**Vector Bundles and Poincaré–Hopf Theorem**

**Euler Characteristic (Knots) of an oriented Vector Bundle**

Let \( E \rightarrow E^{m+k} \) be an oriented \( R^k \)-bundle over a smooth manifold \( M \).

Let \( L = s(M), \) where \( s: M \rightarrow E, \) \( s(p) = (p,0), \) the zero section. \( \pi \circ s = Id_M, \) \( \pi(s(p)) = \pi(p,0) = p, \)

\( L \subseteq E \) is a closed submanifold diffeomorphic to \( M. \)

Let \( K = L \cap \tilde{L} \) be the transverse self-intersection of \( L. \)

The \( K \) is a submanifold in \( E \) of dimension \( \dim(L) + \dim (\tilde{L}) - \dim E = m + m + k = m-k. \)

Clearly, \( K \subseteq L \) is a submanifold of \( L. \) Since \( s: M \rightarrow L \) is a diffeomorphism by identifying \( M \) with \( L \), we may regard \( K \) as a submanifold of \( M. \)

Assuming \( M \) is oriented also, \( K \subseteq M \) is an oriented submanifold of dimension \( m-k. \) The \( \tilde{E} \cap H_{\text{DR}}^k(M) \) the Poincaré dual of \( K \) will be called the Euler class of the vector bundle.

**Notation:** \( e(E) = \{p_2\}. \)

Remark: \( R^n \rightarrow E \rightarrow M^n, \) where both \( M \) and the bundle are oriented. Then the Euler class \( e(E) \in H_{\text{DR}}^n(M) \simeq R. \) By assumption...
that $M$ is compact. The real number
\[ \int_{E} e(E) \in \mathbb{R} \] is called the Euler number of $M$ the vector bundle.

\[ \dim K = m + m - (m + m) = 0. \Rightarrow [p_{K}] \in H_{2m}^{m}(M). \]

\[ \text{Int}(K) / M = \int_{M} e(M) \] the oriented (signed) sum of points in $K$.

This integer is also called the Euler number of the bundle $E \to M$.

2) \[ k = m + 1 \pmod{2} \] then the self intersection \[ \text{Int}(L_{1} \smile L_{2}) = 1 \] is zero.

3) If a bundle $E \to M$ has a section $s: M \to E$ so that $s(p)$ is never zero, then $e(M)$ never intersects the zero section. Hence $e(M) = 0$.

**Examples**
1) $M = T^{2} = S^{1} \times S^{1}$, $E \cong T^{2} \overset{\theta_{1}, \theta_{2}}{\to} T^{2}$

\[ x(\theta_{1}, \theta_{2}) = \frac{\partial}{\partial \theta_{1}} \] is a nonzero zero section of $T^{2}$.

Hence, $e(E) = 0$.

**Definition:** For a manifold $M$ the Euler number of its tangent bundle $TM$ is called also the number number of $M$ and we sometimes denote it by $e(M)$.

\[ e(M) = e(TM). \]
So, $\langle LT^2 \rangle = 0$. Similarly, $e(T^n) = 0$ for any $n$.

2) $T^{\mathbb{C}P} = S^2$,

$$\mathbb{C} \times \mathbb{C}P^1 = \mathbb{T} \times \mathbb{C} \times \mathbb{T} \times \mathbb{C}$$

$$\left( \frac{z}{i}, \frac{w}{i} \right) \sim \left( \frac{z}{i}, -\frac{w}{i} \right) \quad (z \neq 0)$$

$$S_3 = 1$$

$L \leq T^1 \mathbb{C}P^1$ complex submanifold

Here, $S^1 \times i^2$ is a complex submanifold of $\mathbb{C}P^1$.

Therefore, both $I$ and $-I$ have $+1$ orientation.

$e(T^1 \mathbb{C}P^1) = 1 + 1 = 2$.

Remark: For the manifolds $S^1$ and $S^2$, the Euler characteristics and Euler numbers agree. This is not a coincidence. Indeed, it is the content of so-called Poincaré-Hopf Theorem.

Now consider the complex line bundle $\mathcal{O}(k) \to \mathbb{C}P^1$

$$\mathcal{O}(k) = \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$$

$$
\left( \frac{z}{i}, \frac{w}{i} \right) \sim \left( \frac{z}{i}, \frac{w}{i} \right) \quad (z \neq 0)
$$

$$S_3(z) = \frac{1 + \lambda}{2} = -S_3(z)$$

is a section of $\mathcal{O}(k)$.

This section has $k$-complex zeros. Hence,

$$e(\mathcal{O}(k)) = k.$$
3) \( L^2 \leq M^{2e} \), \( L \) non-orientable, \( M \) oriented

\( \mathcal{L} : \) compact submanifold.

As an example, take \( L = \mathbb{RP}^2 \) in \( \mathbb{CP}^2 = M \).

We may define the integer self intersection of \( L \) within itself in \( M \).

\[
\text{Oriented } \mathcal{T}_x L \otimes \mathcal{T}_x L = \mathcal{T}_x M
\]

orientable \( \Rightarrow (\nu_1, \nu_2), (\nu_3, \nu_4), (\nu_5, \nu_6) \Rightarrow (\nu_1, \nu_2)
\]

Then we may define the integer self intersection of a non-orientable compact submanifold \( L^2 \) in an oriented manifold \( M^{2e} \).

**Proposition:** The self intersection of \( \mathbb{RP}^2 \) in its tangent bundle (oriented suitably) is equal one: \( e(\mathbb{RP}^2) = 1 \).

**Proof:** \( S^2 \xrightarrow{\pi} \mathbb{RP}^2 = S^2 \)

\[
(x, y) \sim (-x, -y, 2)
\]

\( e(\mathbb{RP}^2) = 1 \).
4) Let’s compute the self intersection of $\mathbb{RP}^2$ in $\mathbb{R}^3$.

$z_1 = x_1 + x_2$, $z_2 = x_3$. Local chart on $\mathbb{RP}^2$.

$x_1, y_1$ local chart on $\mathbb{R}P^2$.

$p \in \mathbb{R}P^2 \subseteq \mathbb{R}^2$

$T_p \mathbb{R}P^2 = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right)$, $T_p \mathbb{R}^2 = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \right)$.

$T_p \mathbb{R}P^2 \to T_p \mathbb{R}^2$; normal bundle $\nu_{\mathbb{R}P^2}$ in $\mathbb{R}^2$.

$(p, \nu = a \frac{\partial}{\partial x_1} + b \frac{\partial}{\partial x_2}) \mapsto (p, \nu = a \frac{\partial}{\partial x_1} + b \frac{\partial}{\partial y_1})$.

$T_p \mathbb{R}P^2 = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} \right)$

$= - \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} \right)$

$= - T_p \mathbb{R}P^2 \oplus \nu_{\mathbb{R}P^2}$

$= T_p \mathbb{R}P^2 \oplus (- \nu_{\mathbb{R}P^2})$

So the isomorphism $T_p \mathbb{R}P^2 \to T_p \mathbb{R}^2$ given by multiplication with $i$ reverses the orientation. In other words, the problem orientation on $\nu_{\mathbb{R}P^2}$ is given by $\left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_2} \right)$.

Hence, the $e(-\nu_{\mathbb{R}P^2}) = e(\mathbb{R}P^2) = 1$. In other words, the self intersection of $\mathbb{RP}^2$
$CD^2$ is $-1$.

5) Let's compute the self-intersection of $\mathbb{RP}^2$ in $CD^2$ directly.

$\phi_6: \mathbb{RP}^2 \to \mathbb{RP}^2$, $[z_0: z_1: z_2] \mapsto [\frac{z_0}{z_0 + z_1 + z_2}: \frac{z_1}{z_0 + z_1 + z_2}: \frac{z_2}{z_0 + z_1 + z_2}]$.

$\mathbb{RP}^2 = \{[z_0: z_1: z_2] \in \mathbb{RP}^2 : \text{Im}(z_i) = y_i = 0, i = 0, 1, 2\}$.

$\phi_6(\mathbb{RP}^2) = \mathbb{RP}^2_6 = \{[x_0: x_1: x_2] \in \mathbb{RP}^2 : \frac{2t^4}{x_0^2}, \frac{2t^4}{x_0^2}, \frac{2t^4}{x_0^2}, \frac{2t}{x_0} \} \mid x_i \in \mathbb{R}^2$}

$\mathbb{RP}^2 \cap \mathbb{RP}^2_6$ for $t > 0$ and small.

Let's compute the sign of intersection at each point:

1) $[0: 0: 0]$. $U_0 = \{z_0 \neq 0\}$ local chart in $U_0$ for $\mathbb{RP}^2$.

$\mathbb{RP}^2$ and $\mathbb{RP}^2_6$ are given by

$(\frac{x_1}{x_0}, \frac{x_1}{x_0})$ and $(\frac{2t}{x_0}, \frac{2t}{x_0}, \frac{2t}{x_0})$.

Putting these coordinates side by side we get

$(\frac{x_1}{x_0}, \frac{x_1}{x_0}, \frac{2t}{x_0}, \frac{2t}{x_0}, \frac{2t}{x_0})$. Compare this with the complex orientations of $CD^2$ using the chart $(\frac{z_0}{x_0}, \frac{z_0}{x_0}, \frac{z_0}{x_0})$, which is

$(\frac{z_0}{x_0}, \frac{z_0}{x_0}, \frac{z_0}{x_0}, \frac{z_0}{x_0}, \frac{z_0}{x_0})$. Hence, the orientations do not match at this point; hence, the
sign of intersection at $\Gamma(1;0;2)$ is $-1$.

2) $[0;1;0] \quad u_1 = \frac{1}{2} \chi_1 + \chi_2 \quad \left(\frac{\chi_0}{\chi_1}, \frac{\chi_2}{\chi_1}\right)$

$\mathbb{R}^2 \quad \mathbb{R}^2$

$(\frac{x_0}{x_1}, \frac{x_2}{x_1}) \quad (\frac{\chi_1}{\chi_0}, \frac{x_2}{x_1})$

$\Rightarrow \left(\frac{x_0}{x_1}, \frac{x_2}{x_1}, -\frac{\chi_1}{\chi_0}, \frac{x_2}{x_1}\right)$

Now, the orientation for $\mathbb{C}P^2$ in this coordinate is given by:

$(\frac{x_0}{x_1}, \frac{\chi_1}{\chi_0}, \frac{x_2}{x_1}, \frac{\chi_1}{\chi_0})$

$\Rightarrow \begin{pmatrix} \frac{x_0}{x_1} & \frac{x_2}{x_1} \\ -\frac{\chi_1}{\chi_0} & \frac{x_2}{x_1} \end{pmatrix}$

Hence, the sign of the intersection $[0;1;0]$ is $+1$.

3) $[0;0;1] \quad u_2 = \frac{1}{2} \chi_2 + \chi_3 \quad \left(\frac{\chi_0}{\chi_2}, \frac{\chi_3}{\chi_2}\right)$

$\mathbb{R}^2 \quad \mathbb{R}^2$

$(\frac{x_0}{x_2}, \frac{x_3}{x_2}) \quad (\frac{\chi_2}{\chi_0}, \frac{x_3}{x_2})$

$\Rightarrow \left(\frac{x_0}{x_0}, \frac{x_3}{x_2}, -\frac{\chi_2}{\chi_0}, \frac{x_3}{x_2}\right)$

However, the (complex) orientation at this point is:

$\Rightarrow \begin{pmatrix} \frac{x_0}{x_0} & \frac{x_3}{x_2} \\ -\frac{\chi_2}{\chi_0} & \frac{x_3}{x_2} \end{pmatrix}$

Hence, the sign of intersection $[0;0;1]$ is $-1$. 

So the total intersection number is 
\(-1 + 1 - 1 = -1\).
Remark: $e(C_1 R^2) = 1$ i.e., $\text{Int}(C_1 R^2, R^2) = 1$ in $T_x R^2$.

Also, considered as a submanifold of $C_2 R^2$ in $C_1 R^2$, it has $\text{Int}(C_1 R^2, R^2) = -1$, i.e., $e(C_1 R^2) = -1$.

Lemma: Let $M^n$ be a smooth manifold. Then $TM$ is orientable and has a canonical orientation.

Proof: Let $U \subseteq R^n$ be an open subset with coordinates $x_1, ..., x_n$. Then $T_x U = U \times R^n$ has coordinates $x_1, ..., x_n, y_1, ..., y_n$, where $y_i : T_x U \to R, y_i (\frac{\partial}{\partial x_j}) = \delta_{ij}$.

Let $V \subseteq R^n$ be another open subset with coordinates $\tilde{x}_1, ..., \tilde{x}_n$ and $F : U \to V$ a diffeomorphism.

Let $(x_1, ..., x_n) = (\tilde{f}_1(x_1, ..., x_n), ..., \tilde{f}_n(x_1, ..., x_n)) = F(x_1, ..., x_n).

On the other hand, we know that $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$ and $\frac{\partial}{\partial \tilde{x}_1}, ..., \frac{\partial}{\partial \tilde{x}_n}$ are bases for $T_{F(p)} U$ and $T_{F(p)} V$, where $A(x) = (DF_x)^T_0 (\frac{\partial}{\partial x}_i) = (\frac{\partial \tilde{f}_i}{\partial x_j}) (p)$.

Let $\tilde{y}_1, ..., \tilde{y}_n$ be coordinates on $T_{F(p)} V$ given by

$\tilde{y}_i (\frac{\partial}{\partial \tilde{x}_1}, ..., \frac{\partial}{\partial \tilde{x}_n}) = \delta_{ij}.$

Hence, the diffeomorphism $\phi = (F, DF) : T_x U \to T_{F(p)} V$

between the total spaces of tangent bundles is given in coordinates as
\[ \psi(x_1, \ldots, x_n, y_1, \ldots, y_n) = (F(x_1, \ldots, x_n), A(p)(y_1, \ldots, y_n)). \]

So its Jacobian matrix has the form

\[
D\psi_{E,v} = \begin{bmatrix}
  D_F & O \\
  A(p) & O
\end{bmatrix}, \text{ which has determinant } \det(D\psi_{E,v}) = \det A(p) \det(A(p)) = (\det A(p))^2 > 0.
\]

This finishes the proof.

**Example:** \( F(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2)) \), \( F : \mathbb{R}^2 \to \mathbb{R}^2 \).

\[
A = D_F = \begin{bmatrix}
  \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
  \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{bmatrix}, \text{ so } \begin{bmatrix}
  \frac{\partial \psi_1}{\partial x_1} \\
  \frac{\partial \psi_1}{\partial x_2}
\end{bmatrix} = A \begin{bmatrix}
  \psi_1 \\
  \psi_2
\end{bmatrix}
\]

\[
\psi(x_1, x_2, y_1, y_2) = (f_1, f_2, y_1 \frac{\partial f_1}{\partial x_1} + y_2 \frac{\partial f_1}{\partial x_2}, y_1 \frac{\partial f_2}{\partial x_1} + y_2 \frac{\partial f_2}{\partial x_2}).
\]

So

\[
D\psi = \begin{bmatrix}
  \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
  \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{bmatrix} \begin{bmatrix}
  \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
  \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{bmatrix} = \begin{bmatrix}
  \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
  \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{bmatrix} \begin{bmatrix}
  \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
  \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{bmatrix}
\]

where \( \begin{bmatrix}
  \psi_1 \\
  \psi_2
\end{bmatrix} = (\text{Hess}(f_1)(\psi_0))' \) and \( \begin{bmatrix}
  \psi_1 \\
  \psi_2
\end{bmatrix} = (\text{Hess}(f_2)(\psi_0))' \).
For the self-intersection of $\mathbb{RP}^2$ in $\mathbb{CP}^2$ we note the following fact that the tangent bundle of $\mathbb{CP}^2$ restricted to $\mathbb{RP}^2$ is the complexification of the tangent bundle of $\mathbb{RP}^2$:

$$T_*\mathbb{CP}^2 |_{\mathbb{RP}^2} = T_*\mathbb{RP}^2 \otimes \mathbb{C}.$$ 

Now let's consider the computations

$$e(\mathbb{RP}^2) = \text{Int}(\mathbb{RP}^2, \mathbb{RP}^2) \in T_*\mathbb{RP}^2$$

and

$$e(\mathbb{CP}^2) = \text{Int}(\mathbb{RP}^2, \mathbb{CP}^2) \in \mathbb{CP}^2.$$ 

If $x_1, x_2$ are coordinates on $\mathbb{RP}^2$ then the orientation on $T_*\mathbb{RP}^2$ is "given basically" 

$$\frac{2}{2x_1}, \frac{-2}{2x_2}, \frac{2}{2y_1}, \frac{2}{2y_2}, \quad y_1 \left(\frac{\partial}{\partial x_1}\right) = \delta_{15}. $$

On the other hand, the coordinates on $\mathbb{CP}^2$ is given by $\bar{z}_1 = x_1 + iy_1$, $\bar{z}_2 = x_2 + iy_2$ are thus the complex orientation on the tangent bundle is given by 

$$\frac{2}{2\bar{z}_1}, \frac{-2}{2\bar{z}_2}, \frac{2}{2\bar{y}_1}, \frac{2}{2\bar{y}_2}, \quad \bar{y}_1 \left(\frac{\partial}{\partial \bar{z}_1}\right) = \delta_{15}. $$

Note that the two orientations do not match!

Remark: If $X$ is a submanifold of a complex manifold $M^n$ ($\dim M = n$) so that

$$T_*X \otimes \mathbb{R} \mathbb{C} = T_*M |_{X},$$

then the complex orientation on $T_*M$ restricted to $X$ is given...
by \( \frac{2}{\partial x_1}, \frac{2}{\partial y_1}, \frac{2}{\partial \phi_2}, \ldots, \frac{2}{\partial x_n}, \frac{2}{\partial y_n}, \frac{3}{\partial \phi_3} \), when as the orientation on \( \mathbb{T}^n \times \mathbb{R} \) is given by \( \frac{2}{\partial x_1}, \ldots, \frac{2}{\partial x_n}, \frac{2}{\partial y_n}, \frac{3}{\partial \phi_3} \).

Hence, the two orientations differ by

\[ 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2} \text{ transpositions}. \]

Hence,

\[ e(X) = (-1)^{\frac{n(n-1)}{2}} e(V_X) \text{ or equivalently}, \]

\[ \mathbf{Int}_{\mathbb{T}^n \times \mathbb{R}}(X, X) = (-1)^{\frac{n(n-1)}{2}} \mathbf{Int}_M(X, X). \]
Gysin Exact Sequence

$r: E \rightarrow M^n$ smooth oriented vector bundle over $M$.
Put a metric on $E$ and let $P \rightarrow M$ be the unit sphere bundle of $E$.

$\pi^{-1}(p) \cong E_p \cong \mathbb{R}^r \cong S^{-1}$ the unit sphere.

**Theorem:** Assume the above setup. Then we have an exact sequence of the form

$$\cdots \rightarrow H_{DR}^{-r-1}(P) \xrightarrow{S_{-1}} H_{DR}^{-r}(M) \xrightarrow{\pi^*} H_{DR}^{-r}(M) \xrightarrow{\pi^*} H_{DR}^{-r}(P) \xrightarrow{S_{-1}} \cdots$$

Here $S_{-1}$ represents integration along fibers and $e(E)$ is the Euler class of $\pi: E \rightarrow M$.

**Proof:** uses vertically compactly supported cohomology.

\[ \mathbb{R}^r \rightarrow E \]
\[ \downarrow \pi \]
\[ M \supset \text{supp}(\omega) \]

\[ \xrightarrow{?} \]
\[ M \]
\[ \xrightarrow{E} \]

\[ \xrightarrow{d} \]
\[ \Omega_{v+1}^k(E) \]

\[ H_{v+1}^k(E) = \frac{\ker (d: \Omega_{v+1}^k(E) \rightarrow \Omega_{v+1}^{k+1}(E))}{\text{Im} (d: \Omega_{v+1}^{k+1}(E) \rightarrow \Omega_{v+1}^k(E))} \]

\[ H_{v+1}^k(E) \xrightarrow{\int_{E^r}} H_{br}^k(M) \]
 Leray-Hirsch and K"{o}nneth Theorems

**Theorem (Leray-Hirsch)**

Let \( \pi : P \rightarrow M \) be a fiber bundle, whose fibers are diffeomorphic to the manifold \( F \). Assume that a subset \( \{x_1, \ldots, x_N\} \subset H^k_{dR}(P) \) exists so that their restriction to each \( H^k_{dR}(F_x) \), \( F_x = \pi^{-1}(x) \), is a basis for \( H^k_{dR}(F_x) \). Then, \( H^k_{dR}(P) \) is a free module with basis \( \{x_1, \ldots, x_N\} \) over the subalgebra \( \pi^*(H^k_{dR}(M)) \).

![Diagram]

\[ P \xrightarrow{\pi} M \]

**Proof:** uses very similar ideas to that of the proof of Poincaré Duality.

**Special Case:** \( P_M = M \times N \rightarrow M \), where \( M \) and \( N \) are smooth manifolds. \( P_M : M \times N \rightarrow N \)

The map \( P_M^* : H^k_{dR}(N) \rightarrow H^k_{dR}(M \times N) \) is an injective map, and we may consider \( H^k_{dR}(N) \) as a subalgebra of \( H^k_{dR}(M \times N) \). Pick a basis \( \{x_1, \ldots, x_N\} \) of \( H^k_{dR}(N) \). The \( \{x_1, \ldots, x_N\} \) satisfy the condition of Leray-Hirsch Theorem for the fiber bundle \( P_M : M \times N \rightarrow M \).
So by the Lefschetz theorem $H^*_{DR}(M \times N)$ is a free module over $H^*_{DR}(M)$ with basis $\xi^1, \ldots, \xi^n \otimes \eta^1, \ldots, \eta^n$, which is an $R$-basis for the vector space $H^*_{DR}(N)$. Hence, we have

**Theorem (Künneth Formula)**

$$H^*_{DR}(M \times N) = H^*_{DR}(M) \otimes_R H^*_{DR}(N)$$

$$H^k_{DR}(M \times N) = \bigoplus_{i+j=k} H^i_{DR}(M) \otimes_R H^j_{DR}(N).$$

**Definition:** Poincaré Series of a smooth manifold whose cohomology is finite dimensional is defined as the series

$$P_{M, t} = \sum_{k=0}^{\infty} b_k(M) t^k.$$

Clearly, if $\dim M = m$, then $P_{M, t}$ is a polynomial of degree at most $m$.

**Corollary:** If $M$ and $N$ are smooth manifolds whose Poincaré series are defined, then

$$P_{M \times N, t} = P_{M, t} P_{N, t}.$$

$$\text{For } H^k_{DR}(S^1) = \begin{cases} R & i + j = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P_{S^1, t} = 1 + t. \text{ Hence, } P_{S^1 \times S^1, t} = (1 + t)^n.$$
Theorem (Poincaré–Hopf)

For any compact orientable manifold \( M \), the Euler number of \( M \) is equal to the Euler characteristic of \( M \).

**Proof**

Must show: \( \chi(M) = \int_M e(M) \).

First consider the map \( f: M \to M \times M \) given by \( f(x) = (x,x) \), for all \( x \in M \). Let \( \Delta \) denote \( f(M) \):

\[
\Delta = f(M) = \{ (x,x) | x \in M \}.
\]

Clearly, \( f: M \to \Delta \) is a diffeomorphism. Moreover, \( \nu(\Delta) \) is the normal bundle of \( \Delta \) in \( M \times M \). Then

\[
\nu(M \times M) = T_x \Delta \oplus \nu(\Delta).
\]

The normal bundle \( \nu(\Delta) \) is oriented via the equation

\[
T_x \Delta = \{ (v,v) | v \in T_x M \} \quad \text{and} \quad \nu(\Delta) = \{ (-v,v) | v \in T_x M \}.
\]

So if \( T_x M \) has oriented basis \( \{ v_1, v_2 \} \),

\[
T_x(M \times M) = T_x \Delta \oplus \nu(\Delta).
\]
The $T_{(p,p)} \Delta$ is oriented by the basis $\mathfrak{F}(v_i, v_i, \ldots, (v_i, v_i))^3$ and $\mathfrak{T}(\Delta)$ is oriented by $\mathfrak{F}(v_i, v_i, \ldots, (-v_i, v_i))^5$ and $T_{(p,p)} \{M\} \text{ oriented by } \mathfrak{F}(v_i, v_i, \ldots, (v_i, 0), (0, v_i), \ldots, (0, v_i))^3$.

Claim: The orientations of $T_{(p,p)} \{M\}$ induced by $\mathfrak{F}(v_i, v_i, \ldots, (v_i, 0), (0, v_i), \ldots, (0, v_i))^3$ and $\mathfrak{F}(v_i, v_i, \ldots, (v_i, v_i), (-v_i, v_i), \ldots, (-v_i, v_i))^3$ are the same.

Example $M = \mathbb{R}^2$, $T_p M = \{ (1,0), (0,1) \} \phi$

$T_{(p,p)} \Delta M = \mathfrak{F}(1,0,1,0), (0,1,0,1)^3$ and $\mathfrak{T}(\Delta M) = \mathfrak{F}(-1,0,1,0), (1,0,-1,0)^3$.

$$
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

$\rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}$ which is the basis $\{ (0,0,0,0), (0,0,0,1), (0,0,1,0), (0,1,0,0) \}$ for $T_{(p,p)} \{M\}$. 

$\mathfrak{F}(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)^3$. 
Claim: The map $TM \to \gamma(N)$ given by $(p, v) \mapsto (g(p), (-v, v))$ is a vector bundle isomorphism as oriented bundles.

Therefore, the self-intersection of $M$ in $TM$ is the same as the self-intersection of the diagonal in $M \times M$.

$e(M) = \text{Int}(N_0, N_0) = \text{Int}(\Delta, \Delta)$, where $N_0$ is the zero section of $M$ in $TM$.

Let $w \in H^1_{de}(M \times M)$ be Poincaré dual (in $H^1_{de}(M \times M)$) of the submanifold $\Delta$ of $M \times M$.

Then $e(M) = \text{Int}(\Delta, \Delta) = \int_{\Delta} w$.

So, we need to show that $\int_{\Delta} w = \chi(M)$.

Let $\gamma_1, \gamma_2$ be an R-basis of the vector space $H_{dr}^k(M) = \bigoplus_{k=0}^\infty H^k_{dr}(M)$. Now by Poincaré duality, there is another basis, say $\beta_1, \beta_2$ of $H_{dr}^k(M)$ so that

$\int_M a_i \wedge b_j = 0, \quad a_i \wedge b_j = \delta_{ij}.$
Let \( \pi_1 : M \times N \to M \), \( \pi_2 : M \times N \to N \), be the projection maps onto the first and second factors.

\[
\pi_1 (p, q) = p, \quad \pi_2 (p, q) = q, \quad (p, q) \in M \times N.
\]

Claim: \( \omega = \sum_{i=1}^{\infty} (-1)^{\deg(a_i)} \pi_1^* (a_i) \wedge \pi_2^* (b_i) \)  
(This result is called Diagonal Approximation.)

**Proof:** By the Künneth Theorem, the cohomology \( H^*_M(M \times M) \) is generated by the set

\[
\{ \pi_1^* (a_i) \wedge \pi_2^* (b_j) \mid i, j \in \mathbb{Z} \}.
\]

Hence, \( \omega = \sum \sigma_{i,j} \pi_1^* (a_i) \wedge \pi_2^* (b_j) \), for some \( \sigma_{i,j} \in \mathbb{R} \).

Let \( f : M \to M \times M \) denote the diagonal map \( f(p) = (p, p) \). Then

\[
\int_{M} \pi_1^* (b_k) \wedge \pi_2^* (a_l) = \int_{M} f^* (\pi_1^* (b_k) \wedge \pi_2^* (a_l))
\]

\[
\pi_1 \circ f = \text{id}_M \quad \pi_2 \circ f = \text{id}_M
\]

\[
= \sum_{k} b_k \wedge a_k
\]

\[
= (-1)^{\deg(a_k) \deg(b_k)} \int_{M} a_k \wedge b_k
\]

\[
= (-1)^{\deg(a_k) \deg(b_k)} \delta_{k,l}.
\]
On the other hand, since \( \omega \) is the Poincaré dual of the submanifold \( \Delta \) in \( M \times M \) we have

\[
\int_{\Delta} \pi_1^*(b_2) \wedge \pi_2^*(a_1) = \int_{M \times M} \pi_1^*(b_2) \wedge \pi_2^*(a_1) \wedge \omega
\]

\[
= \sum c_{ij} \int_{M \times M} \pi_1^*(b_2) \wedge \pi_2^*(a_1) \wedge \pi_1^*(q_i) \wedge \pi_2^*(q_j) \wedge \omega
\]

\[
= \sum c_{ij} \delta_{i,j} \delta_{q_i,q_j}
\]

\[
= \sum c_{ik} \delta_{i,k} \delta_{q_i,q_k} \delta_{q_i,q_k}
\]

Comparing the two results we obtain

\[
c_{ij} = (-1)^{\deg(a)} \delta_{i,j}.
\]

This finishes the proof of the claim.

Finishing the proof of the theorem:

\[
e_{M} = \sum_{E} (\Delta \times M) = \sum_{M} \omega
\]
\[
= \sum_{\Delta} \frac{\text{deg}(a_i)}{i} \pi_1^*(a_i) \wedge \hat{\pi}_2^*(b_i)
\]
\[
= \frac{1}{i} \sum_{\Delta} \text{deg}(a_i) \int \pi_1^*(a_i) \wedge \hat{\pi}_2^*(b_i)
\]
\[
= \frac{1}{i} \sum_{\Delta} (-1)^{\text{deg}(a_i)} \int \pi_1^*(a_i) \wedge \hat{\pi}_2^*(b_i)
\]
\[
= \frac{1}{i} \sum_{\Delta} (-1)^{\text{deg}(a_i)} \int a_i \wedge b_i
\]
\[
= \frac{1}{i} \sum_{\Delta} (-1)^{\text{deg}(a_i)} \delta_{ij} \wedge \sigma_{ij}
\]
\[
= \frac{1}{i} \sum_{\Delta} (-1)^{\text{deg}(a_i)} \delta_{ij} \wedge \sigma_{ij}
\]
\[
= \frac{1}{i} \sum_{\Delta} (-1)^{i} b_k(M)
\]
\[
= \chi(M).
\]

This finishes the proof of the theorem.
Corollary: Let \( f: M \to N \) be a covering space of compact manifolds of degree \( k \). Then \( \chi(M) = k \chi(N) \), provided that \( M \) and \( N \) are oriented and \( \xi \) is orientation preserving.

**Proof:** By Poincaré-Hopf Theorem, it is enough to show that \( e(M) = k e(N) \).

\[
\begin{align*}
f: M & \to N, \quad f^*(\xi) = \xi_1 - \xi_2. \\
M & \xrightarrow{v_k} \mathbb{R}^k \\
N & \xrightarrow{v_i} M \text{ differentiable},
\end{align*}
\]

\[
\begin{array}{ccc}
\mathbb{R}^k & \xrightarrow{\pi} & \mathbb{R}^2 \\
\downarrow f & & \downarrow f \\
\mathbb{R}^2 & \xrightarrow{\pi} & N \\

f^*(\mathcal{T}_N) & \subseteq & M \\
\end{array}
\]

\[
f^*(\mathcal{T}_N) = \{(p, \omega) \in M \times \mathcal{T}_N \mid f(p) = \pi(\omega)^2\}
\]

\[
(p, \omega) \xrightarrow{1} p, \quad (p, \omega) \xrightarrow{2} \omega
\]

**Claim:** The map \( \Phi: \mathcal{T}_m \to f^*(\mathcal{T}_N) \) defined by

\[
\Phi(p, v) = (p, Df_\xi(p)(v)), \quad v \in T_p M, \quad \xi \text{ a vector bundle isomorphism}
\]

Now let \( s: N \to \mathcal{T}_N \) be a section transverse to the zero section. Then the Euler number \( e(N) \) is the signed count of \( s \).

Thus \( \tilde{e}: M \to f^*(\mathcal{T}_N) \equiv \mathcal{T}_m \), \( \tilde{e}(p) = (p, f(p)) \).
So the Thurston modulus to the boundary point
\[ \mu = 0. \]
\[ \mu = \frac{1}{2} \frac{\| \mathbf{H}_t \|}{\mathbf{H}_t} \]
\[ \mu = \frac{1}{2} \mathbf{H}_t \]

**Conjecture:** Let \( f : \mathbb{P} \to \mathbb{P} \) be a smooth map of a compact

\[ f = w \in \mathbb{C} \]

or this function the proof.

\[ f = w \in \mathbb{C} \]

\[ f \circ f \]

\[ f \circ f \]

\[ f \circ f \]

\[ f \circ f \]
Proof of the Theorem: Recall the form
\[ u_0 = \sum_{i} (-1)^{\deg (a_i)} \pi_i^* (a_i) \wedge \pi_2^* (b_i) \] from the
proof of Poincaré-Hopf. Let \( \varphi : M \to M \times M \)
be given by \( \varphi (p) = (p, \tau (p)) \), which is clearly a
diffeomorphism from \( M \) to its image \( \varphi (M) \).

Then \( \nabla \Delta = (-1)^{\dim M} \Delta \nabla \varphi \)
\[ = (-1)^{\dim M} \sum_{i} u_0 \]
\[ = (-1)^{\dim M} \int_M \varphi^* (u_0), \quad \varphi : M \to \varphi (M) \text{ is a} 
\text{diffeomorphism} \]

Note that \( \varphi^* (b_i) = \sum_{j} a_{i,j} b_j \) for \( a_{i,j} \in \mathbb{R} \).
( \$673 \text{ is a basis for } H^*_D (M) \)
So, \( \sum_{k} (-1)^{\deg (b_k)} \text{Tr} (\varphi^* (H^*_D (M)) \to H^*_D (M)) = \sum_{i} (-1)^{\deg (b_i)} \)

Now let's continue the computation we started above.
\[ \nabla \Delta = (-1)^{\dim M} \sum_{i} \varphi^* (u_0) \]
\[ = \sum_{i} (-1)^{\deg (a_i)} \int_M \varphi^* (\pi_i^* (a_i) \wedge \pi_2^* (b_i)) \]
\[ = \sum_{i} (-1)^{\deg (b_i)} \int_M \varphi^* (\pi_i^* (a_i) \wedge \pi_2^* (b_i)) \]
\[ = \sum \frac{d_{\text{deg}(b_i)}}{i} \int_M \alpha_i \wedge \left( \sum \frac{d_{\text{deg}(b_i)}}{i} b_i \right) \]

\[ = \sum \frac{d_{\text{deg}(b_i)}}{i} \int_M \alpha_i \wedge \delta_{i, j} b_j \]

\[ = \sum \frac{d_{\text{deg}(b_i)}}{i} \int_M \delta_{i, j} \alpha_i \wedge b_j \]

\[ = \sum \frac{d_{\text{deg}(b_i)}}{i} \delta_{i, j} \]

\[ \text{which finished the proof.} \]

**Example:** Let \( M \) be a smooth manifold which compact and orientable. If \( \chi(M) \neq 0 \) and \( f: M \to M \) is any smooth map homotopic to the identity, then \( f \) has a fixed point.

**Proof:**

\[ f \sim \text{id} \implies f^* = \text{id} \quad \text{for } b(M) \]

\[ \Delta_f = \chi(M) \neq 0 \implies f \text{ has a fixed point.} \]

**Remark:** \( f: \mathbb{R} \to \mathbb{R}, \ f(x) = x+1 \), has no fixed points, even though \( \chi(\mathbb{R}) = b_0(\mathbb{R} \setminus \{0\}) = -1 \neq 0. \)
Note that $f^t \equiv \text{id}$ and $f^t = x + t$, $t \in [0, T]$.

$f_0 = \text{id}$, $f_1 = f$.

**Riemann-Hurwitz Theorem:**

$f: \Sigma_1 \rightarrow \Sigma_2$ a holomorphic map between compact Riemann Surfaces. The set of critical points of $f$, say $C = \{ p \in \Sigma_1 \mid f'(p) = 0 \}$.

Clearly, $C$ is closed because

$$C = f^{-1}(0), \quad g = f'.$$

Since $g$ is a holomorphic $C$ cannot have an accumulation point. Since $\Sigma_1$ is compact and $C$ is closed $C$ should be a finite set. Hence, $f: \Sigma_1 \setminus C \rightarrow \Sigma_2 \setminus f(C)$ has no critical values.

On the other hand, any holomorphic map $D$ open and thus $f: \Sigma_1 \rightarrow \Sigma_2$ is onto because,
\( f(\Sigma_1) \) is both open and closed in \( \Sigma_2 \) (assuming both \( \Sigma_1 \) and \( \Sigma_2 \) connected).

But \( N = \text{deg}(f) \). So \( N = f^{-1}(q) \) for any regular value \( q \in \Sigma_2 \setminus f(\Sigma_1) \), because since \( f \) is holomorphic every zero comes with sign \( +1 \) (\( f \) is orientation preserving).

For any \( \rho \in \Sigma_1 \), choose local coordinates at \( \rho \in \Sigma_1 \) and \( q = f(\rho) \in \Sigma_2 \) so that around \( \rho \) \( f \) is given by a power series as

\[
f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots
\]

\( \lambda \in \mathbb{N} \), \( a_n \neq 0 \).

\[
f(z) = \frac{z}{\lambda} \cdot h(z), \quad h(z) = a_0 + a_1 z + \cdots
\]

\( h(0) = a_0 \neq 0 \), \( f(z) = z \cdot e^{\frac{\lambda}{z^\lambda}} \), \( \lambda \in \mathbb{N} \).

\[
\Rightarrow f(z) = \omega \cdot z, \quad \omega = e^{\frac{\lambda}{z^\lambda}}
\]

\[
\omega(0) = a_0 \neq 0. \quad \text{Hence, } \omega \text{ is an holomorphic inverse so that } \omega = \frac{1}{2} \frac{d}{dz} \log(\omega(z)) \text{ is an holomorphic}
\]
coordinate change. So if we replace \( u \) with \( \frac{1}{\sqrt{2}} z \), we get

\[ w \rightarrow z + 2 w \]

\[ \mathbb{Z}_1 \]

\[ \mathbb{Z}_2 \]

\[ \mathbb{Z}_2' \]

\[ \mathbb{Z}_2' \]

\[ f^{-1}(p) = \{ q_j - p_j \} \]

\[ d_i : \text{ramification index at } q_i \text{ and denoted } e_{q_i} \]

\[ N = \deg f = \sum_{j=1}^r \deg (\phi_{q_j}) = \sum_{j=1}^r e_{q_j} d_j \]

**Theorem** Assume the above set up. Then

\[ \chi(Z_1) = \deg(f) \chi(Z_2) - \sum_{q \in Z_2} (e_q - 1) \]

when \( e_q \) is the local degree of \( f \) at \( q \).

**Proof:**

\[ Z_1 \xrightarrow{f} Z_2 \]

\[ \mathbb{Z}_1 \]

\[ \mathbb{Z}_2 \]

\[ \mathbb{Z}_2' \]

\[ \mathbb{Z}_2' \]

\[ \mathbb{Z}_2' \]
Let \( \gamma \) be a vector field on \( \Sigma \) so that near each \( p \in \Sigma \), it is given as \( X(p) = 2 \), so that its index \( \bar{s} = \frac{1}{2} + 1 \) and equals zero outside \( U_0 \). The perturb \( X_0 \) outside \( U_0 \), so that it is transverse to the zero vector fields.

\( f \) is a local diffeomorphism outside \( f^{-1}(\mathbb{R}) \) so that we obtain a vector field \( \tilde{X}_0 \) on \( Z := f^{-1}(\mathbb{R}) \) so that \( f(\tilde{X}_0(q)) = X(f(q)) \) for all \( q \notin \Sigma_1 \). Let \( V_\mathbb{R} \) be \( \frac{1}{2} \) for some \( d \) so that if we define \( \tilde{X}(q) = \frac{1}{d} \) on \( V_\mathbb{R} \), we get

\[
D f_q (\tilde{X}(q)) = d \tilde{X}^{-1} \cdot \tilde{X}(q) = \frac{1}{d} = X(q) = X(f(q)).
\]

Hence, \( \tilde{X} \) is well defined on all of \( \Sigma_1 \) and is a vector field \( D f_q (\tilde{X}(q)) = X(f(q)) \) at all \( q \notin \Sigma_1 \).

The indices of zeros of \( \tilde{X} \) at the points \( f^{-1}(\mathbb{R}) \) are all \( +1 \), since \( D f_q (\tilde{X}(q)) = \frac{1}{d} \) for some \( d \). The index of zeros of \( \tilde{X} \) at a point \( q \) of \( f^{-1}(\mathbb{R}), p \notin \mathbb{R} \), is the sum of that of \( X \) at \( p \), because \( f(q) = p \) and \( f \) is a diffeomorphism near \( q \). Note that for each \( p \notin \mathbb{R} \), \( f^{-1}(p) \) has exactly \( N \) points. Thus we get

\[
X(\Sigma_1) - \mid f^{-1}(\mathbb{R}) \mid = N \left( X(\Sigma_1) - \mid \mathbb{R} \mid \right).
\]
So, \[ x(I_1) = 0 \cdot x(I_2) + \left| f^{-1}(R) \right| - \left| R \right| \]

\[ = x(I_2) = 0 \cdot x(I_2) + \sum_{i=1}^{\infty} \left( \frac{1}{x} - \Sigma_{q_i \in f^{-1}(R)} \frac{1}{y_i} \right) \]

\[ = 0 \cdot x(I_2) + \sum_{i=1}^{\infty} \left( 1 - \varepsilon_{q_i} \right) \]

\[ = \text{deg}(f) x(I_2) + \sum_{q_i \in f^{-1}(R)} \left( 1 - \varepsilon_{q_i} \right) \]


Theorem (Hurwitz)

Let $\mathbb{P}_g$ be a compact Riemann surface of genus $g$, $f: \mathbb{P}_g \rightarrow \mathbb{P}_g$ a finite subgroup of holomorphic automorphisms of $\mathbb{P}_g$ then

\[ |G| \leq 84(g-1). \]

**Proof:** $f: \mathbb{P}_g \rightarrow \mathbb{P}_g$ continuous map. \[ f \]

has infinitely fixed points, \[ p \in \mathbb{P}_g \] if \( f(p) = p \).

Then since $\mathbb{P}_g$ is a compact space, the closed set $\{ p \in \mathbb{P}_g \vert f(p) = p \}$ must have an accumulation point, say $p_0$. \[ f \]

is analytic around the function defined locally near $p_0$, by \( f \). But

there infinitely many new coming to $p_0$ and the $f(z) - z = 0$ on that open set, \( f \) \in $ on $\mathbb{P}_g$. Hence we see that any analytic function $f: \mathbb{P}_g \rightarrow \mathbb{P}_g$ can have
finitely many fixed points.

$G \leq \text{Aut}(\mathbb{Z}_g)$ finite group. It follows that $G$ has no fixed points on $\mathbb{Z}_g \cdot C$, where $C$ is a finite set. Hence $G$ acts freely on $\mathbb{Z}_g \cdot C$. Moreover, $G$ acts on $C$. Known that the $G$-orbits of points of $C$ are

$0, O_1, O_2, \ldots, O_n.$

$C = O_0, O_1, \ldots, O_n.$

$\begin{array}{c}
\text{Cyclic for all } \gamma \in C.
\end{array}$

$2g - 2 = \left| \mathcal{L} \right| (2h - 2 + n) - \left| \Theta \right|$

$= \left| \mathcal{L} \right| (2h - 2 + n) - \sum_{i=1}^{\left| \Theta \right|} \left| \Theta_i \right|$

$= \left| \mathcal{L} \right| (2h - 2 + n) - \frac{\left| \mathcal{L} \right|}{k_i}$, where
\[ k_i = s + \frac{b}{p}, \quad p \leq q_i. \]

\[ 2q - 2 = \log_n \left( 2n - \sum_{k=1}^{q} \frac{1}{k_i} \right) \]
\[ = \log_n \left( 2n - \sum_{k=1}^{\infty} \frac{k}{1 - k} \right) \]

\[ \log_n = \frac{2q - 2}{2n - \sum_{k=1}^{\infty} (1 - \frac{1}{k})} \]

To find an upper bound for \( \log_n \) we need to find a lower bound for the sum
\[ \sum_{k=1}^{n} (1 - \frac{1}{k}) \], when \( k_i \geq 2 \).

**Claim** The lower bound for \( 2n - \sum_{k=1}^{\infty} (1 - \frac{1}{k}) \)

\[-2 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{2}{3} = \frac{1}{6} \] (when \( n=4 \))

For \( n \geq 4 \) and for \( 0 \leq n \leq 3 \)

\[-2 + \frac{1}{2} + \frac{2}{3} + \frac{6}{7} = \frac{1}{42} \] (when \( n=3 \), \( n=2 \), \( n=1 \))

5. Using the claim \( \log_n \leq \frac{(2q - 2)}{\sqrt{42}} = 84(q-1) \).

**Remark**: In our analysis, the number of
branch points of the covering \( \Sigma_g \to \Sigma_h \) is \( \geq 4 \), then we get a better bound

\[ 161 \leq \frac{(2g-2)}{1/6} = 12(3g-1). \]

Example: Klein bottle: \( x^3y + y^3z + z^3x = 0 \). It is \( \mathbb{CP}^2 \).

\( \mathbb{Z}_3 \):

\[ 6 = \text{Aut}(\Sigma_3) \]

Holomorphic automorphisms of \( \mathbb{Z}_3 \).

\[ 161 \leq 84(3-3) = 164. \]

\[ 161 = 168 \text{ and } G \text{ is the unique simple group of order 168}. \]
Some Applications:

1) \( \mathbb{RP}^2 \) does not embed into \( \mathbb{R}^3 \).

Proof: First assume that \( \mathbb{RP}^2 \) embeds into \( \mathbb{R}^3 \).

\[ \mathbb{RP}^2 = \mathbb{S}^1 \cup \mathbb{D}^2 \]

\( C \): center of the Mobius Band.

\( C \subseteq MB \subseteq \mathbb{RP}^2 \subseteq \mathbb{R}^3 \), consider the tubular neighborhood \( \nu \) of \( C \) in \( \mathbb{R}^3 \).

Since \( C \) and \( \mathbb{R}^3 \) are both oriented, the disk bundle \( \nu \) is oriented. Now rotate each disk 90° degrees with the orientation to get another copy of the Mobius band. Note the center circle do not move under rotation.

\( C' \subseteq MB' \) and intersects \( MB \) at one point \( P \), when \( C' \) is a copy of \( C \).
Inside the rotated copy $\mathbb{R}^2$ of $\mathbb{R}^3$. Choosing the tubular neighborhood we see that $C'$, which is a circle intersected, $MB$ and $\mathbb{R}^2$ only at one point transversely. Hence the unoriented intersection of the closed manifold $C' \cong S^1$ and $\mathbb{R}^2$ in $\mathbb{R}^3$ is

$$\text{Int}(C', \mathbb{R}^2) = 1 \pmod{2}$$

However, $C'$ and $\mathbb{R}^2$ are closed submanifolds of $\mathbb{R}^3$. Since $\mathbb{R}^3$ is unbounded by translating $C'$ with vector we can make sure that $C'$ and $\mathbb{R}^2$ do not intersect at all. Let $D$ still a transverse intersection and thus

$$\text{Int}(C', \mathbb{R}^2) = 0 \pmod{2}.$$ 

This is clearly a contradiction!

Hence, $\mathbb{R}^2$ cannot be embedded inside $\mathbb{R}^3$.

Remark: 1) $\mathbb{R}P^2 \subseteq \mathbb{R}P^3$ when $\mathbb{R}P^3$ is also oriented, however and $\text{Int}(C', \mathbb{R}^2) = 1 \pmod{2}$. Since $\mathbb{R}P^3$ is compact it is not possible to translate $C'$ far enough so that $C'$ and $\mathbb{R}^2$ do not intersect any more.

2) $N = \mathbb{R}^2 \times \mathbb{R}$, $\mathbb{R}^2 \times \{p\} \to \mathbb{R}^2 \times \{p\}$

$p \neq p' \Rightarrow$ these two copies do not intersect.

Q: Which part of the above proof does not work in this case?
The tubular neighborhood of the center circle $C$ in $N = \mathbb{R}P^2 \times \mathbb{R}$ is not orientable. Therefore, rotating each disc $\mathbb{R}_2$-radially counterclockwise is not possible.

3) $\mathbb{R}^3 \subseteq S^3 = \mathbb{R}^3 \cup \{\infty\}$ and therefore $\mathbb{R}P^2$ does not embed into $S^3$.

4) Now let $N$ be any closed non-orientable surface. The above argument proves that $N$ cannot embedded into $\mathbb{R}^3$ or $S^3$.

\[ N = 2g \# \mathbb{R}P^2 \text{ or } 2g \# \mathbb{R}P^2 \]

\[ \mathbb{Z}_2 \]

\[ \times \]

\[ \mathbb{R}P^2 \]

KB: $2g \# \mathbb{R}P^2$

\[ \Rightarrow KB \subseteq N \text{ and therefore we can repeat the above proof for } N. \]

KB
2) **Theorem (Gauss–Bonnet Theorem)**

If \( S \) is a genus \( g \) orientable surface \( \subset \mathbb{R}^3 \), then

\[
\int_{S} K \, dS = 2\pi \chi(S) = 4\pi(1 - g)
\]

where \( K : S \to \mathbb{R} \) is the Gaussian curvature function on \( S \).

**Proof:** Step 1: \( \sigma : U \to \mathbb{R}^3 \setminus \{(0,0,0)^3\} \)

\( U \subset \mathbb{R}^2 \), \( \sigma(x,y) = (f_x, f_y, 1) \) the Gauss map of the surface \( S \subseteq \mathbb{R}^3 \) parametrized by a local coordinate system

\[
(x,y) \mapsto (x,y, f(x,y)) \quad \text{and} \quad (x, y, f(x,y)) \mapsto (f_x, f_y, 1)
\]

\( \mathbb{R}^3 \setminus \{(0,0,0)^3\} \)

\( \sigma \)

\( \text{Gauss map} \)

\[
\mathbb{H}_2^2 (\mathbb{R}^3 \setminus \{(0,0,0)^3\}) = R = \langle \mathcal{E}_3 \rangle \quad \text{when}
\]

\[
u_0 = x \, dy \wedge dz + y \, dx \wedge dz + z \, dx \wedge dy \quad \in \mathcal{E}_2(\mathbb{R}^3)
\]

\[
(x^2 + y^2 + z^2)^{3/2}
\]
\[ \int_{\sigma} w = 4\pi \quad (\text{Exercise}) \]

\[ dS = \sqrt{1 + \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2} \quad \text{duly} \]

Claim: \[ \sigma^*(w) = \chi \, dS, \quad \text{when} \]

\[ \chi(x, y) = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_{xx}^2 + f_{yy}^2)^2} \quad \text{the Gaussian curvature.} \]

Proof is left as an exercise.

Step 2) \( \Sigma_3 \hookrightarrow \mathbb{R}^3 \), \( \sigma : \Sigma_3 \rightarrow \mathbb{R}^3 \setminus \{0, 0, 0\} \): Gauss map

\[ \sigma : \Sigma_3 \rightarrow S^2 \quad \text{the span of oriented 2-planes} \]

\[ \text{in } \mathbb{R}^3. \]

\( \sigma \) is homotopic to \( \frac{\sigma}{16\pi} \), we can replace \( \sigma \) by \( \frac{\sigma}{16\pi} \). So we assume that \( \sigma : \Sigma_3 \rightarrow S^2 \).

\( \sigma : \Sigma_3 \rightarrow S^2 = \text{Gr}^+_R(3, 2) \subseteq \text{Gr}^+_R(n, 2) \)

\[ \mathbb{R}^3 = \mathbb{R}^n \]

\( \sigma : \Sigma_3 \rightarrow S^2 = \text{Gr}^+_R(3, 2) \subseteq \text{Gr}^+_R(n, 2) \)

\[ (x, y, z) \mapsto (x, y, z, 0, \ldots, 0) \]

\[ \text{Gr}^+_R(n, 2) = \{ (u, v) \in S^n \times S^{n-1} : u \perp v \} \]

\[ (u, v) \sim (u', v') \iff \begin{cases} u_2 = c \cos \theta u_1 - s \sin \theta v_1, \\ v_2 = s \cos \theta u_1 + c \sin \theta v_1, \end{cases} \]
$Gr^+_\mathbb{R}(n,2)$ is a smooth manifold of dimension $2(n-2)$. (Exercise!)

We have a map $\Phi: Gr^+_\mathbb{R}(n,2) \to \mathbb{CP}^{n-1}$ by

$\Phi([u,v]) = [u+iv]$ \quad u+iv \in \mathbb{C}^n \setminus \{0\}$

$u, v \in \mathbb{R}^n$

Claim: $\Phi(Gr^+_\mathbb{R}(n,2))$ is the quadric hypersurface $z_0^2 + \sum z_i^2 = 0$ in $\mathbb{CP}^{n-1}$.

Example: $n = 3$, $Gr^+_\mathbb{R}(3,2) \cong S^2$, $\Phi(Gr^+_\mathbb{R}(3,2))$ is the quadric curve in $\mathbb{CP}^2$ given by $z_0^2 + z_1^2 + z_2^2 = 0$.

But $F: \Sigma_g \times [0,1] \to \mathbb{R}^n$ be a differentiable map.

$f_t = F(-, t) \quad f_t: \Sigma_g \to \mathbb{R}^n$ homotopy of maps.

Assume that each $f_t$ is an immersion into $\mathbb{R}^n$.

$\sigma: \Sigma_g \to Gr^+_\mathbb{R}(n,2)$, $p \mapsto [e^+_t(T_p\Sigma_g)]$
Consider the composition \( \Phi = \xi : \Sigma_g \to \mathbb{C}P^n \)
\[ \Sigma_g \xrightarrow{\xi} \text{Gr}_n(\nu) \xrightarrow{\Phi} \mathbb{C}P^n \]
but \( \alpha \in H^2_{dR}(\mathbb{C}P^n) \) so that \( \int_\mathbb{C}P^n \alpha = 1/2 \).

\( \Rightarrow \int_\Sigma_g \alpha = 1 \) because \( \Phi : \text{Gr}_n(3, 2) \to \mathbb{C}P^1 \)
\( \Phi(\text{Gr}_n(3, 2)) \) is a double cover.

\( \Rightarrow x_1 dx = \sigma^*(\omega) = 4\pi (\Phi \circ \sigma)_* [\alpha] \) as cohomology classes.

**Conclusion:** For any two immersions of \( \Sigma_g \) into \( \mathbb{R}^n \) the integral
\[ \int_\Sigma_g \omega \] gives the same result.

**Proposition:** Any two immersions of \( \Sigma_g \) into \( \mathbb{R}^n \) \((n \geq 3)\) are homotopic through immersions.

**Proof:** The vector space of all polynomials in
\[ \mathbb{R}[x,y,z] \] of degree at most \( d \) has dimension
\[ s = \left( \begin{array}{c} 3 + d \\ d \end{array} \right). \] Take any point \( P = (x_0, y_0, z_0) \in \mathbb{R}^3 \).
By the linear change of coordinates
\[ (x, y, t+1) \rightarrow (x-x_0, y-y_0, t-t_0) \] we can
assume that \( P = (0, 0, 0) \).

Let \( f_1, \ldots, f_{k+3} \in \mathbb{R}^3 \) the vector space of polynomials
\[ x, y, t \] of degree \( \leq 1 \).

\[ \phi = (f_1, \ldots, f_{k+3}) : \mathbb{R}^3 \rightarrow \mathbb{R}^{k+3} \]

The condition that \((0, 0, \ldots, 0)\) is a critical point
for \( \phi \) is a linear condition on the first
degree terms of \( f_i \).

\[ D \phi (0, 0, 0) = \begin{bmatrix}
\nabla f_1 (0, 0, 0) \\
\vdots \\
\nabla f_{k+3} (0, 0, 0)
\end{bmatrix}, \quad \nabla f_i (0) = \left( \frac{\partial f_i}{\partial x} (0, 0, 0), \frac{\partial f_i}{\partial y} (0, 0, 0), \frac{\partial f_i}{\partial t} (0, 0, 0) \right) \]

\[ f_i = a_0 + a_1 x + a_2 y + a_3 t + O(2), \quad \nabla f_i (0) = (a_0, a_1, a_2) \]
\((0, 0, 0)\) is a critical point for \( \phi \) if and only if
the matrix \( D \phi (0) \) has rank \( \leq 2 \).

\[
\begin{array}{c|ccc}
1 & 2 & 3 \\
2 & 1 & 3 \\
3 & 2 & 1 \\
k+1 & \text{independent conditions.} \\
\end{array}
\]

Hence, the subspace of all \((f_1, \ldots, f_{k+3})\) in \( \mathbb{R}^{k+3} \)
having \((0, 0, 0)\) as a critical point have codimension \( k+1 \).
Let \( E = \mathcal{S}(u, v, t_1, t_2, k_3, t, t_3) \in \mathbb{R}^2 \times \mathbb{R}^2 \leq 2^{3\cdot 3} \). 

\[ \Pi: E \to \mathbb{R}^3, (u, v, t_1, t_2, k_3, t, t_3) \mapsto (x, y, t). \]

All the fibers of \( \Pi \) have the same structure and they can be written as \( (\frac{1}{2}) \times \mathbb{R}^3 \) linear subspace of codimension \( k + 1 \).

Thus the set of all polynomial maps 

\[ \phi = (f_1, f_3): \mathbb{R}^2 \to \mathbb{R}^3, \]

which are not an immersion at some point, form a set in \( \mathbb{R}^{2\cdot(k+3)} \) of codimension 

\[ (k+1) - 3 = k - 2. \]

Here, if \( k \geq 4 \) the set of all polynomial immersion \( \phi = (f_1, f_3): \mathbb{R}^3 \to \mathbb{R}^{k+3} \) is path connected because \( k \geq 4 \Rightarrow k - 2 \geq 4 - 2 = 2. \)

In particular, all immersions (polynomial) 

\[ \mathbb{R}^3 \to \mathbb{R}^7 \] is path connected.

So for our surface \( \Sigma \subseteq \mathbb{R}^3 \) restriction of any immersion \( \mathbb{R}^3 \to \mathbb{R}^7 \) to \( \Sigma \) is also an immersion. Hence, the space of all polynomial immersions of \( \Sigma \) into \( \mathbb{R}^7 \) is path-connected.
\[ \Sigma_g \xrightarrow{\phi_0} \mathbb{R}^2, \quad \Sigma_g \xrightarrow{\phi_i} \mathbb{R}^2 \] two
immersions \( \in \mathcal{F} \) homotopy so that
each \( \phi_t : \Sigma_g \to \mathbb{R}^2 \) is an immersion.

In particular, any two embeddings
\( \phi_0 : \Sigma_g \to \mathbb{R}^3, \quad \phi_i : \Sigma_g \to \mathbb{R}^3 \) are
homotopic through immersions into \( \mathbb{R}^3 \).

\[ \int_{\Sigma_0} \omega = \int_{\Sigma_i} \omega, \quad \text{when} \ \omega \ \text{is the}
\Sigma_g \to \Sigma_0
\]
Gauss map corresponding to the embedding \( \phi_i \).
Step 4 \( \Phi_0 : \Sigma_0 \rightarrow \mathbb{R}^3 \), \( \Phi_1 : \Sigma_1 \rightarrow \mathbb{R}^3 \) two embeddings. \( \Phi_0 \) given embedding of \( \Sigma_0 \). \( \Phi_1 \) is the embedding looks like

\[
\int_{\Phi_0} \omega = \int_{\Phi_1} \sigma^*(\omega)
\]

Replace \( \omega \) be a form so that \( \Phi_0 \) is supported in \( \cup \) with the same integral.

\[
\sigma^{-1}(U) = U_1 \cup U_2 \cup \ldots U_{2^n} \text{ disjoint open sets.}
\]

\( \Phi_i : V_i \rightarrow \cup U_i \) is a diffeomorphism.

Thus

\[
\int_{\Phi_0} \sigma^*(\omega) = \int_{\Phi_1} \sigma^*(\omega) = \sum_{i=1}^{2^n} \int_{\Phi_0} \sigma^*(\omega), \quad \sigma^{-1}(U) \text{ for } U_i
\]

when each

\[
\int_{\Phi_0} \sigma^*(\omega) = \pm \int_{U} \omega = \pm \int_{S^2} \omega = \pm 4\pi
\]

and the sign is \( \pm 1 \) depending on whether
\( \mathbf{c} : \nu_i \to \nu \) is orientation preserving or not.

\[ \nu_i : \quad \begin{array}{c}
\text{orientation} \\
\text{preserving}
\end{array} \quad \begin{array}{c}
\nu \\
\end{array} \]

\[ \nu_j : \quad \begin{array}{c}
\text{orientation} \\
\text{reversing}
\end{array} \quad \begin{array}{c}
\nu \\
\end{array} \]

\( \gamma \geq 2 \)

Hence, \( \sum \frac{1}{2} \mathbf{u}_i \mathbf{v}_i \mathbf{d}s - \sum \mathbf{e} \mathbf{f} = 4\pi \mathbf{g} \mathbf{h} = 2\pi \mathbf{u} (\mathbf{v}) \).
**Characteristic Classes**

**Euler class** \( \mathbb{R}^k \to E \) oriented vector bundle \( \downarrow \) \( M \)

Let \( e_0 : M \to E \) be the zero section.

\( e(E) \) Pontryagin dual of the two section \( E \subseteq \text{so}(M) \subseteq E \). \( \downarrow \) \( M \)

\( e(E) = [\omega] \), with \( \text{supp}(\omega) \) lies in a tubular neighborhood where integral along any \( \omega \) (oriented) is equal 1.

\( \omega \in \Omega^k(M) \) \( e(E) \in \operatorname{H}^k_{\text{D2}}(M) \).

**Some Properties of the Euler Class:**

1) \( E_1 \to M \) \( i = 1, 2 \), oriented vector bundles.

The \( E_1 \oplus E_2 \to M \) is also an oriented vector bundle.

**Proposition:** \( e(E_1 \oplus E_2) = e(E_1) \cdot e(E_2) \).

**Proof:** \( s_i : M \to E_i \) section \( i = 1, 2 \).

\( (s_1, s_2) : M \to E_1 \oplus E_2 \) section.

\( (s_1, s_2)^{-1}(0) = s_1^{-1}(0) \cap s_2^{-1}(0) \)
Hence, $e(E_1 \oplus E_2)$ is the Poincaré dual to the intersection of the submanifolds $S^{-1}_i(0)$ and $S^i_0(0)$.

Hence, $e(E_1 \oplus E_2) = e(E_1) \cdot e(E_2)$. 

2) $f: M \to N$ smooth map, $E \to N$ oriented vector bundle. Then

**Proposition** $e(f^*(E)) = f^*(e(E))$

**Proof**

\[
\begin{array}{ccc}
\tilde{S}^{-1}(0) & \xrightarrow{f} & S^{-1}(0) \\
\| & \| & \|
\end{array}
\]

$f^*(E) = \{ (p, v) \in M \times E | f(p) = \pi_1(v) \}$.

$\tilde{S}^{-1}(0) = \{ (p, 0) | f(p) \}$

Hence, $\tilde{S}^{-1}_i(0) = f^{-1}_i(S^{-1}_i(0))$.

Hence, choosing $f$ such that $\tilde{S}_i(0)$ transverse to each other $S^{-1}_i(0)$ is a submanifold in $M$. 

\[
\begin{array}{ccc}
\tilde{S}^{-1}_i(0) & \xrightarrow{f} & S^{-1}_i(0) \\
\| & \| & \|
\end{array}
\]
1) \[ \sum w = 1 \]

\( S^1(w) = 1 \)

This proves the result.

3) For any oriented vector bundle \( E \) let \( -E \) denote the bundle with opposite orientation. Then \( e(-E) = -e(E) \).

\[ \text{Proof:} \quad E: \quad \begin{array}{c} \mathbb{D}^k \\ \mathbb{D}^k \end{array} \xrightarrow{\partial} \quad M \]

\[ \sum w = 1 \]

\[ \sum_{-w} = +1 \quad e(-E) = [-w] = -[w] = -e(E). \]

4) \( E \to \text{ oriented vector bundle and let } E^* \to M \)

be the dual of \( E \to M \).

\[ E^* = \text{hom} \left( E, \mathbb{R} \right), \quad \text{rank}(E)/2 \]

Then \( e(E^*) = (-1)^{\text{rank}(E)/2} e(E) \).

\[ \text{Proof:} \quad C \to \mathbb{L} \text{ complex line bundle} \]

\[ C = \mathbb{R}^2 \quad \Rightarrow \quad \text{we may regard } \mathbb{L} \to M \text{ as an oriented } \mathbb{R}^2 \text{- bundle.} \]
\[ L^* = \text{hom} (L, \mathfrak{g}) \rightarrow M. \]

\[ \varphi_{\alpha \beta} : U_{\alpha} \cap U_{\beta} \rightarrow \mathbb{C}^* \text{ transition functions for } L. \]

\[ \varphi_{\alpha}^{-1} : U_{\alpha} \times \mathbb{C}^* \rightarrow \mathbb{C}^* \text{ transition functions for } L^*. \]

\[ L_1, L_2 \rightarrow M \text{ ca. line bundle.} \]

\[ L_1 \otimes L_2 \rightarrow M \text{ another complex bundle whose transition functions } \varphi \text{ is the product of the transition functions of } L_1 \text{ and } L_2. \]

\[ s_i : M \rightarrow L_i \text{ section of } L_i. \]

\[ s_i \cdot s_j : M \rightarrow L_1 \otimes L_2 \text{ section of } L_1 \otimes L_2. \]

\[ (s_i \cdot s_j)^{-1}(0) = s_i^{-1}(0) \cup s_j^{-1}(0). \]

\[ \operatorname{Int} (N, s_i^{-1}(0) \cup s_j^{-1}(0)) = \operatorname{Int} (N, s_i^{-1}(0)) + \partial (N \cdot s_j^{-1}(0)) \]

\[ \partial (s_i^{-1}(0) \cup s_j^{-1}(0)) = \partial (s_i^{-1}(0)) + \partial (s_j^{-1}(0)) \]

\[ e (L_1 \otimes L_2) = e (L_1) + e (L_2). \]

\[ L \otimes L^* = \mathfrak{g} \rightarrow M \text{ the trivial bundle.} \]

\[ 0 = e (L \otimes L^*) = e (L) + e (L^*). \]
\[ e(L^k) = -e(L) \]

\[ e(L_1 \oplus L_2 \oplus \cdots \oplus L_k) = e(L^1) \cdots e(L^k) \]

\[ = -e(L_1) \cdots -e(L_k) \]

\[ = -e(L_1 \oplus \cdots \oplus L_k) \]

Hence for an oriented vector bundle \( E \) of rank \( 2n \) we take orientation of \( E^* \) as follows:

\[ e_1, \ldots, e_{2n}, \rightarrow (-1)^n e_1^*, e_2^*, \ldots, e_{2n}^* \]

\[ e(E^*) = (-1)^n e(E), \quad \text{rank}(E) = 2n. \]

**Special Case:** \( \mathbb{T}^* M \rightarrow M \) tangent bundle.

The \( \mathbb{T}^* M \) as a smooth manifold is oriented with orientation:

\[ x_\mu \rightarrow x_\mu, \quad a_\mu \rightarrow \text{coor. system on } \mathbb{T}^* M. \]

on \( M \)

\[ a_i (\sum j \frac{\partial}{\partial x^j}(p) = e_i. \]

**Orientation of** \( \mathbb{T}^* M. \)

\[ x_\mu \rightarrow x_\mu, \quad b_\mu \rightarrow b_\mu \quad \text{This gives an oriented bundle } \mathbb{T}^* M. \]

\[ b_i (\sum j \frac{\partial}{\partial x^j}(p) = e_i) \]
However, this is not compatible with the orientation we considered above.

Instead, we take as the canonical orientation on the cotangent bundle as

\[ x_1, b_1, x_2, b_2, \ldots, x_n, b_n. \]

Remark: The difference of orientations given by \( x_1, -x_1, b_1, -b_1 \) and \( x_1, b_1, x_1, b_1, \ldots, x_n, b_n \) is \( \frac{\sqrt{n(n-1)}}{2\cdot n!} \).

2) \( T^*M \) has a canonical symplectic structure given by

\[ \omega = dx_1 \wedge db_1 + dx_2 \wedge db_2 + \ldots + dx_n \wedge db_n. \]
CHERN CHARACTERISTIC CLASSES

\[ \tilde{\Pi}_i : E \to M, \text{ } E \text{ complex vector bundle of rank } n. \]

\[ r = 1, \text{ } \tilde{\Pi}_i : L \to M \text{ complex line bundle.} \]

\[ 0 \leq r^2 = \sigma \leq n \text{. } L \to M \text{ can be viewed as an oriented } \mathbb{R}^2 \text{-bundle, denoted } L_{\mathbb{R}} \to M. \]

\[ (GL^+(2, \mathbb{R}) \hookrightarrow GL^+(\mathbb{R}^2) \xrightarrow{\approx} \mathbb{U}(1) = S^1 \hookrightarrow \mathbb{C}^{	imes}) \]

The first Chern class of \( L \) is defined as

\[ c_1(L) = \mathcal{E}(L_{\mathbb{R}}) \]

Let \( s_0, s_1, \ldots, s_k \) be sections of \( L \to M \). Introduce tame, piecewise linearly independent transversely \( \mathbb{R} \)-spanned sections having no common zeros.

\[ M \]

\[ \xymatrix{ & S_i^{-1}(0) \ar@{.>}[dr] & S_i^{-1}(0) \ar@{.>}[dl] \\ & S_i^{-1}(0) \ar@{.>}[dr] & S_i^{-1}(0) \ar@{.>}[dl] \\ M & \ar@{-->}[rr] \dim = n - 4 & \ar@{-->}[rr] \dim = n - 2 \}
\]

\[ n - 2(k + 1) < 0 \Rightarrow \bigwedge^k S_i^{-1}(0) = 0. \]

Let \( f : M \to \mathbb{CP}^k \) be defined by \( p \mapsto [s_0(p) : \ldots : s_k(p)] \)

\[ \hat{i} = 0, \ldots, k, \text{ } U_i = \{ p \in M \mid S_i(p) \neq 0 \} \subseteq M \text{ open submanifolds.} \]

Clearly, \( M = \bigcup U_0 \cup U_1 \cup \ldots \cup U_k. \)
The $\mathbb{R}$ follows that $L$ is isomorphic to $\mathbb{C}^k$ when $\mathbb{C}^k$ to the tautological line bundle over $\mathbb{C}D^k$.

$$L \ni u_i \rightarrow u_i \times \mathbb{C} \rightarrow \mathbb{C}D^k \setminus \mathbb{S}^0$$

$$\left( p, u \right) \rightarrow \left( \frac{\sum \left( \delta_{ij} u_i \right)}{\delta_{ij}} \ldots, 1 \ldots, \frac{\delta_{kj} u_j}{\delta_{kj}} \right)$$

$$u_i \rightarrow \mathbb{C}D^k$$

The function $f: M \rightarrow \mathbb{C}D^k$ is called a classifying map for the line bundle $L \rightarrow M$.

$$[x_0: \ldots: x_k] \in \mathbb{C}D^k, \ [y_0: \ldots: y_k] \in \mathbb{C}D^k \rightarrow [x_0: \ldots: x_k] \cdot [y_0: \ldots: y_k] \in \mathbb{C}D^k$$

$$\frac{k}{r} \sum_{i=0}^{k} x_i t^i$$

$$\frac{k}{r} \sum_{i=0}^{k} y_i t^i$$

$$\frac{k}{r} \sum_{i=0}^{k} z_i t^i$$

$$\sum_{i=0}^{k} z_i t^i = \left( \frac{k}{r} \sum_{i=0}^{k} x_i t^i \right) \left( \frac{k}{r} \sum_{i=0}^{k} y_i t^i \right)$$

$$z_0 = x_0 y_0, \ z_1 = x_0 y_1 + x_1 y_0, \ z_2 = x_0 y_2 + x_1 y_1 + x_2 y_0 \ldots$$

$$z_{k-1} = x_0 y_{k-1}$$

This gives us embedding.
$\phi: \mathbb{C}P^k \times \mathbb{C}P^l \to \mathbb{C}P^{k+l}$

$\phi^*: H^2_{dR}(\mathbb{C}P^{k+l}) \to H^2_{dR}(\mathbb{C}P^k \times \mathbb{C}P^l)$

Let $H = \{ z_0 = 0 \}$ be the hyperplane in $\mathbb{C}P^{k+l}$ given $z_0 = 0$. Then the Poincaré dual of $H$

$PD(H) = a \in H^2_{dR}(\mathbb{C}P^{k+l})$ so that

$$\int_N a = \text{Int}(H \cap N),$$

for any oriented submanifold $N$ of dimension 2.

$H$

$\phi^{-1}(H) = \phi^{-1}(z_0 = 0) = \phi^{-1}(x_0 y_0 = 0)$

$$= \phi^{-1}(x_0 = 0) \cup \phi^{-1}(y_0 = 0)$$

$$= \{ x_0 = 0 \} \times \mathbb{C}P^l \cup \mathbb{C}P^k \times \{ y_0 = 0 \}$$

$\phi^*(a)$ is the Poincaré dual of this union of submanifolds.

$\mathbb{C}P^k \times \mathbb{C}P^l$

$\phi^{-1}(H)$

Poincaré dual of $Sx_0 = 0$ in $\mathbb{C}P^k$ is a 0-cycle $a$ and Poincaré dual $\phi^*(a)$ in $\mathbb{C}P^k \times \{ y_0 = 0 \}$ is 0-cycle. Hence, $\phi^*(a) = a \oplus 0$.
Recall that $\overline{H}_D^k(CD^k \times C\mathbb{P}^l) = \overline{H}_D^k(CD^k) \otimes \overline{H}_D^k(C\mathbb{P}^l)$

Also consider the map $\overline{j} : M \to M \times M$, $p \mapsto (p, p)$, p.e. M. Then we have

\[ j^* (u \otimes 1) = u, \quad j^* (1 \otimes v) = v, \quad j^* (u \otimes 1 + 1 \otimes v) = u + v. \]

Exercise: From this using projections

\[ \overline{p}_1 : M \times M \to M, \quad (p, q) \mapsto p \]
\[ \overline{p}_2 : M \times M \to M, \quad (p, q) \mapsto q. \]

Let $L_1 \to M$ and $L_2 \to M$ be two complex line bundles over $M$. Also let $s_i : M \to L_i$, \( i = 1, 2 \), be sections of $L_i$.

Then $s(p) = s_1(p) s_2(p)$ is a section of $L_1 \otimes L_2 \to M$.

Let $f : M \to \mathbb{C}P^k$ and $g : M \to \mathbb{C}P^l$ be two classifying maps for the bundles $L_1 \to M$ and $L_2 \to M$, respectively.

\[ f = (s_0, \ldots, s_k), \quad \bigcap s_i^{-1}(0) = \emptyset \]
\[ g = (s_0, \ldots, s_l), \quad \bigcap s_i^{-1}(0) = \emptyset \]

\[ \phi = \bigotimes_{i=0}^{k+l} (s_i) : \mathbb{C}P^k \times \mathbb{C}P^l \twoheadrightarrow \mathbb{C}P^{k+l} \]
\[ \Phi \circ \Phi^{-1} \quad (g) = \left[ \sigma_0(g) \hat{\sigma}(u) : \sigma_0(g) \hat{\sigma}(1) + \sigma_1(g) \hat{\sigma}(2) \right] : \ldots \]

which is a classifying map for the line bundle \( L_1 \otimes L_2 \to M \).

Let \( x = c_1(\mathcal{F}_g) = c^0(\mathcal{F}) \) be the 1st Chern class of the line bundle \( \mathcal{F}_g \to \mathbb{C} \mathbb{P}^1 \).

\[
\begin{align*}
\chi_1(L_1 \otimes L_2) &= c^0((L_1 \otimes L_2)_{\mathbb{R}}) \\
&= c^0((\Phi \circ \Phi^{-1})^* (\mathcal{F}_{\mathbb{R}})) \\
&= (\Phi \circ \Phi^{-1})^* c^0(\mathcal{F}_{\mathbb{R}}) \\
&= (\Phi \circ \Phi^{-1})^* (x) \\
&= \overline{J}^{*} \circ (\Phi^* x) \\
&= \overline{J}^{*} \circ (\Phi^* c_1(\mathcal{F}_g) + \overline{J}^* \circ (\Phi^* c_1(\mathcal{F}_g))) \\
&= \overline{J}^{*} \circ (\Phi^* c_1(\mathcal{F}_g) + \Phi^* c_1(\mathcal{F}_g)) \\
&= \Phi^* c_1(\mathcal{F}_g) + \Phi^* c_1(\mathcal{F}_g) \\
&= \Phi^* (c_1(\mathcal{F}_g) + \Phi^* (c_1(\mathcal{F}_g))) \\
&= \Phi^* (\Phi^* (c_1(\mathcal{F}_g)) + \Phi^* (c_1(\mathcal{F}_g))) \\
&= \Phi^* (c_1(\mathcal{F}_g) + c_1(\mathcal{F}_g)) \\
&= \Phi^* (c_1(\mathcal{F}_g) + c_1(\mathcal{F}_g)) \\
&= c_1(L_1) + c_1(L_2).
\end{align*}
\]

\[ \chi_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2). \]
Remark: \( s_i : \mathcal{M} \to L_i \) Section

\[ s = s_1 \cdot s_2 : \mathcal{M} \to L_1 \otimes L_2 \text{ Section} \]

\[ s^{-1}(c) = s_1^{-1}(c) \cup s_2^{-1}(c) \]

Section Diagram

Proposition: If \( k \) is a positive integer with \( 2(k+1) > n+1 \) and there is a 1-1 correspondence between the homotopy classes of smooth maps from \( \mathcal{M} \to \mathbb{CP}^k \) and the isomorphism classes of complex line bundles over \( \mathcal{M} \):

\[ \left[ \mathcal{M}, \mathbb{CP}^k \right] \rightarrow \mathfrak{l}(\mathcal{M}) \]

\[ \Gamma \rightarrow \mathfrak{s}_k \]

\[ \left[ \pi : \mathcal{M} - \mathbb{CP}^k \rightarrow \mathbb{CP}^k \right] \rightarrow \pi^* \mathfrak{s}_k \rightarrow \mathbb{CP}^k \]

Definition of higher Chern Classes

\( r > 1 \), \( \pi^0 : E \rightarrow \mathcal{M} \) complex vector bundle of rank \( r \)

\[ C^r \rightarrow \mathcal{E} \]

\[ C^r : \mathcal{E} / \mathcal{E}^* = \mathbb{CP}^{r-1} \rightarrow \mathcal{E} \]

\[ \pi^* (E) \rightarrow E \]

\[ \mathbb{P}(E) \rightarrow \mathcal{M} \]

\[ \pi^0 \]
\[ C \rightarrow \pi^*(E) = \{ (p, v) \in P(E) \times E \mid \pi(p) = \pi(v), p \in \pi^{-1}(v) \} \]

The vector bundle \( \pi^*(E) \rightarrow P(E) \) has a natural line subbundle:

\[ L = \{ (p, v) \in \pi^*(E) \mid v \in l_p, \pi(p) \} \rightarrow P(E) \]

where \( l_p = E_p \), rank 1 subspace. Let \( G_2 = E_p / l_p \) be the quotient vector space. The underlying family of vector bundles

\[ 0 \rightarrow L \rightarrow \pi^*(E) \rightarrow Q \rightarrow 0 \]

Exercise: Construct \( Q \rightarrow P(E) \) explicitly using transition functions.

Now define \( a \in c_1(L^*) \in H^2(P(E)) \).

Note that \( c_1(L) = -a \).

Recall that the Euler class and the 1st Chern class of a complex line bundle are natural. Thus the first Chern class of the restriction of \( L \) to any fiber of \( P(E) \) is \(-a\). Since \( L \rightarrow \mathbb{C}P^{n+1} \) is the tautological bundle...
\( a \in H^2_{DR}(\mathbb{CP}^{r-1}) \) is a generator. Hence, the cohomology algebra \( H^*_{DR}(\mathbb{CP}^{r-1}) \) has \( R \)-basis \( \delta_1, a, a^2, \ldots, a^{r-1} \).

Now \( \mathbb{CP}^{r-1} \rightarrow \mathbb{P}(E) \) \( \delta_1, a, a^2, \ldots, a^{r-1} \) \( \subseteq \) \( M \)

Now by Leray-Hirsch the set \( \delta_1, a, a^2, \ldots, a^{r-1} \) makes \( H^*_{DR}(\mathbb{P}(E)) \) a free \( H^*_{DR}(M) \)-module.

Consider the elements \( a^k \in H^k_{DR}(\mathbb{P}(E)) \). Now \( a^k = c_1(\mathcal{E})a^{k-1} + \cdots + c_i(\mathcal{E})a^{k-i} + \cdots + c_r(\mathcal{E}) = 0 \)

for some unique elements \( c_1(\mathcal{E}), \ldots, c_r(\mathcal{E}) \) in \( H^*_{DR}(M) \). Now we call \( c_i(\mathcal{E}) \) as the \( i \)th Chern class of the complex vector bundle \( E \rightarrow M \).

Remark: If \( r = 1 \) then the two definitions of \( c_1(\mathcal{E}) \) agree.

\[ E = L \rightarrow M, \quad \mathcal{E}(E) = H \]

\[ L \simeq \mathcal{L} \times L(1) \rightarrow M, \quad a = c_1(\mathcal{L}) = -c_1(L) = -\varepsilon(\mathcal{L}) \]

= \( a + \varepsilon(\mathcal{L}) = 0 \).

= \( \varepsilon(\mathcal{L}) = \varepsilon(\mathcal{L}_R) \) in the new definition.

2) Assume that \( E \rightarrow M \) is the trivial \( \mathbb{C} \)-bundle.

\[ E = \mathbb{C} \times M \]
The \( \mathfrak{P}(E) = \pi : \mathbb{C} \mathbb{P}^{r-1} \rightarrow M \)

\[ H^s_{\text{Dol}}(\mathfrak{P}(E)) \cong H^s_{\text{Dol}}(M) \otimes H^s_{\text{Dol}}(\mathbb{C} \mathbb{P}^{r-1}) \]

\[ a^r = a^0 = 0 \]

\[ a^r + 0 = 0 = 0 \implies c_i(E) = 0 \text{ for all } i \geq 1. \]

**Definition:** For any complex vector bundle \( E \rightarrow M \), the \( c_0(E) \) is defined to be the class \( [c^* E] \in H^0(M). \)

\[ c_0(E) = 1. \]

**Definition:** The total Chern class of a complex vector bundle \( E \rightarrow M \) is defined to be the element

\[ c(E) = c_0(E) + c_1(E) + \cdots + c_r(E), \quad r = \text{rank } E. \]

**Proposition:** Chern classes are natural in the sense, if \( f : M \rightarrow N \) is a smooth map and \( E \rightarrow M \) is a complex vector bundle, then

\[ c_i(f^* E) = f^* (c_i(E)), \quad \text{for all } i. \]

This can be proved directly using the definitions, whose details are left as an exercise.

We'll prove this using the so called splitting principle.
**Theorem (Splitting Principle)**

Let \( \pi : E \to M \) be a complex vector bundle of rank \( r \). Then there is a manifold \( F(E) \) and a map \( \phi : F(E) \to M \) so that the following hold:

1. \( \phi^*(E) \to F(E) \) is a direct sum complex line bundles:
   \[
   F(E) \cong L_1 \oplus L_2 \oplus \cdots \oplus L_r
   \]

2. the homomorphism \( \phi^*: H_{DR}^k(M) \to H_{DR}^k(F(E)) \) is injective.

Proof left as an exercise.

**Conclusion:** To prove a polynomial identity among the Chern classes of a vector bundle \( E \to M \), we may assume that \( E \) is a sum of line bundles.

**Theorem (Whitney Product Formula)**

Let \( E_i \to M \), \( i = 1, \ldots, r \), be complex vector bundles. Then
\[
c(E_1 \otimes E_2 \otimes \cdots \otimes E_r) = c(E_1) \cdot c(E_2) \cdots c(E_r)
\]

\[
(c(E) = c_0(E) + c_1(E) + \cdots + c_k(E), \text{ rank } E = k)
\]

Proof: Special Case: rank \( E_i = 1 \). \( E_i = L_i \)

\[
E = E_1 \otimes E_2 \otimes \cdots \otimes E_r = L_1 \otimes L_2 \otimes \cdots \otimes L_r.
\]
\[ \pi: \mathbb{P}(E) \rightarrow M \] 

Before and similarly, let

\[ \bar{E} = \pi^*(E) = \{ (p, v) \in \mathbb{P}(E) \times E | \pi^0(v) = p, p \in M, v \in \mathbb{P}(V_p) \} \]

\[ \pi^0: E \rightarrow M \]

\[ \bar{E} \rightarrow \mathbb{P}(E), \quad \bar{L} = \pi^*(L) \]

\[ L = \delta^0 \circ \pi^*(E) | v \in \mathbb{P}(V_p), \exists L \subset \bar{E} \text{ a subline bundle.} \]

\[ L \subset \bar{E} = \bar{L}_1 \oplus \cdots \oplus \bar{L}_r, \quad s_i: L \rightarrow \bar{L}_i \text{ the restriction of the projection } \bar{E} \rightarrow \bar{L}_i \text{ to } L. \]

\[ s_i \in \text{Hom}(L_i, \bar{L}_i) = L^* \otimes \bar{L}_i, \quad s_i(q): \mathbb{P}(E) \rightarrow L^* \otimes \bar{L}_i \]

\[ \forall i \in \{ \delta \in \mathbb{P}(E) | s_i(q) \neq 0 \} \]

Note that for any \( q \in \mathbb{P}(E) \) at least one \( s_i(q) \neq 0 \). Hence, \( \mathbb{P}(E) = V_1 \cup V_2 \cup \cdots \cup V_r \).

Consider \( L^* \otimes \bar{E} = L^* \otimes (\bar{L}_1 \oplus \cdots \oplus \bar{L}_r) \)

\[ = (L^* \otimes \bar{L}_1) \oplus \cdots \oplus (L^* \otimes \bar{L}_r). \]

Now we have sections

\[ s: \mathbb{P}(E) \rightarrow L^* \otimes \bar{E}, \quad s(q) = (s_1(q), \ldots, s_r(q)). \]

By construction \( s(q) \neq 0 \) for all \( q \in \mathbb{P}(E) \).

Hence \( \pi^0(L^* \otimes \bar{E}) = 0 \).
\begin{align*}
&\text{For the general case } E = E_1 \otimes \cdots \otimes E_N, \text{ it is enough to prove this for } r = 2. \\
&\text{Moreover, by the Splitting Principle we may assume that } E_1 \text{ and } E_2 \text{ are sums of linear bundles.}
\end{align*}
Line bundles:

\[ E_1 = L_1 \oplus \cdots \oplus L_k, \quad E_2 = L'_1 \oplus \cdots \oplus L'_k \]

must show: \( c(E_1 \oplus E_2) = c(E_1) c(E_2) \).

\[
\begin{align*}
    c(E_1 \oplus E_2) &= c(L_1 \oplus \cdots \oplus L_k \oplus L'_1 \oplus \cdots \oplus L'_k) \\
                    &= \left[ c(L_1) \cdots c(L_k) \right] \left[ c(L'_1) \cdots c(L'_k) \right] \\
                    &= c(L_1 \oplus \cdots \oplus L_k) c(L'_1 \oplus \cdots \oplus L'_k) \\
                    &= c(E_1) c(E_2).
\end{align*}
\]

Applications: 1) \( E = L_1 \oplus \cdots \oplus L_r \), rank \( E = r \).

\[
    c(E) = c(L_1) \cdots c(L_r)
\]

\[
1 + c_1(E) + \cdots + c_r(E) = (1 + c_1(L_1)) \cdots (1 + c_1(L_r))
\]

\[
= 1 + c_1 + \cdots + c_r, \quad \text{where}
\]

\[
\sigma_1 = c_1(L_1) + \cdots + c_1(L_r) \quad \text{1st elementary symmetric polynomial}
\]

\[
\sigma_2 = \sum_{1 \leq i < j \leq r} c_1(L_i) c_1(L_j) \quad 2^{nd} \text{sym. poly. in } c_1(L_i)
\]

\[
\sigma_r = c_1(L_1) \cdots c_1(L_r) \quad r^{th} \text{elem. sym. poly. in } c_1(L_i)
\]

Hence, \( c_k(E) = k^{th} \text{ elem. sym. poly. of } c_1(L_i) \).
Proposition: Let $E \to M$, rank $E = r$, complex vector bundle. Then $c_r(E) = c(E |_R)$.

Proof: Both $c_i$'s are $c_i$'s on natural coord, and two by the Splitting Principle we assume that $E$ is sum of complex line bundles:

$$E = L_1 \otimes L_2 \otimes \cdots \otimes L_r.$$ 

$$c_r(E) = c_r(L_1 \otimes L_2 \otimes \cdots \otimes L_r)$$ 

$$= c_1(L_1) c_1(L_2) \cdots c_1(L_r)$$ 

$$= e(L_1)e(L_2) \cdots e(L_r)$$ 

$$= e(L_1 \otimes L_2 \otimes \cdots \otimes L_r)_R$$ 

$$= c_r(E |_R).$$

Proposition: Let $E \to M$ be a rank $r$ complex vector bundle and $E^* \to M$ denote the conjugate bundle. Then

$$c_i(E^*) = (-1)^i c_i(E).$$

Proof: The conjugate bundle is defined as follows:

Let $\phi : U \to GL(r, C)$ be a transition function for the bundle $E \to M$; then

$$c_i(E^*) = (-1)^i c_i(E).$$
the bundle whose transition functions

\[ \overline{\rho}_{\beta \beta} : U_{\alpha} \cap U_{\beta} \to GL(\mathcal{V}(\ell)), \quad \overline{\rho}_{\beta \beta}(p) = \overline{\rho}_{\alpha \beta}(p). \]

If \( E = L \) is a line bundle then

\[ \overline{\rho}_{\alpha \beta}(p) = \overline{\rho}_{\beta \alpha}(p) \quad \text{and} \quad c_1(L) = -c_1(L). \]

\[ \overline{\rho}_{\alpha \beta}(p) = \frac{x + iy}{y - x} \quad \text{and} \quad \overline{\rho}_{\beta \alpha}(p) = \frac{x - iy}{y + x}. \]

Remark \quad \overline{\mathcal{T} \mathcal{E}} = L_{\lambda} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \otimes \overline{L}_{\nu} \ot times
Chern Classes of $\mathbf{CP}^n$

\[ H^*_{dR}(\mathbf{CP}^n) = R[\alpha] / (\alpha^{n+1}) \quad \alpha \in H^2_{dR}(\mathbf{CP}^n) \]

\[ a = PD(H), \quad H = \mathbf{CP}^n \quad (Z = 0), \quad \sum a = 1 \]

\[ \sum a^k = 1, \quad \mathbf{CP}^k = \mathbf{CP}^n, \quad x_0 = 0, \ldots, x_{n-1} = 0. \]

\[ \text{Claim: } c(T_X \mathbf{CP}^n) = (1 + \alpha)^{n+1} = 1 + (n+1)\alpha + \cdots + (n+1)\alpha^{n+1} \]

\[ c_0 = 1, \quad c_1 = (n+1)a, \ldots, c_n = (n+1)a^n \]

\[ \text{Proof: } \text{by induction on } n. \]

\[ n = 1 \quad \mathbf{CP}^1 = \mathbb{S}^1 \quad c_0 = 1, \quad c_1 = c_1(T_\mathbf{CP}^1) = c(T_\mathbb{S}^1) = 2\alpha. \]

\[ c(T_X \mathbf{CP}^n) = (1 + 2\alpha) = (1 + c)^2 \]

Now assume the result for $n$, and let's prove it for $n+1$.

\[ c(T_X \mathbf{CP}^{n+1}) = (1 + c)^{n+2}. \]

\[ \text{must show } c(T_X \mathbf{CP}^{n+1}) = (1 + c)^{n+2}. \]

\[ H = \mathbf{CP}^{n+1}, \quad H: x_0 = 0 \quad H = \mathbf{CP}^n \]

\[ a = PD(H), \quad T_X \mathbf{CP}^{n+1}|_H = N \oplus T_X H \text{ as complex bundles.} \]

\[ H \xrightarrow{\alpha} \quad \text{rank}_\mathbb{C} N = 1, \quad c_1(N) = c(T_X \mathbf{CP}^n) = \alpha \]
because $\nu_\perp$ is the Bôcher dual of the submanifold $H \subseteq \mathbb{C}^n$.

i: $H = \mathbb{C}^n \rightarrow \mathbb{C}^{n+1}$, $\mathbb{T}_* \mathbb{C}^{n+1}|_{\mathbb{C}^n} = N \oplus \mathbb{T}_* \mathbb{C}^n$

$\iota^*(c(\mathbb{T}_* \mathbb{C}^{n+1})) = c(\mathbb{T}_* \mathbb{C}^n) c(N) = (1 + a)^n (1 + a)$

Since $\mathbb{T}$ is homogeneous, for $0 < k < n$, we get

$c_k(\mathbb{T}_* \mathbb{C}^{n+1}) = \binom{n+2}{k} a^k$, for $0 < k < n$.

So we need to show that $c_{n+1}(\mathbb{T}_* \mathbb{C}^{n+1}) = (n+2)^{n+1} a^{n+1}$.

$c_{n+1}(\mathbb{T}_* \mathbb{C}^{n+1}) = c(\mathbb{T}_* \mathbb{C}^n) = x(\mathbb{C}^{n+1}) a^{n+1} = (n+2)^{n+1} a^{n+1}$

This finishes the proof.

Adjoint Formula $M = \mathbb{C} \oplus \mathbb{C}^2$, $C : (t = 0) \subseteq \mathbb{C}^2$

deg $f = d$ homogeneous polynomial in $x, y, z, t$.

Assume that $C$ is a smooth connected curve. $C \subseteq \mathbb{C}^2$ is an oriented surface.

$C = z \circ \bigcirc \bigcirc \bigcirc \bigcirc$ $g ? f$

$\mathbb{T}_* \mathbb{C}^2|_C = N \oplus \mathbb{T}_* C$

$\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc$

$N \subseteq C$
\[ c_i \left( \mathbb{P}^2 \right) = (1 + a)^3 = 1 + 3a + 3a^2 \]
\[ c_i \left( \mathbb{P}^2 \right) = 3a \]
\[ c_i \left( \mathbb{P}^2 \right) = 3a^2 \]

\[ c_i \left( T_x \mathbb{P}^2 \right) = c_i \left( \mathbb{P}^2 \right) + c_i (T_x \mathbb{P}^2) \]
\[ (2a) = c_i (N) + c_i (T_x \mathbb{P}^2) \]
\[ = e(N) + e(T_x \mathbb{P}^2) \]

\[ \int_{C} \int_{(-a-b)} \int_{C} e(N) = 2 \mathcal{X} (C) = 2 - 2g \]
\[ \int_{C} \int_{(-a-b)} \int_{C} e(T_x \mathbb{P}^2) = 3 \mathcal{X} (C) = 3 - 3d \]

\[ H = 4 \mathbb{P}^1 \text{ has degree } 2 \text{ and } a. \]

So, we get \( 3d = d^2 + 2 - 2g \).
\[ \Rightarrow \quad 2g = d^2 - 3d + 2 = (d-1)(d-2) \]
\[ \Rightarrow \quad g = \frac{1}{2} (d-1)(d-2). \]

\[ \text{For } d = 1, 2 \Rightarrow g = 0 \Rightarrow C = \mathbb{P}^2 \]
\[ d = 3 \Rightarrow g = 1 \Rightarrow C = T^2 \text{ Elliptic curve} \]
\[ d = 4 \Rightarrow g = 3 \]

This is called the degree genus formula for curves in \( \mathbb{P}^2 \).
Degree-Genus Formula for $\mathcal{M} = \mathcal{C}^{2l} \times \mathcal{C}^{2l}$

$f(z_0, z_1, w_0, w_1)$ homogeneous of bidegree \( d \) and \( d_2 \) in \( z_0 \) and \( w_0, w_1 \), respectively.

\[
\frac{f}{z_0^d + z_1^d - w_0 w_1 + w_1^d}
\]

$C : (f = 0) \cap \mathcal{M}$. \( \pi : \mathcal{C}^{2l} \times \mathcal{C}^{2l} \to \mathcal{C}^{2l} \), projection.

\[
\pi_* \left( \mathcal{C}^{2l} \times \mathcal{C}^{2l} \right) = \pi_*^\mathcal{C} \left( \mathcal{T}_\pi \mathcal{C}^{2l} \right) \oplus \pi_*^\pi \left( \mathcal{T}_\pi \mathcal{C}^{2l} \right)
\]

\[
c_1 \left( \mathcal{T}_\pi \left( \mathcal{C}^{2l} \times \mathcal{C}^{2l} \right) \right) = \pi_*^\mathcal{C} \left( 2\gamma \right) + \pi_*^\pi \left( 2\delta \right)
\]

$\alpha_1$ is the Poincaré dual of $\delta \in \mathcal{H}(\mathcal{C}^{2l})$.

\[
\int \alpha_1 = \int + \left( C, \mathcal{C}, \mathcal{C}^{2l} \times \mathcal{C}^{2l} \right) = d_2
\]

\[
\int \alpha_2 = \int + \left( C, \mathcal{C}, \mathcal{C}^{2l} \times \mathcal{C}^{2l} \right) = d_1.
\]

\[
\int c_1 \left( \mathcal{T}_\pi \left( \mathcal{C}^{2l} \times \mathcal{C}^{2l} \right) \right) = 2(d_1 - d_2).
\]

Also, we have $\mathcal{T}_\pi \left( \mathcal{C}^{2l} \times \mathcal{C}^{2l} \right) |_C = \pi \oplus \mathcal{T}_\pi C$.

\[
c_1 (N) = e (N_{\mathbb{R}}), \quad \int c_1 (N) = \int e (N_{\mathbb{R}}) = \int + (C, C) = 2d_1 - d_2.
\]
\[ f=0 \quad \text{Diagram} \quad c = \frac{5}{2}. \]

\[ c_i(T_x \mathbb{P}^1 \times \mathbb{P}^1)|_c = c_i(N) + c_i(T_x \mathbb{P}^1) \]

\[ 2(d_1 + d_2) = 2d_1d_2 + 2 - 2\alpha \]

\[ \Rightarrow \alpha = d_1d_2 - d_1 - d_2 + 1 = (d_1 - 1)(d_2 - 1). \]

The lens space formula for smooth bideg \((d_1, d_2)\) curves in \(\mathbb{P}^1 \times \mathbb{P}^1\).
Portraying Characteristic Classes

$E \to M$ real vector bundle of rank $k$.

$F = E \otimes \mathbb{C} \to M$ complexification of $E$.

Let $\psi_{ab}: U_a \cap U_b \to GL(k, \mathbb{R})$ be the transition function of $E \to M$, then the transition function of $F = E \otimes \mathbb{C} \to M$ are the same functions considered into $GL(k, \mathbb{C})$.

$\psi_{ab}: U_a \cap U_b \to GL(k, \mathbb{R}) \leq GL(k, \mathbb{C})$.

$E = U_a \times \mathbb{R}^k \underrel{\sim}{\left/ x \sim (x, y_{ab}(x, v)) \right.}$

$x \in U_a \cap U_b, v \in \mathbb{R}^k, \psi_{ab}$

$F = U_a \times \mathbb{C}^k \underrel{\sim}{\left/ C \right.}$

$x \sim (x, y_{ab}(x, v))$

$a x \in U_a \cap U_b, v \in \mathbb{C}^k, \psi_{ab}$

Therefore, the transition function for $E, F = \otimes \mathbb{C}$ and $F = E \otimes \mathbb{C}$ are the same.

In particular, $F$ and $\overline{F}$ are isomorphic as complex vector bundles. Thus

$c_{2i+1}(F) = c_{2i+1}(\overline{F}) = (-1)^{2i+1} c_{2i+1}(F) = -c_{2i+1}(\overline{F})$
So, \( c_{2i+1}(F) = 0 \) for all \( i \).

**Definition:** The \( i \)-th Pontryagin class \( \mathcal{P}_i(F) \) of a real vector bundle \( E \rightarrow M \) is defined to be the class

\[
\mathcal{P}_i(E) = (-1)^i c_i(E \otimes \mathcal{C}) \in H^{2i}_{\text{dR}}(M).
\]

**Proposition:** Let \( E \rightarrow M \) be a complex vector bundle of rank \( r \). Let \( E^r \) denote the underlying real rank \( 2r \) bundle. Then

\[
E^r \otimes \mathcal{C} = E \oplus \overline{E} \text{ as complex vector bundles.}
\]

**Proof:** \( \mathfrak{C}^r, \omega \in \mathfrak{C}^r, \omega = (u_1 + iv_1, \ldots, u_r + iv_r) \)

\( u_i, v_i \in \mathbb{R} \quad (u_1, v_1, \ldots, u_r, v_r) \in \mathbb{R}^{2r} \)

\( \mathfrak{C}^r \cong \mathbb{R}^{2r} \)

Let \( z = e^{i \theta} \in \mathbb{C} \). Then \( z : \mathbb{C}^r \rightarrow \mathbb{C}^r \) \( \omega \mapsto z \omega \)

\( \omega : (u_1, v_1, \ldots, u_r, v_r) \)

\[
= (\ldots, r \cos \theta u_k - r \sin \theta v_k, r \sin \theta u_k + r \cos \theta v_k, \ldots)
\]

\( z \omega : (u_k) = \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right) (u_k) \)

\( (v_k) \)
Let's diagonalize this operator:

\[ A_z = r \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]

\[ \text{tr}(A_z) = 2r \cos \theta \]

\[ \text{det}(A_z) = r^2 \]

\[ -\text{tr}(A_z) = -2r \cos \theta = -2z \overline{z} \]

Hence, the roots of the eigenvalue equation for \( A_z \) on \( z \) are \( z \) and \( \overline{z} \).

In other words, eigenvalues of \( w \in \mathbb{R} \) are \( z \) and \( \overline{z} \).

\[ \mathbb{R}^{2n} \otimes \mathbb{C} \xrightarrow{z} \mathbb{R}^{2n} \otimes \mathbb{C} \]

\[ w_{\mathbb{R}} \xrightarrow{z} z \cdot w_{\mathbb{R}} \]

\[ A_z = r \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]

\[ A_z - z \mathbb{I} = r \begin{pmatrix} \cos \theta - z & -\sin \theta \\ \sin \theta & \cos \theta - z \end{pmatrix} \]

\[ = r \begin{pmatrix} (1-z)z & -(\sin \theta) \\ (\sin \theta) & (1-z)z \end{pmatrix} \begin{pmatrix} e_1 \\ e_{\overline{z}} \end{pmatrix} \]

If \( z \sin \theta + y \sin \theta b = 0 \) \( \Rightarrow \) \( z \overline{a} + \overline{b} = 0 \)

\[ a = 1, \quad b = -i \quad \begin{pmatrix} 1 \\ -i \end{pmatrix} \sim e_k - i f_k \]
\[ z \mapsto e^{i \frac{k}{3}} \quad \mathcal{B} = \{ e^{i \frac{1}{3}}, \ldots, e^{i \frac{2}{3}} \} \]
\[ \bar{z} \mapsto e^{i \frac{k}{3}} \quad \bar{\mathcal{B}} = \{ e^{i \frac{1}{3}}, \ldots, e^{i \frac{2}{3}} \} \]

When \( \mathcal{B} = e^{i \frac{k}{3}} \), \( \bar{\mathcal{B}} \) is the standard basis for \( \mathbb{C}^r \).

The \( \mathbb{C}^r \otimes \mathbb{C} = \mathcal{B} \otimes \bar{\mathcal{B}} \)

\[ z = \bar{z} \quad \bar{z} = \bar{z} \quad \bar{z} = -z \]

This proves the property of our fiber. Since the complex structure varies from fiber to fiber smoothly, this procedure gives

\[ E \otimes \mathcal{B} = \mathcal{E} \otimes \bar{\mathcal{E}} \] as vector bundles.

For \( E \rightarrow M \) \( \mathcal{T} \mathcal{E} \) the total

Borel equi-class of \( E \). Hence, \( \mathcal{T}_0 \mathcal{E} = \mathcal{E} \mathcal{H}_0 \mathcal{E} \).

Also define \( \mathcal{T} \mathcal{E} \) \( \mathcal{E} = \sum (-) \mathcal{T} \mathcal{E} \).

Corollary \( E \rightarrow M \) \( E \) is complex vector

bundle then \( \mathcal{T} \mathcal{E} \mathcal{E} = \mathcal{E} \mathcal{E} \mathcal{E} \).
Proof: \( c(E)c(\overline{E}) = (\sum c_i(E)) \cdot (\sum c_j(\overline{E})) \)

\[ = (\sum c_i(E)) \cdot (\sum (-1)^{i-j} c_j(\overline{E})) \]

\[ = \sum_{i,j} (-1)^{i-j} c_i(E)c_j(\overline{E}) \]

\( \mathbb{E} \otimes \mathbb{C} = \mathbb{E} \otimes \overline{\mathbb{E}} \)

\( \overline{i}(\mathbb{E} \otimes \mathbb{C}) = \sum_k (-1)^k \overline{i}_k(\mathbb{E} \otimes \mathbb{C}) \)

\[ = \sum_k (-1)^k c_{2k}(\mathbb{E} \otimes \mathbb{C}) \]

\[ = \sum_k c_{2k}(E \otimes \overline{E}) \]

\[ = \sum_k \sum_{i,j} c_i(E)c_j(\overline{E}) \]

\[ = \sum_k \sum_{i,j} (-1)^{i-j} c_i(E)c_j(\overline{E}) \]

\[ = \sum_k \sum_{i,j} (-1)^{i-j} c_i(E)c_j(\overline{E}) \]

This finishes the proof.

Examples
1) M complex manifold of complex dimension 2 (\( \dim_{\mathbb{C}} M = 4 \)).

The \( \overline{\mathbb{D}}(M) = \overline{\mathbb{D}}(\mathbb{C} \otimes M) = 1 - p_1(M) \)

\[ = (1 + c_1(M) + c_2(M)) \]

\[ (1 - c_1(M) + c_2(M)) \]
\[ 1 - P_1(M) = 1 + 2c_2(M) - c_1^2(M) \]

So \[ -P_1(M) = 2c_2(M) - c_1^2(M) \]

\[ P_1(M) = c_1^2(M) - 2e(M). \]

For example, if \( M = S\mathbb{P}^2 \), then

\[ P_1(M) = c_1^2(M) - 2e(M) \]

\[ = (3a)^2 - 2 \cdot (3 \cdot a^2) \]

\[ = 9a^2 - 6a^2 = 3a^2 \]

**Proposition:** \( f: M \rightarrow N \) differentiable map and \( E \rightarrow N \) is a real vector bundle then

\[ P(f^*(E)) = f^*(P(E)). \]

Moreover, if \( E_i \rightarrow M \) are vector (real) bundles for \( i = 1, 2 \), then

\[ P(E_i \otimes E_j) = p(E_i) \cdot p(E_j). \]

Proof follows from naturality of Chern classes and related formulas for the Chern classes.

**Example:** Suppose that \( M \subseteq \mathbb{R}^n \) an submanifold.

\[ T_x(R^n \mid _M) = T_xM \otimes \mathbb{R} \]

Hence, \( p(T_x(R^n \mid _M)) = p(M) \cdot p(x) \).
Since $T_x \mathbb{R}^n|_M = M \times \mathbb{R}^n$ is trivial, $\pi_1(T_x \mathbb{R}^n|_M) = 1$.

Here $u_1 = \pi_1(M \times \mathbb{R}^n)$.

Let for example $M = \mathbb{R} \times \mathbb{R}^2 \leq \mathbb{R}^n$. Then we have

$$1 = p(x \mathbb{R}^2) \cdot p(x \mathbb{R}^2).$$

$$1 = (1 + 3 a^2) p(x \mathbb{R}^2) \text{ in } H^*(\mathbb{R} \times \mathbb{R}^2) = \mathbb{R}[x]/(x^2).$$

Then $p(x \mathbb{R}^2) = (1 - 3 a^2)$. So $p_1(x \mathbb{R}^2) = -3a^2$.

$x \mathbb{R}^2$ is an oriented bundle of rank $n - n = n - 4$.

If $n = 5$, then $x \mathbb{R}^2$ is an oriented rank $1$ bundle.

So $x \mathbb{R}^2 = M \times \mathbb{R}$ is trivial.

Thus $p(x \mathbb{R}^2) = 1$, a contradiction.

If $n = 6$, then $x \mathbb{R}^2$ is an oriented rank $2$ bundle.

Hence, $x \mathbb{R}^2$ can be regarded as a complex line bundle.

$x \mathbb{R}^2 = L \to \mathbb{R}^2 = M$.

Say $c_1(L) = x(x \mathbb{R}^2) = x a$, for some integer $a$.

Then $p_1(L) = (x - t)^2 c_2(L \oplus C)$

$$= - c_2(L \oplus C)$$

$$= - c_1(L) c_1(C)$$

$$= -(a - 1)(-a)$$

$$= 2a^2.$$
Therefore, we must have $k^2 = -3$, a contradiction.

Hence, $M = CD^2$ cannot be embedded into $\mathbb{R}^6$.

Remark: It is known that $CD^2$ embeds in $\mathbb{R}^7$.

Proposition: If $E \to M$ is an oriented real vector bundle of rank $2k$, then $\pi_k(E) = c(E)^k$.

Proof: $E$ oriented rank $2k$ vector bundle. Say \( \{e_1, \ldots, e_n\} \) be an oriented basis at a fiber $yF$.

The $E \oplus \mathbb{C} \bar{E}$ is also a real vector bundle of rank $4k$.

The orientation coming from the complex bundle is given by the basis \( \{e_1, ie_1, \ldots, e_{2k}, ie_{2k}\} \).

On the other hand, $E \oplus \mathbb{C} \bar{E} = E \oplus E$ with orientation \( \{e_1, ie_1, \ldots, e_{2k}, ie_{2k}\} \).

It follows that any real oriented bundles

\[
E \oplus \mathbb{C} \bar{E} = (-1)^{k(2k-1)} E \oplus E
\]

\[= (-1)^k E \oplus \bar{E}.\]
\[
\text{So, } \beta_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C}) \\
= (-1)^k e(E \otimes \mathbb{C}), \text{ since } \text{var}_F E \otimes \mathbb{C} = 2k \\
= (-1)^k e((E^k) \otimes \mathbb{C}) \\
= (-1)^k (-1)^k e(E \otimes \mathbb{C}) \\
= e(E) e(E) \\
= e(E)^2.
\]

**Bochner–Bastiaanssen Numbers**

If a dimension 4n, let \( k_1, k_2, \ldots, k_r \geq 0 \) be integers with \( k_1 + 2k_2 + \ldots + rk_r = n \). Then

\[
p_1(M), p_2(M), \ldots, p_r(M) = H^{2k_1 + \ldots + 2k_r}(M)
\]

So we may define the real number

\[
p_k, \beta_k = -\kappa M \\
p_k(M) = \int_M p_1^{k_1}(M) \cdots p_r^{k_r}(M).
\]

This real number is an integer and called a **Bochner–Bastiaanssen number of** \( M \).

Clearly, each Bochner–Bastiaanssen number is a different hom-

**Invariant of** \( M \).

**Proposition**

If \( M = \partial N \) is the boundary of an oriented manifold, \( \text{co-rank} \) then all
Pontryagin numbers of $M$ are zero.

Proof: $P^k_{x^{-1} M} = \int \frac{\Omega^k(M)}{x^k(M)^i} = \int \frac{\Omega^k(M)}{x^k(M)^i} = 0, x^k(M)^i = 0.$

Example: $M = \mathbb{C}P^2, P_1(M) = P_1(\mathbb{C}P^2) = 3a^2 \in H^4_{\text{dr}}(\mathbb{C}P^2).$

$P_2 = \int P_1(M) = \int 3a^2 = 3.$

$\mathbb{C}P^2 = \mathbb{C}P^2$

$P_2 = \int P_2(\mathbb{C}P^2) = \int e(\mathbb{C}P^2) - c^k = 0, k = 2$

$\mathbb{C}P^2 = \mathbb{C}P^2$

In particular, $\mathbb{C}P^2$ is not the boundary of any compact orientable smooth manifold.

**Theorem (René Thom)**

Let $M$ be a compact oriented 4n-dimensional smooth manifold. If all Pontryagin numbers of $M$ are zero, then $M$ is a smooth compact oriented manifold $W$ so that $\partial W = \frac{H^k - nM}{k}$ for some integer $k > 0.$
Milnor's Exotic Sphere:

Aim: Construct smooth manifolds that are homeomorphic but not diffeomorphic to the sphere $S^2$.

Let $M=M^n$ be a smooth oriented closed manifold $B=B^n$ a smooth oriented compact manifold so that $\partial B=M$ as oriented manifolds.

Hence, 1) Assume that $M$ and $B$ are as above and $\int_B \partial \alpha = 0 = \int_B \partial \alpha'$. Then the quadratic form

$$
\int_B \partial \alpha \wedge \alpha'
$$

is well-defined.

Proof: Step 1. Let $[\alpha]=[\alpha']$. Then we must show

$$
\int_B \partial \alpha \wedge \alpha = \int_B \partial \alpha' \wedge \alpha'.
$$

Since $[\alpha]=[\alpha']$, $\alpha-\alpha'=d\beta$ for some $\beta \in \Omega^2(B)$. Then $\alpha = \alpha' + d\beta$ and

$$
\alpha^2 = \alpha'^2 + 2 \alpha' \wedge d\beta + d\beta \wedge d\beta
$$

so that

$$
\int_B \alpha^2 = \int_B \alpha'^2 + \int_B 2 \alpha' \wedge d\beta + \int_B d\beta \wedge d\beta.
$$

Thus

$$
\int_B (\alpha^2 - \alpha'^2) = \int_B d(2 \alpha' \wedge d\beta) = \int_B 2 \alpha' \wedge d\beta.
$$
\[ \int (\alpha^2 - \alpha'^2) = \int (2\alpha^1 + \alpha \beta) \wedge \beta. \]

Hence, to finish the proof it is enough to show that \( \beta \) can be chosen so that \( B' = 0 \).

\[ \begin{equation}
M = 2B \quad \nu = M \times (0,1]
\end{equation} \]

**Step 2:**

\[ \begin{equation}
U = B - (M \times [1/2, 1])
\end{equation} \]

\( U \cup V = B \) and \( U \cap V = M \times (0,1/2) \).

Now \( M \times \mathbb{R} \) is diffeomorphic to \( M \times [0,1/2) \) via a proper diffeomorphism (an exercise).

\[ \text{Hence,} \quad H^k_c(U \cup V) = H^k_c(M \times [0,1/2)) = H^k_c(M \times \mathbb{R}) = H^k_c(M) \quad \text{(Poincaré lemma for compact support cohomology)} \]

Also, \( H^k_c(U \cup V) = H^k_c(B) = H^k_{DR}(B) \) and \( H^k_c(U) = H^k_c(B - M) \) since \( M \subset B \).

\( M \) is diffeomorphic to \( U \) via a proper diffeomorphism.
Now consider the local cohomology sequence for compactly supported cohomology:

\[ 0 \to \Omega^k_c(U, U) \to \Omega^k_c(U) \to \Omega^k_c(U)/\Omega^k_c(U)^0 \to 0 \]

is a short exact sequence. It induces a long exact sequence as follows:

\[ \cdots \to H^k_c(U) \to H^k_c(M) \to H^k_c(M \setminus U) \to H^k_c(U) \to \cdots \]

Take \( U = M \setminus L, L \subseteq M \) closed manifold. Then

\[ H^k_c(M, U) = H^k_c(M, M \setminus L) \cong H^k_c(L). \]

Now, \( H^k_c(U, M) \cong H^k_c(M) = H^k_{DR}(M). \)

**Claim:** \( H^k_c(U) = 0. \)

**Proof:** Local cohomology sequence for the pair \((U, M):\)

\[ \cdots \to H^k_c(U \setminus M) \to H^k_c(U) \to H^k_c(U, U \setminus M) \to H^k_c(U) \to \cdots \]

\[ H^k_c(U \setminus M) = H^k_c(M - \{0, 1\}) = H^k_c(M - \{0, 1\}) = H^k_{DR}(M) = H^k_{DR}(M) \]

Thus, \( \cdots \)

\[ H^k_{DR}(U) \to H^k_c(U) \to H^k_{DR}(U) \to H^k_{DR}(M) \to \cdots \]

\[ \to 0 \]
Step 3: Mayer–Vietoris Exact Sequence for locally compactly supported cohomology for $B = U \cup V$.

\[ \begin{array}{c}
\cdots \to H^k_c(U \cap V) \to H^k_c(U) \oplus H^k_c(V) \to H^k_c(B) \to H^{k+1}_c(U \cap V) \to \cdots 
\end{array} \]

Combining the above results we get
\[ \begin{array}{c}
\cdots \to H^3_{DR}(M) \to H^4_c(B \setminus M) \to H^4_{DR}(B) \to H^4_{DR}(M) \to \cdots 
\end{array} \]

By assumption $H^3_{DR}(M) = 0 = H^4_{DR}(M)$ and thus $H^4_{DR}(B) = H^4_c(B \setminus M)$. Hence, the form $\beta$ in Step 1 can be chosen so that $\beta = 0$ on $M$.

This finishes the proof of the lemma.

Now we define the index of the quadratic form by $T(B)$.

\[ H^4_{DR}(B) \to \mathbb{R}, \quad [\alpha] \mapsto \int_B \alpha^2. \]

Remark: \[ T(-B) = -T(B). \]

Define Pontryagin numbers of $B$ as follows:

\[ q_i(B) = q_i(\pi^2(B)) = \int_B p_i(B), \] which is well-defined by Lemma 1.
We'll see later that \( q(B) \) is an integer.

Finally, define \( \lambda(M) = 2q(B) - T(B) \pmod{7} \).

**Theorem:** \( \lambda(M) \) is independent of the choice of \( B \) and determined only by \( M \).

**Corollary:** If \( \lambda(M) \neq 0 \) then \( M \) cannot be the boundary of a compact manifold \( B \) with \( H^4_{DR}(B) = 0 \).

Similar to \( T(B) \), \( \lambda(M) = -\lambda(M) \), and thus we get

**Corollary:** If \( \lambda(M) \neq 0 \) then \( M \) does not admit an orientation reversing diffeomorphism.

**Proof:** Suppose not: let \( \phi : M \to -M \) be a diffeomorphism. Then \( \lambda(M) = \lambda(-M) = -\lambda(M) \) and thus \( \lambda(M) \neq 0 \), a contradiction.

**Proof of the Theorem:**

**Step 1:** Let \( B_1 \) and \( B_2 \) be two compact oriented manifolds with \( \partial B_i = \Sigma_{j=1}^2 \).
Let $\mathcal{C} = B_1 \cup B_2$, which is an oriented manifold with $\partial \mathcal{C} = \emptyset$. Then by Theorem Hintenbruch Signature Theorem

$$\tau(\mathcal{C}) = \frac{1}{45} \int \tau p^2(\mathcal{C}) - p^4(\mathcal{C}).$$

$$\int_{H^4_{tr}(\mathcal{C})} \rightarrow \mathbb{R}, \quad [\alpha] \mapsto \int_{\mathcal{C}} \alpha^2 \text{ (quadratic form)}$$

$\tau(\mathcal{C})$ is the signature of this quadratic form.

Then we get

$$45 \tau(\mathcal{C}) + 9(\mathcal{C}) = \int \tau p^2(\mathcal{C}) - p^4(\mathcal{C}) + \int p^2(\mathcal{C})$$

$$= 0 \text{ (mod 7).}$$

$$24(\mathcal{C}) + 90 \tau(\mathcal{C}) = 0 \text{ (mod 7)}$$

$$2q(\mathcal{C}) - \tau(\mathcal{C}) = 0 \text{ (mod 7).}$$

**Step 2:** We will prove that

$$\tau(\mathcal{C}) = \tau(B_1) - \tau(B_2) \text{ and}$$

$$q(\mathcal{C}) = q(B_1) - q(B_2).$$

Note that Step 2 finishes the proof of the theorem.
Using similar ideas used in the proof of Lemma 1 we obtain a commutative diagram where each arrow is an isomorphism:

\[ H^4_e(C) \leftarrow H^4_e(B_1-M) \oplus H^4_e(B_2-M) \]

\[
\downarrow \quad \circ \quad \downarrow \\
H^4_{db}(M) \rightarrow H^4_{db}(B_1) \oplus H^4_{db}(B_2) \\
\alpha \quad \longrightarrow \quad B_1 + B_2
\]

So, for any \( \alpha \in H^4_{db}(M) \) we can write \( \alpha = \beta_1 + \beta_2 \) for some \( \beta_i \in H^4_{db}(B_i) \), \( i = 1, 2 \) so that each \( \beta_i \) restricts to zero on \( \partial B_i = M \).

\[ \alpha = \beta_1 + \beta_2 \quad \beta_i \in H^4_{db}(B_i), \quad \beta_i|_{\partial B_i} = 0. \]

\[ \alpha^2 = \beta_1^2 + \beta_2^2 + 2\beta_1 \beta_2 \]

\[ \Rightarrow T(C) = \int \alpha^2 = \int \beta_1^2 + \beta_2^2 = \int \beta_1^2 - \int \beta_2^2 \\
C = B_1 - B_2 \quad B_1 \quad B_2 \]

\[ = T(B_1) - T(B_2). \]

So \( q(C) = q(B_1) - q(B_2) \) \( \text{not note that} \)

Therefore denotes an oriented and then we get \( C = B_1 - B_2 \Rightarrow q(C) = q(B_1) + q(B_2) \)

\[ \Rightarrow p_1^2(C) = p_1^2(B_1) + p_1^2(B_2) \Rightarrow q(C) = q(B_1) - q(B_2). \]
Tangent Bundle of $\mathbb{S}^4$.

$\mathbb{S}^4 = H \cup H_0$, $p \in \mathbb{H}^* = H \setminus \mathbb{D}^2$.

$T_\mathbb{S}^4 = T_\mathbb{H} \cup T_{\mathbb{H}} / \langle p, v \rangle \sim \langle \mathbb{I} / p, D_p (\mathbb{I} / p) \rangle$, when

$\varphi: \mathbb{H}^* \to \mathbb{H}^*$, $\varphi(p) = \mathbb{I} / p$.

Let's compute $D_p (\mathbb{I} / p)$, for $p \in \mathbb{H}^*$, $v \in \mathbb{H} = \mathbb{I} \mathbb{H}^*$.

$D_p (\mathbb{I} / p) = \lim_{h \to 0} \frac{\varphi(p+pv) - \varphi(p)}{h}$

$= \lim_{h \to 0} \frac{\mathbb{I} / p + hv - \mathbb{I} / p}{h}$

$= \lim_{h \to 0} \frac{\mathbb{I} / p + hv}{h} \frac{\mathbb{I} / p - \mathbb{I} / p}{h}$

$= \lim_{h \to 0} \frac{\mathbb{I} / p + hv}{h} \frac{\mathbb{I} / p - \mathbb{I} / p}{h}$

$= \lim_{h \to 0} \frac{(\mathbb{I} / p + hv) ||p||^2 - (\mathbb{I} / p + hv) (\mathbb{I} / p + hv) \mathbb{I} / p}{h} \frac{||p||^2}{h}$

$= \lim_{h \to 0} \frac{(\mathbb{I} / p + hv) ||p||^2 - (\mathbb{I} / p + hv) (\mathbb{I} / p + hv) \mathbb{I} / p}{h} \frac{||p||^2}{h}$

$= \lim_{h \to 0} \frac{(\mathbb{I} / p + hv) \left[ \frac{||p||^2 - (\mathbb{I} / p + hv) \mathbb{I} / p}{h} \right]}{h}$
\begin{align*}
\lim_{h \to 0} & \frac{(\bar{p} + h\bar{v}) \cdot (\chi \cdot \bar{v})}{\|\bar{p} + h\bar{v}\|^2 \cdot \|\bar{p}\|^2} \\
&= \lim_{h \to 0} -\left(\frac{\bar{p} + h\bar{v}}{\|\bar{p} + h\bar{v}\|^2}\right) \cdot \left(\frac{\bar{v}}{\|\bar{p}\|^2}\right) \quad \text{(denominators are non-zero real numbers so they commute)} \\
&= -\frac{\bar{p}}{\|\bar{p}\|^2} \cdot \frac{\bar{v}}{\|\bar{p}\|^2} \\
&= -\frac{1}{\bar{p}} \cdot \frac{\bar{v}}{\bar{p}}.
\end{align*}
$\mathbb{R}^4$-bundle over $S^4$:

$S^2 = \mathbb{C}P^1 = \mathbb{C} \cup \mathbb{C} / z \sim \phi(z) = \frac{1}{z}, \; \phi(0) = 0$.

$\Rightarrow S^4 = H\mathbb{P}^1 = H \cup H / \phi \sim \frac{1}{\phi} = \frac{\overline{p}}{\|p\|^2} = \phi(p)$

$p \in H = \mathbb{R}^4, \; p = (x, y, z, w) = x + iy + z + kw.$

$\overline{p} = x - iy - z + kw.$

$T^*S^4 = T^*H \cup T^*H / (p, v) \sim (\frac{1}{\phi}, \Phi(p)(v))$

$\Phi(p) : T_pH \to T_{\phi(p)}H$

$\Phi(p)(v) = \lim_{h \to 0} \frac{\Phi(p + hv) - \Phi(p)}{h}$

$= -\frac{1}{p} v \frac{1}{p} \left( \mp \frac{1}{p^2} v \right)$

In $H^* = \mathbb{R}^4 \setminus \{0\}$ then there is a path joining -1 to 1. Using this path we can define a homotopy joining the map

$\Phi(p, v) \to -\frac{1}{p} v \frac{1}{p}$

to the map

$\Phi(p, v) \to \frac{1}{p} v \frac{1}{p}$

so that these two maps give isomorphic bundles.
Now for any pair of integers \((h, J)\) define the bundle
\[
\mathbb{F}_{h, J} \to S^4
\]
\[
\mathbb{F}_{h, J} = H \times H \cup H \times H
\]
\[
(l, v) \sim (l_1, v_1 + v_2)
\]
\[
(l, v) \in H^* \times S^4.
\]

Note that \(\mathbb{F}_{-1, -1} = T_x S^4\).

**Lemma:** The characteristic classes \(p_i\) and \(e\) of \(\mathbb{F}_{h, J} \to S^4\) are given by
\[
p_1(\mathbb{F}_{h, J}) = 2(h - J) v \quad \text{and} \quad e(\mathbb{F}_{h, J}) = -(h + J) v
\]
where \(v \in H^*_D(\mathbb{S}^4)\) with \(\int v = 1\).
\[
\mathfrak{S}_{h,j} \rightarrow \mathfrak{S}^4, \quad h, j \in \mathbb{Z}
\]

\[
\mathfrak{S}_{h,j} = H \times H \cup H \times H
\]

\[
(p, v) \sim (\frac{1}{p}, p^h v^j), \quad p \neq 0.
\]

\[
1 \star \mathfrak{S}^4 = \mathfrak{S}_{-1, -1}.
\]

**Lemma:** \( p_1(\mathfrak{S}_{h,j}) = 2(h-j)\nu \) and \( e(\mathfrak{S}_{h,j}) = -(h+j)\nu \)

where \( \nu \in H^4_{\mathrm{DR}}(S^4) \) so that \( \int_{S^4} \nu = 1 \).

**Proof:**

1. \( e(\mathfrak{S}_{h,j}) = -(h+j)\nu \)

**Case 1:** \( h+j \leq 0 \)

Note that the functions \( s_i : H \rightarrow T_a H \)

\[
s_i(p) = (p, 1 + \frac{h-j}{p}) \quad \text{for } i = 1, 2, \quad p \in H,
\]

satisfy the identity \( \frac{1}{p} s_i(p) p = s_2(p) \), \( \forall p \neq 0 \)

and therefore they define a section \( s_{h,j} \)

\[
s : \mathfrak{S}^4 = H_1 \cup H_2 / \sim \quad \overset{p \neq 0}{\rightarrow} \mathfrak{S}_{h,j}
\]

\[
s(p) = \begin{cases} s_1(p) & p \in H_1, \\ s_2(p) & p \in H_2 \end{cases}
\]

\[
(p, v) \sim (\frac{1}{p}, p^h v^j)
\]
\[ \rho = \frac{\beta}{\rho} \rho^h = \rho^h (1 + \frac{\beta}{\rho} - 1) \rho^h \]
\[ = \rho^h \rho^h + 1 \]
\[ = \left( \frac{1}{\rho} \right)^{-h-\frac{1}{2}} + 1 \]
\[ = s_2 \left( \frac{1}{\rho} \right). \]

\[ \int e(\xi_{(h)}, \frac{1}{2}) = \text{Number of zeros of a section transverse to the zero section.} \]

Fact (Eilenberg-Niren, Bull. Amn. Math. Soc. 1944)

The degree of the polynomial map
\[ \mathbb{N} \rightarrow \mathbb{N}, \quad \rho \mapsto p^k, \quad k \in \mathbb{Z}^+, s = k. \]

Hence \( \text{deg} \left( 1 + \frac{\beta}{\rho} - 1 \right) \mathbb{N} \rightarrow \mathbb{N} = -(h+\frac{1}{2}). \)

So, \( \int e(\xi_{(h)}, \frac{1}{2}) = -(h+\frac{1}{2}) \) and then
\[ e(\xi_{(h)}, \frac{1}{2}) = -(h+\frac{1}{2}) \nu, \quad \int \nu = 1. \]

Case 2: \( h + \frac{1}{2} > 0. \)

Note that the homotopy, \( t \in [0, 1], \) given by
\[ (t, (x, y)) \mapsto \left( \frac{1}{\rho}, \frac{p^h}{\rho + (1-t)(\rho + 1 \rho^h)} \right)^{\frac{1}{3}} \]

\[ + (h+\frac{1}{2}) \nu \]
the maps
\[ t = 0, \quad (p, v) \mapsto \left( \frac{1}{p^1}, \left( \frac{p}{\|p\|} \right) \cdot \left( \frac{q}{\|q\|} \right) \right) \]
and
\[ t = 1, \quad (p, v) \mapsto \left( \frac{1}{p}, p^h v, p^h \right) \]
\[ S^h = D^4 \cup D^4 / p \sim \frac{1}{p}, \quad p \in \partial D^4 = S^3. \]
Now let's reverse the orientation of the fibers of \( S_{h_1, \overline{5}} \).

\[ v \mapsto \overline{v} = \overline{v} \]
and the gluing function of the bundle becomes
\[ (p, v) \mapsto \left( \frac{1}{p}, p^h v, p^h \right) \]
\[ (p, \overline{v}) \mapsto \left( \frac{1}{p^1}, \overline{p}^h v, \overline{p}^h \right) \quad \overline{a b} = \overline{b a} \]
\[ \overline{p} = \frac{1}{p} = p^{-1} \]

Hence, we get \( (p, v) \mapsto \left( \frac{1}{p}, \overline{p}^h v, p^h \right) \) so that the effect of changing the orientation of the fiber of \( S_{h_1, \overline{5}} \) results in the bundle \( \overline{3}_{-5}, -\overline{h} \).

\[ \therefore \quad -\overline{3}_{h_1, \overline{5}} = \overline{3}_{-5}, -\overline{h}. \]
Thus \( e(\xi_{h,j}) = -e(-\xi_{h,j}) = -e(\xi_{-h}) \)
\[= -(l + j) \mathcal{V} \]
\[= -(l + j) \mathcal{V}. \]

This finishes the proof of part (A).

(B) \( p_1(\xi_{h,j}) = 2(h - j) \mathcal{V}. \)

Now let's change the orientation of both the base and the fiber of \( \xi_{h,j} \rightarrow \mathbb{S}^4 \).

\((p, q) \mapsto (\bar{p}, \bar{q}) = (q, \bar{q}).\)

\(\xi_{h,j} \mapsto (p, q) \mapsto (\frac{1}{p}, p^h q^j)\)

This becomes in the \((q, \bar{q})\) coordinates

\((q, \bar{q}) \mapsto (\frac{1}{q}, q^p \bar{q}^q)\)

\((p, q) \rightarrow (\frac{1}{p}, p^h q^j) \)

\(\bar{p} \)

\((\bar{p}, \bar{q}) \rightarrow (\frac{1}{\bar{p}}, \bar{p}^\bar{q} \bar{q}^\bar{p})\)

\((q, \bar{q}) \rightarrow (\frac{1}{q}, q^p \bar{q}^q)\)

So the bundle \( \xi_{h,j} \) becomes \( \xi_{j,h} \).

What about the effect of the change of orientation on \( p_1 \)?
As oriented real vector spaces $H \otimes C$ and $-H \otimes C$ are isomorphic.

$H \otimes C : \{ e_1, e_2, e_3, e_4 \} \rightarrow H \otimes C : \{ -e_1, e_2, e_3, e_4 \}$

$-H \otimes C : \{ e_1, e_2, e_3, e_4 \} \rightarrow -H \otimes C : \{ -e_1, e_2, e_3, e_4 \}$

On the other hand, changing the orientation $\nu \rightarrow -\nu$ replaces $H \otimes C$.

$H \rightarrow \frac{\mathbb{S}^n}{\bar{H}} \quad H \rightarrow \frac{\mathbb{S}^n}{H}$

$\nu \rightarrow -\nu$

$p_1(\frac{\mathbb{S}^n}{H}) = c \nu \quad p_1(\frac{\mathbb{S}^n}{\bar{H}}) = c (-\nu)$

$S_0 \quad p_1(\frac{\mathbb{S}^n}{H}) = -p_1(\frac{\mathbb{S}^n}{\bar{H}})$.

Example $SO \quad p_1(\frac{\mathbb{S}^n}{H}) = p_1(\frac{\mathbb{S}^n}{\bar{H}}) = -p_1(\frac{\mathbb{S}^n}{H})$

$\Rightarrow p_1(\frac{\mathbb{S}^n}{H}) = 0$.

Indeed, $p_1(\frac{\mathbb{S}^n}{H}) = 0$, for all $n \in \mathbb{Z}$.

**Lemma:** $p_1(\frac{\mathbb{S}^n}{H}, \bar{H}) = p_1(\frac{\mathbb{S}^n}{H}, -\bar{H})$.

**Proof:** $\Psi : H = \mathbb{R}^n \rightarrow \mathbb{R}^n = H, \quad v \rightarrow -v \quad \text{and} \quad \Psi_2 : H = \mathbb{R}^n \rightarrow \mathbb{R}^n = H, \quad v = v_p = \bar{v}$ (p $\neq 0$)
when considered as $\mathbb{R}$-linear maps from $\mathbb{R}^4$ to $\mathbb{R}^4$ are not homotopic in $GL(4, \mathbb{R})$ because the $\Psi_2 = \Psi_1$ and conjugation changes orientation so that the determinants in a fixed basis have different signs.

$$B = \{e_1, e_2, e_3, e_4\}, \quad p = a+ib+jc+kd = a_e_1 + b_e_2 + c_e_3 + d_e_4$$

$$v = \sum_{i=1}^{4} v_i e_i, \quad e_i p = a_i p = a - 2b + ic + kd$$

$$e_i p = i. p = -b + ia + jd + kc$$

$$\Rightarrow v, \quad e_i p = i. p = -c + kd + a j + bk$$

$$[\Psi_1]^B = \begin{bmatrix} a - b & -c & -d & 0 \\ b & a & d & e \\ c & d & a & b \\ d & c & b & a \end{bmatrix}, [\Psi_2]^B = \begin{bmatrix} a - b & -c & -d & 0 \\ -b & -a & -d & c \\ -c & -d & -a & b \\ -d & -c & -b & a \end{bmatrix}$$

$$e_4 \cdot p = -d + ci + b j + ak$$

However, the linear maps

$$\Psi_1 \otimes \Psi_2 : C^4 = H \otimes \mathbb{C} \rightarrow H \otimes \mathbb{C} = C^4$$

are homotopic as maps into $GL(4, \mathbb{C})$.

$$(\Psi_1 \otimes \Psi_2)^B : \mathbb{R}^4 \rightarrow \mathbb{R}^4$$

$$a = a + i b \begin{bmatrix} a \cos \theta & -a \sin \theta \\ a \sin \theta & a \cos \theta \end{bmatrix}, \quad \theta \in [0, \pi]$$

$$\theta = \pi \rightarrow \begin{bmatrix} -a & 0 \\ 0 & -a \end{bmatrix} \rightarrow -a.$$
Hence, replacing $\nu p$ by $\overline{\nu p} = \overline{\nu} \overline{p}$ does not change the isomorphism type of the bundle.

$$\tilde{S}_{n+1} ((p, \nu)) \cong \left( \frac{1}{p}, \frac{h}{p^*} \nu p^{-1} \right) \cong \left( \frac{1}{p}, \frac{h^{-1}}{p} \overline{\nu} p^{-1} \right)$$

$$p \frac{h}{p^*} \nu p^{-1} = \frac{1}{p} \overline{\nu} \overline{p} p^{-1} = \overline{p} \frac{1}{p} \overline{\nu} \overline{p}$$

$$\left( p, \overline{\nu} \right) \rightarrow (p, \nu) \rightarrow \left( \frac{1}{p}, \frac{h^{-1}}{p} \overline{\nu} p^{-1} \right)$$

$$\left( p, \nu \right) \rightarrow \left( \frac{1}{p}, \frac{h^{-1}}{p} \nu p^{-1} \right) \rightarrow \tilde{S}_{n+1}, \varepsilon.$$ 

Changing the orientation on the fiber does not change the orientation of the complexified bundle and the two bundles $\tilde{S}_{n+3}$ and $\tilde{S}_{n-1}, \varepsilon$ are isomorphic once they are complexified.

Hence, $p_1 (\tilde{S}_{n+3}) = p_1 (\tilde{S}_{n-1}, \varepsilon).$

Using this result $3$ times consecutively we obtain

$$p_1 (\tilde{S}_{n, 0}) = p_1 (\tilde{S}_{n-3}, 0).$$

Also note that the map $\varphi_n : \tilde{S}_{n} \rightarrow \tilde{H}_{n+1} \cup \tilde{H}_{n-1} \cup \tilde{H}_{n}$ given by $\varphi_n (p) = p^n$ satisfies

$$\varphi_n (\tilde{S}_{n, 0}) \cong \tilde{S}_{n, 0} \quad (Exercise!)$$

and thus

$$p_1 (\tilde{S}_{n, 0}) = p_1 (\varphi_n (\tilde{S}_{n, 0})) = \varphi_n (p_1 (\tilde{S}_{n, 0})) = \deg (\varphi_n) p_1 (\tilde{S}_{n, 0}).$$
\[ p, (\xi_{h,0}) = h p, (\xi_{1,0}) \] since \( \deg (\Theta) = h \).
Hence, we just need to compute \( p, (\xi_{1,0}) \).

Claim: The bundle \( \xi_{1,0} \to S^4 \) admits a complex structure.

Proof: \( p = a + i b + j c + k d = A + j B, \quad A = a + i b \)
\[ B = c - d \]
\[ v = e + i f + j g + k h = c + j D, \quad C = e + i f, \quad D = g - h \]
\[ p \cdot v = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} \]
\[ v \cdot z = \begin{bmatrix} C \\ D \end{bmatrix} z = \begin{bmatrix} C \bar{z} \\ D \bar{z} \end{bmatrix}, \quad z \in C. \]

This finishes the proof.

So, \( p, (\xi_{1,0}) = c_1(\xi_{1,0}) - 2 c_2(\xi_{1,0}) \) since \( \xi_{1,0} \) is a complex vector bundle of rank two.

\[ p, (\xi_{1,0}) = \frac{2}{3} \chi(\xi_{1,0}) - 2 c_2(\xi_{1,0}) = -2 (1 - 0) \chi = 2 \chi \]
when \( \chi \in H^2_{DR}(S^4) \) with \( \int_{S^4} \chi = 1. \)

Corollary: For the bundle \( \xi_{h,5} \to S^4 \) oriented by the natural orientation of \( H \), we have
\[ c(\xi_{h,5}) = -(h + 5) \chi \] and \( p, (\xi_{h,5}) = 2(h - 4) \chi. \)
Milnor's Exotic Spheres

For a given odd integer $k$, choose $h, j \in \mathbb{Z}$ so that $h + j = -1$, $h - j = k$.

Let $B_k = B_k^3$ be the total space of the disk bundle

$$\mathbb{D}^h \rightarrow S^4 \rightarrow S^3$$

Let $H_k = \partial B_k^3$ be the total space of the corresponding unit sphere bundle.

$$D^4 \rightarrow B_k \rightarrow S^3 \rightarrow H_k$$

Theorem \( \chi(H_k) = k^2 - 1 \equiv 0 \pmod{2} \).

Proof: Consider the projection map $\pi: B_k \rightarrow S^4$.

Then $\Gamma_\pi B_k = \pi^*(\Gamma_\pi S^4) \oplus \pi^*(S_{h,j})$

Let $\alpha = \pi^*(\gamma) \in H_4^{DR}(B_k)$.

By the Whitney Product Formula

$$p_1(B_k) = p_1(\pi^*(\Gamma_\pi S^4)) + p_1(\pi^*(S_{h,j}))$$

$$= \pi^*(p_1(\Gamma_\pi S^4)) + \pi^*(p_1(S_{h,j}))$$

$$= \pi^*(2(h+j)\gamma) = 2(h+j)\alpha = 2k\alpha.$$
On the other hand, $h + \tilde{f} = -1$ implies that $c(L_{h, \tilde{f}}) = 1 \cdot 1 = 1$.

Claim: $\int_B \alpha^2 = 1$.

Proof: $\pi^* : H^4_{dR}(S^4) \to H^4_{dR}(B_k)$ is an isomorphism ($B_k$ is homotopy equivalent to $S^4$) and thus $H^4_{dR}(B_k) = \langle \alpha \rangle$, $\alpha = \pi^*(1)$. Let $S^4$ denote the zero section of the bundle $D^4 \to B_k \to S^4$.

Also let $B$ be the 2-frame field of $S^4$ in $B_k$. Hence, $B = \alpha d$ for some $\alpha \in \mathbb{R}$. Since the number of $\pi : B_k \to S^4$ is one then the self-intersection of $S^4$ in $B_k$ is one:

$$1 = \text{Int}(S^4, S^4) = \int_{B_k} B = \int_{B_k} \alpha^2."
\[ q(B_k) = \int_{\mathcal{B}_k} \rho^2(B_k) \, d\mathcal{B}_k = \int_{\mathcal{B}_k} (2k\alpha)^2 \, d\mathcal{B}_k = 4k^2 \int_{\mathcal{B}_k} \alpha^2 \, d\mathcal{B}_k = 4k^2. \]

Hence, \( \Delta(M_k) = 2q(B_k) - I(B_k) = 8k^2 - 1 = k^2 - 1 \) (7).
Theorem: For any odd integer $k \in \mathbb{Z}$ the manifold $M_k$ is homeomorphic to $S^3$. However, $M_k$ is not homeomorphic to $S^3$ provided that $\chi(M_k) = k^2 - 1 \neq 0 \mod 7$.

Proof: If $S^3$ is homeomorphic to $M_k$ then $\chi(M_k) = \chi(S^3) = 0$ because $S^3 = \partial D^4$ and $H^4_{dR}(D^4) = 0$, which is a contradiction.

So we just need to show that $M_k$ is homeomorphic to $S^3$. $M_k$ is the total space of the unit sphere bundle $\Sigma_{h,\sqrt{k}}$, $(h+\sqrt{k} = -1, \quad h-\sqrt{k} = k)$.

$\Sigma_{h,\sqrt{k}} = \mathbb{H} \times S^3 / (p, v) \sim (\frac{1}{\sqrt{k}}, \sqrt{k} v, p \sqrt{k})$, $p \neq 0$

Thus

$M_k = \mathbb{H} \times S^3 / (p, v) \sim \left( \frac{1}{\sqrt{k}}, \frac{1}{|p| \sqrt{k} |v|} \sqrt{k} v, p \right), p \neq 0$

$(p, v) \sim (\frac{1}{\sqrt{k}}, \frac{1}{|p| \sqrt{k} |v|} \sqrt{k} v, p) = (q, w)$

Define a function $F: M_k \rightarrow \mathbb{R}$ as follows:

$F(p, v) = \frac{\text{Re}(v)}{\sqrt{|1 + |p||^2}}$ and on the other coordinate

$\text{chart} F(q, w) = \frac{\text{Re}(w)}{\sqrt{|1 + |q||^2}}$, where
\( q = \frac{1}{p}, \ w = q \frac{1}{u} \). We must check that \( F \) is well-defined:

\[
F(q, u) = \frac{\text{Re}(w)}{\sqrt{1 + \|q\|^2}} = \frac{\frac{1}{\|p\|} \text{Re}(p^{\frac{1}{2}} \overline{v} p^{\frac{1}{2}})}{\sqrt{1 + \text{Re}(p^{\frac{1}{2}} \overline{v} p^{\frac{1}{2}})^2}} = \frac{\text{Re}(p^{\frac{1}{2}} \overline{v} p^{\frac{1}{2}})}{\sqrt{1 + \|p\|^2}} = \frac{\text{Re}(\overline{v})}{1 + \|p\|^2} = \frac{\text{Re}(v)}{1 + \|p\|^2}.
\]

\[
\text{Re}(ab) = \text{Re}(ba).
\]

\( F: \mathbb{R}^n \to \mathbb{R} \) smooth function.

Exercise: \( F \) has exactly two critical points which are both non-degenerate.

Say \( P_{\text{min}} \) and \( P_{\text{max}} \) are the critical points of \( F \).

\( \text{Hess}(F)(P_{\text{min}}) \) is positive definite, and \( \text{Hess}(F)(P_{\text{max}}) \) is negative definite.
**Morse lemma:** Around a point \( p_0 \) (resp. \( p_{\text{max}} \)) there is a coordinate chart on which \( F \) is given by

\[
F(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2
\]

(resp. \(-x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2\))

The gradient vector field \(-\nabla F\) is near zero on \( M_k \setminus S_{\text{crit}}, \text{Dmax}^2\).

This vector field has flow, i.e., a family of diffeomorphisms \( \phi_t: M_k \to M_k \), \( \phi_0 = \text{id} \) with

\[
\dot{\phi}_t(p) = -\nabla F(p)
\]

Hence \( M_k \) is homeomorphic to \( S^6 \times [a, b] \cup S^2 \cup S^2 \)

which \( \partial \sim S^2 \) (topologically).

(Reed's Sphere Theorem, 1946).