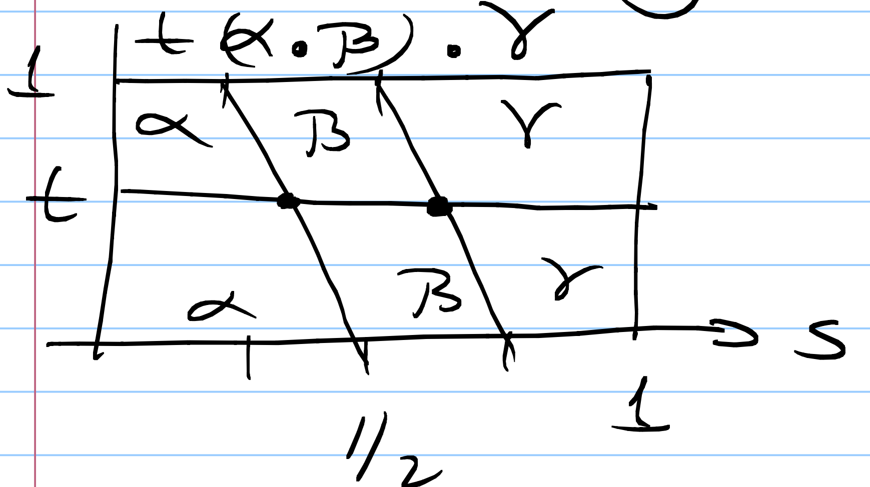
Group structure on π_1:

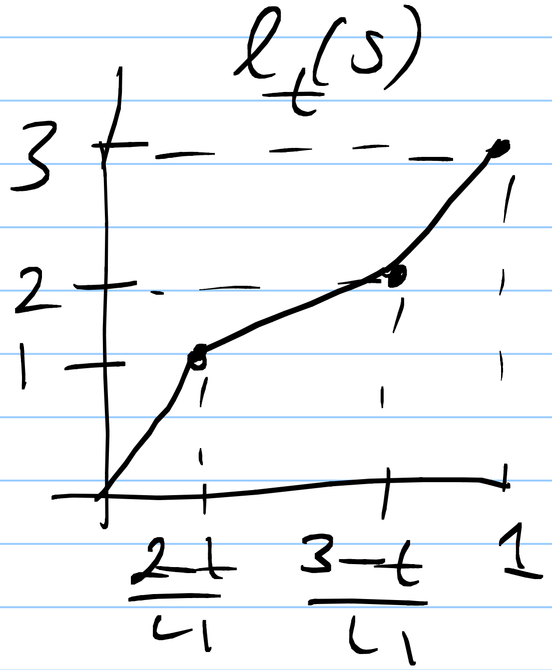
For $\alpha, \beta \in \Omega$ let $\alpha \cdot \beta$ denote the product loop

$$(\alpha \cdot \beta)(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq 1/2 \\ \beta(2s-1), & 1/2 \leq s \leq 1. \end{cases}$$

This descends to a multiplication on $\pi_1(X, x_0)$ which makes π_1 a group with the identity element $[e]$, where $e:[0,1]\to X$, $e(s)=x_0$, the constant loop.

Associativity:





$\alpha\cdot(\beta\cdot\gamma)$

Let $l_t(s) = \begin{cases} \dfrac{4s}{2-t}, & 0 \le s \le \dfrac{2-t}{4} \\ 4s+t-1, & \dfrac{2-t}{4} \le s \le \dfrac{3-t}{4} \\ \dfrac{4s+3t-1}{1+t}, & \dfrac{3-t}{4} \le s \le 1 \end{cases}$

$$H(s,t)=\begin{cases}\alpha(\ell_t(s)) & 0\le s\le \frac{2-t}{4}\\ \beta(\ell_t(s)-1) & \frac{2-t}{4}\le s\le \frac{3-t}{4}\\ \gamma(\ell_t(s)-2) & \frac{3-t}{4}\le s\le 1.\end{cases}$$

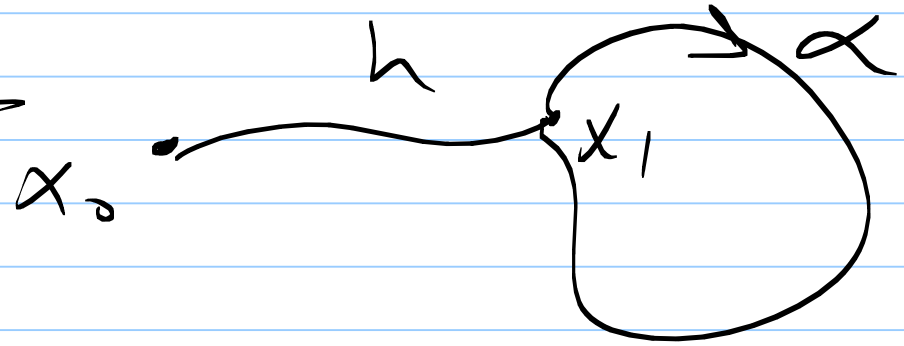
Example For a convex subset X of $\mathbb{R}^n_{std}$ and any $x_0\in X$ $\pi_1(X,x_0)=(e)$.

Proposition: Let X be a space $x_0,x_1\in X$ points and $h:[0,1]\to X$ is a path joining x_0 to x_1: $h(0)=x_0$, $h(1)=x_1$. Then the map $\beta_h:\pi_1(X,x_1)\to\pi_1(X,x_0)$ defined by

$\beta_h([\alpha])=[h\cdot\alpha\cdot\bar{h}]$ is an

Is an Isomorphism.

<u>Proof</u>



<u>Definition</u>: A path connected space X is called simply connected if $\pi_1(X, x_0) = \{e\}$ for some (and thus all) $x_0 \in X$.

<u>Proposition</u>: A space X is simply connected if and only if for any two points $x_0 \in X$ and $x_1 \in X$ there is a unique

homotopy class of paths joining x_0 to x_1, when homotopies fix the end points at all times.

<u>Remark:</u> If X is simply connected and $x_0 \in X$ then there is a bijection

$$\left\{\alpha:[0,1]\to X \mid \alpha(0)=1\right\} \Big/ \sim$$

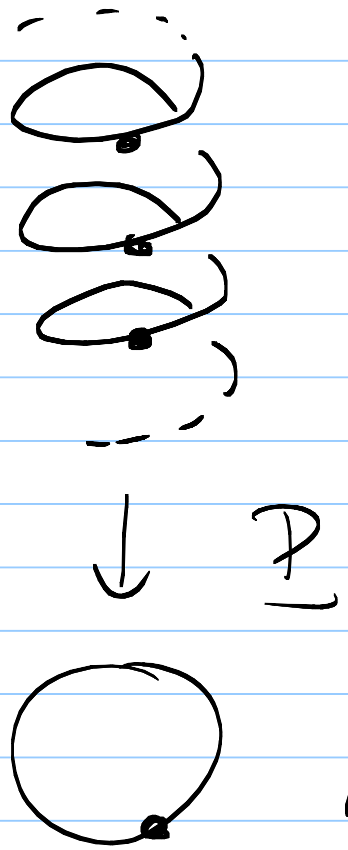
$$\begin{array}{ccc} \tilde{\alpha} & & \\ \downarrow & & \updownarrow \\ \alpha(1) & \in & X \end{array}$$

when $\alpha \sim \beta$ if and only if they are homotopic through homotopies fixing the end points.

2) The fundamental Group of the circle

<u>Theorem</u>: The map $\Phi : \mathbb{Z} \to \pi_1(S^1)$ sending an integer n to the homotopy class of the loop $\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$ based at $(1,0)$ is an isomorphism.

<u>Proof</u>: i) Consider the map $p : \mathbb{R} \to S^1$, $p(s) = (\cos 2\pi ns, \sin 2\pi ns)$



Let $\tilde{\omega}_n(s) = ns$ satisfy $\omega_n(s) = p(\tilde{\omega}_n(s))$.

$\tilde{\omega}_n$ is a lift of ω_n.

Note that $\Phi(n)$ can be defined as the homotopy class of the loop $p \circ \tilde{f}$ for any path $\tilde{f}$ in $\mathbb{R}$ from 0 to n, because any such $\tilde{f}$ is homotopic to $\tilde{\omega}_n$, keeping end points fixed: $t \longmapsto (1-t)\tilde{f} + t\tilde{\omega}_n$ and thus $p \circ \tilde{f}$ is homotopic to ω_n:

$$[p \circ \tilde{f}] = [\omega_n].$$

i) <u>Claim:</u> ϕ is a homomorphism.

<u>pnf</u> Let $\tau_m : \mathbb{R} \to \mathbb{R}$ be the translation $\tau_m(x) = x + m$. Then $\tilde{\omega}_m \cdot (\tau_m(\tilde{\omega}_n))$ is a path in $\mathbb{R}$ from 0 to $m+n$, so $\Phi(m+n)$ is the homotopy class of a loop in S^1, which is the image of this path under p. This image is just $\omega_m \cdot \omega_n$, so that

$$\phi(m+n) = \Phi(m) \cdot \phi(n).$$

To show Φ is an isomorphism we shall use two facts:

a) For each path $f: I \to S^1$ starting at a point $x_0 \in S^1$ and each $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lift $\tilde{f}: I \to \mathbb{R}$ starting at $\tilde{x}_0$.

b) For each homotopy $f_t: I \to S^1$ of paths starting at x_0 and $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lifted homotopy $\tilde{f}_t: I \to \mathbb{R}$ of path starting at $\tilde{x}_0$.

Ti) (a) and (b) prove the theorem.

$\underline{\Phi \text{ is surjective}}$: Let $f: I \to S^1$ be a loop at the base point $(1,0)$ representing a given element of $\pi_1(S^1)$. By (a) there is a lift $\tilde{f}$

starting at 0. This path $\tilde{f}$ ends at some integer since $p \circ \tilde{f}(1) = f(1) = (1,0)$ and $p^{-1}(1,0) = \mathbb{Z} \subseteq \mathbb{R}$. By the extended definition of Φ we have $\Phi(n) = [p\tilde{f}] = [f]$. Hence, Φ is surjective.

Φ is injective: Suppose $\Phi(m) = \Phi(n)$, which implies $\omega_m \simeq \omega_n$. Let f_t be a homotopy from $\omega_m = f_0$ to $\omega_n = f_1$. By (b) this homotopy lifts to a homotopy $\tilde{f}_t$ of paths starting at 0. The uniqueness part of (a) implies that $\tilde{f}_0 = \tilde{\omega}_m$ and $\tilde{f}_1 = \tilde{\omega}_n$. Since $\tilde{f}_t$ is a homotopy of paths the endpoints $\tilde{f}_t(1)$

is independent of t. For $t=0$ the endpoint is m and for $t=1$ it is n, so $m=n$.

Instead of (a) and (b) we'll prove

<u>(c)</u>: Given a map $F: Y \times I \to S^1$ and a map $\tilde{F}: Y \times \{0\} \to \mathbb{R}$ lifting $F|_{Y \times \{0\}}$, then there is a unique map $\tilde{F}: Y \times I \to \mathbb{R}$ lifting F and restricting to the given $\tilde{F}$ on $Y \times \{0\}$.

(c) $\Rightarrow$ (a) : $Y = \{pt\}$.

(c) $\Rightarrow$ (b) $Y = I$

In the proof of (c) we make use the following property

of the map $p : \mathbb{R} \to S^1$:
There is an open cover $\{U_\alpha\}$ of S^1 so that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open subsets each of which is homeomorphically mapped onto U_α via p.

<u>Defn:</u> Let $p : X \to Y$ be an onto map. If Y has an open cover $\{U_\alpha\}$ so that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open subsets each of which is mapped homeomorphically onto U_α then p is called a covering map.

3) Proof of c) First construct a lift $\tilde{F} : N \times I \to \mathbb{R}$ for N

some neighborhood in Y of a point $y_0 \in Y$. Since F is continuous every point $(y_0, t) \in Y \times I$ has a product neighborhood $N_t \times (a_t, b_t)$ s.t. $F(N_t \times (a_t, b_t)) \subset U_\alpha$ for some α. By compactness of $\{y_0\} \times I$ finitely many such products cover $\{y_0\} \times I$, say $N_{\alpha_i} \times (a_i, b_i)$ $i = 1, \dots, k$. Let $N = \bigcap_{i=1}^{k} N_{\alpha_i}$ and $0 = t_0 < t_1 < t_2 < \dots < t_m = 1$.

$F(N \times [t_i, t_{i+1}]) \subseteq U_i \doteq U_{\alpha_i}$.

Assume that $\tilde{F}$ has been constructed on $N \times [0, t_i]$. Since $F(N \times [t_i, t_{i+1}]) \subseteq U_i$ there

to some $\tilde{U}_i \subseteq \mathbb{R}$ projecting homeomorphically onto U_i by p and containing the point $\tilde{F}(y_0, t_i)$. If necessary replace N by a smaller open neighborhood so that $\tilde{F}(N \times \{t_i\}) \subseteq \tilde{U}_i$.
(Replace $N \times \{t_i\}$ with
$N \times \{t_i\} \cap (\tilde{F}|_{N \times \{t_i\}})^{-1}(\tilde{U}_i)$.)

Now define $\tilde{F}$ on $N \times [t_i, t_{i+1}]$ to be the composition of F with the homeomorphism $p^{-1} : U_i \longrightarrow \tilde{U}_i$. Repeating this finitely many times we get the required lift $\tilde{F} : N \times I \to \mathbb{R}$.

Now let's prove the uniqueness part of (c) in the case $Y=\{p\}$. We'll drop Y from the notation. So suppose $\tilde{F}$ and $\tilde{F}'$ are two lifts of $F: I \longrightarrow S^1$ so that $\tilde{F}(0)=\tilde{F}'(0)$. As above choose a partition $0=t_0<t_1<\dots<t_m=1$ of I so that $F([t_i,t_{i+1}])\subseteq U_i$ for some i. Assume inductively that $\tilde{F}=\tilde{F}'$ on $[0,t_i]$. Since $[t_i,t_{i+1}]$ is connected $\tilde{F}([t_i,t_{i+1}])$ must lie in a single component of $p^{-1}(U_i)$, say $\tilde{U}_i$. Similarly, $\tilde{F}'([t_i,t_{i+1}])$ must lie in a single component say $\tilde{\tilde{U}}_i$.

But $\tilde{F}(t_i) = \tilde{F}'(t_i)$ and thus $\tilde{U}_i = \tilde{U}_i$. Since p is injective on $\tilde{U}_i$ and $p \circ \tilde{F} = p \circ \tilde{F}'$ it follows that $\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$. So by induction $\tilde{F} = \tilde{F}'$ on $[0, t_m] = [0, 1]$.

Finishing the proof: Since $\tilde{F}$'s constructed above on the sets $N \times I$ are unique when restricted to each segment $\{y\} \times I$, they must agree whenever two such $N \times I$'s overlap. Thus we get a well defined lift $\tilde{F}$ on all of $Y \times I$. This $\tilde{F}$ is continuous since

$\tilde{H}$ is continuous on each $N \times I$, and $\tilde{H}$ is unique since $\tilde{H}$ is unique on each segment $\{y\} \times I$.

4) <u>Applications</u>:

<u>Theorem</u>: Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

<u>Proof</u> Let $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$ and assume that $p(z)$ has no roots in $\mathbb{C}$. Then for any real number $r \geq 0$ the formula

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$$

defines a loop in the unit circle $S^1 \subseteq \mathbb{C}$, based at 1.
In fact, as r varies f_r is a homotopy of loops based at 1. f_0 is the trivial loop at 1.
Hence $[f_r] = [f_0] = 0 \in \pi_1(S^1)$ the trivial element.
Let $r = |a_1| + |a_2| + \cdots + |a_n| + 1$.
Now for $|z| = r$ we have

$$|z^n| = r^n = r\, r^{n-1}$$
$$> (|a_1| + \cdots + |a_n|)\, |z|^{n-1}$$
$$\geq |a_1 z^{n-1} + \cdots + a_{n-1} z + a_n|.$$

So the polynomial
$P_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$ has no zero on the circle $|z| = r$ when $0 \leq t \leq 1$.

Then the formula

$$\frac{P_t(re^{2\pi i s})/P_t(r)}{\left|\qquad\qquad\quad\right|}$$

for

$t \in [0,1]$ defines a homotopy from w_n to f_r, where

$w_n(s) = e^{2\pi i n s}$. So

$0 = [f_r] = [w_n] = n$

$\Rightarrow p(z) = a_0$ is a constant.

$\blacksquare$

Theorem: Every continuous map $f : D^2 \to D^2$ has a fixed point

Proof (Brouwer proved this for D^n in 1910).

Easy!

<u>Theorem</u> (Borsuk–Ulam)
For every continuous map $f: S^2 \to \mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$.

<u>Proof</u> Suppose $f(x) \neq f(-x)$ for all $x \in S^n$.
Define $g: S^2 \to S^1$ as $g(x) = \dfrac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$.

Let $\eta: [0,1] \to S^1$, $\eta(s) = (\cos 2\pi s, \sin 2\pi s)$ and $h = g \circ \eta$. Since $g(-x) = -g(x)$ we have $h(s + \frac{1}{2}) = -h(s)$.
Let $\tilde{h}: [0,1] \to \mathbb{R}$ be a lift of h.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \arrow[ur, "\tilde{h}"] \arrow[r, "h"'] & S^1 \end{tikzcd}$$

Since $h(s + \frac{1}{2}) = -h(s)$ we have
$$\tilde{h}(s + \tfrac{1}{2}) = \tilde{h}(s) + \frac{q}{2}$$

for some odd integer q. Note that since $q = 2\left(\tilde{h}(s+\frac{1}{2}) - \tilde{h}(s)\right)$ and $\tilde{h}$ is continuous q is independent of s. In particular,

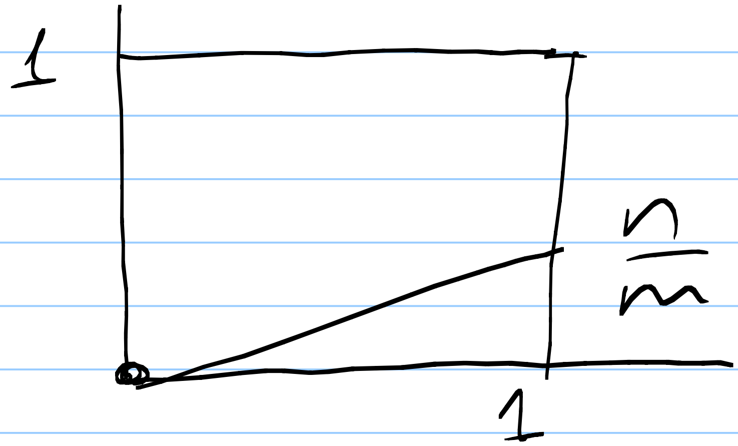
$$\tilde{h}(1) = \tilde{h}\left(\tfrac{1}{2}\right) + \frac{q}{2} = \tilde{h}(0) + q$$

$\Rightarrow [h] = q \in \mathbb{Z} = \pi_1(S^1)$. Since q is an odd integer it is not zero and thus h is <u>not</u> null homotopic. However, $h = g \circ \eta$ and η is clearly null homotopic. Thus $h = g \circ \eta$ is null homotopic. This is a contradiction.

$\blacksquare$

Proposition: If X and Y are path connected then $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

Corollary $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$.



$(m,n) \to (e^{2\pi i s m}, e^{2\pi i s n})$

5) **Induced Homomorphisms:**

$\varphi : X \to Y$, $x_0 \in X$, $y_0 = \varphi(x_0) \in Y$

$\Rightarrow \varphi_\# : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

a) $\psi : (Y, y_0) \to (Z, z_0)$ then

$(\psi \circ \varphi)_\# = \psi_\# \circ \varphi_\#$

b) $1 = 1_X$, then $1_\# = 1_{\pi_1(X,x_0)}$.

6) Theorem $\pi_1(S^n, x_0) = (0)$, $n \geq 2$.

proof Let $f : [0,1] \to S^n$ be any loop based at x_0. Let $x \in S^n$, $x \neq x_0$ and choose a small open ball B^n around x. The inverse image $f^{-1}(B)$ is a disjoint union of open intervals (a_i, b_i) in $\mathbb{R}$. Since $f^{-1}(x)$ is a compact set covered by the union of (a_i, b_i), we see that $f^{-1}(x)$ is covered by finitely many (a_i, b_i). Now for each of these finitely many (a_i, b_i) choose some $g_i : [a_i, b_i] \to S^n$

so that g_i is homotopic to $f_i|_{[a_i,b_i]}$, $f_i(a_i)=g_i(a_i)$, $f_i(b_i)=g_i(b_i)$ and $g_i([a_i,b_i])\subseteq \partial B$.

Replacing $f_i|_{[a_i,b_i]}$ with g_i for these finitely many i, we get a continuous map g homotopic to f and satisfying

$$g(I)\subseteq S^n-\{x\}=\mathbb{R}^n.$$

Since $\mathbb{R}^n$ is contractible g is homotopic to a constant and thus $[f]$ is trivial in $\pi_1(S^n)$.

This finishes the proof.

__Examples__ $x \in \mathbb{R}^n$

$\mathbb{R}^n \setminus \{x\} \simeq \mathbb{R} \times S^{n-1}$

$\Rightarrow \pi_1(\mathbb{R}^n \setminus \{x\}) = \begin{cases} \mathbb{Z} & \text{if } n=2 \\ 0 & \text{if } n>2. \end{cases}$

__Corollary__ $\mathbb{R}^2$ is __not__ homeomorphic to $\mathbb{R}^n$ if $n \neq 2$.

__Proof__ If $f : \mathbb{R}^n \to \mathbb{R}^2$ were a diffeomorphism then so would be any

$f|_{S^n - \{x\}} : S^n \setminus \{x\} \to S^2 - f(x)$.

In particular,

$\pi_1(S^n \setminus \{x\}) = \pi_1(S^2 - f(x))$

$\Rightarrow n = 2$, which is a contradiction. $\blacksquare$

Proposition: If a space X retracts onto a subspace A then the induced homomorphism

$i_*: \pi_1(A) \to \pi_1(X)$ is injective and

$r_*: \pi_1(X) \to \pi_1(A)$ is surjective, where $i: A \to X$ is the inclusion and $r: X \to A$ is the retraction.

Corollary S^1 is not a retract of D^2.

Proposition: Let $\varphi: X \to Y$ be a homotopy equivalence. Then $\varphi_\#: \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorphism.

7) Van Kampen's Theorem:

Free products of groups: let $G_1, \dots, G_k$ be groups. Then the free product of $G_1, \dots, G_k$, denoted by $G_1 * G_2 * \dots * G_k$ is the set of elements of the form $g_1 g_2 \dots g_n$, $g_i \in G_j$ with obvious group operations.

Example 1) $\mathbb{Z}_2 = \langle a \rangle$, $\mathbb{Z}_2 = \langle b \rangle$

$\mathbb{Z}_2 * \mathbb{Z}_2 = \{1, a, b, ab, ba, aba, bab, \dots\}$

$$1 \to \underset{\ker f}{\overset{\parallel}{N}} \to \mathbb{Z}_2 * \mathbb{Z}_2 \xrightarrow{f} \mathbb{Z}_2 \to 0$$

$a \longmapsto 1$

$b \longmapsto 1$

2) $\mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \mid a^2 = 1, b^3 = 1 \rangle$

$\mathbb{Z}_2 * \mathbb{Z}_3 \longleftrightarrow PSL(2, \mathbb{Z})$ isomorphism

$a \leftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $b \leftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$.

Universal Property of Free products:

Let $f_i : G_i \to K$ $i = 1, \ldots, k$ be homomorphisms. Then there exists a unique homomorphism $F : G_1 * G_2 * \cdots * G_k \to K$ such that $F \circ i_j = f_j$ for all j, where

$$i_j : G_j \longrightarrow G_1 * \cdots * G_k$$
$$g \longmapsto g$$

Group presentation:

$\mathbb{Z} = \langle a \mid - \rangle$

$\mathbb{Z} * \mathbb{Z} = \langle a, b \mid - \rangle$

$\mathbb{Z}_n = \langle a \mid a^n \rangle$

$S_3 = \langle \sigma, \tau \mid \sigma^2, \tau^3, \sigma\tau\sigma\tau \rangle$

$F_n = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z} = \langle a_1, \ldots, a_n \mid - \rangle$

$\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid aba^{-1}b^{-1} \rangle$

Theorem (Seifert, Van Kampen)
Let X be a topological space and U, V path connected open subsets of X so that $U \cap V$ is also path connected. Then the map $\Phi : \pi_1(U) * \pi_1(V) \longrightarrow \pi_1(X)$, where $\Phi(g) = i_{U\#}(g)$ $\forall g \in \pi_1(U)$ and $\Phi(g) = i_{V\#}(g)$ $\forall g \in \pi_1(V)$, is

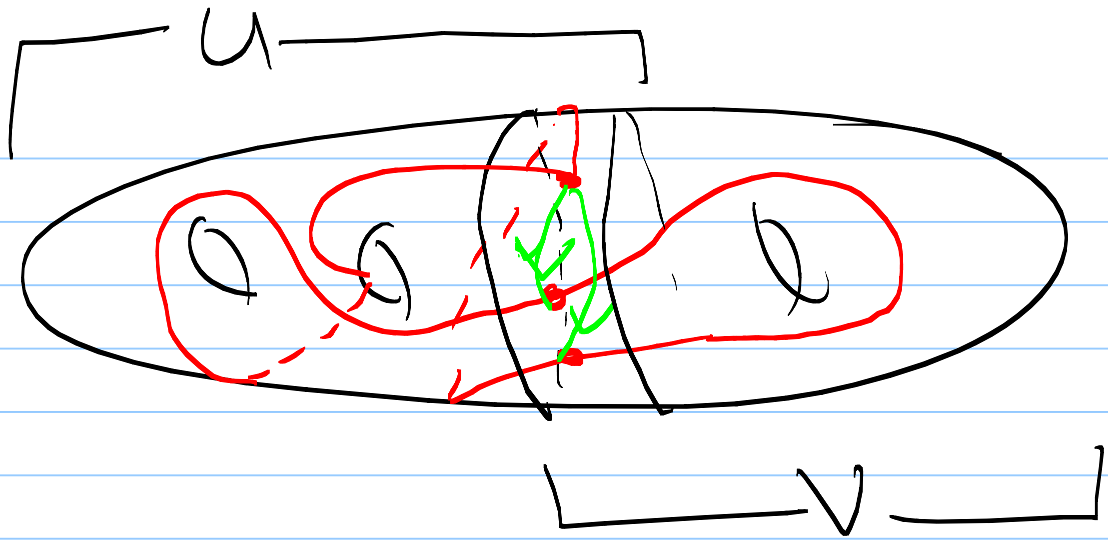
surjective. Moreover, the kernel $N \trianglelefteq \mathcal{B}$ is generated by all elements of the form $J_{U\#}(\omega) J_{V\#}(\omega)$, where $\omega \in \pi_1(U \cap V)$:

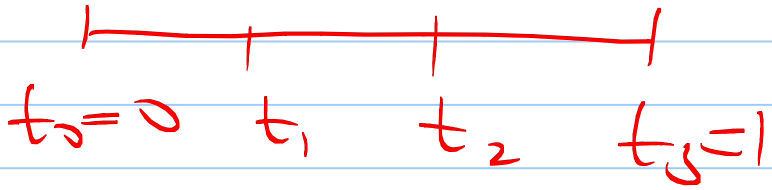
$$\pi_1(X) = \pi_1(U) \underset{\pi_1(U\cap V)}{*} \pi_1(V)$$ **OR**

$$\begin{tikzcd}
 & \pi_1(U) \arrow[dr, "I_{U\#}"] & \\
\pi_1(U\cap V) \arrow[ur, "J_{U\#}"] \arrow[dr, "J_{V\#}"'] & & \pi_1(U\cup V) \\
 & \pi_1(V) \arrow[ur, "I_{V\#}"'] &
\end{tikzcd}$$

<u>Idea of the proof of the first part:</u>

Given a map $f:[0,1] \to X$ with $f(0) = x_0 = f(1) \in U \cap V$ choose a partition of $[0,1]$ $0 = t_0 < t_1 < \cdots < t_m = 1$ so that $f([t_i, t_{i+1}]) \subseteq U$ or $\subseteq V$.





$$f \simeq f_1 \cdot f_2 \cdot f_3$$

$f_1 \in \pi_1(U)$, $f_2 \in \pi_1(V)$, $f_3 \in \pi_1(U)$.

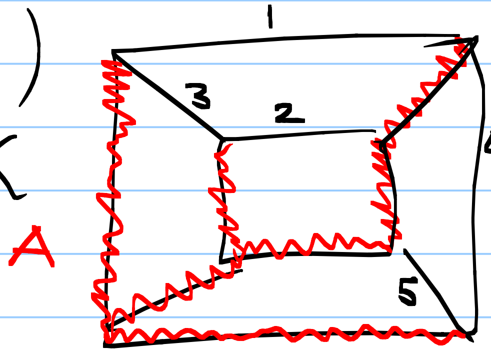
8) <u>Examples</u>: 1) $X = \bigvee_{\alpha} X_\alpha$

Suppose each X_α is path connected and $x_\alpha \in X_\alpha$ so that there is some open subset $x_\alpha \in U_\alpha \subseteq X_\alpha$ which deformation retracts onto $\{x_\alpha\}$.

Then $\pi_1(X) = *_{\alpha} \pi_1(X_\alpha)$

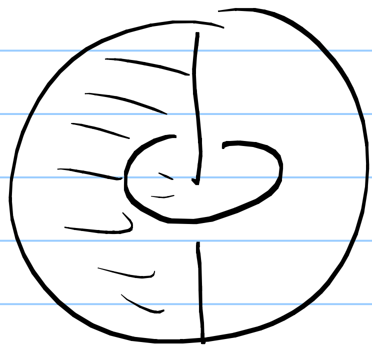
In particular, $\pi_1(\bigvee_n S^1) = *_n \mathbb{Z}$

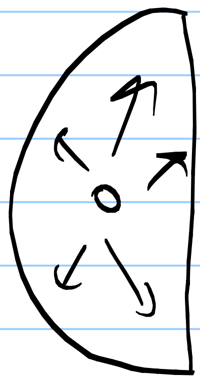
2) X



$\pi_1(X) \simeq \pi_1(X/A) \simeq \pi_1(\bigvee_5 S^1) \simeq F_5$

3) $\mathbb{R}^3 \setminus S^1 \simeq$

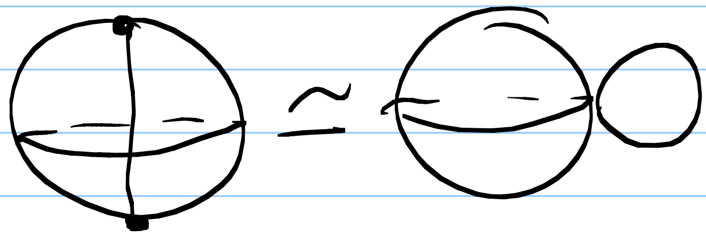




So $\mathbb{R}^3 \setminus S^1 \simeq S^2 \vee S^1$

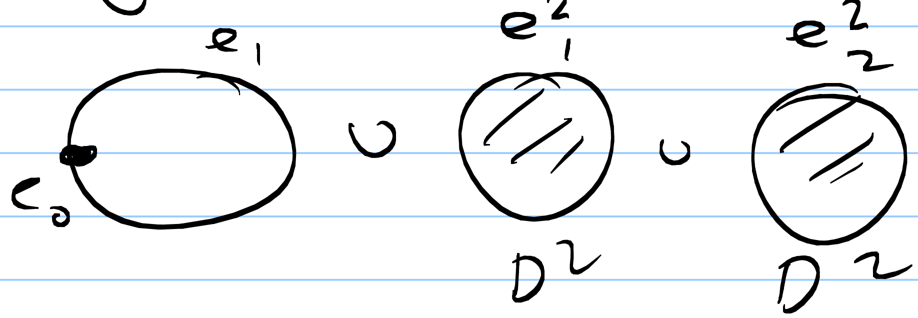
1S

$\Rightarrow \pi_1 \simeq \mathbb{Z}$.



4) T^2, $\mathbb{RP}^2$, KB, Σ_g, N_h.

5) More general cell complexes.

6) $X =$ 

$\partial e_1^2 = S^1 \longrightarrow S^1 = e_0 \cup e_1$

$z \longmapsto z^2$

$\partial e_2^2 : S^1 \longrightarrow S^1$, $z \longmapsto z^2$.

$\pi_1(X) \simeq \mathbb{Z}_2$.

7) $\pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2) = \mathbb{Z}_2 * \mathbb{Z}_2$.

8) Construct a cell complex X such that $\pi_1(X) = \mathbb{Z}_2 * \mathbb{Z}_3$.

9) <u>Exercise</u>: Compute π_1 of space containing ∞'ly many circles!

COVERING SPACES

1) <u>Definition</u> A covering space of a space X is a space together with a map $p: \tilde{X} \to X$ so that there exists an open covering $\{U_\alpha\}$ of X, where each $p^{-1}(U_\alpha)$ is a disjoint union of open subsets of $\tilde{X}$ each of which is mapped homeomorphically onto U_α via p.

2) <u>Examples</u>

a) $\mathbb{R}$

$$\downarrow p(t) = e^{2\pi i t}$$

S^1

Regular

$G = \mathbb{Z}$

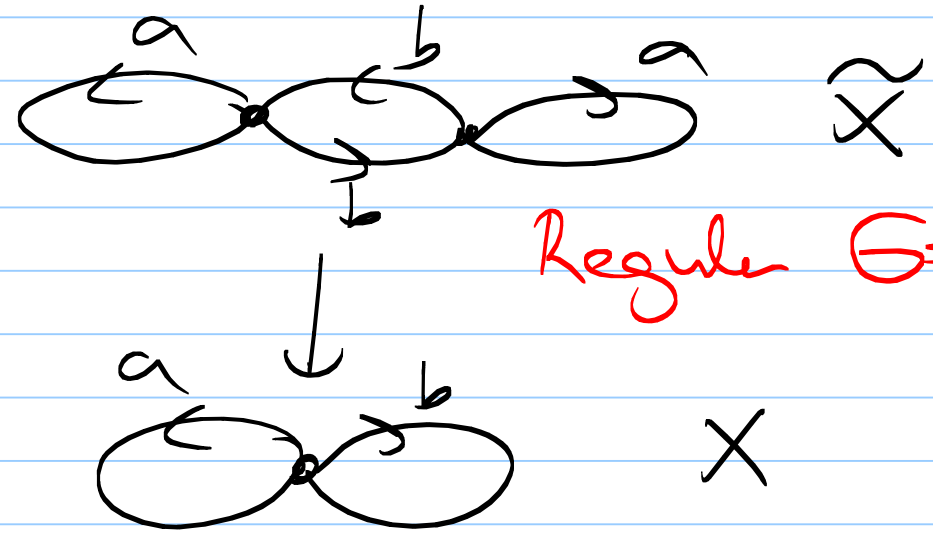
b) $p: S^1 \longrightarrow S^1$, $p(e^{2\pi i t}) = e^{2n\pi i t}$

$z \longmapsto z^n$ Regular $G = \mathbb{Z}_n$

c) $p: S^n \longrightarrow \mathbb{RP}^n = S^n/_{p\sim -p}$

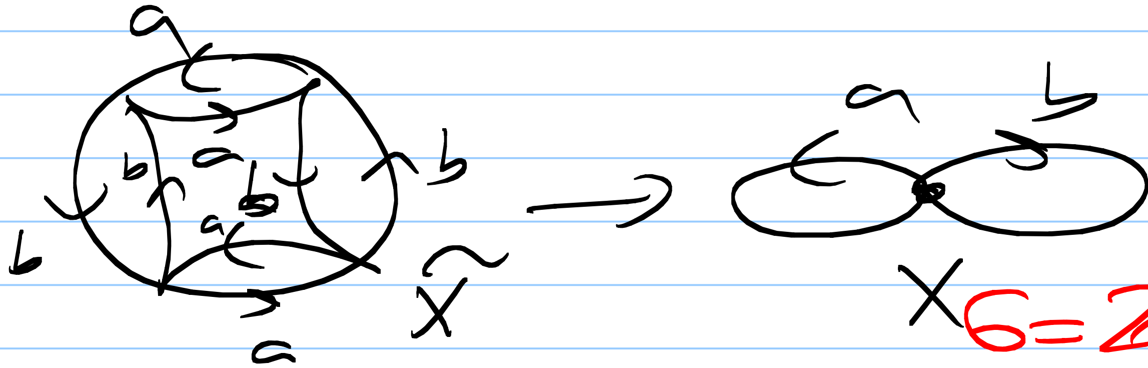
$p \longmapsto [p]$ Regular $G = \mathbb{Z}_2$

d)



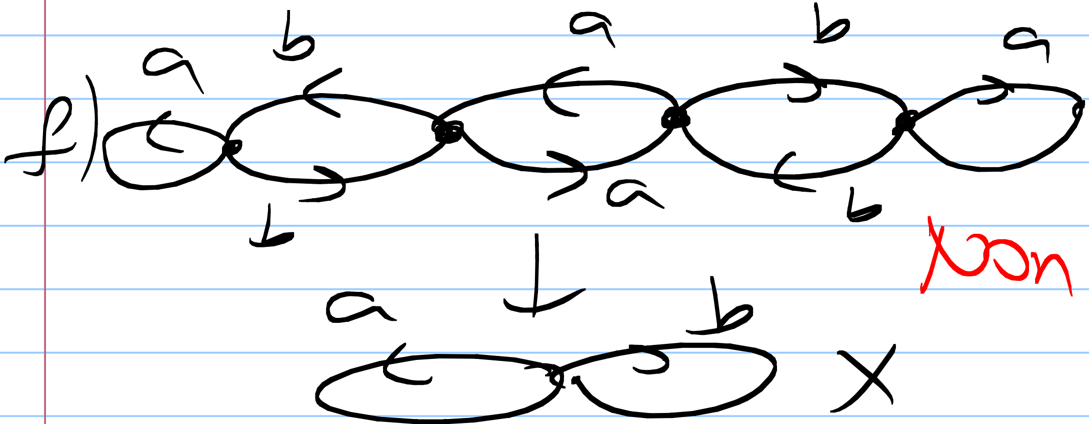
Regular $G = \mathbb{Z}_2$

e)



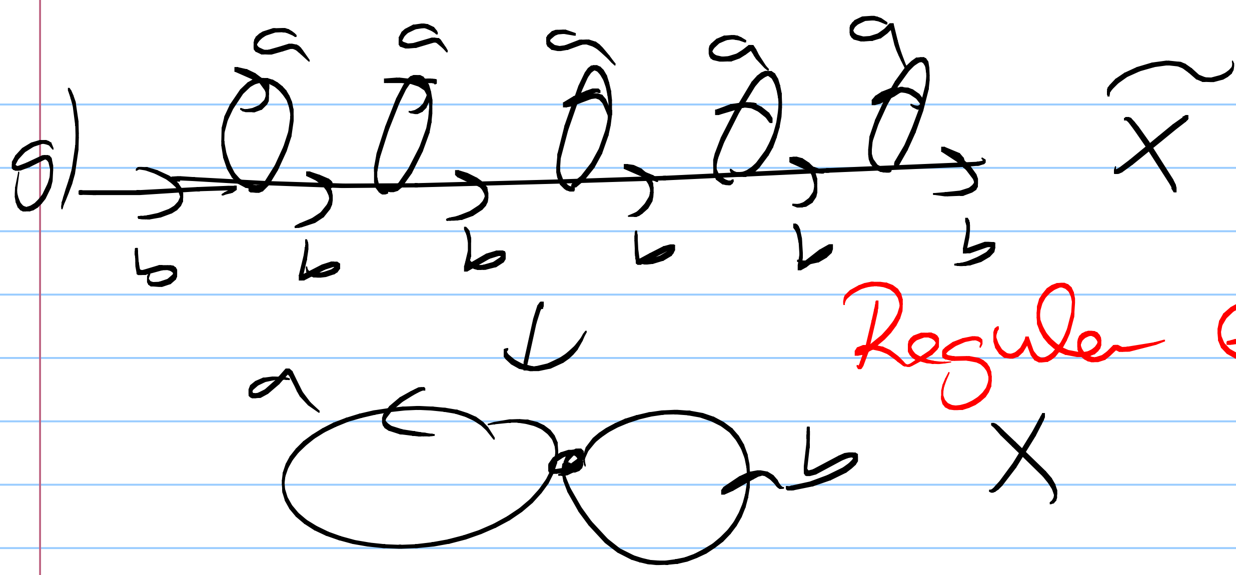
$G = \mathbb{Z}_2 \times \mathbb{Z}_2$

f)



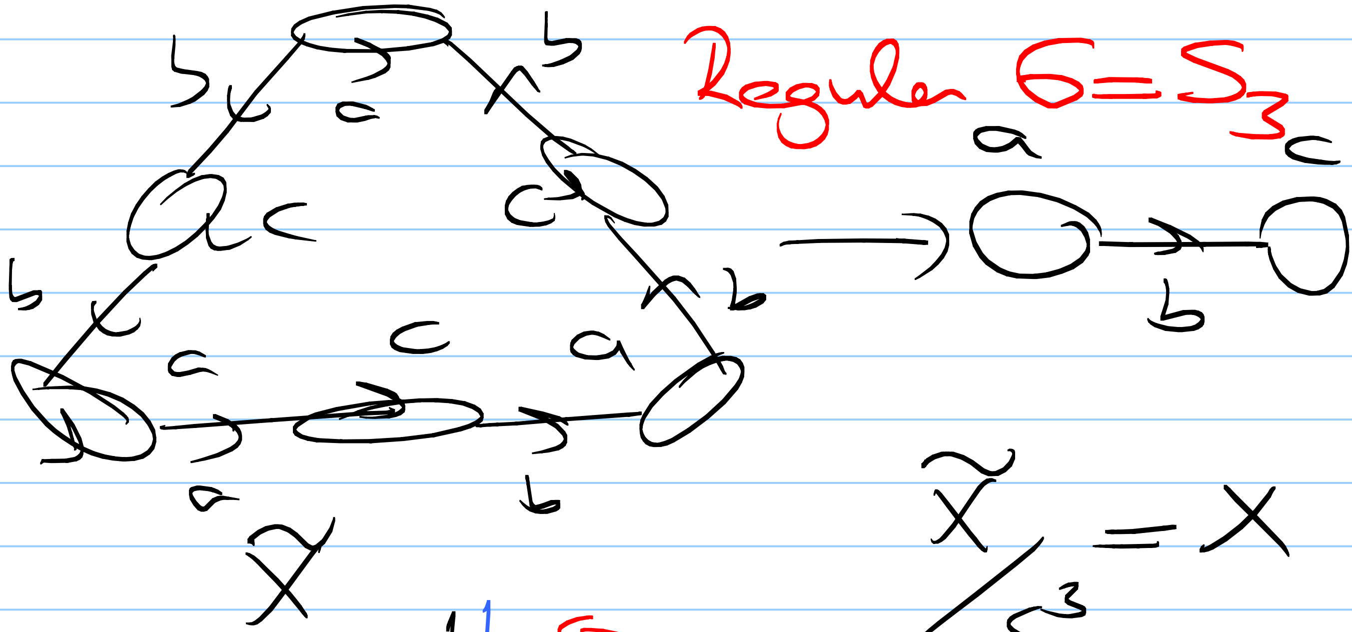
Non regular

g)



Regular $G = \mathbb{Z}$

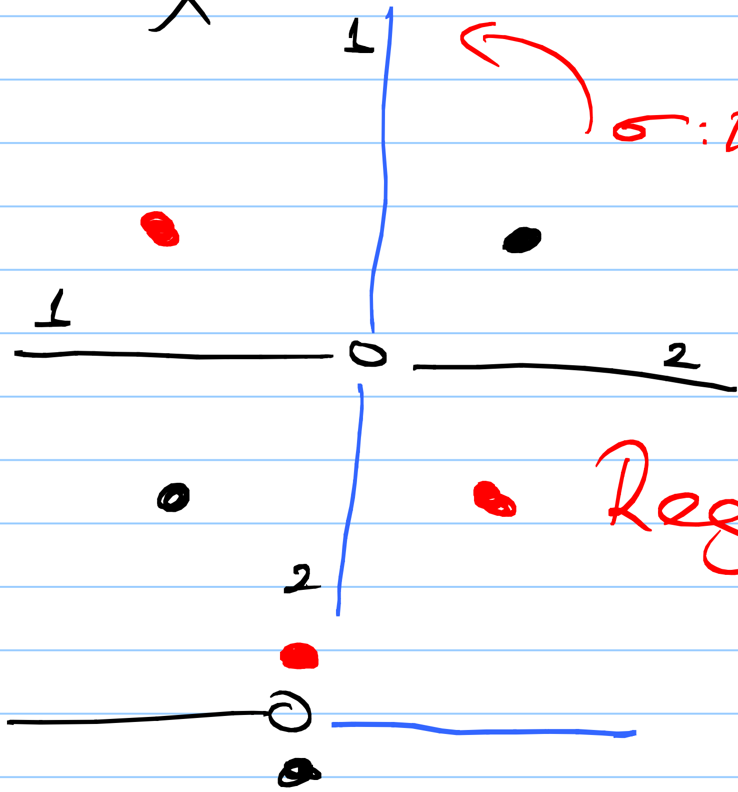
h)



Regular $G = S_3$

$\tilde{X} / S^3 = X$

i)

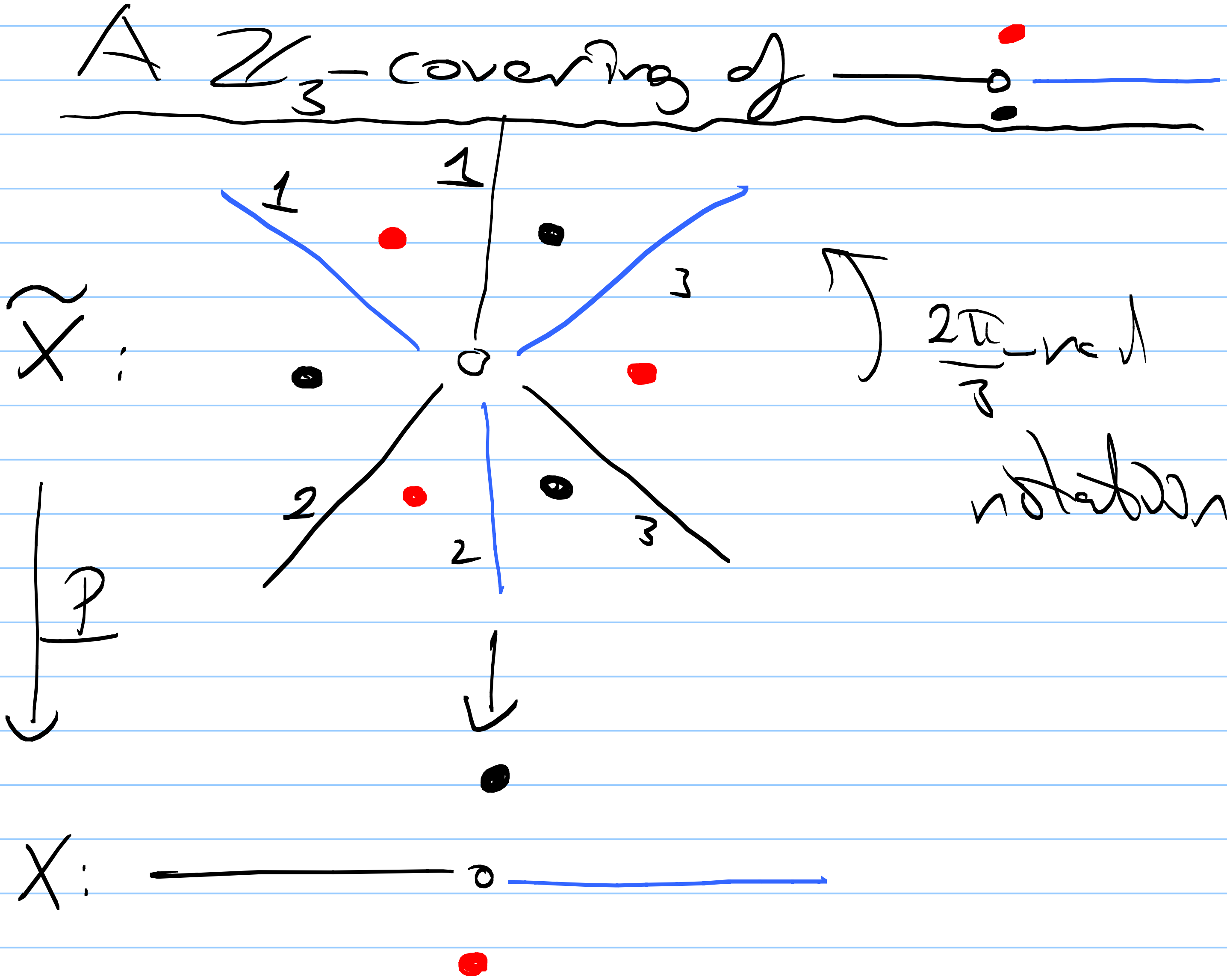


σ: π-rad rotation

Regular

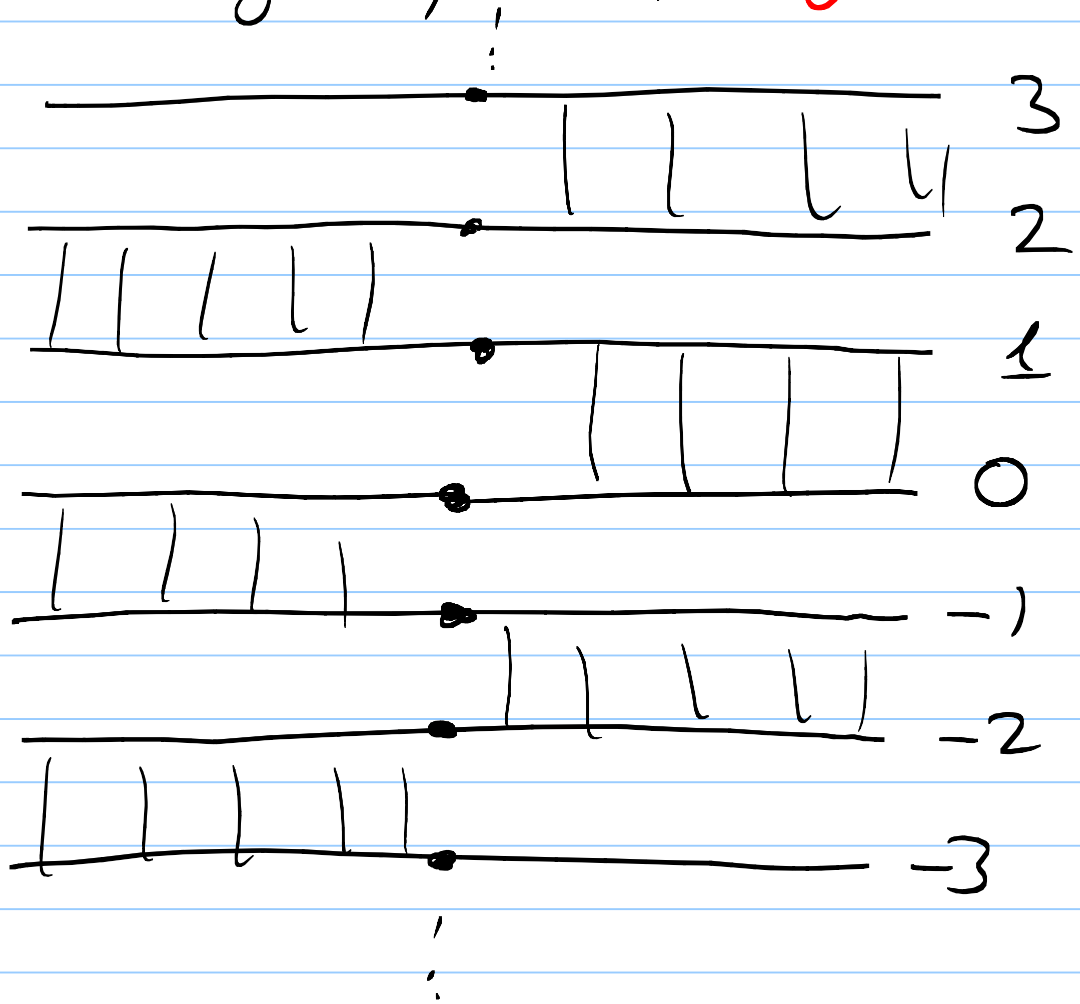
$G = \mathbb{Z}_2 = \langle \sigma \rangle$

A $\mathbb{Z}_3$-covering of



Both spaces have $\pi_1 \simeq \mathbb{Z}$.

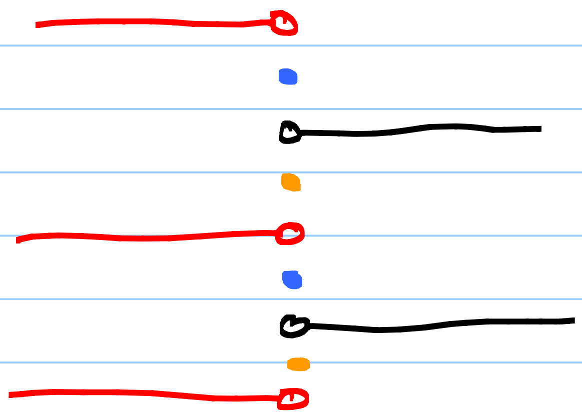
7) Let $\tilde{X} = \mathbb{R} \times \mathbb{Z} / \sim$ be the following space: Regular $G = \mathbb{Z}$



$(x, n) \sim (x, n+1)$ for $x > 0$ and n even

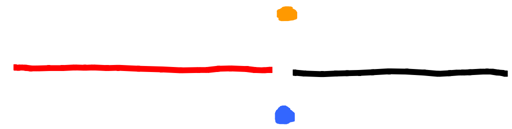
$(x, n) \sim (x, n+1)$ for $x < 0$ and n odd

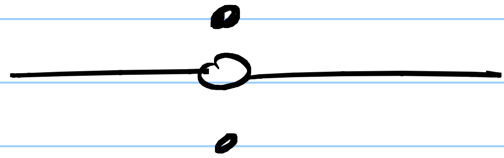
$\tilde{X}$:



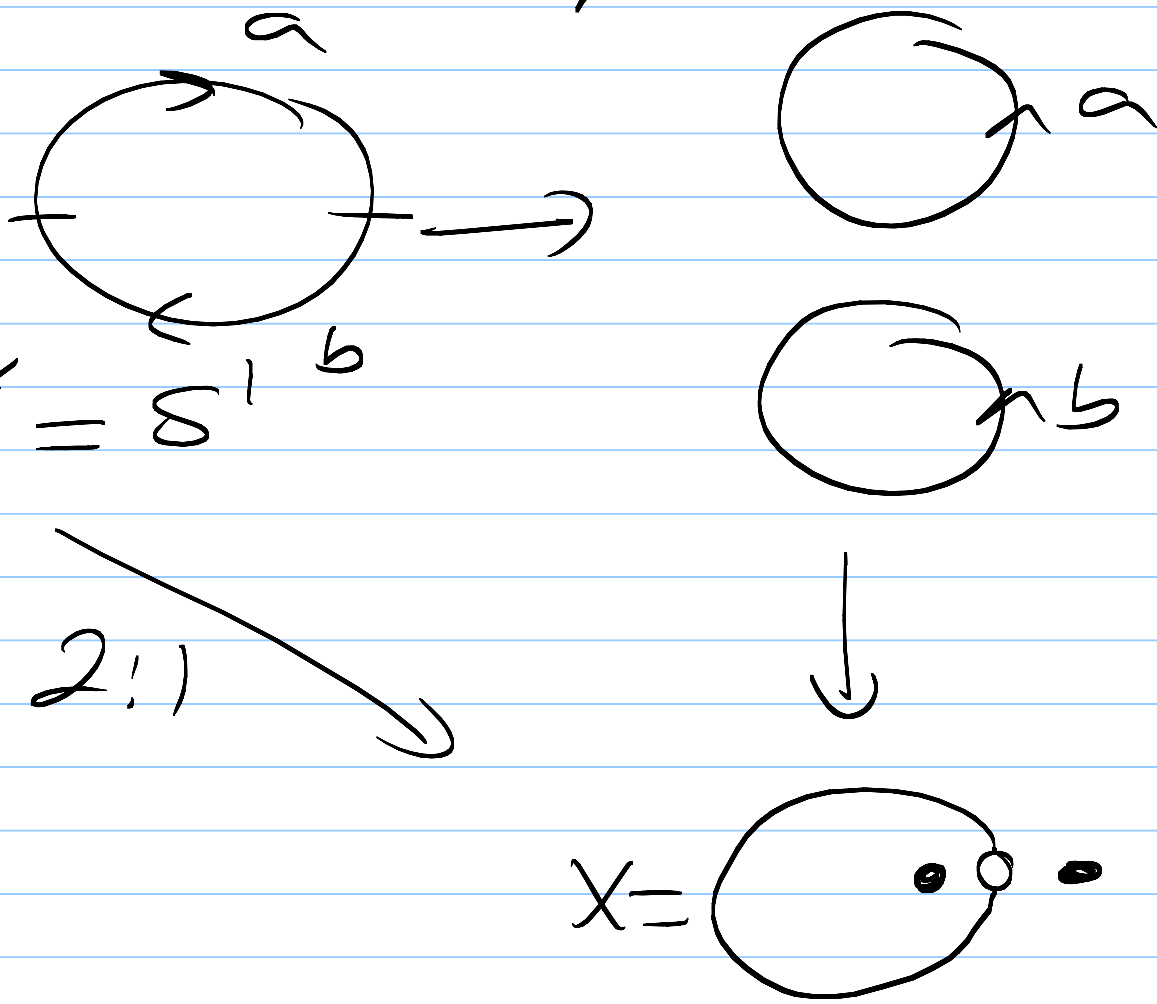
Show that $\pi_1(\tilde{X}) = \{e\}$.

(See p. 25)

X : 

Then $\tilde{X} \longrightarrow X$: 

is a covering space.

b) 

$$X = S^1 \times \{0,1\} / (x,0) \sim (x,1)$$
if $x \neq 1$

This is a 2 to 1 map and a local homeomorphism, but it is not a covering map.

Exercise: Prove the following:
Let $p: \tilde{X} \to X$ be a local homeomorphism, where $\tilde{X}$ and X are compact, connected Hausdorff spaces. Show that p is onto and a covering space.

Remark: A covering space $p: \tilde{X} \to X$ is called regular if $X = \tilde{X}/G$, where G is a group acting freely and properly discontinuously on $\tilde{X}$.

3) Lifting Properties:

Let $p: \tilde{X} \to X$ be a covering space and $f: Y \to X$ is a map. A lifting of $f: Y \to X$ is a map $\tilde{f}: Y \to \tilde{X}$ s.t. $f = p \circ \tilde{f}$:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "f"] \arrow[r, "f"'] & X \end{tikzcd}$$

Proposition: (Homotopy Lifting)
Given a covering space $p: \tilde{X} \to X$, a homotopy $f_t: Y \to X$, and a map $\tilde{f}_0: Y \to \tilde{X}$ lifting of f_0, then there is a unique homotopy $\tilde{f}_t: Y \to \tilde{X}$ of $\tilde{f}_0$

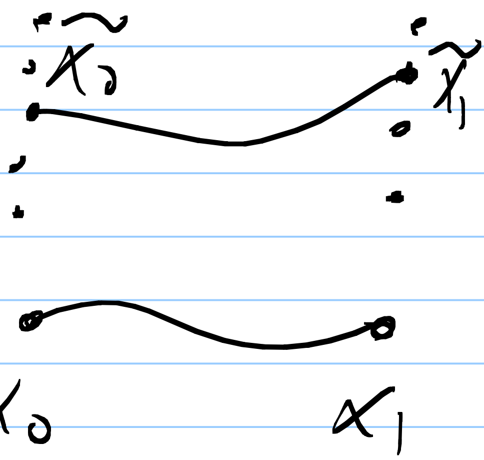
that lifts f_t.

Proof: Property (c) in the proof of $\pi_1(S^1)=\mathbb{Z}$ gives the proof. ∎

Special Cases

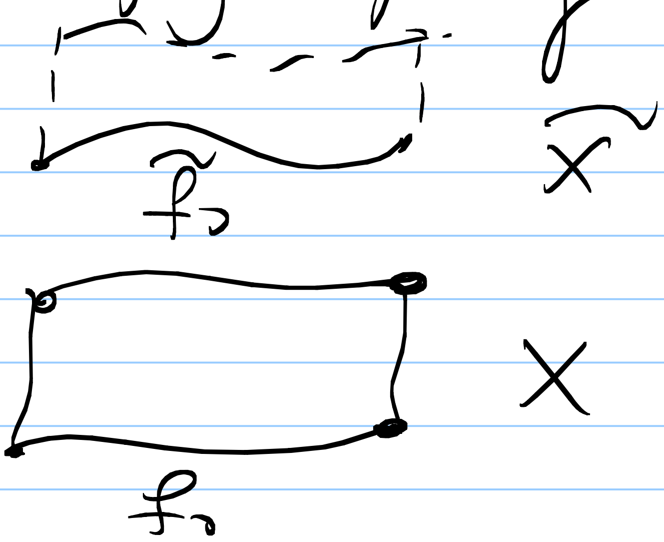
1) $Y=\{pt\}$ path lifting

$$\begin{tikzcd} & \tilde{X} \arrow[d] \\ Y\times I = I \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$



2) $Y=I$ homotopy lifting

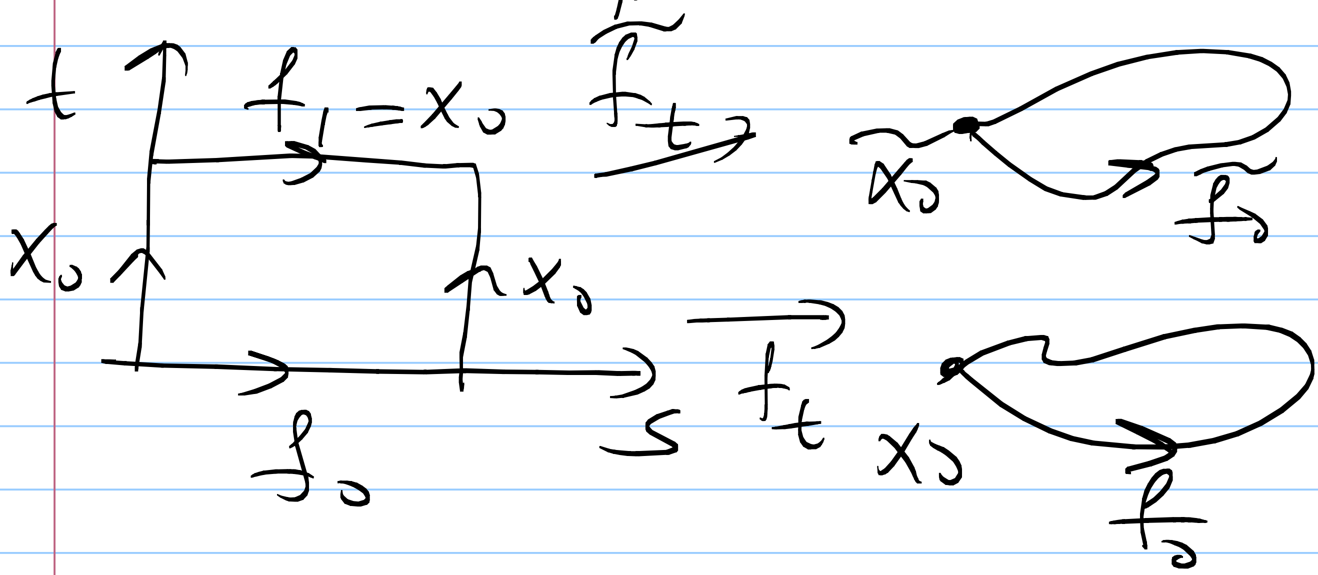
$Y\times I = I\times I$



Proposition: The map $p_{\#}: \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ induced by a covering projection is injective. The image subgroup $p_{\#}(\pi_1(\tilde{X}, \tilde{x}_0))$ consists of the homotopy classes of loops in X based at x_0 whose lifts to $\tilde{X}$ starting at $\tilde{x}_0$ are loops.

Proof: Let $\tilde{f}_0 : I \to \tilde{X}$ be a loop based at $\tilde{x}_0$ which represents a class in the kernel of the homomorphism $p_{\#} : \pi_1(\tilde{X}, x_0) \to \pi_1(X, x_0)$. Then $\tilde{f}_0$ is the unique lifting of $f_0 \doteq p \circ \tilde{f}_0 : I \to X$, a loop in X based at x_0. By assumption f_0 is homotopic to a constant. Hence there is a homotopy

$f_t : I \to X$ from f_0 to the constant loop at x_0.



By the previous proposition there is a homotopy $\tilde{f}_t$ of $\tilde{f}_0$ to $\tilde{f}_1$ so that

$$p \circ \tilde{f}_t = f_t \text{ for all } t.$$

In particular, $p \circ \tilde{f}_1(s) = f_1(s) = x_0$, for all $s \in [0,1]$. Hence, $\tilde{f}_1(s) \in p^{-1}(x_0)$ for all $s \in [0,1]$.

Since $p^{-1}(x_0)$ is a discrete set and $\tilde{f}_1(0) = \tilde{x}_0$, $\tilde{f}_1(s) = \tilde{x}_0$ $\forall s \in [0,1]$

Hence, $\tilde{f}_1$ is a constant loop.
$\Rightarrow [\tilde{f}] = e$ in $\pi_1(\tilde{X}, \tilde{x_0})$.

The second statement is easy.

$\equiv$

<u>Remark:</u> If $p: \tilde{X} \to X$ is a covering map then the cardinality of $p^{-1}(x)$ is a locally constant function. Hence, if X is connected $|p^{-1}(x)|$ is independent of x. In this case it is called the degree of the covering:
double covering, 3-fold, n-fold covering.
The degree of the covering is also said to be the number of sheets

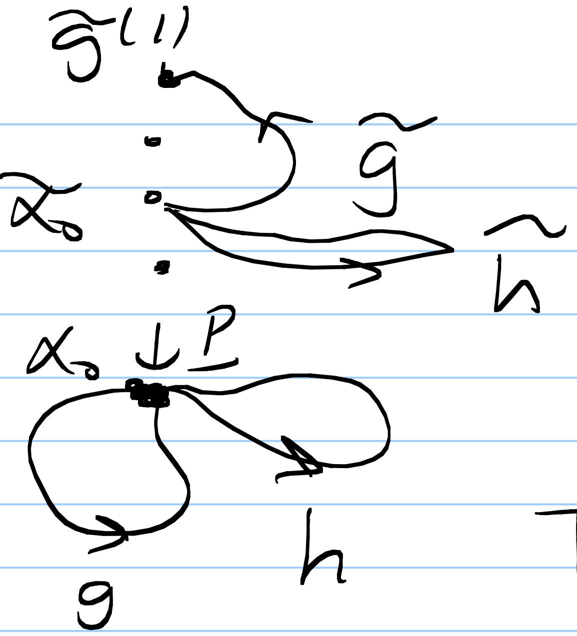
of the covering.

Proposition: The number of sheets of a covering space $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ of path connected spaces is equal to the index of $p_\#(\pi_1(\tilde{X},\tilde{x}_0))$ in $\pi_1(X,x_0)$.

Proof: Let g be a loop at x_0 and $\tilde{g}$ be its lift to $\tilde{X}$ starting at $\tilde{x}_0$.

If $[h]\in H=p_*(\pi_1(\tilde{X},\tilde{x}_0))$ then the lift $\tilde{h}\tilde{g}$ of hg starting at $\tilde{x}_0$ has the same end point with $\tilde{g}$.

So we get a well defined function $\Phi:\{H[g]\mid [g]\in\pi_1(X,x_0)\}\to p^{-1}(x_0)$ by sending the coset $H[g]$ to $\tilde{g}(1)$.



Since $\tilde{X}$ is path connected Φ is surjective. To see that Φ is injective assume that $\Phi(H[g_1]) = \Phi(H[g_2])$. Then $g_1 g_2^{-1}$ has a lift which is a loop at $\tilde{x}_0$. So $[g_1][g_2]^{-1} \in H$ and thus $H[g_1] = H[g_2]$. $\equiv$

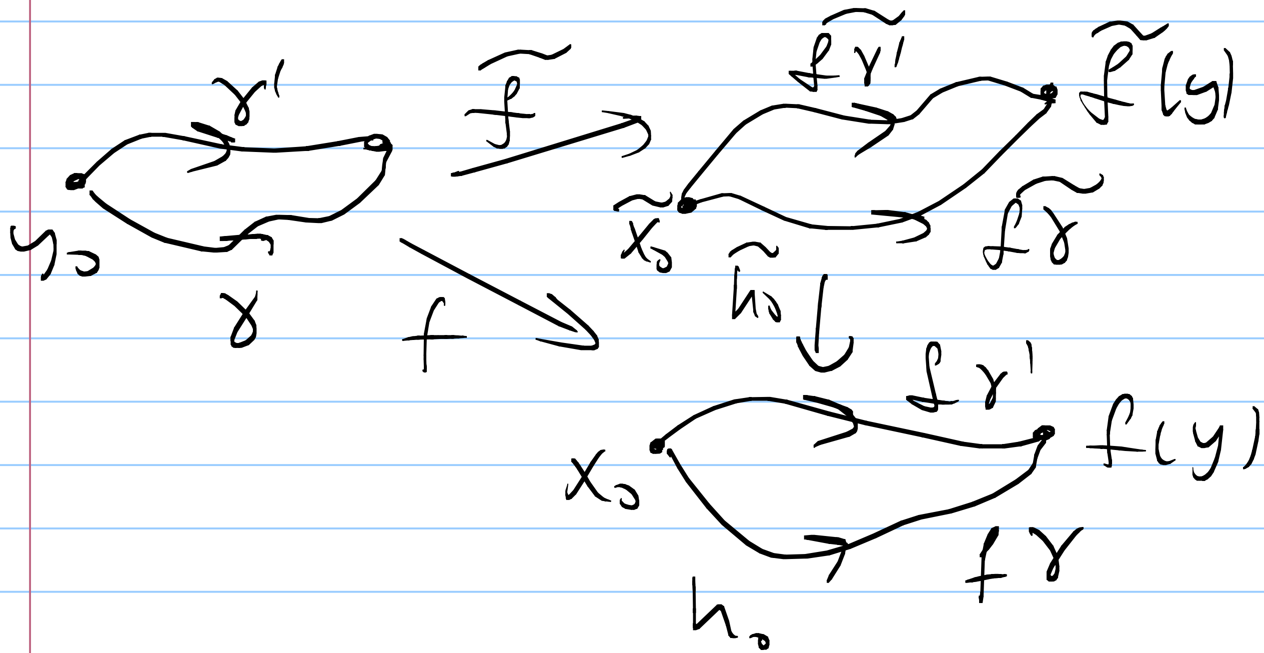
<u>Proposition</u> (Lifting Criterion)
Let $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be a covering space, $f : (Y, y_0) \to (X, x_0)$ a map where $f(y_0) = x_0$ and Y is path connected and locally path connected. Then f has a lift

$\tilde{f}:(Y,y_0)\to(\tilde{X},\tilde{x}_0)$ if and only if
$f_\#(\pi_1(Y,y_0))\subseteq p_\#(\pi_1(\tilde{X},\tilde{x}_0))$.

Proof ($\Rightarrow$) is clear since $f_\# = p_\# \circ \tilde{f}_\#$.
($\Leftarrow$) Let $y\in Y$ and γ a path in Y from y_0 to y. The path $f\gamma$ in X starting at x_0 has a unique lift $\widetilde{f\gamma}$ starting at $\tilde{x}_0$. Define $\tilde{f}(y)=\widetilde{f\gamma}(1)$.

Well definedness of $\tilde{f}$:
Let γ' be another path from y_0 to y. Then $(f\gamma')\cdot(f\gamma)^{-1}$ is a loop h_0 at x_0 with $[h_0]\in f_\#(\pi_1(Y,y_0))$ $\subseteq p_\#(\pi_1(\tilde{X},\tilde{x}_0))$. Hence, there is a homotopy h_t of h_0 to a loop h_1 that has a lift $\tilde{h}_1$ in $\tilde{X}$ based at $\tilde{x}_0$. By the homotopy

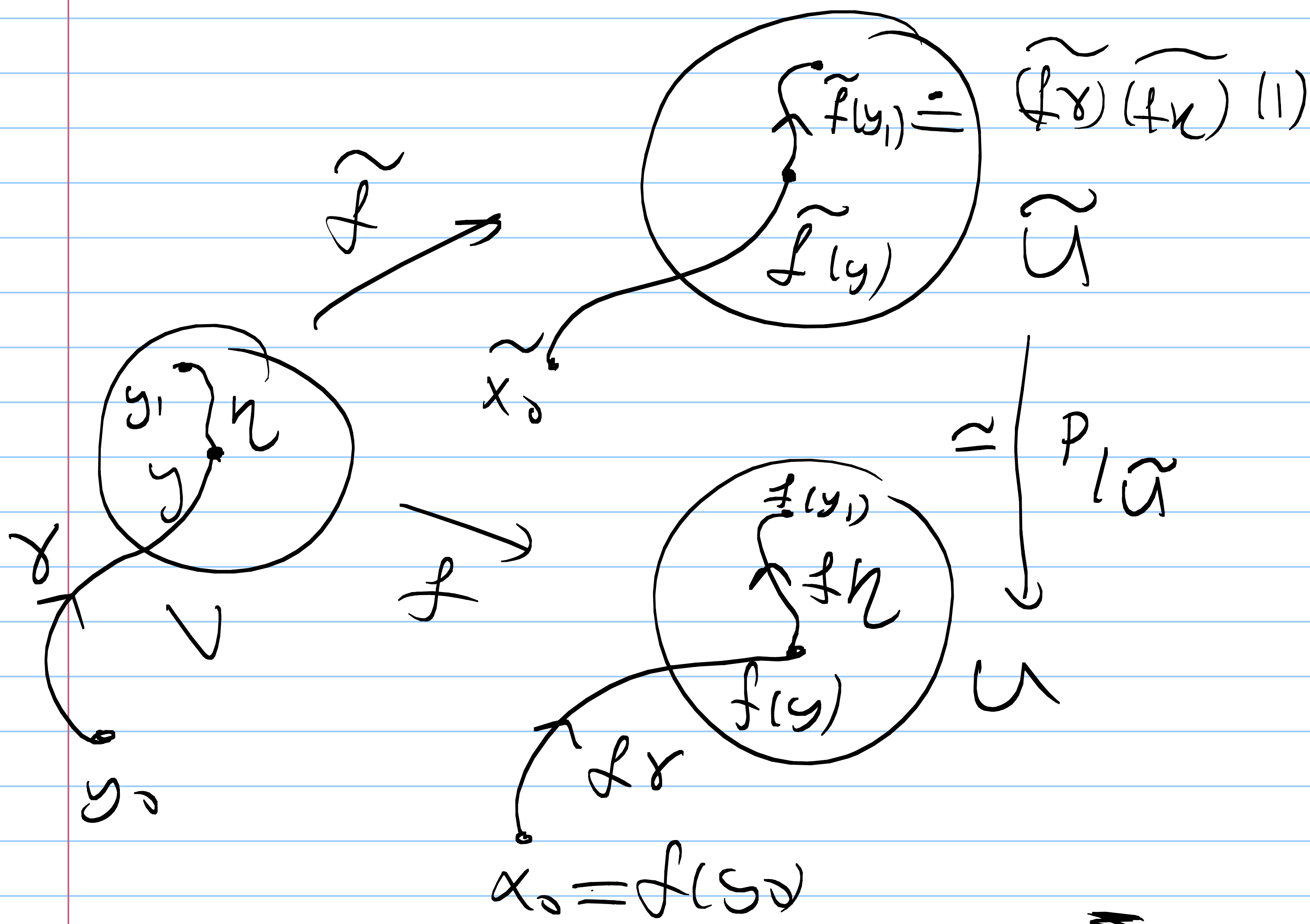
lifting h_t have a lift $\tilde{h}_t$. Since $\tilde{h}_1$ is a loop at $\tilde{x}_0$ so is $\tilde{h}_0$.



By the uniqueness of lifted paths, the first half of $\tilde{h}_0$ is $\widetilde{f\gamma'}$ and the second half is $(\widetilde{f\gamma})^{-1}$ with the common midpoint $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$. This shows that $\tilde{f}$ is well defined.

<u>$\tilde{f}$ is continuous</u>: Let $y \in Y$ and $U \subseteq X$ an open subset containing

$f(y)$ such that there is some $\tilde{U} \subseteq \tilde{X}$ with $p: \tilde{U} \to U$ is a homeomorphism. Choose a path connected open neighborhood V of y with $f(V) \subseteq U$. Fix a path γ from y_0 to y. The diagram below shows that $\tilde{f}(V) \subseteq \tilde{U}$ so that $\tilde{f}$ is continuous:



∎

Proposition Given a covering space $p: \tilde{X} \to X$ and a map $f: Y \to X$ with two lifts $\tilde{f}_1$ and $\tilde{f}_2$ from Y to $\tilde{X}$ that agree at one point y_0 of Y, then if Y is connected these two lifts agree on all of Y.

Proof: Let A be the set of points in Y on which the two lifts agree. Since $p: \tilde{X} \to X$ is a local homeomorphism A is both open and closed. On the other hand, A is nonempty since $y_0 \in A$. Finally, since A is connected $A = Y$ and thus the proof finishes. $\blacksquare$

4) Classification of Covering Spaces:

First we will construct a universal covering space of a given space X provided that X is semilocally simply connected. In other words, every $x \in X$ has a neighborhood U such that

$$\pi_1(U, x) \longrightarrow \pi_1(X, x) \text{ is trivial.}$$

Example  is not semilocally simply connected.

Example: CW-complexes are locally contractible and thus they are locally simply connected $\Rightarrow$ semilocally simply connected.

Theorem: Let X be a path connected, locally path connected and semilocally simply connected. Then X has a universal covering $p: \tilde{X} \to X$, i.e., a simply connected covering space.

Idea of the proof:

Let $\tilde{X} = \{[\gamma] \mid \gamma$ is a path in X starting at $x_0\}$, where $[\gamma]$ denote the homotopy class of γ with respect to homotopies fixing the endpoints $\gamma(0)$ and $\gamma(1)$.

Also define $p: \tilde{X} \longrightarrow X$.

$$[\gamma] \longmapsto \gamma(1)$$

Clearly, p is surjective.

Next, let $\mathcal{U}$ denote the collection of path connected open sets $U \subseteq X$

s.t. $\pi_1(U) \to \pi_1(X)$ is trivial.
Notice that if $V \subseteq U \in \mathcal{U}$ then
$\pi_1(V) \to \pi_1(U) \to \pi_1(X)$ is also
trivial and thus $V \in \mathcal{U}$.
Hence, $\mathcal{U}$ is a basis for
the topology on X.
Now given a set $U \in \mathcal{U}$ and
a path γ in X from x_0 to a
point in U, let

$$U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1)\}.$$

- $U_{[\gamma]}$ depends only on $[\gamma]$.
- Since U is path connected $p: U_{[\gamma]} \to U$ is onto
- p is also injective.

This is because η and η' are homotopic in X via homotopy fixing the end points, because $\pi_1(U) \to \pi_1(X)$ is trivial.

Thus $[\gamma \cdot \eta] = [\gamma \eta']$.

[<u>Conclusion</u> $p : U_{[\gamma]} \to U$ is a bijection. Using p we can carry the topology on U to $U_{[\gamma]}$ making $p|_{U[\gamma]}$ a homeomorphism.]

- The sets $U_{[\gamma]}$ form a basis for a topology on $\tilde{X}$ so that each $p|: U_{[\gamma]} \to U$ is a homeo morphism.

- $p: \tilde{X} \to X$ is a continuous map.
- $p: \tilde{X} \to X$ is a covering projection.
- $\tilde{X}$ is simply connected.

Theorem (Classification)

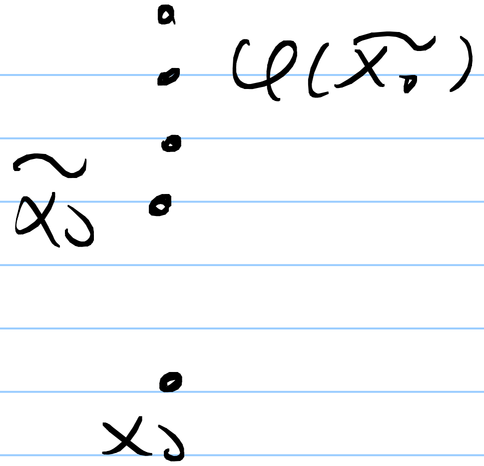
Suppose X is a path connected, locally path connected and semilocally simply connected. Then there is a bijection between the set of base point preserving isomorphism classes of path connected covering spaces $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$

associating the covering space $(\tilde{X}, \tilde{x}_0)$ to $p_\#(\pi_1(\tilde{X}, \tilde{x}_0))$. If base points are ignored, this correspondence gives a bijection between the isomorphism classes of covering spaces and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

5) Deck Transformations and Group Actions

Given a covering space $p: \tilde{X} \to X$ of path connected spaces its isomorphisms are called Deck transformations. Clearly, they form a group denoted $G(\tilde{X})$.

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$



The unique lifting property implies that φ is determined by its image at a single point.
Hence, $|G(\tilde{X})| \leq |p^{-1}(x_0)| =$ the number of sheets of the covering.

<u>Defn</u> A covering $p : \tilde{X} \to X$ is called normal if $p_{\#}(\pi_1(\tilde{X}, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$ is a normal subgroup.

<u>Proposition</u> Given a covering of path connected spaces $p : \tilde{X} \to X$ the following are equivalent:

a) $p:\tilde{X}\to X$ is normal

b) For any $\tilde{x}$ and $\tilde{x}'$ in $\tilde{X}$ with $p(\tilde{x})=p(\tilde{x}')$ then is a deck transformation $\varphi\in G(\tilde{X})$ s.t. $\varphi(\tilde{x})=\tilde{x}'$.

c) For any $[\gamma]\in\pi_1(X,x_0)$ either all lifts of γ are loops or all lifts of γ are non loops.

Moreover, in this case, $\tilde{X}=X/G(\tilde{X})$ and $\pi_1(X)/p_\#(\pi_1(\tilde{X}))\simeq G(\tilde{X})$.

<u>Proposition</u>: For a covering spaces of path connected spaces $G(\tilde{X})$ is isomorphic to $N(H)/H$ where $H=p_\#(\pi_1(\tilde{X},\tilde{x}_0))$ and $N(H)$ is its normalizer.

6) Applications to Group Theory

Construct subgroups of free groups of a given index.
See examples at page 57.

Exercise 1) Find a normal subgroup N of F_2 such that F_2/N is isomorphic to the group of isometries of a cube/tetrahedron or any other platonic solid.

2) If $F_n \leq F_m$ then

then $m-1 \mid n-1$ and $[F_n : F_m] = \frac{n-1}{m-1}$.

7) Branched Coverings of Surfaces

Let G be a finite group acting freely on a simplicial complex K via simplicial homeomorphisms. Then the quotient space $X = |K|/G$ has a cell structure, say $L = K/G$.

Proposition If K, G and L are as above then $\chi(K) = |G|\,\chi(L)$.

Now suppose that there is a finite set of vertices $\{x_1, \ldots, x_N\}$ in K so that the action of G on $X - \{x_1, \ldots x_N\}$ is free. Suppose that $\{x_1, \ldots, x_n\} = O_1 \cup \ldots \cup O_n$ where each $O_i = Gx_j$ for some $j \in \{1, \ldots, N\}$, and $O_i \cap O_j = \emptyset$ if $i \neq j$.

Then the same arguments shows that $\chi(K) - N = |G|(\chi(L) - n)$

This is known as <u>Riemann–Hurwitz Theorem</u> in Algebraic Geometry.

In this case, we say that $K \to L$ is a branched cover with branch locus $\{x_1 \ldots x_n\}$.

<u>Theorem (Hurwitz)</u>

If G is a finite group acting on a finite 2-dim'l simplicial complex K so that $|K| = \Sigma_g$ and the action is free outside finitely many vertex. Then

$|G| \leq 84(g-1)$ provided that $g \geq 2$.

Proof: $C=\{p\in\Sigma_g \mid |\mathrm{Orb}_G(p)|<|G|\}$

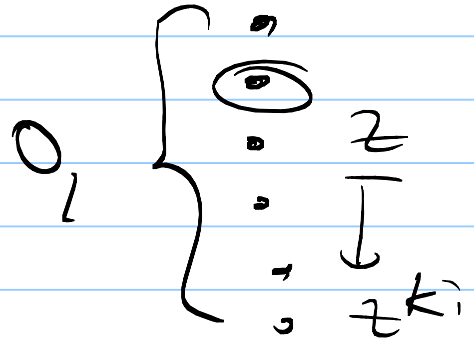
i.e. $\mathrm{Stab}_G(p)\neq \mathrm{id}$.

Then G acts freely on $\Sigma_g - C$.

By assumption C is a finite set and G acts on C also.

$$C = O_1\cup O_2\cup \cdots \cup O_n$$

$$\begin{array}{c}\Sigma_g \\ \downarrow \\ \Sigma_h=\Sigma_g/G\end{array}$$



$$k_i=\frac{|G|}{|O_i|}$$

Let $N=|O_1|+\cdots+|O_n|=|C|$

Then by the Riemann-Hurwitz theorem

$$2-2g-N=|G|(2-2h-n)$$

$$\Rightarrow 2-2g-\sum_{i=1}^{n}|O_i|=|G|\left(2-2h-\sum_{i=1}^{n}1\right)$$

$$\Rightarrow 2-2g-\sum_{i=1}^{n}\frac{|G|}{k_i}=|G|\left(2-2h-\sum_{i=1}^{n}1\right)$$

$$\Rightarrow 2(1-g)=|G|\left(2-2h-\sum_{i=1}^{n}(1-1/k_i)\right)$$

$$\Rightarrow 2(g-1)=|G|\left(2(h-1)+\sum_{i=1}^{n}(1-1/k_i)\right)$$

$$|G|=\frac{2(g-1)}{2(h-1)+\sum_{i=1}^{n}(1-1/k_i)}$$

Here $h\geq 0$ and all $k_i\geq 2$. So to find an upper bound for $|G|$ we need to minimize

$$2(h-1)+\sum_{i=1}^{n}(1-1/k_i).$$

Smallest value is obtained for $n=4$ is obtained at

$h=0$, $k_1=2$, $k_2=2$, $k_3=2$, $k_4=3$
which gives

$$-2+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{2}{3}=\frac{1}{6}$$

$\Rightarrow |G| \le 12(g-1)$.

and for $0<n\le 3$ at $h=0$, $n=3$,
$k_1=2$, $k_2=3$, $k_3=7$ which gives

$$-2+\frac{1}{2}+\frac{2}{3}+\frac{6}{7}=\frac{1}{42}$$

$\Rightarrow |G|\le 42\cdot 2(g-1)=84(g-1)$.

<u>Remark</u> Indeed one can show that if there are more that $n\geq 4$ singular fibers then $|G|\le 12(g-1)$.

Proposition: Under the assumptions of Riemann-Hurwitz theorem $g(K) \geq g(L)$.

Proof: We have as before $2-2g-N = |G|(2-2h-n)$. Also note that $|G|n \geq N$ (in fact $|G|n > N$ if $N>0$). Then $2-2g = (2-2h)|G| + N - n|G|$.

$\Rightarrow 2-2g \leq (2-2h)|G|$.

Since $|G| \geq 2 \Rightarrow$

$g-1 \geq 2(h-1) = 2h-2$

$\Rightarrow g-h \geq h-1$.

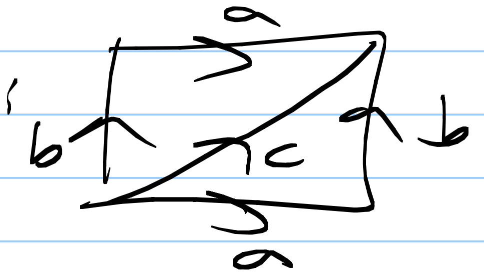
If $h \geq 1$ then $g \geq h$ and if $h=0$ then $g \geq 0 = h$. $\blacksquare$

HOMOLOGY

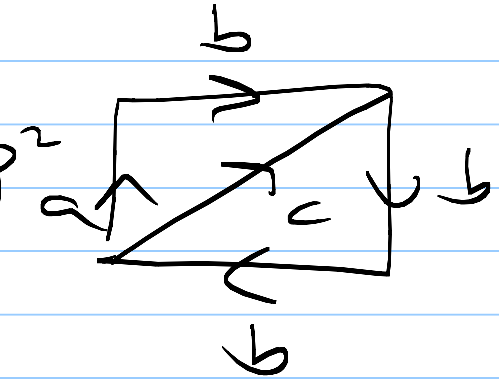
1) Simplicial Homology

Δ-complex

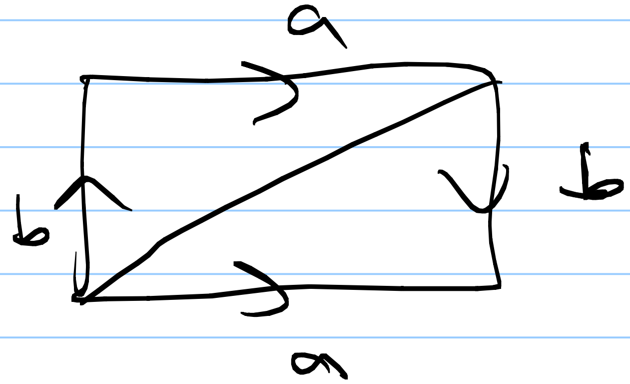
T^2:



$\mathbb{RP}^2$:



KB:



Recall that

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_{i=0}^{n} t_i = 1\}$$

is the standard n-simplex.

More generally, if $\{v_0, v_1, \dots, v_n\}$ is a set of vectors in $\mathbb{R}^m$ such that $\{v_1 - v_0, \dots, v_n - v_0\}$ is a linearly independent set then the n-simplex determined by $\{v_0, \ldots, v_n\}$ is defined by

$$[v_0, \ldots, v_n] \doteq \left\{ \sum_{i=0}^{n} t_i v_i \;\middle|\; t_i \geq 0, \sum_{i=0}^{n} t_i = 1 \right\}.$$

Note that

$$\Delta^n \longrightarrow [v_0, \ldots, v_n]$$

$$(t_0, \ldots, t_n) \longmapsto \sum_{i=0}^{n} t_i v_i \quad \text{is a}$$

homeomorphism. Any simplex $[v_{i_1}, \ldots, v_{i_k}]$, $i_1, \ldots, i_k \in \{0, 1, \ldots, n\}$ is called a face on $[v_0, \ldots, v_n]$.

A delta complex is a quotient space of some disjoint union of simplicies, where certain faces of simplices are identified by linear isomorphisms. Note that Δ-complexes are naturally

CW-complexes.

<u>Simplicial Homology</u>: Let X be a Δ-complex. Define

$\Delta_n(X)$ = free abelian group with basis the open n-simplices e^n_α of X.

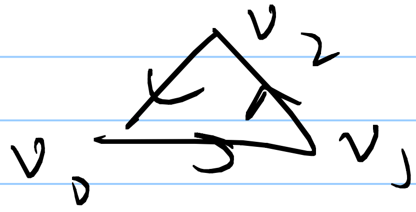
Elements of $\Delta_n(X)$ are called n-chains: $\sum_\alpha n_\alpha e^n_\alpha$, $n_\alpha \in \mathbb{Z}$, $n_\alpha = 0$ for all but finitely many α.

<u>Boundary of a simplex</u>

$$\partial([v_0, \ldots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n].$$

<u>Ex</u> $\partial[v_0, v_1] = [v_1] - [v_0]$

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1].$$



Lemma $\partial_n \circ \partial_{n-1} = 0$.

Remark $\partial_n \circ \partial_{n-1} = 0 \iff$
$\operatorname{Im} \partial_{n-1} \subseteq \ker \partial_n$.

Definition The nth simplicial homology of Δ-complex X is defined to be the quotient group $H_n^{\Delta}(X) = \dfrac{\ker \partial_n}{\operatorname{Im} \partial_{n-1}}$

Example $S^1, T^2, S^n, \mathbb{RP}^2$.

2) **Singular Homology** Consider maps $\sigma : \Delta^n \to X$ (continuous)
$C_n(X)$ = the free abelian group with basis $\{\sigma : \Delta^n \to X \mid \sigma \text{ is continuous}\}$.

$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$$
$$\sum n_\alpha \sigma_\alpha \longmapsto \sum n_\alpha \partial \sigma_\alpha$$
where if $\sigma_\alpha : [v_0, v_1, \ldots, v_n] \to X$

then $\partial \sigma_\alpha = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}$

Clearly, $\partial_{n+1} \circ \partial_n = 0$ and thus the nth singular homology of X is defined by

$$H_n(X) = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}.$$

<u>Remark</u> Compare the two homology theories.

<u>Proposition</u>: If X is nonempty and path connected then $H_0(X) \cong \mathbb{Z}$.

Hence if $X = \bigcup_\alpha X_\alpha$, where each X_α is path connected, then

$H_0(X) \simeq \bigoplus_\alpha \mathbb{Z}$.

3) If $f: X \to Y$ is a continuous map then $f_\#: C_n(X) \to C_n(Y)$ by $f_\#(\sigma) = f \circ \sigma : \Delta^n \to Y$ and then

$$f_\#\left(\sum_i n_i \sigma_i\right) = \sum_i n_i f_\#(\sigma_i)$$

$$= \sum_i n_i (f \circ \sigma_i).$$

Note that $f_\#(\partial\sigma) = \partial f_\#(\sigma)$

and thus we get a commutative diagram of chains complexes:

$$\begin{array}{ccccccc}
\cdots \to & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \to \cdots \\
& \downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# & \\
\cdots \to & C_{n+1}(Y) & \longrightarrow & C_n(Y) & \longrightarrow & C_{n-1}(Y) & \to \cdots
\end{array}$$

$\Rightarrow f_* : H_n(X) \longrightarrow H_n(Y).$

Note that

i) $(fg)_* = f_* g_*$

ii) $(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$, for all n.

<u>Theorem</u>: If two map $f, g : X \to Y$ are homotopic then $f_* = g_*$.

<u>Corollary</u>: If $f : X \to Y$ is a homotopy equivalence then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism.

<u>Definition</u> (Reduced homology)

$$\cdots \to C_n(X) \to C_{n-1}(X) \to \cdots$$

$$\to C_1(X) \to C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

$\epsilon: C_0(X) \to \mathbb{Z}$ by

$\epsilon(\sum n \sigma_\alpha) = \sum n$.

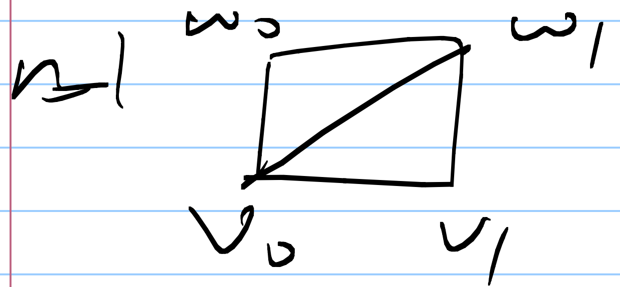
Its homology is called the <u>reduced</u> homology of X and denoted by $\tilde{H}_n(X)$.

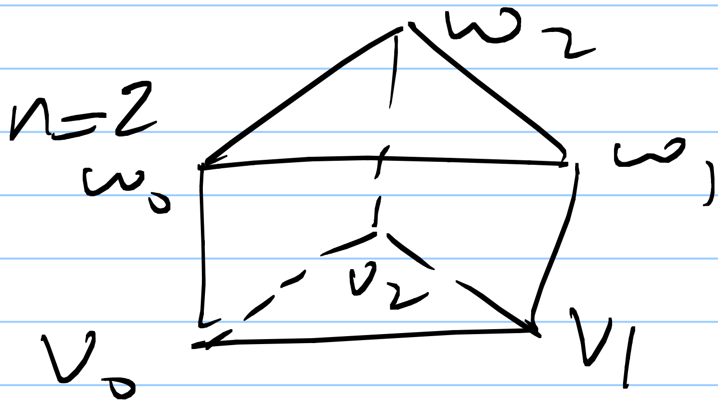
<u>Proposition</u> $\tilde{H}_n(X) = H_n(X)$ if $n > 0$ and $\tilde{H}_0(X) \oplus \mathbb{Z} \cong H_0(X)$.

<u>Proposition</u> If X is contractible then $\tilde{H}_n(X) = 0$ for all n.

<u>Idea</u> of <u>the proof</u> of <u>the Theorem</u>,

Subdivide $\Delta^n \times I$ into $(n+1)$-simplices





$$\Delta^n \times I = \bigcup_{i=0}^{n} [v_0, \ldots, v_i, w_i, \ldots, w_n]$$

$$\Delta^n \times \{0\} = [v_0, \ldots, v_n]$$

$$\Delta^n \times \{1\} = [w_0, \ldots, w_n]$$

Given a homotopy $F: X \times I \to Y$ define the Prism operator

$$P: C_n(X) \longrightarrow C_{n+1}(Y)$$

<u>Claim</u> $\partial P = g_\# - f_\# - P\partial$

where $f(x) = F(x,0)$, $g(x) = F(x,1)$.

<u>Claim</u> <u>proves the theorem:</u>

For any cycle $\alpha \in C_n(X)$,

$$g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + \underbrace{P\partial(\alpha)}_{0}$$

$$= \partial P(\alpha)$$

$\Rightarrow [g_\#(\alpha)] = [f_\#(\alpha)]$

$\Rightarrow g_*(\alpha) = f_*(\alpha)$ $\blacksquare$

4) Exact Sequences and Excision:

X space, $A \subseteq X$ subspace.

We'll relate $H_*(X)$, $H_*(A)$ and $H_*(X/A)$.

Chain complex:

$$\to A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to$$

$\partial_n \circ \partial_{n+1} = 0$ or equivalently

$\mathrm{Im}\, \partial_{n+1} \subseteq \ker \partial_n$.

It is called exact if

$$\mathrm{Im}\, \partial_{n+1} = \ker \partial_n.$$

Remark

1) $0 \to A \xrightarrow{\alpha} B$ is exact iff

$\ker \alpha = 0 \iff \alpha$ is injective.

2) $A \xrightarrow{\alpha} B \to 0$ is exact iff $\operatorname{Im} \alpha = B \iff \alpha$ is surjective.

3) $0 \to A \xrightarrow{\alpha} B \to 0$ is exact iff α is an isomorphism

4) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact iff α is injective, β is surjective and $\operatorname{Im} \alpha = \ker \beta$.

<u>Theorem</u> If X is a space and A is a nonempty closed subset that is deformation retract of some neighborhood in X, then there is an exact sequence

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{i_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to$$

where $i : A \hookrightarrow X$ is the inclusion, $j : X \to X/A$ is the quotient map and ∂ is the boundary homomorphism.

Remark If X is CW complex and $A \subseteq X$ is subcomplex then (X, A) is a such pair, called a good pair.

Corollary $\tilde{H}_n(S^n) \simeq \mathbb{Z}$ and $\tilde{H}_i(S^n) = 0$ if $i \neq n$.

Corollary ∂D^n is <u>not</u> a retract of D^n and thus any map $f : D^n \to D^n$ has a fixed point.

Idea of proof $A \subseteq X$

Define $C_n(X,A) \doteq \dfrac{C_n(X)}{C_n(A)}$.

Note that $\partial : C_n(X) \to C_{n-1}(X)$ induces a homomorphism $\partial : C_n(X,A) \to C_{n-1}(X,A)$.

Clearly $\partial^2 = 0$ in

$$\to C_{n+1}(X,A) \xrightarrow{\partial_{n+1}} C_n(X,A) \xrightarrow{\partial_n} C_{n-1}(X,A)$$

and thus we may define

$$H_n(X,A) = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$$

Lemma If $0 \to A_\alpha \to B_\alpha \to C_\alpha \to 0$ is an exact sequence of chain complexes then there is a long exact sequence in homology

$\to H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\partial} H_{n-1}(C)$

Remark

$0 \to C_*(A) \to C_*(X) \to C_*(X,A) \to 0$

$\Rightarrow \cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A)$

Remark $B \subseteq A \subseteq X$

$0 \to C_n(A,B) \to C_n(X,B) \to C_n(X,A) \to 0$

$\Rightarrow \cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A)$

$\to H_{n-1}(A,B) \to \cdots$

Theorem (Excision)

Let $Z \subseteq A \subset X$ subspaces such that $\overline{Z} \subset \operatorname{Int}(A)$. Then the inclusion $(X-Z, A-Z) \hookrightarrow (X,A)$ induces isomorphism

$H_n(X-Z, A-Z) \xrightarrow{\sim} H_n(X,A), \ \forall n.$

Remark Note that

$Z \subseteq A \subseteq X$, $\overline{Z} \subseteq \mathrm{Int}(A)$

implies $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$

where $B = X - Z$.

Idea of the proof uses

$\boxed{C_n(X) \simeq C_n(A+B)}$ & critical point.

$$\Rightarrow \frac{C_n(X)}{C_n(A)} \simeq \frac{C_n(A+B)}{C_n(A)}$$

$$\Rightarrow H_n(X,A) \simeq H(\quad)$$

Also $\dfrac{C_n(B)}{C_n(A \cap B)} \longrightarrow \dfrac{C_n(A+B)}{C_n(A)}$

is also an isomorphism since both quotient groups are free with basis the singular n-simplices in B that do not lie in A. Then

$$H_n(B, A\cap B) \simeq H_n(X,A)$$

$$\|$$

$$H_n(X-Z, A-Z)$$

$\blacksquare$

<u>Proposition</u> For good pairs (X,A) the quotient map

$q:(X,A) \longrightarrow (X/A, A/A)$ induces isomorphism

$$q_*: H_n(X,A) \to H_n(X/A, A/A) \simeq \tilde{H}_n(X/A)$$

$\forall n.$

5) Singular Homology is equal to the simplicial Homology. For a simplicial complex X

$$H_n^{\Delta}(X) \simeq H_n(X).$$

6) Betti numbers and Euler Characteristic.