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MATH 349
An Introduction to Real Analysis

METU Math. Dept.

Fall of 2020

Textbook: An Introduction to Real Analysis
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(Available upon request.)

Note Title — 13.10.2020

MATH 349 - Int. to Real Analysis — METU MATH DEPT - Fall of 2020 — **Video 1**

Real Number System

$(\mathbb{R}, +, \cdot)$ "$<$" order relation.

Least Upper Bound Propery

A subset $A \subseteq \mathbb{R}$ is called bounded from above if there is a real number $M \in \mathbb{R}$ with

$$x \leq M \text{ for all } x \in A.$$

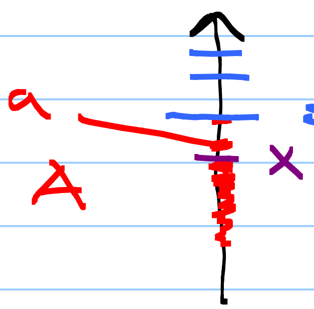
Such M is called an upper bound for A.

Example: $A = (-\infty, 10)$ $M = 20, 15, 10$ are upper bounds for A.

We know the construction of real numbers that any nonempty subset A of $\mathbb{R}$ which is bounded from above has a least upper bound, denoted $\sup(A)$, satisfying the conditions:

1) $\sup(A)$ is an upper bound for A;

2) If $x \in \mathbb{R}$ with $x < \sup(A)$, then x is not an upper bound for A. In other words there is some $a \in A$ with $a > x$.



$\sup(A)$: supremum of A.

Fact: If $A \subseteq \mathbb{R}$ is a nonempty bounded subset from above then $\sup(A)$ is unique.

Proof: Say $y \in \mathbb{R}$ is another least upper bound.

If $y \neq \sup A$ then without loss of generality we may assume that $y < \sup A$. Hence, y is not an upper bound for A, which is a contradiction.

Hence, $\sup A$ is unique. ∎

Similarly, one can define infimum (or the greatest lower bound) for a non empty subset A of $\mathbb{R}$, which is bounded from below.

Let $A \subseteq \mathbb{R}$, $A \neq \emptyset$. Define $-A \doteq \{x \in \mathbb{R} \mid -x \in A\}$.

$a, b \in \mathbb{R}$, with $a < b$ then $-a > -b$.

Hence, if M is an upper bound for A then $-M$ is a lower bound for $-A$. Similarly, if M is bounded from below that $-A$ is bounded from above.

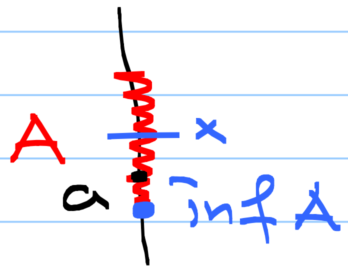
Definition: If $A \subseteq \mathbb{R}$ is a nonempty subset which is bounded from below then the infimum of A (the greatest lower bound) is defined as follows:

$\inf A = -\sup(-A)$.

<u>Fact:</u> $\inf A$ satisfies the followings:

1) $\inf A$ is a lower bound

2) If $x > \inf A$ then x is <u>not</u> a lower bound for A. In other words, there is some $a \in A$ with $a < x$.



<u>Proof</u>: $\inf A = -\sup(-A)$.

1) $\inf A$ is a lower bound: Let $a \in A$. Then $-a \in -A$. Then $-A$ is bounded from above and $-a \leq \sup(-A)$. Hence,

$$a \geq -\sup(-A) = \inf A, \text{ so that}$$

$\inf A$ is a lower bound for A.

2) $\inf A$ is the greatest lower bound.

Let $x \in \mathbb{R}$ with $x > \inf A = -\sup(-A)$. Then $-x \in \mathbb{R}$ satisfies $-x < \sup(-A)$. Hence, $-x$ cannot be an upper bound for $-A$. Hence, there is some $-y \in -A$ so that $-y > -x$. Hence, $x > y$, where $y \in A$, so that x is <u>not</u> a lower bound for A. This shows that $\inf A$ is the greatest lower bound. This finishes the proof.

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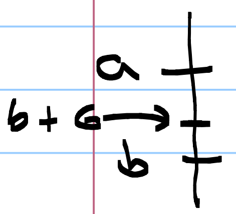
Some Applications

Lemma: If for all $\epsilon > 0$ we have $a \leq b + \epsilon$, then
$$a \leq b.$$

Proof: Assume on the contrary that $a > b$.



$\epsilon = \frac{a-b}{2}$

Let $\epsilon = \frac{a-b}{2}$, then $\epsilon > 0$ because $a > b$.

Then $b + \epsilon = b + \frac{a-b}{2} = \frac{a+b}{2} < \frac{a+a}{2} = a$

$\Rightarrow b + \epsilon < a$, which is a contradiction to the assumption.

Hence, we must have $a \leq b$. ∎

Proposition: Let A and B be two nonempty subsets of $\mathbb{R}$. If A and B are bounded from above that the subset
$$A + B \doteq \{a + b \mid a \in A,\ b \in B\}$$
is bounded from above and
$$\sup(A+B) = \sup A + \sup B.$$

Proof: Since A and B are both bounded from above $\sup A$ and $\sup B$ exist and they are upper bounds for A and B, respectively.

$a \in A$ then $a \leq \sup A$, and
$b \in B$ then $b \leq \sup B$.

Thus $a+b \leq \sup A + \sup B$. Hence, $\sup A + \sup B$ is an upper bound for $A+B$. Thus $A+B$ is bounded from above and $\sup(A+B)$ exists.

In particular, $\boxed{\sup(A+B) \leq \sup A + \sup B.}$

Let $\epsilon > 0$ be any real number. I will show that
$$\sup(A+B) + \epsilon \geq \sup A + \sup B.$$

Since $\epsilon > 0$, $\frac{\epsilon}{2} > 0$. Hence, $\sup A - \frac{\epsilon}{2}$ is

not an upper bound for A. So there is some $a \in A$ with $a > \sup A - \frac{\epsilon}{2}$.

Similarly, $\sup B - \frac{\epsilon}{2}$ is not an upper bound for B and thus there is some $b \in B$ with
$b > \sup B - \frac{\epsilon}{2}$.

So $a + b > \sup A + \sup B - \epsilon$.

Moreover, $a+b \in A+B$ and thus
$\sup(A+B) \geq a+b$.

$\Rightarrow \sup(A+B) \geq a+b > \sup A + \sup B - \epsilon$.
$\Rightarrow \sup(A+B) + \epsilon > \sup A + \sup B$.
$\Rightarrow \sup(A+B) + \epsilon \geq \sup A + \sup B$.

Since, $\epsilon > 0$ was arbitrary we deduce by the previous lemma that

$$\boxed{\sup(A+B) \geq \sup A + \sup B.}$$

Hence, we obtain

$$\sup(A+B) = \sup A + \sup B.$$

■

<u>Proposition</u>: Let A and B be two subsets of $\mathbb{R}$ such that for every $a \in A$, there is some $b \in B$ with $a \leq b$. If B is bounded from above, then A is also bounded from above and

$$\sup A \leq \sup B.$$

<u>Proof</u>: Let $a \in A$, then there is some $b \in B$ with $a \leq b$. Since B is bounded from above $\sup B$ exists and $b \leq \sup B$.

Hence, $a \leq b \leq \sup B$ so that $\sup B$ is an upper for A. Finally, we deduce that A is bounded from above by $\sup B$ and thus $\sup A$ exists and satisfies

$$\sup A \leq \sup B.$$

■

<u>Exercises</u>: 1) If A and B are non empty subsets, which are bounded from below then $A+B$ is bounded from below and

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$\operatorname{Inf}(A+B) = \operatorname{Inf} A + \operatorname{Inf} B.$

2) If A and B are subsets of $\mathbb{R}$ so that for any $a \in A$ there is some $b \in B$ with $b \leq a$. Then if B is bounded from below that A is bounded from below and $\inf B \leq \inf A$.

Prove these statements.

Examples: 1) $A = \{-3, 2, 5, 8\} \cup (-\infty, 0)$

$\sup A = 8$, $\inf A$ does not exist.

2) $A = \mathbb{Q} \cap \{x \in \mathbb{R} \mid x^2 < 2\}$.

$= \mathbb{Q} \cap (-\sqrt{2}, \sqrt{2})$

$\sup A = \sqrt{2}$, $\operatorname{Inf} A = -\sqrt{2}$

3) $A = \mathbb{Z} \cap (-\sqrt{2}, \sqrt{2}) = \{-1, 0, 1\}$

$\sup A = 1$, $\inf A = -1$.

Remark: The set of complex number does not have an order.

$P \subseteq \mathbb{C}$, 1) $\mathbb{C} = P \cup \{0\} \cup -P$ and $P \cap (-P) = \emptyset$.

2) If $z, w \in P$ then $z+w, z \cdot w \in P$.

$i \in \mathbb{C}$, $i \neq 0$, $i \in P$ or $-i \in P$.

$$i \cdot i = (-i) \cdot (-i) = -1 \in P$$

$$(-1) \cdot (-1) = 1 \in P \Rightarrow P \cap -P \neq \emptyset.$$

__Fact:__ The set of natural numbers is __not__ bounded from above.

__Proof:__ If $\mathbb{N}$ is bounded then let $a = \sup \mathbb{N}$.

Then $a-1$ is __not__ an upper bound for $\mathbb{N}$ and there is some $n \in \mathbb{N}$ so that
$$n > a-1.$$

Then $n+1 > a$ and $n+1 \in \mathbb{N}$, which is a contradiction since $a = \sup \mathbb{N}$ must be an upper bound for $\mathbb{N}$. This finishes the proof. ∎

__Proposition:__ If $0 < x$ and y are real numbers then there is some natural number $n \in \mathbb{N}$ so that $nx > y$.

__Proof:__ Since $x \neq 0$, $y/x \in \mathbb{R}$ and thus it cannot be an upper bound for $\mathbb{N}$. So there is some $n \in \mathbb{N}$ with $n > y/x$. Hence, we get $nx > y$. ∎

Greatest Integer Functions:

Let $x \in \mathbb{R}$ and define $A = \{n \in \mathbb{Z} \mid n \leq x\}$. Then A is bounded from above by x and thus $\sup A$ exists.

Claim: $\sup A$ is an integer.

Define the greatest integer part of x as

$$[x] \doteq \sup A.$$

Proof of the claim: $\sup A - 1$ is not an upper bound for A and there is some $n \in A$ so that $n > \sup A - 1$. So $n+1 > \sup A$. Now if $k \in A$ then $\sup A \geq k$ and thus $n+1 > k$. However, both n and k are integers and therefore $n \geq k$. Since $k \in A$ is arbitrary we see that n is an upper bound for A and hence, $n \geq \sup A$.

On the other hand, $n \in A$ and thus $n \leq \sup A$. Thus $n = \sup A$, which is an integer. ∎

Examples $[2.3] = 2$, $[0.8] = 0$,

$[-3.1] = -4$.

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Proposition: Given any real number x, for each $n \in \mathbb{N}$, we can find a rational number $r_n \in \mathbb{Q}$ such that

$$r_n \leq r_{n+1} \text{ and } r_n \leq x < r_n + \frac{1}{10^n}, \text{ for}$$

all $n \in \mathbb{N}$.

Example: $x = 17.382096\ldots$

$r_1 = 17.3$, $r_2 = 17.38$, $r_3 = 17.382$

$r_n \leq x < r_n + \frac{1}{10^n}$ $\qquad a_0 = 17$, $a_1 = 3$

$r_1 = 17 + \frac{3}{10} = 17.3$

Proof: Let $a_0 = [x] \in \mathbb{N}$, then

$a_0 \leq x < a_0 + 1. \Rightarrow 0 \leq x - a_0 < 1$ and

$0 \leq 10x - 10a_0 < 10$. Let $a_1 = [10x - 10a_0]$ and

set $r_1 = a_0 + \frac{a_1}{10}$. Similarly, let $a_n = [10^n x - 10^n r_{n-1}]$,

where $r_{n-1} = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_{n-1}}{10^{n-1}}$.

Set $r_n = r_{n-1} + \frac{a_n}{10^n} = a_0 + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n}$.

Then clearly, $r_{n-1} \leq r_n$.

Claim: $r_n \leq x < r_n + \frac{1}{10^n}$.

Proof by induction on n:

$n=1$, $a_1 = [10x - 10a_0] \leq 10x - 10a_0 < a_1 + 1.$

$\Rightarrow a_1 + 10a_0 \leq 10x < 10a_0 + a_1 + 1$

$$\Rightarrow a_0 + \frac{a_1}{10} \leq x < a_0 + \frac{a_1}{10} + \frac{1}{10}$$

$$r_1 \leq x < r_1 + \frac{1}{10^1}$$

This finishes the proof for $n=1$.

Now assume the result for $n=k$:

$$r_k \leq x < r_k + \frac{1}{10^k}.$$

must prove the result for $n=k+1$:

$$r_{k+1} \leq x < r_{k+1} + \frac{1}{10^{k+1}}.$$

Now, $a_{k+1} = [10^{k+1}x - 10^{k+1} r_k]$. Hence,

$a_{k+1} \leq 10^{k+1}x - 10^{k+1} r_k < a_{k+1} + 1.$

$a_{k+1} + 10^{k+1} r_k \leq 10^{k+1} x < 10^{k+1} r_k + a_{k+1} + 1$

$$r_k + \frac{a_{k+1}}{10^{k+1}} \leq x < r_k + \frac{a_{k+1}}{10^{k+1}} + \frac{1}{10^{k+1}}$$

$r_{k+1} \leq x < r_{k+1} + \frac{1}{10^{k+1}}$ and the proof finishes. ■

Theorem (Density Theorem)

If $a, b \in \mathbb{R}$ with $a<b$, then there is some rational number $r \in \mathbb{Q}$ with
$$a < r < b.$$

Proof: Since $a<b$ we have $b-a>0$.

Since $\mathbb{N}$ is unbounded there is some $n_0 \in \mathbb{N}$ so that $n_0 > 1/_{b-a}$.

Clearly, $10^{n_0} \geq n_0 > \frac{1}{b-a}$. Now by previous Proposition there is a sequence for $x=a$ (r_k) so that
$$r_k \leq a < r_k + \frac{1}{10^k}, \text{ for all } k.$$

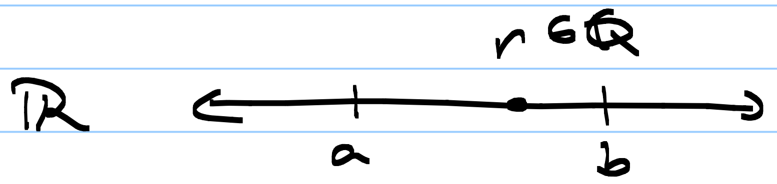
Let $r = r_{n_0} + \frac{1}{10^{n_0}}$. Then we have

$$a < r \text{ and } 10^{n_0} > \frac{1}{b-a} \Rightarrow b-a > \frac{1}{10^{n_0}}$$

and thus, $b > a + \frac{1}{10^{n_0}} \geq r_{n_0} + \frac{1}{10^{n_0}} = r$.

So, $a<r<b$ and the proof finishes. ■

Exercise: Prove that $10^n \geq n$ for all $n \in \mathbb{N}$.



We rephrase this theorem as follows: The set of rationals is dense in the set real numbers.

Absolute Value Function:

$$|\cdot| : \mathbb{R} \longrightarrow \mathbb{R}^+ \cup \{0\}$$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Properties: 1) $|x| = |-x|$
2) $|xy| = |x||y|$
3) $|x+y| \leq |x| + |y|$ (Triangle Inequality)
4) $||x| - |y|| \leq |x-y|$

for all $x, y \in \mathbb{R}$.

Proof: of 4) $|x| = |(x-y)+y| \leq |x-y| + |y|$

$\Rightarrow |x| - |y| \leq |x-y|$.

Similarly, $|y| - |x| \leq |y-x| = |x-y|$.

Hence, both $|x|-|y|$ and $-(|x|-|y|)$ are less than or equal to $|x-y|$. Therefore,

$||x|-|y|| \leq |x-y|$. ∎

Exercise: Prove by induction that for any real numbers $x_1, x_2, \ldots, x_n$ we have $|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$.

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Sequences of Real Numbers

A sequence of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$.

Notation: $f : \mathbb{N} \to \mathbb{R}$, $f(n)$

$f_n \doteq f(n)$

$x : \mathbb{N} \to \mathbb{R}$, $x_n \doteq x(n)$ or $a : \mathbb{N} \to \mathbb{R}$, $a_n \doteq a(n)$

Examples $x : \mathbb{N} \to \mathbb{R}$, $x(n) = (-1)^n$ and

$a : \mathbb{N} \to \mathbb{R}$, $a(n) = \dfrac{n+1}{2^n}$ are some sequences.

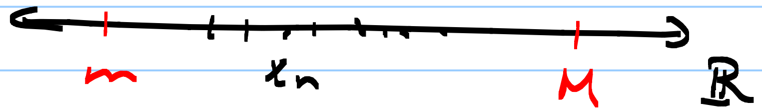
$x \leftrightarrow (x_n) = ((-1)^n)$, $\quad a \leftrightarrow (a_n) = \left(\dfrac{n+1}{n}\right)$.

We'll see that why sequences are important in the construction of real numbers.

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Definition: A sequence (x_n) of real number is called bounded from above (from below) if there is some $M \in \mathbb{R}$ (resp. $m \in \mathbb{R}$) so that

$$x_n \leq M \quad (\text{resp. } x_n \geq m), \text{ for all } n \in \mathbb{N}.$$



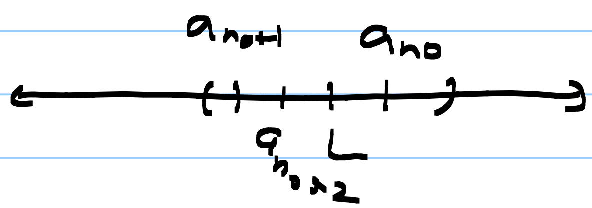
A sequence is said to be bounded if it bounded from above and below.
In this case, let $K = \max\{|m|, |M|\}$. Then
$$-K \leq x_n \leq K \text{ for all } n, \text{ or equivalently}$$
$$|x_n| \leq K.$$

<u>Examples</u>: 1) $(x_n) = (n^2) = (1, 4, 9, 16, \ldots)$ is bounded from below by 0.

2) $(x_n) = \left(\frac{n}{n+1}\right)$ is bounded.

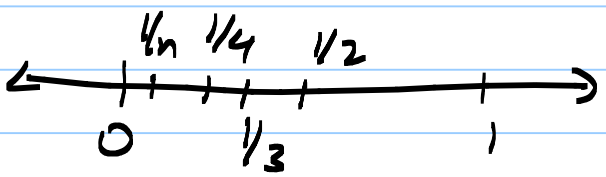
$$0 \leq \frac{n}{n+1} \leq 1 \text{ so } \left|\frac{n}{n+1}\right| \leq 1 \text{ bounded by } 1.$$

<u>Convergence of Sequences</u>: Let (a_n) be a sequence of real numbers and $L \in \mathbb{R}$ any real number.
We say that (a_n) converges to L if for any $\epsilon > 0$ given there is some $n_0 \in \mathbb{N}$ so that
$$n \geq n_0 \text{ implies } |a_n - L| < \epsilon.$$



In this case, we write $\lim a_n = L$ or $\lim\limits_{n\to\infty} a_n = L$

<u>Examples</u> 1) $(a_n) = \left(\frac{1}{n}\right) = (1, 1/2, 1/3, 1/4, \ldots)$



Claim: $\lim 1/n = 0$.

Proof: Let $\epsilon > 0$ be given. Then $1/\epsilon \in \mathbb{R}$.

Since $\mathbb{N}$ is unbounded there is some $n_0 \in \mathbb{N}$ $n_0 > 1/\epsilon$. Then $\forall\ n \geq n_0$ we have.

$$|x_n - L| = \left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{n_0} < \epsilon.$$

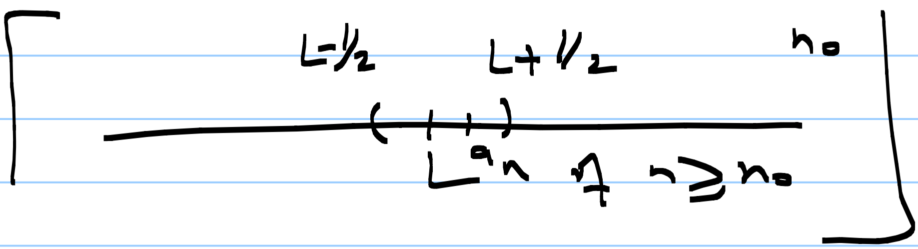
Hence, $\lim a_n = L = 0$. ∎

2) Let (a_n) be the sequence given by

$a_n = (-1)^n$, $n \in \mathbb{N}$. So $(a_n) = (-1, 1, -1, 1, \dots, 1, -1, \dots)$.

Claim: $\lim a_n$ does not exist.

Proof: Suppose on the the contrary that $\lim a_n$ exists and that $\lim a_n = L$ for some real number $L \in \mathbb{R}$.



Let $\epsilon = 1/2 > 0$. Since we have $\lim a_n = L$ there is some $n_0 \in \mathbb{N}$ so that

$n \geq n_0$ implies $|a_n - L| < \epsilon$.

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$\Rightarrow |a_n - L| < 1/2$, for all $n \geq n_0$.

In particular, $|1-L| < 1/2$ and $|-1-L| < 1/2$.

Then $2 = |1-(-1)| = |(1-L)+(L-(-1))|$

$$\leq |1-L| + |L-(-1)|$$

$$\leq |1-L| + |L+1|$$

$$\leq |1-L| + |-1-L|$$

$$< 1/2 + 1/2 = 1$$

$\Rightarrow 2 < 1$, which is a clear contradiction.

Hence, $\lim a_n = \lim (-1)^n$ does <u>not</u> exist. ∎

<u>Definition</u>: A sequence (a_n) of real numbers is called Cauchy if for any $\epsilon > 0$ given there is some $n_0 \in \mathbb{N}$ so that

$m, n \geq n_0$ implies that $|a_n - a_m| < \epsilon$.

<u>Proposition</u>: A convergent sequence is Cauchy.

<u>Proof</u> Given $\epsilon > 0$. Then $\epsilon/2 > 0$ and since (a_n) is convergent to some $L \in \mathbb{R}$ there is some $n_0 \in \mathbb{N}$ so that

$n \geq n_0$ implies $|a_n - L| < \epsilon/2$.

Hence, if $m, n \geq n_0$ then we have

$$|a_n - a_m| = |(a_n - L) + (L - a_m)|$$
$$\leq |a_n - L| + |L - a_m|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus (a_n) is a Cauchy sequence. ∎

Proposition: Any Cauchy sequence (a_n) is bounded.

Proof: Let $\epsilon = 1 > 0$. Then since (a_n) is Cauchy there is some $n_0 \in \mathbb{N}$ so that $m, n \geq n_0$ implies $|a_n - a_m| < \epsilon = 1$.

In particular, $|a_n - a_{n_0}| < 1$, for all $n \geq n_0$.

Let $M = \max\{1, |a_n - a_{n_0}|, n \leq n_0 - 1\}$

Now $|a_n - a_{n_0}| \leq M$ $\forall$ $n \leq n_0 - 1$ and

$|a_n - a_{n_0}| < 1 \leq M$ $\forall$ $n \geq n_0$.

Hence $|a_n - a_{n_0}| \leq M$, for all $n \in \mathbb{N}$.

$\Rightarrow$ $-M \leq a_n - a_{n_0} \leq M$, for all $n \in \mathbb{N}$.

$\Rightarrow$ $-M + a_{n_0} \leq a_n \leq a_{n_0} + M$, for all $n \in \mathbb{N}$

$\Rightarrow$ $|a_n| \leq K$, for all $n \in \mathbb{N}$,

where $K = \max\{|-M + a_{n_0}|, |a_{n_0} + M|\}$.

Hence, (a_n) is bounded. ∎

<u>Example</u> A constant sequence is convergent and thus Cauchy.

$(a_n) = (c, c, c, \ldots)$

<u>Claim</u> $\lim a_n = c$.

<u>Proof</u>: Given $\epsilon > 0$ just choose $n_0 = 1$.
Then if $n \geq n_0 = 1$, then

$|a_n - c| = |c - c| = 0 < \epsilon$. ∎

<u>Definition</u>: A sequence of real numbers (a_n) is called increasing (decreasing) if

$a_n \leq a_{n+1}$ (resp. $a_n \geq a_{n+1}$) for all n.

<u>Examples</u> 1) $(a_n) = \left(\frac{1}{n}\right) = (1, 1/2, 1/3, 1/4, \ldots)$ decreasing

2) $(b_n) = \left(\frac{n}{n+1}\right) = \left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right)$ increasing

(<u>Exercise</u>: Show that (b_n) is an increasing sequence!)

3) $(c_n) = (c, c, c, \ldots, c, \ldots)$ constant sequence.

This is both increasing and decreasing.

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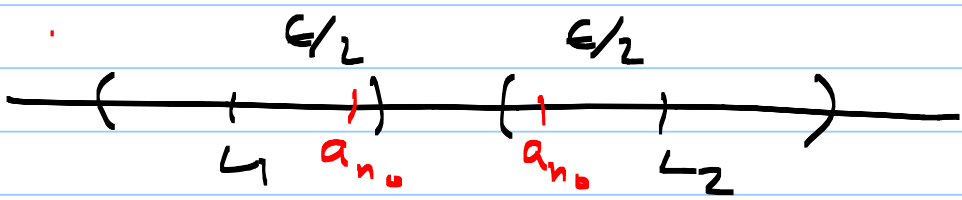
4) $(d_n) = ((-1)^n) = (-1, 1, -1, 1, -1, \dots)$ is neither increasing nor decreasing.

<u>Definition:</u> A sequence (a_n) is called eventually increasing (eventually decreasing) if there is some index $n_0 \in \mathbb{N}$ so that $(a_{n_0}, a_{n_0+1}, a_{n_0+2}, \dots)$ is increasing (resp. decreasing).

<u>Proposition:</u> A sequence (a_n) may converge to at most one limit value.

<u>Proof:</u> Assume that (a_n) has two limits say L_1 and L_2. Let $\epsilon > 0$ be given. Then since $\lim a_n = L_1$ there is some $n_1 \in \mathbb{N}$ so that $n \geq n_1$ implies $|a_n - L_1| < \epsilon/2$. Similarly, since $\lim a_n = L_2$ there is some $n_2 \in \mathbb{N}$ so that $n \geq n_2$ implies $|a_n - L_2| < \epsilon/2$.



Let $n_0 = \max\{n_1, n_2\}$. Then $n_0 \geq n_1$ and $n_0 \geq n_2$. So $|a_{n_0} - L_1| < \epsilon/2$ and $|a_{n_0} - L_2| < \epsilon/2$.

Finally, $|L_1 - L_2| = |(L_1 - a_{n_0}) - (L_2 - a_{n_0})|$

$$\leq |L_1 - a_{n_0}| + |L_2 - a_{n_0}|$$

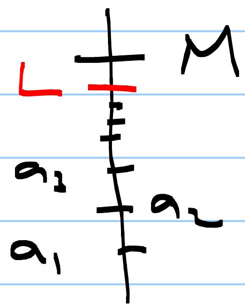
$\Rightarrow |L_1 - L_2| < \epsilon/2 + \epsilon/2 = \epsilon.$

Hence, $|L_1 - L_2| < \epsilon$, for all $\epsilon > 0$.
Then $|L_1 - L_2| = 0$ and thus $L_1 = L_2$.
This finishes the proof. ■

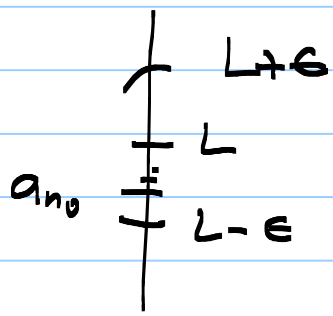
<u>Proposition</u>: An increasing sequence is convergent if and only if it is bounded from above.
A decreasing sequence is convergent if and only if it is bounded from below.

<u>Proof</u>: Let's prove only the first statement.

Let (a_n) be an increasing sequence.
First assume that it is convergent. Then (a_n) is Cauchy and thus it is bounded from above and below.
Conversely, assume that (a_n) is bounded from above. Then we must show that (a_n) is convergent.



Let $L = \sup\{a_n \mid n = 1, 2, \ldots\}$.



<u>Claim</u>: $\lim a_n = L$.
<u>Proof</u>: Let $\epsilon > 0$ be given.
Since $L - \epsilon$ is <u>not</u> an upper bound there

is some $n_0 \in \mathbb{N}$ so that $L-\epsilon < a_{n_0} \leq L$.
So if $n \geq n_0$ then $L-\epsilon < a_{n_0} \leq a_n \leq L$.

Hence $|a_n - L| < \epsilon$ if $n \geq n_0$. This finishes the proof. ∎

Proposition: Let (a_n) and (b_n) be convergent sequences. Then (a_n+b_n) and $(a_n \cdot b_n)$ are also convergent and

$\lim(a_n+b_n) = \lim a_n + \lim b_n$ and

$\lim(a_n \cdot b_n) = (\lim a_n) \cdot (\lim b_n)$.

Proof: Let $\lim a_n = a$ and $\lim b_n = b$.

must prove: $\lim(a_n+b_n) = a+b$.

Let $\epsilon > 0$ be given. Then $\epsilon/2 > 0$. Since $\lim a_n = a$ there is some $n_1 \in \mathbb{N}$ so that
$n \geq n_1$ implies $|a_n - a| < \epsilon/2$. Similarly, $\lim b_n = b$ and thus there is some $n_2 \in \mathbb{N}$ such that $n \geq n_2$ implies $|b_n - b| < \epsilon/2$.

Set $n_0 = \max\{n_1, n_2\}$. Then if $n \geq n_0$ we $n \geq n_1$ and $n \geq n_2$. Now if $n \geq n_0$,

$$
\begin{aligned}
|(a_n+b_n) - (a+b)| &= |(a_n - a) + (b_n - b)| \\
&\leq |a_n - a| + |b_n - b| \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}
$$

Hence, $\lim(a_n+b_n)=a+b$. ∎

For the second statement let $\epsilon>0$ be given. Since (a_n) is convergent (a_n) is bounded, say by some $M>0$. Hence $|a_n|\leq M$ for all $n\in\mathbb{N}$. Then $\epsilon/2M>0$ and since (b_n) has limit value b, there is some index $n_1\in\mathbb{N}$ so that

$$n\geq n_1 \Rightarrow |b_n-b|<\epsilon/2M.$$

Suppose first $b\neq 0$. Then $\epsilon/2|b|>0$. Since $\lim a_n=a$ there is some $n_2\in\mathbb{N}$ so that

$$n\geq n_2 \Rightarrow |a_n-a|<\epsilon/2|b|.$$ Now let

$n_0=\max\{n_1,n_2\}$. Then if $n\geq n_0$ then

$$\begin{aligned}|a_nb_n-ab| &= |(a_nb_n-a_nb)+(a_nb-ab)| \\ &\leq |a_n(b_n-b)|+|b(a_n-a)| \\ &= |a_n||b_n-b|+|b||a_n-a| \\ &\leq M|b_n-b|+|b||a_n-a| \\ &< M\cdot\frac{\epsilon}{2M}+|b|\cdot\frac{\epsilon}{2|b|}=\epsilon.\end{aligned}$$

Now assume that $b=0$. Then

$$|a_nb_n-ab|=|a_nb_n|=|a_n||b_n|\leq M|b_n|$$

Since $\lim b_n=b=0$ there is some $n_0\in\mathbb{N}$

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so that $n \geq n_0 \Rightarrow |b_n - 0| < \frac{\epsilon}{M}$.

Hence, if $n \geq n_0$ then

$|a_n b_n - ab| \leq M|b_n| < M\frac{\epsilon}{M} = \epsilon$.

This finishes the proof. ∎

Exercise: Assume that $\lim a_n = a$, where $a_n \neq 0$ and $a \neq 0$ for all n. Then prove that $\lim (1/a_n)$ exists and equal $1/a$.

Example: Compute the limit $\lim \frac{n^2+1}{2n^2+3n+4}$.

$$\frac{n^2+1}{2n^2+3n+4} = \frac{\overset{\to 1}{1} + \overset{\to 0}{1/n^2}}{\underset{\to 2}{2} + \underset{\to 0}{\frac{3}{n}} + \underset{\to 0}{\frac{4}{n^2}}} \to \frac{1+0}{2+0+0} = 1/2.$$

We've proved earlier that $\lim \frac{1}{n} = 0$.

Hence, $\lim \frac{1}{n^2} = (\lim \frac{1}{n})(\lim \frac{1}{n}) = 0.0 = 0$.

$\lim \frac{3}{n} = (\lim 3)(\lim \frac{1}{n}) = 3 \cdot 0 = 0$

$\lim \frac{4}{n^2} = (\lim 4) \lim (1/n^2) = 4 \cdot 0 = 0$.

Proposition: Every real number is the limit of a sequence of rational numbers.

Ex: $x = 27.0183591600275\ldots$

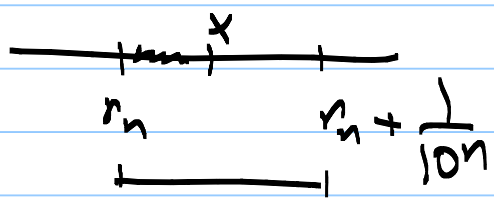
$x_1 = 27$, $x_2 = 27.0$, $x_3 = 27.01$, $x_4 = 27.018$

$(x_n) \longrightarrow x$.

Proof: Recall that we've proved the existence of a sequence of rational numbers $r_1, r_2, \ldots, r_n, \ldots$ so that

$$r_1 \leq r_2 \leq r_3 \leq \cdots \quad \text{and} \quad r_n \leq x < r_n + \frac{1}{10^n}$$

for all $n = 1, 2, \ldots$.



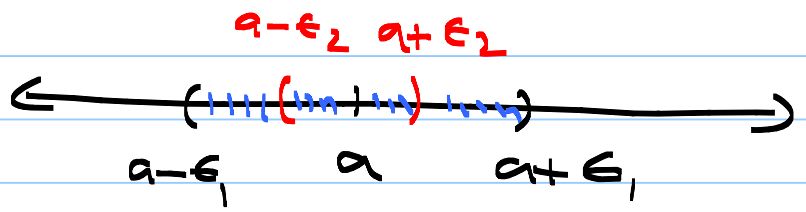
Hence, $|x - r_n| \leq \frac{1}{10^n}$.

Given $\epsilon > 0$, choose $n_0 \in \mathbb{N}$ so that $10^{n_0} > \frac{1}{\epsilon}$.

So, if $n \geq n_0$ then $|x - r_n| \leq \frac{1}{10^n} \leq \frac{1}{10^{n_0}} < \epsilon$. $\blacksquare$

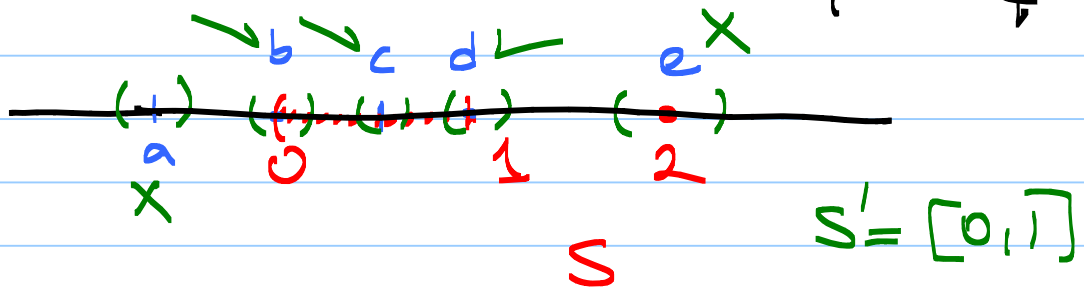
Accumulation Point.

Let S be a set of real numbers. A real number a is called an accumulation point of $\mathbb{R}$ if for every $\epsilon > 0$ the interval $(a-\epsilon, a+\epsilon)$ contains infinitely many points of S.



Example: $S = (0,1] \cup \{2\}$.

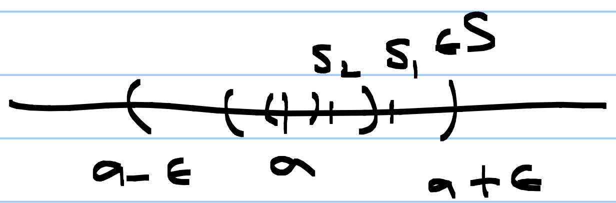
What are the accumulation points of S.



The set of all accumulation points of S will be denoted as S', called the derived set of S.

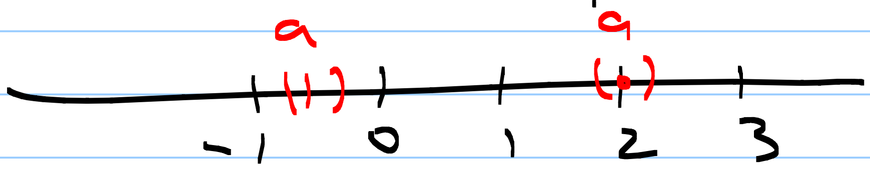
Equivalent Statement: A real number a is an accumulation point of S if

$$((a-\epsilon, a+\epsilon)\setminus\{a\}) \cap S \neq \emptyset$$



Example Any finite set has no accumulation point.

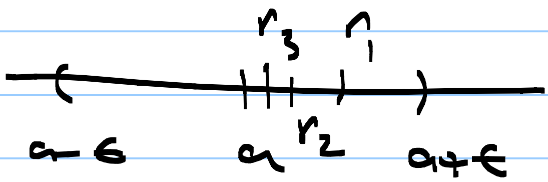
Example $S = \mathbb{Z}$, $S' = \phi$



Let $a \in \mathbb{R}$, then $(a - 1/2, a + 1/2) \cap \mathbb{Z}$ can have at most one element and thus a is not an accumulation point. Hence, $\mathbb{Z}' = \phi$.

Example $S = \mathbb{Q}$, $S' = \mathbb{R}$.

$a \in \mathbb{R}$,



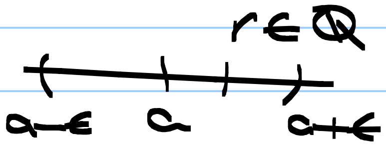
$a < r_1 < a + \epsilon$

$a < r_2 < r_1$

$a < r_3 < r_2$

$r_1 > r_2 > r_3$

$\mathbb{Q} \cap (a - \epsilon, a + \epsilon)$ contains the set $\{r_1, r_2, r_3, \ldots\}$, which is infinite.



$r \in (a - \epsilon, a + \epsilon) \cap (S \setminus \{a\})$

$\Rightarrow a \in S'$.

Hence, $S' = \mathbb{Q}' = \mathbb{R}$

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Proposition: A real number $a \in \mathbb{R}$ is an accumulation of S if and only if there is a sequence (a_n) of elements of S with $a_n \neq a_m$ if $n \neq m$ and $\lim a_n = a$.

Proof: ($\Longrightarrow$) Assume that $a \in \mathbb{R}$ is an accumulation point of S.

Let $r_1 = 1$ then the interval $(a - r_1, a + r_1)$ satisfy

$((a - r_1, a + r_1) \setminus \{a\}) \cap S \neq \phi$. So choose

$a_1 \in (a - r_1, a + r_1) \setminus \{a\} \cap S$. Then $a_1 \in S$ and

$|a_1 - a| < r_1 = 1$. Since $a_1 \neq a$, $|a_1 - a| > 0$.

Let $r_2 = \min\{|a_1 - a|, 1/2\}$ and choose some a_2

from $(a - r_2, a + r_2) \setminus \{a\} \cap S$.

Since $|a_2 - a| < r_2 \leq |a_1 - a|$ we get $a_2 \neq a_1$.

Moreover, $|a_2 - a_1| < r_2 \leq 1/2$.

Inductively, choose $a_n \in (a - r_n, a + r_n) \setminus \{a\} \cap S$

where $r_n = \min\{|a_{n-1} - a|, 1/n\} \cap S$. Then $a_n \in S$ and satisfies

$|a_n - a| < r_n \leq |a_{n-1} - a| \Rightarrow a_n \neq a_{n-1}$ and indeed, $|a_n - a| < |a_k - a|$ for all $k \leq n-1$.
So that $a_n \neq a_k$ for all $k \neq n$.

Moreover, $|a_n - a| < r_n \leq 1/n$.

Claim: Let $\lim a_n = a$.

Proof: Given $\epsilon > 0$, choose $n_0 \in \mathbb{N}$ with $n_0 > \frac{1}{\epsilon}$.

Then $n \geq n_0 \Rightarrow |a_n - a| < 1/n \leq 1/n_0 < \epsilon$.

Hence, we've constructed a sequence (a_n) of elements of S with

1) $a_n \neq a_m$ if $n \neq m$, and
2) $\lim a_n = a$.

($\Leftarrow$) For this direction assume that there is a sequence (a_n) of elements of S with $a_n \neq a_m$ if $n \neq m$ and $\lim a_n = a$.

Let $\epsilon > 0$ be given. Then since $\lim a_n = a$ there is some $n_0 \in \mathbb{N}$ s.t. $n \geq n_0$ implies $|a_n - a| < \epsilon$.

So $a_n \in (a-\epsilon, a+\epsilon) \cap S$, if $n \geq n_0$.

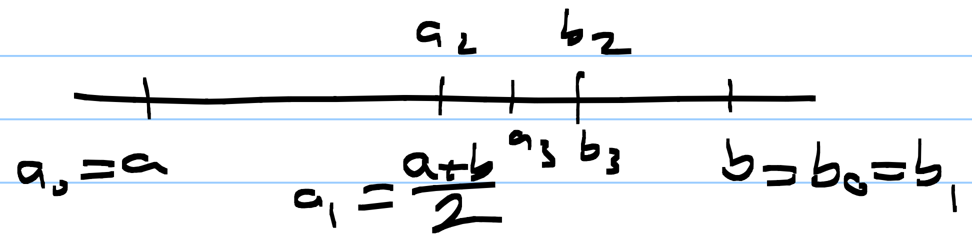
If $a_{n_0} \neq a$ then $a_{n_0} \in ((a-\epsilon, a+\epsilon) \setminus \{a\}) \cap S$ and we are done.

If $a_{n_0} = a$ then $a_{n_0+1} \neq a_{n_0} = a$ and $a_{n_0+1} \in ((a-\epsilon, a+\epsilon) \setminus \{a\}) \cap S$. This finishes the proof.

Theorem (Bolzano – Weierstrass)

If S is a bounded subset of $\mathbb{R}$ and if S has infinitely many elements, then S has at least one accumulation point.

Proof: Since S is bounded both $b = \sup S$ and $a = \inf S$ exist and for any $s \in S$ we have $a \le s \le b$.



$a = a_0 \le a_1 \le a_2 \le \cdots \qquad b_2 \le b_1 \le b_0 = b$

$|a_1 - b_1| = \frac{|a_0 - b_0|}{2}, \; |a_2 - b_2| = \frac{|a_0 - b_0|}{4},$

$|a_n - b_n| = \frac{|a_0 - b_0|}{2^n}$

First we construct two sequences (a_n) and (b_n) so that

i) $a_n \le a_{n+1}$, ii) $b_{n+1} \le b_n$, iii) $|b_n - a_n| = \frac{|b_0 - a_0|}{2^n}$

and $S \cap [a_n, b_n]$ is infinite.

(a_n) is increasing and bounded from above, and (b_n) is decreasing and bounded from below. Hence, both limits

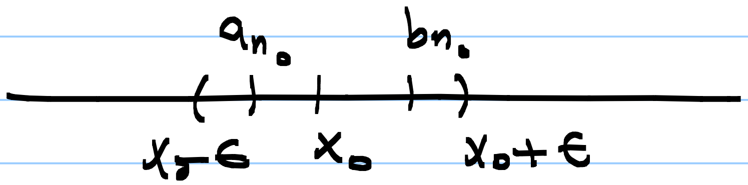
$\lim a_n$ and $\lim b_n$ exist.

However, $|a_n - b_n| = \dfrac{|a_0 - b_0|}{2^n}$ and thus

$|\lim a_n - \lim b_n| = 0 \Rightarrow \lim a_n = \lim b_n$.

Let's call this common point x_0.

$x_0 = \lim a_n = \lim b_n$.



Claim $x_0 \in S'$.

Proof Let $\epsilon > 0$ be given. Since $\lim a_n = x_0$ there is some n_1 so that $n \geq n_1$ implies $|a_n - x_0| < \epsilon$. Similarly, since $\lim b_n = x_0$ there is some n_2 so that $n \geq n_2$ implies $|b_n - x_0| < \epsilon$.

Set $n_0 = \max\{n_1, n_2\}$. Then we get $|a_{n_0} - x_0| < \epsilon$ and $|b_{n_0} - \epsilon| < \epsilon$. Finally, $[a_{n_0}, b_{n_0}] \subseteq (x_0 - \epsilon, x_0 + \epsilon)$ and $[a_{n_0}, b_{n_0}] \cap S$ is infinite and thus $(x_0 - \epsilon, x_0 + \epsilon) \cap S$ is infinite. This finishes the proof. ■

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Definition: Let (a_n) be any sequence of real numbers. For any increasing sequence of positive integers

$$k_1 < k_2 < k_3 < \cdots < k_n < \cdots$$

the sequence (a_{k_n}) is called a subsequence of (a_n).

Ex: $(a_n) = (-1, 0, 3, 5, 8, 9, -1, 2, \ldots)$

$k_1 = 2,\ k_2 = 3,\ k_3 = 5,\ k_4 = 7, \ldots$

$(a_{k_n}) = (a_2, a_3, a_5, a_7, \ldots) = (0, 3, 8, -1, \ldots)$

Lemma: If (a_n) is convergent then so is any subsequence (a_{k_n}).

Proof: Note that $k_n < k_{n+1}$ and $k_n \geq n$, for all n.

Say $\lim a_n = L$. Then given $\epsilon > 0$ choose n_0 so that $n \geq n_0$ implies $|a_n - L| < \epsilon$.

Then $k_n \geq n \geq n_0$ and then $|a_{k_n} - L| < \epsilon$.

This finishes the proof. ▬

Lemma: If (a_n) is a Cauchy sequence and if it has a subsequence converging to some $a \in \mathbb{R}$ then (a_n) also converges to a.

Proof: Assume (a_{k_n}) is a subsequence of (a_n)

with $\lim a_{k_n} = a$.

<u>must show</u>: $\lim a_n = a$.

Given $\epsilon > 0$. Then $\frac{\epsilon}{2} > 0$. Since (a_n) is Cauchy there is some n_1 so that

$$m, n \geq n_1 \Rightarrow |a_m - a_n| < \epsilon/2.$$

Similarly, since $\lim a_{k_n} = a$ there is some $n_2 \in \mathbb{N}$ so that

$$n \geq n_2 \Rightarrow |a_{k_n} - a| < \epsilon/2.$$

Choose $n_0 = \max\{n_1, n_2\}$. Then if $n \geq n_0$

$$\begin{aligned} |a_n - a| &= |(a_n - a_{k_n}) + (a_{k_n} - a)| \\ &\leq |a_n - a_{k_n}| + |a_{k_n} - a| \\ &< \frac{\epsilon}{2} + \epsilon/2 \end{aligned}$$

$$\boxed{\begin{aligned} &n \geq n_0 \Rightarrow \\ &n \geq n_1, \ k_n \geq n \geq n_1 \end{aligned}}$$

so that $|a_n - a| < \epsilon$. This finishes the proof. ∎

<u>Theorem</u>: A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

<u>Proof</u> We've already proved that any convergent sequence is Cauchy.
For the other direction let (a_n) be a Cauchy sequence.
Let $S = \{a_n \mid n = 1, 2, 3, \ldots\}$.

We have two cases:

Case 1: S is finite. Then (a_n) must have a constant subsequence, say (a_{k_n}). Since a constant sequence is convergent, (a_{k_n}) is convergent and thus (a_n) is convergent to the same limit point.

Case 2: S is infinite. Since (a_n) is Cauchy it is a bounded sequence and thus the set $S = \{a_n \mid n \in \mathbb{N}\}$ is a bounded infinite set.

Now by Bolzano-Weierstrass S has an accumulation point, say $x_0 \in \mathbb{R}$.

Aim: Construct a subsequence (a_{k_n}) of (a_n) converging to x_0.

Let $\epsilon_1 = 1$. Then choose $a_{k_1} \in (x_0 - 1, x_0 + 1) \cap S$. Next let $\epsilon_2 = 1/2$ and choose a_{k_2} with

$a_{k_2} \in (x_0 - 1/2, x_0 + 1/2)$ and $k_2 > k_1$. Such k_2

exists since $(x_0 - 1/2, x_0 + 1/2) \cap S$ is infinite.

Inductively, choose a_{k_n} so that

$a_{k_n} \in (x_0 - 1/n, x_0 + 1/n) \cap S$ and $k_n > k_{n-1}$.

Hence, (a_{k_n}) is subsequence of (a_n) and

$|a_{k_n} - x_0| < 1/n$ for all n.

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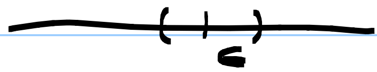
In particular, (a_{k_n}) is convergent with $\lim a_{k_n} = x_0$. However, (a_n) is Cauchy and thus (a_n) is convergent with $\lim a_n = x_0$. This finishes the proof. ∎

Definitions: A subset S of $\mathbb{R}$ is called closed if the set S' of accumulation points of S is contained in S: $S' \subseteq S$.

A subset U of $\mathbb{R}$ is called open if $\mathbb{R} \setminus U$ is closed.

Examples: 1) $\phi' = \phi \Rightarrow \phi$ is closed.

2) $\mathbb{R}' = \mathbb{R} \Rightarrow \mathbb{R}$ is closed.



3) If S is finite then $S' = \phi$ and $\phi \subseteq S$. So S is closed.

4) Since ϕ is closed, $\mathbb{R} = \mathbb{R} \setminus \phi$ is open. Similarly, $\mathbb{R}$ is closed and thus $\phi = \mathbb{R} \setminus \mathbb{R}$ is open.

Remark: We'll see later that $\mathbb{R}$ and ϕ are the only subsets of $\mathbb{R}$ which are both open and closed.

5) If S is finite then $U = \mathbb{R} \setminus S$ is open.

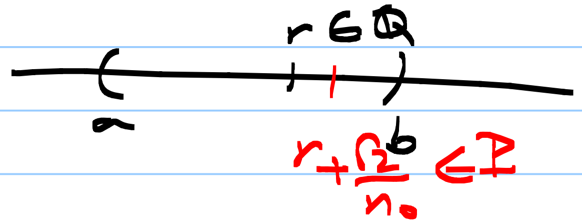
6) $S = \mathbb{Z}$, $S' = \phi$, $S \subseteq \mathbb{Z}$ and thus $\mathbb{Z}$ is closed.

7) $S = \mathbb{Q}$, $S' = \mathbb{R}$, $\mathbb{R} \not\subseteq \mathbb{Q}$ and thus $\mathbb{Q}$ is not closed.

$\mathbb{R}\setminus\mathbb{Q} = \mathbb{I}$ the set of irrational numbers is also dense in $\mathbb{R}$ and thus $\mathbb{I}' = \mathbb{R}$ so that $\mathbb{I}' \not\subseteq \mathbb{I}$ and thus $\mathbb{I}'$ is not closed. Hence $\mathbb{Q} = \mathbb{R}\setminus\mathbb{I}$ is <u>not</u> open.

Hence both $\mathbb{Q}$ and $\mathbb{I}$ are neither closed nor open.

<u>Exercise</u>: Show that $\mathbb{I}$ is dense in $\mathbb{R}$.



$$r + \frac{\sqrt{2}}{n_0}$$

<u>Proposition</u>: A subset B of $\mathbb{R}$ is open if and only if for every $x \in B$ there is some $\delta > 0$ such that $(x-\delta, x+\delta)$ is a subset of B.

<u>Proof</u>: ($\Longrightarrow$) Assume that B is open. Let $x \in B$ be given. $x \notin \mathbb{R}\setminus B$, where $\mathbb{R}\setminus B$ is closed. Let $S = \mathbb{R}\setminus B$. Since S is closed $S' \subseteq S$ and thus $x \notin S'$. In other words, x is not an accumulation point of S. Hence, there is some $\delta > 0$ so that the deleted interval

$$\big((x-\delta, x+\delta)\setminus\{x\}\big) \cap S = \emptyset.$$

Hence, $(x-\delta, x+\delta)\setminus\{x\} \subseteq \mathbb{R}\setminus S = B$. Finally, since $x \in B$ we see that $(x-\delta, x+\delta) \subseteq B$.
This finishes the proof of the "$\Longrightarrow$" direction.

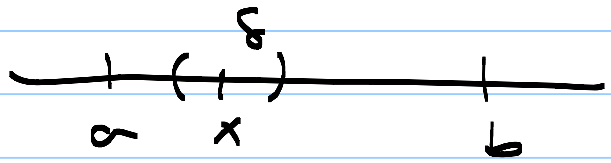
($\Longleftarrow$): Now assume that for any $x \in B$

there is some $\delta > 0$ with $(x-\delta, x+\delta) \subseteq B = \mathbb{R}\setminus S$

<u>must show:</u> $S = \mathbb{R}\setminus B$ is closed.

Enough to prove that $S' \subseteq S$. Let $x \in S'$.
If $x \in B$ then the interval $(x-\delta, x+\delta)$ must contain infinitely many elements from $S = \mathbb{R}\setminus B$, a contradiction since $(x-\delta, x+\delta) \subseteq B$.
Hence, $x \in \mathbb{R}\setminus B = S$. Thus $S' \subseteq S$ and therefore S is closed. ∎

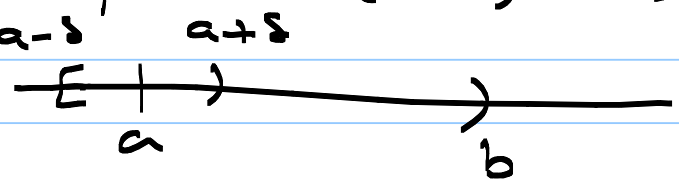
<u>Examples</u> 1) $U = (a,b)$ is open. If $x \in (a,b)$



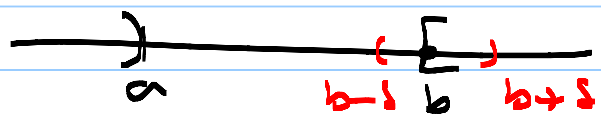
then $a < x < b$.

Let $\delta = \min\left\{ \frac{x-a}{2}, \frac{b-x}{2} \right\}$. Then clearly,
$(x-\delta, x+\delta) \subseteq (a,b)$. Hence, (a,b) is open.

2) $[a,b)$ is not open because no interval of the form $(a-\delta, x+\delta) \not\subseteq [a,b)$.



$\mathbb{R}\setminus[a,b) = (-\infty, a) \cup [b, \infty)$ is <u>not</u> open either.



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$[a,b)$ is not closed.

3) $[a,b]$, $(-\infty,a]$, $[a,\infty)$ are all closed.

<u>Theorem</u>: Let (A_k) be a sequence of non-empty closed subsets of $\mathbb{R}$ such that A_1 is bounded and $A_{k+1} \subseteq A_k$ for each k.

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots\cdots$$

Then $\bigcap_{k=1}^{\infty} A_k$ is <u>not</u> empty.

<u>Example</u> $\quad A_n = (0, 1/n)$

$$\underset{A_1}{\underset{\parallel}{(0,1)}} \supseteq \underset{A_2}{\underset{\parallel}{(0,1/2)}} \supseteq \underset{A_3}{\underset{\parallel}{(0,1/3)}} \supseteq \cdots \supseteq \underset{A_n}{\underset{\parallel}{(0,1/n)}} \supseteq \cdots$$

Note that $A_n' = [0, 1/n] \not\subseteq A_n$ and thus A_n is <u>not</u> closed.

$$\bigcap_{n=1}^{\infty} A_n = \{x \in \mathbb{R} \mid 0 < x < 1/n \text{ for all } n = 1, 2, \ldots\}$$
$$= \emptyset.$$

This is <u>not</u> a contradiction because A_n's are <u>not</u> closed.

Proof: $A_n \neq \emptyset$ for all n.
A_n is closed for all n.
$A_{n+1} \subseteq A_n$ for all n.

Choose $x_n \in A_n$, which is possible since each $A_n \neq \emptyset$. Then (x_n) is a sequence in A_1. Indeed, $x_k \in A_n$ for all $k \geq n$, because

$$x_k \subseteq A_k \subseteq A_{k-1} \subseteq \cdots \subseteq A_{n+1} \subseteq A_n.$$

Consider the set $B = \{x_n \mid n = 1, 2, \dots\}$. We have two cases.

Case 1 B is finite. Then there is so constant subsequence (x_{k_n}) of (x_n). Say $x_{k_n} = x_{n_0}$. Then

$x_{n_0} \subseteq A_n$ for all n. Fix some $m \in \mathbb{N}$

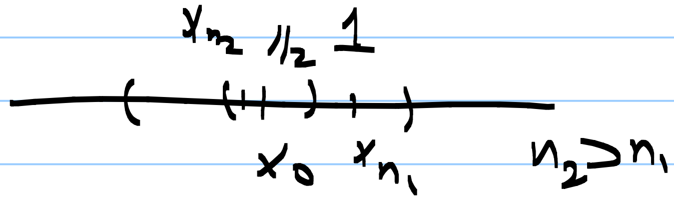
$k_n \to \infty$, $x_{n_0} = x_{k_n} \in A_{k_n} \subseteq A_m$ if $k_n \geq m$.

Hence, $x_{n_0} \in \bigcap_{n=1}^{\infty} A_n$ so that $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

Case 2. B is infinite.

Now $B \subseteq A_1$ and A_1 is bounded. Hence, B has an accumulation point, say x_0.

$B = \{x_1, x_2, x_3, \ldots\}$.



Choose $x_{n_1} \in (x_0 - 1, x_0 + 1)$. Then choose $x_{n_2} \in B$ with

$x_{n_2} \in (x_0 - 1/2, x_0 + 1/2)$, $x_{n_2} \neq x_{n_1}$ and $n_2 > n_1$.

Similarly, choose $x_{n_3} \in (x_0 - 1/3, x_0 + 1/3)$ with $x_{n_3} \neq x_{n_2}$, $x_{n_3} \neq x_{n_1}$ and $n_3 > n_2$.

Hence, we obtain a sequence say (y_n) with $y_1 = x_{n_1}$, $y_2 = x_{n_2}$, $y_3 = x_{n_3}, \ldots$ with

$|y_n - x_0| < 1/n$. So $\lim y_n = x_0$.

The sequence $(y_n) \subseteq A_1$ and $\lim y_n = x_0$.
So, $x_0 \in A_1'$. A_1 is closed and thus $A_1' \subseteq A_1$.

Hence $x_0 \in A_1$.

Let $m \in \mathbb{N}$. If $k \geq m$ then $y_k = x_{n_k} \in A_m$

because $n_k \geq k \geq m$. ($x_{n_k} \subseteq A_{n_k} \subseteq A_m$.)

Now $x_0 = \lim y_m \in A_m'$ and since A_m is closed $x_0 \in A_m' \subseteq A_m$.

In particular, $x_0 \in \bigcap_{m=1}^{\infty} A_m$ so that $\bigcap_{m=1}^{\infty} A_m \neq \phi$.

$\blacksquare$

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Limit Superior and Limit Inferior

Let (a_n) be any sequence. Assume that it is bounded from above.

Define for any $n \in \mathbb{N}$, $b_n = \sup\{a_k \mid k \geq n\}$.

Then (b_n) is a decreasing sequence of real numbers.

$b_1 = \sup\{a_1, a_2, a_3, a_4, \ldots\}$
$\qquad\qquad\quad \cup\!\mid$
$b_2 = \sup\{a_2, a_3, a_4, \ldots\}$ $\qquad b_1 \geq b_2$
$\qquad\qquad\quad \cup\!\mid$
$b_3 = \sup\{a_3, a_4, \ldots\}$ $\qquad b_2 \geq b_3$

$b_1 \geq b_2 \geq b_3 \geq \cdots \geq b_n \geq b_{n+1} \geq \cdots$

Now assume (a_n) is also bounded from below and define

$c_n = \inf\{a_k \mid k \geq n\}$. Now (c_n) is a increasing sequence.

$c_1 = \inf\{a_1, a_2, a_3, \ldots\}$
$\wedge\!\mid \qquad\qquad \cup\!\mid$
$c_2 = \inf\{a_2, a_3, \ldots\}$
$\wedge\!\mid \qquad\qquad \cup\!\mid$
$c_3 = \inf\{a_3, \ldots\}$

Examples

1) $(a_n) = (0, 1, 0, 1/2, 0, 1/3, 0, 1/4, \ldots, 0, 1/n, \ldots)$

$b_1 = \sup\{0, 1, 1/2, 1/3, \ldots\} = 1$

$b_2 = \sup\{0, 1, 1/2, 1/3, \ldots\} = 1$

$b_3 = \sup\{0, 1/2, 1/3, \ldots\} = 1/2$

$b_4 = \sup\{1/2, 0, 1/3, \ldots\} = 1/2$

$\limsup a_n = 0$
$\liminf a_n = 0$

$(b_n) = (1, 1, 1/2, 1/2, 1/3, 1/3, \ldots)$

$c_1 = \inf\{0, 1, 1/2, 1/3, \ldots\} = 0$

$c_2 = \inf\{1, 0, 1/2, 1/3, \ldots\} = 0$

$c_3 = \inf\{0, 1/2, 1/3, \ldots\} = 0$

$c_n = 0$ for all n.

$\lim b_n = 0 = \lim c_n$.

2) $(a_n) = (1, -1, 1/2, -1/2, 2/3, -2/3, 3/4, -3/4, \ldots, \frac{n}{n+1}, \frac{-n}{n+1}, \}$

$b_1 = \sup\{1, -1, 1/2, -1/2, \ldots\} = 1$

$b_2 = \sup\{-1, 1/2, -1/2, 2/3, \ldots\} = 1$

$b_3 = \sup\{1/2, -1/2, 2/3, -2/3, \ldots\} = 1$

$\limsup a_n = 1$

$\liminf a_n = -1$

So $(b_n) = (1, 1, 1, \ldots)$

Similarly, $c_n = \inf\{a_n, a_{n+1}, \dots\} = -1$ for all n.

$(b_n) = (1, 1, 1, \dots)$, $(c_n) = (-1, -1, -1, \dots)$

$\lim b_n = 1$ and $\lim c_n = -1$.

If (a_n) is bounded then (b_n) is a decreasing sequence bounded from below and thus it is convergent. Similarly, (c_n) is convergent.

Say $m \leq a_n \leq M$ for some $m, M \in \mathbb{R}$ and for all $n \in \mathbb{N}$.

$M \geq b_1 \geq b_2 \geq b_3 \geq \dots \geq b_n \geq \dots \geq c_n \geq \dots c_3 \geq c_2 \geq c_1 \geq m$

$\lim b_n$ is called $\limsup a_n$ and $\lim c_n$ is called $\liminf a_n$.

Clearly, $\limsup a_n \geq \liminf a_n$.

<u>Proposition</u>: Assume (a_n) is a bounded sequence. Then $\limsup a_n$ and $\liminf a_n$ exist. Moreover, $\limsup a_n = \liminf a_n$ if and only if $\lim a_n$ exists and equals $\limsup a_n = \liminf a_n$.

<u>Proof</u>: First assume that $\limsup a_n = \liminf a_n$.

<u>must prove</u>: $\lim a_n$ exist and equals

$\limsup a_n = \liminf a_n$

Let $L = \limsup a_n = \liminf a_n$.

$b_n = \sup\{a_n, a_{n+1}, \ldots\} \geq a_n \geq \inf\{a_n, a_{n+1}, \ldots\} = c_n$

$\Rightarrow$ $b_n \geq a_n \geq c_n$, for all n.

<u>Claim:</u> Assume we've three sequence (a_n), (b_n), (c_n) with
$b_n \geq a_n \geq c_n$ for all n and

$\lim b_n = \lim c_n = L$. Then $\lim a_n$ exist, and equals L.

<u>Proof:</u> Let $\epsilon > 0$ be given. Then since $\lim b_n = L$ there is some $n_1 \in \mathbb{N}$ with $n \geq n_1$ implies $|b_n - L| < \epsilon$. Similarly, there is $n_2 \in \mathbb{N}$ so that $n \geq n_2$ implies $|c_n - L| < \epsilon$.

So, if $n \geq n_0 = \max\{n_1, n_2\}$, then

$$L + \epsilon > b_n \geq a_n \geq c_n \geq L - \epsilon.$$

Thus $|a_n - L| < \epsilon$ if $n \geq n_0$.

Hence, $\lim a_n = L$. ∎

($\Leftarrow$) Assume that $\lim a_n$ exists, say $\lim a_n = L$.

<u>must show:</u> $L = \limsup a_n = \lim b_n = \lim c_n = \liminf a_n$.

Let $\epsilon > 0$ be given. Then there is some $n_0 \in \mathbb{N}$

so that $n \geq n_0$ implies $|a_n - L| < \epsilon$. In other words, for any $n \geq n_0$ we have $L - \epsilon < a_n < L + \epsilon$.

Hence, $L - \epsilon$ is a lower bound and $L + \epsilon$ is an upper bound for the subset $\{a_n \mid n \geq n_0\}$.

Thus, $b_n = \sup\{a_n \mid n \geq n_0\} \leq L + \epsilon$ and

$$c_n = \inf\{a_n \mid n \geq n_0\} \geq L - \epsilon, \text{ for any } n \geq n_0.$$

In particular, $\limsup a_n = \lim b_n \leq L + \epsilon$ and
$\liminf a_n = \lim c_n \geq L - \epsilon$.

Hence, $L - \epsilon \leq \liminf a_n \leq \limsup a_n \leq L + \epsilon$, which implies

$$|\liminf a_n - L| \leq \epsilon \text{ and } |\limsup a_n - L| \leq \epsilon.$$

Since $\epsilon > 0$ were arbitrary we deduce that

$\liminf a_n = L = \limsup a_n$, which finishes the proof. ■

CHAPTER TWO: METRIC SPACES

Definition A metric (or a distance function) on a set X is a real valued function
$$d : X \times X \longrightarrow \mathbb{R},$$ which satisfies the following properties:

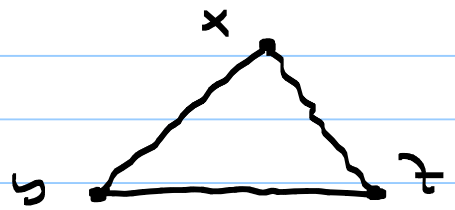
M1) $d(x,y) \geq 0$ for all $x, y \in X$ and

$d(x,y) = 0$ if and only if $x = y$.

M2) $d(x,y) = d(y,x)$, for all $x, y \in X$.

M3) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in X$.

M3) is called the Triangle Inequality.



Definition If d is a metric on a set X, then the pair (X,d) will be called a metric space.

Proposition: For any $x_1, x_2, \ldots, x_n$ in a metric space (X,d) we have

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n).$$

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Proof Induction on n.

$n=2$, $d(x_1,x_2) \leq d(x_1,x_2)$ ✓

Assume the result for $n=k$. Now let $n=k+1$. Let $x_1, x_2, \dots, x_k, x_{k+1} \in X$. Then

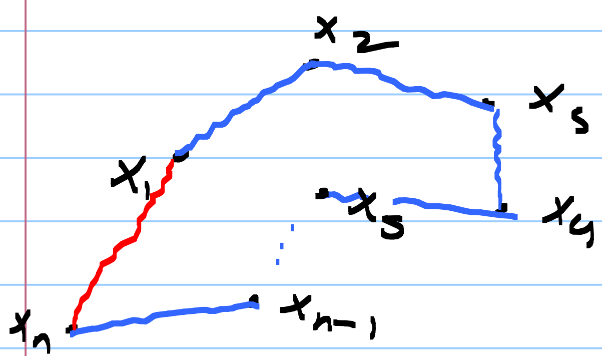
$d(x_1,x_{k+1}) \leq d(x_1,x_k) + d(x_k,x_{k+1})$ by the triangle Inequality for x_1, x_k, x_{k+1}.

On the other hand, by the Induction hypothesis we have

$d(x_1,x_k) \leq d(x_1,x_2)+d(x_2,x_3)+\dots+d(x_{k-1},x_k)$.

So, $d(x_1,x_{k+1}) \leq d(x_1,x_2)+d(x_2,x_3)+\dots+d(x_{k-1},x_k)+d(x_k,x_{k+1})$.

Hence, we are done. ∎



Proposition: Let (X,d) be a metric space. Then for any point x, y, z and w in X we have

$|d(x,w) - d(y,z)| \leq d(x,y) + d(w,z)$.

Proof: By the previous proposition we have

$d(x,w) \leq d(x,y) + d(y,z) + d(z,w)$. Hence,

$d(x,w) - d(y,z) \leq d(x,y) + d(z,w)$.

Let replace the letter x with y and z with w.
Then the above inequality becomes

$d(y,z) - d(x,w) \leq d(y,x) + d(w,z)$.

$-(d(x,w) - d(y,z)) \leq d(x,y) + d(z,w)$

Hence, $|d(x,w) - d(y,z)| \leq d(x,y) + d(z,w)$. $\blacksquare$

Examples 1) $X = \mathbb{R}$, $d = |\cdot|$

$d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $d(x,y) \doteq |x-y|$.

M1) $d(x,y) = |x-y| \geq 0$, for any $x, y \in X$ and
$d(x,y) = 0$ if and only $x = y$.

M2) $d(x,y) = |x-y| = |y-x| = d(y,x)$.

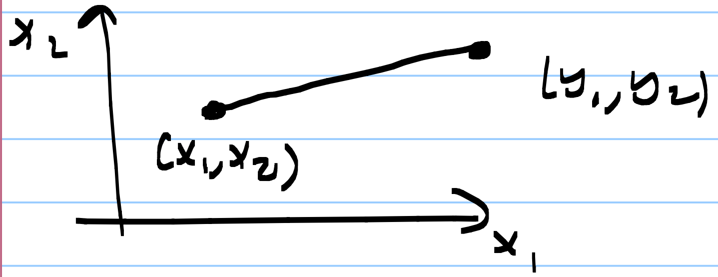
M3) $d(x,y) = |x-y| = |(x-z) + (z-y)| \leq |x-z| + |z-y|$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in \mathbb{R}$.

Hence, $(\mathbb{R}, |\cdot|)$ is a metric space.

2) $X = \mathbb{R}^2$, $d: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$$



More generally, let $X = \mathbb{R}^n$ and set

$d: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \doteq \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$$

<u>Claim</u>: d is a metric on $\mathbb{R}^n$.

<u>Proof</u> M1) $d(p,q) \geq 0$, for all $p, q \in \mathbb{R}^n$ and

if $p = q$ then $x_i = y_i$ for all $i = 1, \ldots, n$.

Thus, $d(p,q) = \sqrt{0^2 + 0^2 + \cdots + 0^2} = 0$.

Moreover, if $d(p,q) = \left(\sum_{i=1}^{n} |y_i - x_i|^2 \right)^{1/2} = 0$, then

$\sum_{i=1}^{n} |y_i - x_i|^2 = 0$, which implies $|y_i - x_i|^2 = 0$

for all $i = 1, \ldots, n$. Hence, $y_i = x_i$ for all $i = 1, \ldots, n$,

so that $p = (x_1, \ldots, x_n) = (y_1, \ldots, y_n) = q$.

M2) $d(p,q) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2} = \left(\sum_{i=1}^{n} |y_i - x_i|^2\right)^{1/2} = d(q,p)$.

M3) $d(p,q) \leq d(p,r) + d(r,q)$ for all $p, q, r \in \mathbb{R}^n$.

We need so called the Cauchy-Schwarz Inequality:

$(\mathbb{R}^n, (\cdot,\cdot))$ Inner product space

$p = (x_1, \ldots, x_n)$, $q = (y_1, \ldots, y_n)$

$(p,q) = \sum_{i=1}^{n} x_i y_i$.

$|(p,q)| \leq \|p\| \, \|q\|$, where $\|p\| = \sqrt{(p,p)}$.

$\Rightarrow \left|\sum_{i=1}^{n} x_i y_i\right| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} y_i^2\right)^{1/2}$.

Cauchy-Schwarz Inequality.

Let $p = (x_1, \ldots, x_n)$, $q = (y_1, \ldots, y_n)$, $r = (z_1, \ldots, z_n)$.

$d(p,r) = \left(\sum_{i=1}^{n} (x_i - z_i)^2\right)^{1/2} = \|p - r\|$.

$d(p,r) = \|p - r\| = \|(p-q) + (q-r)\|$

$\leq \|p-q\| + \|q-r\| = d(p,q) + d(q,r)$.

Here, Cauchy-Schwarz is used to prove the triangle inequality as follows:

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Let $u, v \in \mathbb{R}^n$ be two vectors. Then

$$\|u+v\|^2 = (u+v)\cdot(u+v)$$
$$= u\cdot u + u\cdot v + v\cdot u + v\cdot v$$
$$= \|u\|^2 + \|v\|^2 + 2\, u\cdot v$$
$$\leq \|u\|^2 + \|v\|^2 + 2\,\|u\|\,\|v\|$$
$$= (\|u\| + \|v\|)^2$$ and thus we obtain

the triangle inequality, $\|u+v\| \leq \|u\| + \|v\|$.

Hence, $(\mathbb{R}^n, d_2)$ is a metric space, when

$$d_2(p,q) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2}.$$

Similarly, we may define d_1, d_p and d_∞ as follows, all metrics on $\mathbb{R}^n$.

$$d_1 : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$d_1(p,q) \doteq \sum_{i=1}^{n} |x_i - y_i|$$

$$d_p(p,q) \doteq \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p} \qquad (p \geq 1)$$

$$d_\infty(p,q) \doteq \max\{|x_1-y_1|, |x_2-y_2|, \ldots, |x_n-y_n|\}$$

<u>Definition:</u> We'll call two metrics d_1 and d_2 on a set X equivalent if there are positive real numbers $m, M \in \mathbb{R}^+$ so that

$$m\, d_1(x,y) \leq d_2(x,y) \leq M\, d_1(x,y) \quad \text{for all } x,y \in X.$$

Proposition: Being equivalent is an equivalence relation on the set of all metrics on X.

Proof: Reflexive: let $m = M = 1$ then

$$m \cdot d(x,y) = d(x,y) \leq d(x,y) \leq d(x,y) = M \cdot d(x,y)$$

Symmetric: Assume d_1 and d_2 are equivalent.

Then $m\, d_1(x,y) \leq d_2(x,y) \leq M\, d_1(x,y)$ for some $m, M \in \mathbb{R}^+$ and for all $x,y \in X$.

$$\frac{1}{M} d_2(x,y) \leq d_1(x,y) \leq \frac{1}{m} d_2(x,y), \text{ for all } x,y \in X.$$

Transitivity: Let d_1 and d_2 be equivalent

and d_2 and d_3 be equivalent. So there are constants m_1, M_1, m_2 and M_2 so that

$$m_1 d_1 \leq d_2 \leq M_1 d_1 \quad \text{and} \quad m_2 d_2 \leq d_3 \leq M_2 d_2.$$

$$m_1 m_2 d_1 \leq m_2 d_2 \leq d_3 \leq M_2 d_2 \leq M_1 M_2 d_1$$

$$\Rightarrow m_1 m_2 d_1 \leq d_3 \leq M_1 M_3 d_1.$$

Hence, d_1 and d_3 are equivalent. ∎

What about our metrics?

$X=\mathbb{R}^n$, $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$

$d_1(x,y)=|x_1-y_1|+\dots+|x_n-y_n|$

$d_2(x,y)=(|x_1-y_1|^2+\dots+|x_n-y_n|^2)^{1/2}$

$d_p(x,y)=(|x_1-y_1|^p+\dots+|x_n-y_n|^p)^{1/p}$ $\quad p\geq 1.$

$d_\infty(x,y)=\max\{|x_1-y_1|,\dots,|x_n-y_n|\}.$

Notice that

$$\begin{aligned}
d_\infty(x,y) &= \max\{|x_1-y_1|,\dots,|x_n-y_n|\} \\
&= |x_k-y_k| = (|x_k-y_k|^p)^{1/p} \\
&\leq (|x_1-y_1|^p+\dots+|x_k-y_k|^p+\dots+|x_n-y_n|^p)^{1/p} \\
&= d_p(x,y) \qquad p=2,m\in\mathbb{N} \text{ (Exercise!)} \\
&\leq |x_1-y_1|+\dots+|x_k-y_k|+\dots+|x_n-y_n| \\
&= d_1(x,y). \\
&\leq |x_k-y_k|+\dots+|x_k-y_k|+\dots+|x_k-y_k| \\
&= n\cdot|x_k-y_k| \\
&= n\,d_\infty(x,y).
\end{aligned}$$

$d_\infty \leq d_p \leq d_1 \leq n\,d_\infty.$

Video 16

Hence, d_∞ is equivalent to d_p for any $p \geq 1$. Finally, since being equivalent is an equivalence relation all d_p's are equivalent $p \in [1, \infty)$ or $p = \infty$.

Uniform Metric:

Let S be any nonempty set. A function $f: S \to \mathbb{R}$ is called bounded if there is some $M \in \mathbb{R}^+$ so that

$$|f(s)| \leq M \text{ for all } s \in S.$$

Let $B(S) = \{f: S \to \mathbb{R} \mid f \text{ is bounded}\}$.

Define a metric on $B(S)$, called the uniform metric as follows:

For $f, g \in B(S)$ let

$$d_{\sup}(f, g) = \sup\{|f(s) - g(s)| \mid s \in S\}.$$

$d_{\sup}$ is indeed a metric on $B(S)$.

Proposition: $d_{\sup}$ is a metric on $B(S)$.

Proof: Since f and g are bounded say by M_1 and M_2 we have

$-M_1 \leq f(s) \leq M_1$ and $-M_2 \leq g(s) \leq M_2$, for all $s \in S$.

$$-M_2 \leq -g(s) \leq M_2$$

Then $-M_1 - M_2 \leq f(s) - g(s) = f(s) + (-g(s)) \leq M_1 + M_2$

so that $|f(s) - g(s)| \leq M_1 + M_2$, for all $s \in S$.
Hence, $d_{sup}(f, g) = \sup\{|f(s) - g(s)| \mid s \in S\}$ exists.

M1) $d_{sup}(f, g) \geq 0$ since each $|f(s) - g(s)| \geq 0$.

Moreover, if $d_{sup}(f, g) = 0$, then $0 \leq |f(s) - g(s)| \leq 0$ and then $f(s) = g(s)$ for all $s \in S$. Hence, $f = g$.

M2) $d_{sup}(f, g) = \sup\{|f(s) - g(s)| \mid s \in S\}$

$= \sup\{|g(s) - f(s)| \mid s \in S\}$

$= d_{sup}(g, f)$.

M3) Let f, g and $h \in B(S)$. Then, for any $s \in S$

$$|f(s) - h(s)| = |(f(s) - g(s)) + (g(s) - h(s))|$$
$$\leq |f(s) - g(s)| + |g(s) - h(s)|$$
$$\leq d_{sup}(f, g) + d_{sup}(g, h).$$

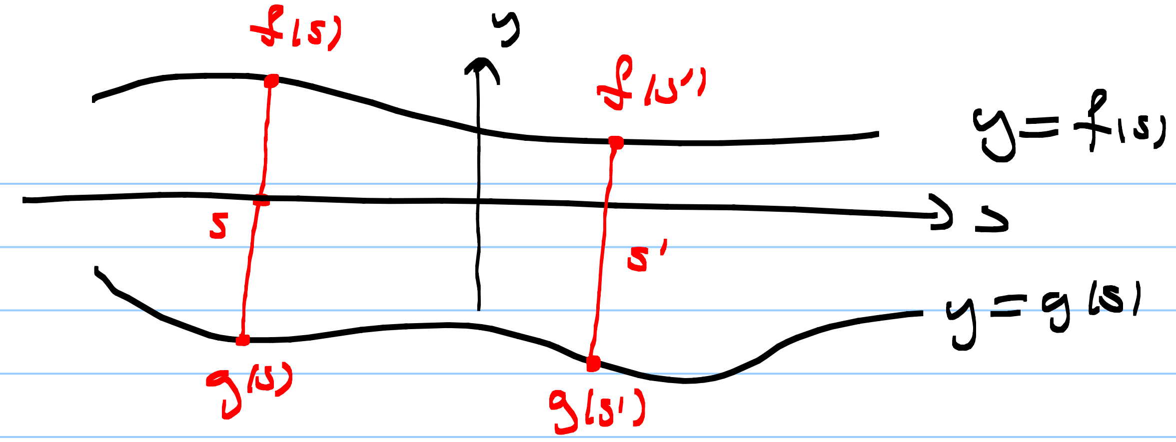
Hence, $d_{sup}(f, g) + d_{sup}(g, h)$ is an upper bound for

$\{|f(s) - h(s)| \mid s \in S\}$. Hence,

$d_{sup}(f, h) = \sup\{|f(s) - h(s)| \mid s \in S\} \leq d_{sup}(f, g) + d_{sup}(g, h)$.

<u>Example</u>: $S = \mathbb{R}$. Then

$B(S) = B(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is bounded}\}$.



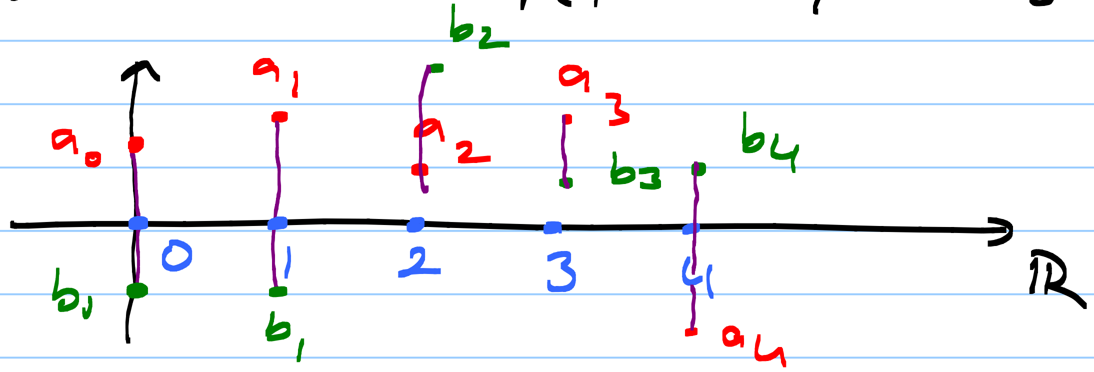
$d_{sup}(f,g) = \sup\{ |f(s) - g(s)| \mid s \in \mathbb{R} \}$.

Example: $S = \mathbb{N} = \{0,1,2,3,\ldots\}$

A function $a : S = \mathbb{N} \to \mathbb{R}$ is just a sequence. The value of a at any $n \in \mathbb{N}$ is denoted $a(n)$ by a_n.

$B(S) = \{ (a_n) \mid (a_n) \text{ is a bounded sequence} \}$.

$d_{sup}((a_n),(b_n)) = \sup\{ |a_n - b_n| \mid n \in \mathbb{N} \}$.



Definition: Let (X,d) be a metric space and Y a non-empty subset of X. Then (Y,d) is also a metric space, called a subspace of (X,d).

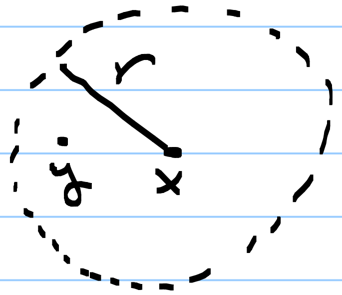
Example: Let $I = [0,1]$, then $(B(I), d_{sup})$ is the metric space of all bounded functions on I. Recall that any continuous function on I is bounded. Thus the set of all continuous functions

$C(I)$ is contained in $B(I)$. Hence, $(C(I), d_{sup})$ becomes a metric space.

<u>Definition</u>: Let (X, d) be a space, $r>0$ a real number and $x \in X$ an element. Then the set

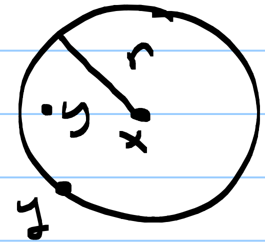
$$B(x,r)=\{y\in X \mid d(x,y)<r\}$$ is called the open

ball in (X,d) with center x and radius $r>0$.



Similarly, for $r \geq 0$, the closed ball with center x and radius r is defined as

$$B[x,r]=\{y\in X \mid d(x,y)\leq r\}.$$



<u>Examples</u>: 1) $(X, d)=(\mathbb{R}, |\cdot|)$

$$B(x,r)=\{y\in\mathbb{R} \mid d(x,y)<r\}=\{y\in\mathbb{R} \mid |x-y|<r\}$$

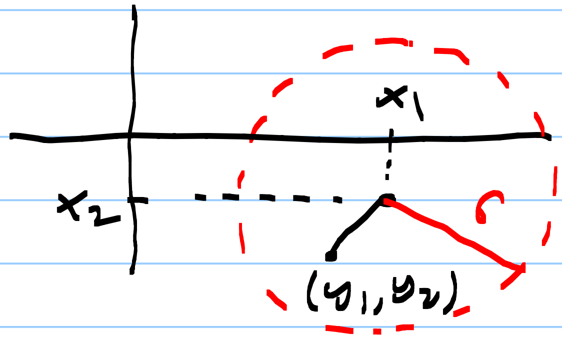
$$=(x-r, x+r).$$

Similarly, $B[x,r]=[x-r, x+r]$.

2) $X=\mathbb{R}^2$, $d=d_2$, $d((x_1,x_2),(y_1,y_2))=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$

Hence, if $x=(x_1,x_2)$, $r>0$, then

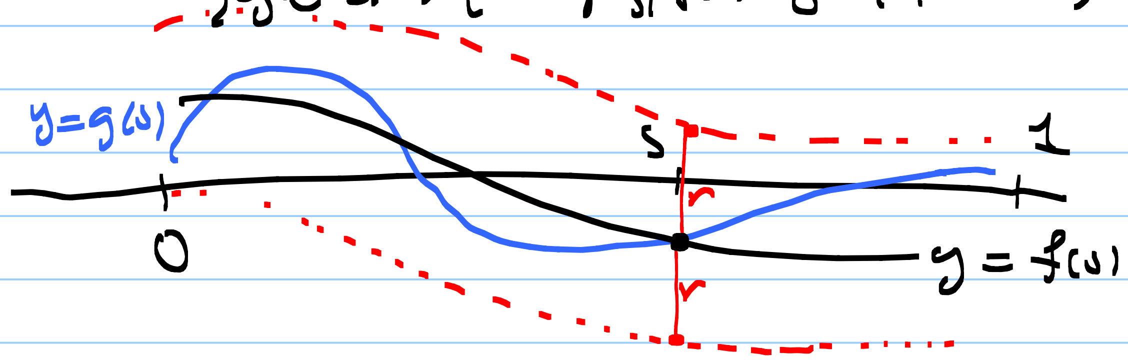
$$B(x,r)=\{y=(y_1,y_2)\in\mathbb{R}^2 \mid d((x_1,x_2),(y_1,y_2))<r\}.$$
$$=\{(y_1,y_2)\in\mathbb{R}^2 \mid \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}<r\}$$
$$=\{(y_1,y_2)\in\mathbb{R}^2 \mid (x_1-y_1)^2+(x_2-y_2)^2<r^2\}$$



(Remark: $r=0$, $B(x,0)=\{y\in X \mid d(x,y)<0\}=\emptyset$)

3) $S=[0,1]$, $f\in B(S)$ any function, $r>0$.

$$B(f,r)=\{g\in B(S) \mid d_{sup}(f,g)<r\}.$$
$$=\{g\in B(S) \mid \sup\{|f(s)-g(s)| \mid s\in S\}<r\}.$$



Definition: Let X be any nonempty set and d be the function defined as

$$d: X\times X\to\mathbb{R},\quad d(x,y)=\begin{cases}0 & \text{if } x=y,\\ 1 & \text{if } x\neq y.\end{cases}$$

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Claim: d is a metric on X.

Proof: M1) $d(x,y) \geq 0$ for all $x,y \in X$ and $d(x,y)=0$ if and only if $x=y$.

M2) $d(x,y) = d(y,x)$, for all $x,y \in X$, trivially.

M3) Let $x,y,z \in X$, then note that if $x \neq z$ then either $x \neq y$ or $y \neq z$.

Hence, $$d(x,z) = \begin{cases} 0 & \text{if } x=z \\ 1 & \text{if } x \neq z \end{cases} \leq \begin{cases} 1 & \text{if } x=z \\ 1 & \text{if } x \neq z \end{cases} = 1$$

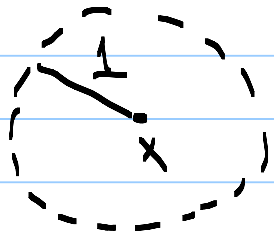
$$\leq d(x,y) + d(y,z)$$

Hence, d is a metric on X, called the discrete metric.

Example: (X,d) be a discrete metric space.

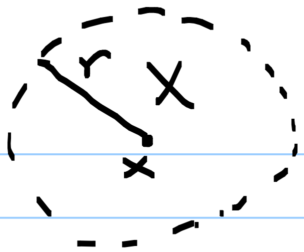
Take any $x \in X$, $r > 0$. If $r \leq 1$, then

$$B(x,r) = \{y \in X \mid d(x,y) < r \leq 1\}$$
$$= \{y \in X \mid d(x,y) = 0\}.$$
$$= \{x\}.$$



If $r > 1$, then $B(x,r) = \{y \in X \mid d(x,y) < r\} = X$.

$r > 1$

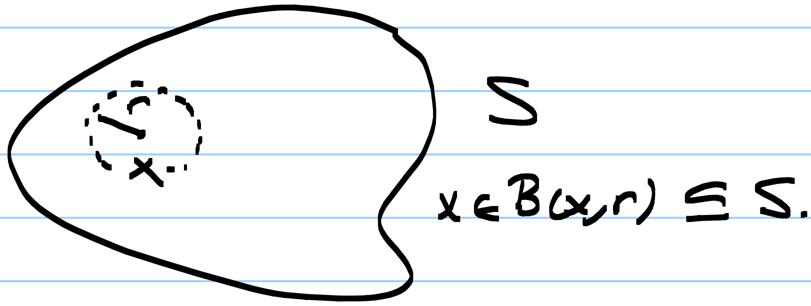


$B(x,r) = X.$

Definition: Let (X,d) be any metric space and $S \subseteq X$ any subset. A point $x \in X$ is called an interior point of S if there is some $r > 0$ so that

$$B(x,r) \subseteq S.$$

In this case, we write $x \in \mathrm{Int}(S)$.



Remark: Clearly, any $x \in \mathrm{Int}(S)$ belongs to S.

Hence, $\mathrm{Int}(S) \subseteq S$.

A subset U of X is called <u>open</u> if

$U = \mathrm{Int}(U)$.

Finally, a subset C of X is called <u>closed</u> if

$X \setminus C$ is open.

Remark: Note that a subset U of X is open if and only if every point of U is an interior point of U. Equivalently, U is open if and only if for any $x \in U$ there is some

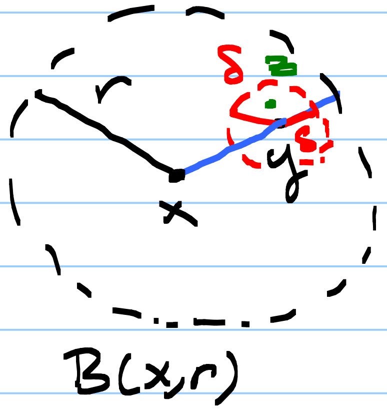
$r > 0$ so that $B(x,r) \subseteq U$.

Proposition: Any open ball is open and any closed ball is closed.

Proof: Let (X,d) be a metric space and let $U = B(x,r)$ be any open ball ($x \in X$, $r > 0$).

must show: $U = B(x,r)$ is an open subset.

Let $y \in U$, then we must find some $\delta > 0$ so that $B(y,\delta) \subseteq U$.



Let $\delta = r - d(x,y)$.

Claim $B(y,\delta) \subseteq B(x,r)$.

Proof Let $z \in B(y,\delta)$. Then $d(y,z) < \delta$.

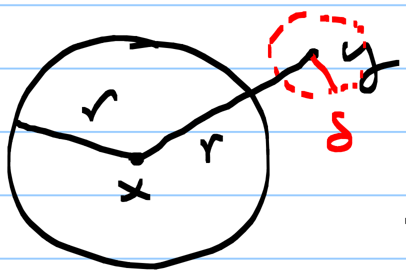
Now,
$$\begin{aligned} d(x,z) &\le d(x,y) + d(y,z) \\ &= (r-\delta) + d(y,z) \\ &< (r-\delta) + \delta = r. \end{aligned}$$

Hence, $z \in B(x,r)$. Thus $B(y,\delta) \subseteq B(x,r)$.

This finishes the proof of the statement that any open ball is open subset.

For the second statement consider any closed ball, say $B[x,r]$.
To show that $B[x,r]$ is a closed subset we must prove that $U = X \setminus B[x,r]$ is an open subset.

Let $y \in U$. Let $\delta = d(x,y) - r$.



Claim: $B(y,\delta) \subseteq U$.

Proof: Assume on the contrary that $B(y,\delta) \not\subseteq U$. Hence, there is some $z \in B(y,\delta)$ with $z \notin U = X \setminus B[x,r]$.
Hence, $z \in B[x,r]$.

So, $d(z,y) < \delta$ and $d(x,z) \leq r$. Thus, by the triangle inequality we get

$d(x,y) \leq d(x,z) + d(z,y) < r + \delta$ so that

$d(x,y) < r + (d(x,y) - r) = d(x,y)$, a

contradiction. Hence $B(y,\delta) \subseteq U$ and this finishes the proof. ∎

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Proposition: a) The union of a family of open sets is open.

b) The intersection of a family of closed sets is closed.

c) The intersection of finitely many open sets is open.

d) The union of finitely many closed sets is closed.

Proof: Let (X,d) be a metric space.

a) Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be any family of open subsets of X.

<u>must prove:</u> $U = \bigcup_{\alpha\in\Lambda} U_\alpha$ is an open subset.

Let $x\in U$. Then $x\in U_{\alpha_0}$ for some $\alpha_0\in\Lambda$. Since U_{α_0} is open there is some $\delta>0$ so that $B(x,\delta)\subseteq U_{\alpha_0}$. In particular, $B(x,\delta)\subseteq U$. Hence, U is open.

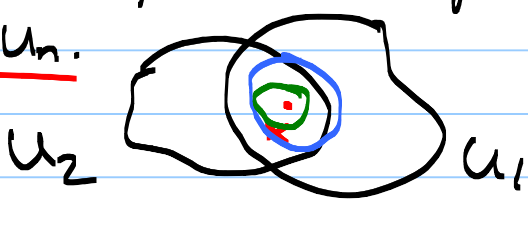
b) Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of closed subsets of X.

Then $U_\alpha \doteq X\setminus A_\alpha$ are all open for all $\alpha\in\Lambda$. Now by Part $\bigcup_\alpha U_\alpha$ is open in X. Hence,

$$\bigcap_{\alpha\in\Lambda} A_\alpha = \bigcap_{\alpha\in\Lambda}(X\setminus U_\alpha) = X\setminus\Big(\bigcup_{\alpha\in\Lambda} U_\alpha\Big)$$ is closed since

$\bigcup_{\alpha\in\Lambda} U_\alpha$ is open.

c) Let $U_1,\dots,U_n$ be open subsets of (X,d). Take any $x\in U_1\cap\dots\cap U_n$.



Since $x \in U_i$ and U_i is open there is some $r_i > 0$ so that $B(x, r_i) \subseteq U_i$, $i = 1, \ldots, n$.
Let $r = \min\{r_1, \ldots, r_n\}$. Then $0 < r \leq r_i$, for all $i = 1, \ldots, n$, and thus

$$B(x, r) \subseteq B(x, r_i) \subseteq U_i, \text{ for all } i = 1, \ldots, n.$$

Thus $B(x,r) \subseteq \bigcap_{i=1}^{n} U_i$, so that $\bigcap_{i=1}^{n} U_i$ is open.

d) Let $A_1, \ldots, A_n$ be closed subsets of X.

Aim: Show that $A_1 \cup \ldots \cup A_n$ is closed.

As before let $U_i = X \setminus A_i$. Then U_i is open since A_i is closed. Thus

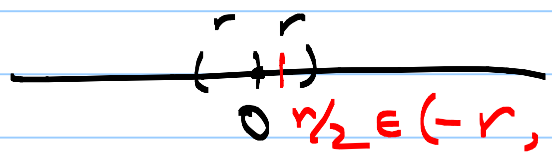
$$A_1 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} (X \setminus U_i) = X \setminus \left(\bigcap_{i=1}^{n} U_i \right),$$

where $\bigcap_{i=1}^{n} U_i$ is open by Part (c). Hence, $A_1 \cup \ldots \cup A_n$ is closed. ∎

Remark: 1) $U_n = (-1/n, 1/n)$, $n = 1, 2, \ldots$. Clearly, each U_n is open. However,

$$\bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\} \text{ is not open.}$$

$(\mathbb{R}, |\cdot|)$



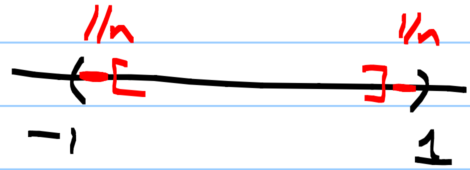
$r/2 \in (-r, r)$ but $r/2 \notin \{0\}$.

Hence, arbitrary intersection of open sets may not be open.

2) Similarly, arbitrary union of closed set may not be closed:

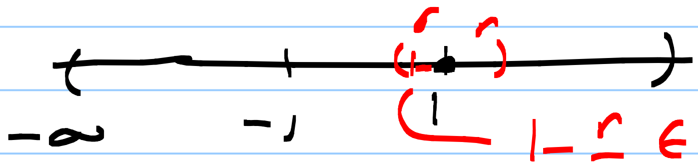
Let $A_n = [-1+\frac{1}{n}, 1-\frac{1}{n}]$, which is closed in $(\mathbb{R}, |\cdot|)$

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} [-1+\frac{1}{n}, 1-\frac{1}{n}]$$



$= (-1,1)$ and $(-1,1)$ is not closed, because

$\mathbb{R} \setminus (-1,1) = (-\infty,-1] \cup [1,\infty)$ is not open.



$1-\frac{r}{2} \in (1-r, 1+r)$, $1-\frac{r}{2} \notin \mathbb{R}\setminus(-1,1)$.

Example Let X be any non empty set and d be the discrete metric on X.

$$d: X\times X \to \mathbb{R}, \quad d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

If $x \in X$, then $B(x, 1/2) = \{y \in X \mid d(x,y) < 1/2\}$

$$= \{x\}.$$

Hence, $\{x\}$ is open, for any $x \in X$.

Now, if $A \subseteq X$ is any subset, then

$A = \bigcup_{x\in A} \{x\}$ is an open subset since each $\{x\}$ is open

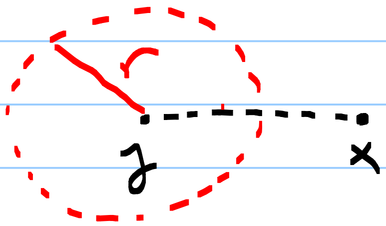
Moreover, any subset A of X is closed because $A = X \setminus (X \setminus A)$ and $X \setminus A$ is an open.

Clearly as a subset X of X is open. Hence, $\emptyset = X \setminus X$ is closed.
$\emptyset \subseteq X$ is also open! Hence, $X = X \setminus \emptyset$ is closed.

$\therefore$ In a discrete metric space every subset is both open and closed.

Remark: In any metric space X and $\emptyset$ are always both open and closed.

Example: Let (X,d) be a metric space. Then $X \setminus \{x\}$ is open for any $x \in X$. Let $U = X \setminus \{x\}$.



If $y \in U$ then $y \neq x$ and then $d(x,y) > 0$. Let $r = \frac{1}{2} d(x,y) > 0$.

Then $x \notin B(y, \frac{r}{2})$ and thus $B(y, \frac{r}{2}) \subseteq X \setminus \{x\}$.

Hence, $X \setminus \{x\}$ is open. Thus, $\{x\}$ is closed.

In particular, if $x_1, \ldots, x_n$ are points in X, then $\{x_1, \ldots, x_n\} = \bigcup_{i=1}^{n} \{x_i\}$ is also closed.

Recall that a point $x \in A \subseteq X$ is called an interior point of A if there is a ball $B(x,\epsilon)$, $(\epsilon > 0)$ so that

$$x \in B(x,\epsilon) \subseteq A.$$

In this case, we write $x \in \mathrm{Int}(A)$, the set of interior points of A.

$$\mathrm{Int}(A) = \{x \in A \mid \exists \epsilon > 0 \ni B(x,\epsilon) \subseteq A\}.$$

Definition: For any subset A of (X,d) the exterior of A is defined as follows:

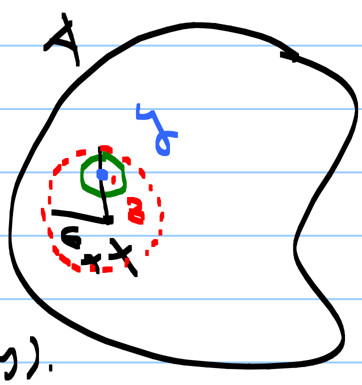
$$\mathrm{Ext}(A) \doteq \mathrm{Int}(X \setminus A).$$

Proposition: $\mathrm{Int}(A)$ is always open.

Proof: Let $x \in \mathrm{Int}(A)$, then there is some $\epsilon_x > 0$

so that $B(x,\epsilon_x) \subseteq A$.

Claim: $B(x,\epsilon_x) \subseteq \mathrm{Int}(A)$



proof of the claim:

Let $y \in B(x,\epsilon_x)$. Let $r = \epsilon_x - d(x,y)$.

Then we have $B(y,r) \subseteq B(x,\epsilon_x)$, because if $z \in B(y,r)$, then

$$d(z,x) \le d(z,y) + d(y,x) < r + d(x,y) = \epsilon_x - d(x,y) + d(x,y))$$

Hence, $d(z,x) < \epsilon_x$ so that $z \in B(x,\epsilon_x) \subseteq A$.

So, $B(y,r) \subseteq A$ which implies that $y \in \mathrm{Int}(A)$.

Thus $B(x,\epsilon_x) \subseteq \mathrm{Int}(A)$. ∎

Now, $\mathrm{Int}(A) = \bigcup_{x \in \mathrm{Int}(A)} \{x\} \subseteq \bigcup_{x\in \mathrm{Int}(A)} B(x,\epsilon_x) \subseteq \mathrm{Int}(A)$.

This implies that $\mathrm{Int}(A) = \bigcup_{x \in \mathrm{Int}(A)} B(x,\epsilon_x)$. In particular, $\mathrm{Int}(A)$ is a union of open subsets and thus $\mathrm{Int}(A)$ is an open subset. ∎

Since interior of any set is open the exterior of any set is open: For any subset A of X

$\mathrm{Ext}(A) = \mathrm{Int}(X \setminus A)$ is open.

Moreover, $\mathrm{Int}(A) \subseteq A$ and thus

$\mathrm{Ext}(A) = \mathrm{Int}(X\setminus A) \subseteq X \setminus A$. In particular,

$\mathrm{Int}(A) \cap \mathrm{Ext}(A) \subseteq A \cap (X\setminus A) = \phi$.

<u>Definition</u>: For any subset A of X the boundary of A is defined to be the subset

$$\partial A \doteq X \setminus (\mathrm{Int}(A) \cup \mathrm{Ext}(A)).$$

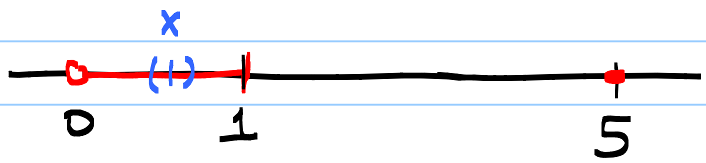
Thus clearly, $\mathrm{Int}(A)$, $\mathrm{Ext}(A)$ and ∂A are all disjoint and ∂A is always closed, because its complement is open.

Hence we can write X as the disjoint union

$$X = \text{Int}(A) \cup \text{Ext}(A) \cup \partial A.$$

<u>Examples:</u> 1) $X = \mathbb{R}$, $d = |\cdot|$

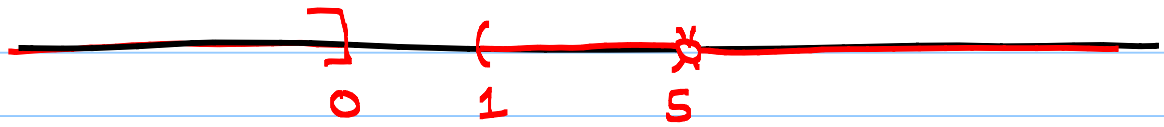
$A = (0,1] \cup \{5\}$



$\text{Int}(A) = (0,1)$

$\text{Ext}(A) = \text{Int}(\mathbb{R} \setminus A) = \text{Int}((-\infty,0] \cup (1,5) \cup (5,\infty))$

$$= (-\infty,0) \cup (1,5) \cup (5,\infty)$$



$\partial A = \mathbb{R} \setminus (\text{Int}(A) \cup \text{Ext}(A)) = \{0,1,5\}.$

<u>Remark:</u> If A is open then for any $x \in A$, there is some $\epsilon > 0$ so that $B(x,\epsilon) \subseteq X$. Hence, $x \in \text{Int}(A)$, so that

$$A \subseteq \text{Int}(A) \subseteq A \implies A = \text{Int}(A).$$

In particular, we see that a subset $A \subseteq X$ is open if and only if $A = \text{Int}(A)$.

Moreover, for any set A, $\mathrm{Int}(A)$ is the largest open subset of A and $\mathrm{Ext}(A)$ is the largest open subset of $X \setminus A$:

$$\mathrm{Int}(A) = \bigcup_{x \in \mathrm{Int}(A)} B(x, \epsilon_x) \subseteq A$$

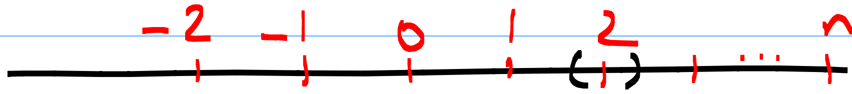
<u>Example 2)</u> Let (X, d) be a discrete metric space.

Let $A \subseteq X$ be any subset. Then A is open and thus $A = \mathrm{Int}(A)$. Similarly,

$\mathrm{Ext}(A) = \mathrm{Int}(X \setminus A) = X \setminus A$. In particular,

$\partial A = X \setminus (\mathrm{Int}(A) \cup \mathrm{Ext}(A)) = X \setminus (A \cup (X \setminus A)) = X \setminus X = \emptyset$.

<u>Example 3)</u> $\mathbb{Z} \subseteq (\mathbb{R}, |\cdot|)$



$\mathrm{Int}(\mathbb{Z}) = \emptyset$, $\mathbb{R} \setminus \mathbb{Z} = \bigcup_{n=-\infty}^{\infty} (n, n+1)$, which is open.

$\mathrm{Ext}(\mathbb{Z}) = \mathrm{Int}(\mathbb{R} \setminus \mathbb{Z}) = \mathbb{R} \setminus \mathbb{Z}$, since it is open.

$$\begin{aligned}\partial \mathbb{Z} &= \mathbb{R} \setminus (\mathrm{Int}(\mathbb{Z}) \cup \mathrm{Ext}(\mathbb{Z})) \\ &= \mathbb{R} \setminus (\emptyset \cup (\mathbb{R} \setminus \mathbb{Z})) \\ &= \mathbb{Z}.\end{aligned}$$

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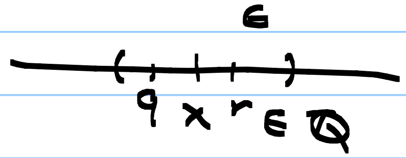
Example 4) Let $A = \mathbb{Q}$ in $(\mathbb{R}, |\cdot|)$

Compute $\mathrm{Int}(A)$, $\mathrm{Ext}(A)$ and ∂A.



$q \in \mathbb{R} \setminus \mathbb{Q}$ $\qquad\qquad q = r - \frac{\sqrt{2}}{n}$

$\mathrm{Int}(A) = \emptyset$ because there is no $x \in \mathbb{R}$ so that $(x-\epsilon, x+\epsilon) \subseteq A = \mathbb{Q}$ for some $\epsilon > 0$.

$\mathrm{Ext}(A) = \emptyset$ because there is no $x \in \mathbb{R}$ so that $(x-\epsilon, x+\epsilon) \subseteq \mathbb{R} \setminus \mathbb{Q}$, for some $\epsilon > 0$.

Hence, $\partial \mathbb{Q} = \partial A = \mathbb{R} \setminus (\mathrm{Int}(A) \cup \mathrm{Ext}(A)) = \mathbb{R}$.

Proposition: Let A be a subset of a metric space X. Then the following are true:

a) $x \in \mathrm{Int}(A)$ if and only if there is some $r > 0$ so that $B(x, r) \subseteq A$;

b) $x \in \mathrm{Ext}(A)$ if and only if there is some $r > 0$ so that $B(x, r) \subseteq X \setminus A$.

c) $x \in \partial A$ if and only if for all $r > 0$ we have $B(x, r) \cap A \neq \emptyset$ and $B(x, r) \cap (X \setminus A) \neq \emptyset$.

Definition: Let $A \subseteq X$ be any subset. The closure of A is the defined to be intersection of all closed subsets of X containing A and is denoted as $\overline{A}$.

$$\overline{A} = \bigcap_{\substack{K \subseteq X \text{ closed} \\ A \subseteq K}} K$$

Hence $A \subseteq \overline{A}$ and $\overline{A}$ is closed. Also note that any closed subset K of X containing A contains $\overline{A}$. Therefore, $\overline{A}$ is the smallest closed subset of X containing A.

Note that $A \cap \mathrm{Ext}(A) = \emptyset$ because $\mathrm{Ext}(A) \subseteq X \setminus A$. Hence, $A \subseteq X \setminus \mathrm{Ext}(A)$. Since $X \setminus \mathrm{Ext}(A)$ is a closed subset and it contains A we must have

$$\overline{A} \subseteq X \setminus \mathrm{Ext}(A) = \mathrm{Int}(A) \cup \partial A.$$

<u>Proposition</u>: $\overline{A} = \mathrm{Int}(A) \cup \partial A$. Moreover, a point $x \in X$ belongs to $\overline{A}$ if and only if $B(x,r) \cap A \neq \emptyset$ for any $r > 0$.

<u>Proof</u>: We've already the inclusion

$$\overline{A} \subseteq \mathrm{Int}(A) \cup \partial A.$$

<u>must prove</u>: $\mathrm{Int}(A) \cup \partial A \subseteq \overline{A}$.

Let C be any closed subset containing A. Then since $A \subseteq C$ we have $\mathrm{Int}(A) \subseteq \mathrm{Int}(C)$. Since C is closed $X \setminus C$ is open. Then $X \setminus C = \mathrm{Ext}(C)$ because $X \setminus C$ is the largest open subset contained in $X \setminus C$. Thus taking complement we get

$$C = \mathrm{Int}(C) \cup \partial C.$$

Now let $x \in \partial A$. $X \setminus C$ is open and it is contained in $X \setminus A$, because $A \subseteq C$. So

$X \setminus C \subseteq \mathrm{Ext}(A)$. Thus $x \notin X \setminus C$ and thus $x \in C$.

So, $\partial A \subseteq C$. Clearly, since $A \subseteq C$ we have

$$\mathrm{Int}(A) \subseteq \mathrm{Int}(C) \subseteq C.$$

Hence, $\mathrm{Int}(A) \cup \partial A \subseteq C$. In other words, any closed subset C of X containing A contains $\mathrm{Int}(A) \cup \partial A$.

Thus $\overline{A} = \bigcap_{\substack{C \subseteq X \text{ closed} \\ A \subseteq C}} C$ contains $\mathrm{Int}(A) \cup \partial A$.

Therefore, $\mathrm{Int}(A) \cup \partial A \subseteq \overline{A}$.
This finishes the proof of the first statement.

For the second statement note the following. If $x \in \overline{A} = \mathrm{Int}(A) \cup \partial A$, then $x \in A$ or $x \in \partial A$. So for any $r > 0$, the intersection

$$B(x, r) \cap A \neq \emptyset.$$

Similarly, if $B(x,r) \cap A \neq \emptyset$ for all $r>0$, then $x \notin \operatorname{Ext}(A)$ because if $x \in \operatorname{Ext}(A)$ then there is some $r>0$ so that $B(x,r) \subseteq X \setminus A$, which implies $B(x,r) \cap A = \emptyset$. Thus, we must have $x \in \operatorname{Int}(A) \cup \partial A$.

This finishes the proof. ▪

__Examples:__ 1) $(\mathbb{R}, |\cdot|)$

$\operatorname{Int}(\mathbb{Z}) = \emptyset$, $\partial \mathbb{Z} = \mathbb{Z} \Rightarrow \overline{\mathbb{Z}} = \operatorname{Int}(\mathbb{Z}) \cup \partial \mathbb{Z}$
$= \mathbb{Z}$.

$\operatorname{Int}(\mathbb{Q}) = \emptyset$, $\partial \mathbb{Q} = \mathbb{R}$, $\overline{\mathbb{Q}} = \operatorname{Int}(\mathbb{Q}) \cup \partial \mathbb{Q}$
$= \mathbb{R}$.

2) (X,d) discrete metric space.

If $A \subseteq X$ any subset A is open and $\operatorname{Int}(A) = A$, $\operatorname{Ext}(A) = X \setminus A$ and $\partial A = \emptyset$. Hence, $\overline{A} = A$.

__Proposition:__ A subset A of a metric space (X,d) is closed if and only if $A = \overline{A}$.

__Proof:__ If A is closed then $\overline{A} \subseteq A$, because $\overline{A}$ is the smallest closed set containing A. Clearly, $A \subseteq \overline{A}$ and thus $A = \overline{A}$.

On the other hand, if $A = \overline{A}$ then A is closed because $\overline{A}$ is a closed set. ▪

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Examples: 1) $(\mathbb{R}, |\cdot|)$ $A = [-2,5) \cup (9,12) \cup \{25\}$

$Int(A) = (-2,5) \cup (9,12)$, largest open set contained in A.

$Ext(A) = (-\infty,-2) \cup (5,9) \cup (12,25) \cup (25,\infty)$, largest open set contained in $\mathbb{R} \setminus A$.

$\partial A = \{-2, 5, 9, 12, 25\}$.

2) $S = [0,1]$, $(B(S), d_{sup})$ the metric space of bounded functions on $[0,1]$. Recall from 1st year Calculus course that any continuous function $f : [0,1] \to \mathbb{R}$ has a maximum and minimum. In particular, f is bounded.

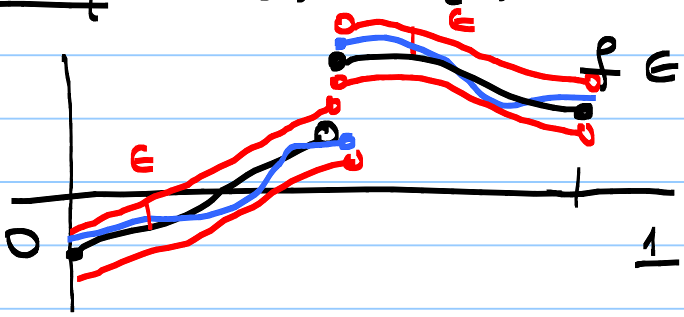
Hence, the set of continuous functions on $[0,1]$ is contained in $B(S)$.

$C([0,1]) = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$,

$C(S) \subseteq B(S)$.

Claim: $C(S)$ is a closed subset.

Proof: $B(S) \setminus C(S)$ is open.



$f \in B(S) \setminus C(S)$.

$g \in B(S) \setminus C(S)$

$B(f, \epsilon) \subseteq B(S) \setminus C(S)$

$\Rightarrow B(S) \setminus C(S)$ is open!

Sequences in Metric Spaces:

Let (X, d) be a metric space. A sequence in X is a function $f: \mathbb{N} \longrightarrow X$. Usually, we denote the value $f(n)$ as f_n and write (f_n) to denote the sequence.

Definition: Let (x_n) be a sequence in (X, d). We say that (x_n) converges to some element $x \in X$ if for any $\epsilon > 0$, there is some $n_0 \in \mathbb{N}$ so that

$$n \geq n_0 \text{ implies } d(x_n, x) < \epsilon.$$

In this case, we write $\lim x_n = x$.

Proposition: Any sequence (x_n) can have at most one limit.

Proof: Assume that $\lim x_n = x$ and $\lim x_n = y$ for some $x, y \in X$.

must show: $x = y$.

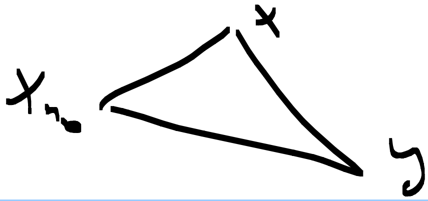
Let $\epsilon > 0$ be given. Then $\epsilon/2 > 0$. So by the definition there is some $n_1 \in \mathbb{N}$ so that

$$n \geq n_1 \implies d(x_n, x) < \epsilon/2.$$

Similarly, since $\lim x_n = y$ there is some $n_2 \in \mathbb{N}$ so that

$$n \geq n_2 \implies d(x_n, y) < \epsilon/2.$$

Let $n_0 = \max\{n_1, n_2\}$. Then $d(x_{n_0}, x) < \epsilon/2$ and



$d(x_{n_0}, y) < \epsilon/2$. Hence, by the triangle inequality we get

$$d(x, y) \leq d(x, x_{n_0}) + d(x_{n_0}, y)$$
$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

$0 \leq d(x,y) < \epsilon$, where $\epsilon > 0$ were arbitrary. Hence, $d(x,y) = 0$ or $x = y$. ∎

Similarly we have the following

Proposition: Let (X, d) be a metric space and (x_n) a sequence in X with $\lim x_n = x$. Then, for any subsequence (x_{k_n}) we have $\lim x_{k_n} = x$.

Proof $1 \leq k_1 < k_2 < \cdots < k_n < k_{n+1} < \cdots \quad k_n \in \mathbb{N}$

Given $\epsilon > 0$. Since $\lim x_n = x$ there is some $n_0 \in \mathbb{N}$ so that $n \geq n_0$ implies $d(x_n, x) < \epsilon$.

Note that since (k_n) is increasing $k_n \geq n$. Hence, if $n \geq n_0$ then $k_n \geq n \geq n_0$ so that

$d(x_{k_n}, x) < \epsilon$. Thus, $\lim x_{k_n} = x$. ∎

Definition: A sequence (x_n) in (X, d) called a Cauchy sequence if for any $\epsilon > 0$ there is some $n_0 \in \mathbb{N}$ so that $m, n \geq n_0$ implies $d(x_n, x_m) < \epsilon$.

Remark: Unlike the metric space $(\mathbb{R}, |\cdot|)$ in a general metric spaces the concepts of being convergent and being Cauchy are not the same.

Proposition: In any metric space any convergent sequence is Cauchy.

Proof is left as an exercise.

Example: Consider the metric space $(\mathbb{Q}, |\cdot|)$

Consider the sequence (r_n) in $\mathbb{Q}$ with

$\lim r_n = \sqrt{2}$. Hence (r_n) is convergent in $(\mathbb{R}, |\cdot|)$. Thus (r_n) is Cauchy in $(\mathbb{R}, |\cdot|)$ and thus it is Cauchy in $(\mathbb{Q}, |\cdot|)$

However, (r_n) is not convergent in the metric space $(\mathbb{Q}, |\cdot|)$. This is because if $\lim r_n = r$ in $(\mathbb{Q}, |\cdot|)$ then $\lim r_n = r$ in $(\mathbb{R}, |\cdot|)$ (since $\mathbb{Q} \subseteq \mathbb{R}$) which would imply that $r = \sqrt{2}$ since limit of a sequence is unique. However, $\sqrt{2} \notin \mathbb{Q}$ and this would be a contradiction.

Definition: A metric space (X, d) is called **complete** if every Cauchy sequence in (X, d) is convergent.

Example: $(\mathbb{R}, |\cdot|)$ is complete and $(\mathbb{Q}, |\cdot|)$ is

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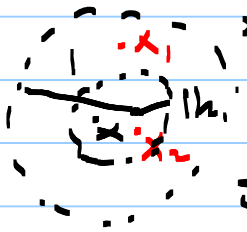
not complete.

Proposition: Let A be a subset of a metric space (X, d). A point $x \in X$ belongs to $\bar{A}$ if and only if there is a sequence (x_n) in A converging to x:
$$\lim x_n = x.$$

Proof: Let $x \in \bar{A}$. We must construct a sequence (x_n) in A with $\lim x_n = x$. Let $\epsilon_1 = 1$. Since $x \in \bar{A}$ the intersection $B(x,1) \cap A \neq \emptyset$.



Choose some $x_1 \in B(x,1) \cap A$. Then for $\epsilon_2 = 1/2$ choose $x_2 \in B(x,1) \cap A \neq \emptyset$. Similarly, choose $x_n \in B(x, 1/n) \cap A \neq \emptyset$.

Now (x_n) is a sequence in X with $x_n \in A$ for all n and $d(x, x_n) < 1/n$. In particular, $\lim x_n = x$. This proves the "$\Rightarrow$" direction.

For the other direction assume that there is a sequence (x_n) in A with $\lim x_n = x$. We must show that $x \in \bar{A}$. Consider any $r > 0$. Since $\lim x_n = x$ there is some $n_0 \in \mathbb{N}$ so that $n \geq n_0$ implies $d(x, x_n) < r$. In particular, $x_{n_0} \in B(x, r) \cap A$ so that $B(x,r) \cap A \neq \emptyset$. Hence, $x \in \bar{A}$.

■

Recall that a metric space (X,d) is called complete if every Cauchy sequence in X is convergent.

Example: Any discrete metric space (X,d) is complete. To see this let (x_n) be a Cauchy sequence in X. Let $\epsilon = 1/2 > 0$. Then there is some $n_0 \in \mathbb{N}$ so that

$$m, n \geq n_0 \text{ implies } d(x_n, x_m) < \epsilon = 1/2.$$

Hence, $d(x_n, x_m) = 0$ if $m, n \geq n_0$. In particular,

$x_n = x_{n_0}$ if $n \geq n_0$. Hence, $\lim x_n = x_{n_0}$, so that (x_n) is convergent. Thus, (X,d) is a complete metric space.

Proposition: Let (X,d) be a complete metric space. A subspace (A,d) is complete if and only if A is a closed subset of X.

Proof: First assume that (A,d) is complete metric space.
must show: A is a closed subset of X.
Let $x \in \overline{A}$. Then there is a sequence (x_n) in A with $\lim x_n = x$. In particular (x_n) is a Cauchy sequence in (X,d). Hence, (x_n) is Cauchy in the subspace (A,d). By the assumption (A,d) is complete and thus (x_n) is convergent in A. In other words, $\lim x_n = y$ for some $y \in A$. Hence, in the metric space (X,d) we have both $\lim x_n = x$ and $\lim x_n = y$. Since a

sequence can have at most one limit we must have $x=y$. Thus
$x = y \in A$.

Now assume that A is a closed subset of X. We must show that the subspace (A,d) is complete. Let (x_n) be a Cauchy sequence in (A,d). In particular, (x_n) is Cauchy in (X,d). Since (X,d) is complete the Cauchy sequence (x_n) must be convergent in (X,d). In other words, $\lim x_n = x$ for some $x \in X$. Since $x_n \in A$ for each n, the limit point $x \in \bar{A}$. Finally, since A is closed, $x \in A = \bar{A}$. Hence, (A,d) is complete. ∎

Now let S be a non empty set and consider the metric space of bounded real valued functions on S, $\mathcal{B}(S)$, equipped with the supremum metric.

$\mathcal{B}(S) = \{ f : S \to \mathbb{R} \mid f \text{ is bounded} \}$

$f \in \mathcal{B}(S)$, then there is some $M > 0$ so that

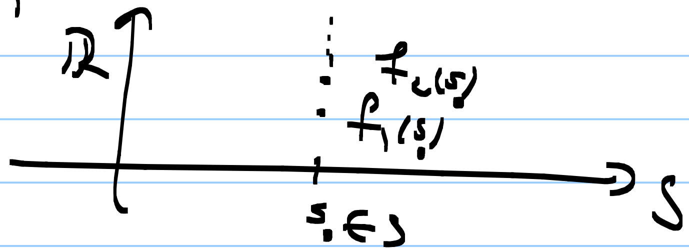
$|f(s)| \leq M$ for all $s \in S$.

$f, g \in \mathcal{B}(S)$, $d_{sup}(f,g) = \sup\{ |f(s) - g(s)| \mid s \in S \}$.

Theorem The metric space $(\mathcal{B}(S), d_{sup})$ is complete.

<u>Proof</u>: Take any Cauchy sequence (f_n) in $B(S)$.

We must construct an element $f \in B(S)$ so that $\lim f_n = f$.



Fix any $s_0 \in S$. Take any $\epsilon > 0$. Since (f_n) is Cauchy in $(B(S), d_{sup})$ there is some $n_0 \in \mathbb{N}$ so that $m,n \geq n_0$ implies

$$d_{sup}(f_n, f_m) < \epsilon.$$

$$\sup\{|f_n(s) - f_m(s)| \mid s \in S\} < \epsilon.$$

In particular, $|f_n(s_0) - f_m(s_0)| < \epsilon$ for all $m,n \geq n_0$.

Hence, the sequence of real numbers $(f_n(s_0))$ is Cauchy. Since $(\mathbb{R}, |\cdot|)$ is complete the Cauchy sequence $(f_n(s_0))$ must be convergent.

Hence, $\lim_{n\to\infty} f_n(s_0) = f(s_0)$, for some real number $f(s_0)$. In particular, we have a function $f : S \to \mathbb{R}$ so that

$$f(s) = \lim_{n\to\infty} f_n(s), \text{ for each } s \in S.$$

<u>Claim</u>: $f \in B(S)$.

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<u>Proof</u>: Since (f_n) is a Cauchy sequence in $(B(S), d_{sup})$ and $\epsilon = 1 > 0$ there is some $n_0 \in \mathbb{N}$ so that

$$\sup_{s \in S} |f_n(s) - f_m(s)| < \epsilon = 1, \text{ for all } m, n \geq n_0.$$

In particular, $|f_n(s) - f_{n_0}(s)| < 1$, for all $s \in S$ and $n \geq n_0$. Hence,

$$f_{n_0}(s) - 1 \leq f_n(s) \leq f_{n_0}(s) + 1, \text{ for all } n \geq n_0 \text{ and } s \in S.$$

However, f_{n_0} is a bounded function and thus there is some $M > 0$ so that

$$-M \leq f_{n_0}(s) \leq M, \text{ for all } s \in S.$$

In particular, $-M-1 \leq f_n(s) \leq M+1$, for all $s \in S$ and $n \geq n_0$.

Hence, $-M-1 \leq \lim_{n \to \infty} f_n(s) = f(s) \leq M+1$, for all $s \in S$, so that $f : S \to \mathbb{R}$ is a bounded function.

To finish the proof we must show $\lim f_n = f$ in the metric space $(B(S), d_{sup})$:

<u>Given $\epsilon > 0$</u>. Since (f_n) is Cauchy there is <u>some $n_0 \in \mathbb{N}$</u> so that $m, n \geq n_0$ implies $|f_m(s) - f_n(s)| < \epsilon/3$ for all $s \in S$.

Fix any $s_0 \in S$. Since $\lim f_n(s_0) = f(s_0)$ there is some $n_0(s_0) \in \mathbb{N}$ so that
$$n \geq n_0(s_0) \Rightarrow |f_n(s_0) - f(s_0)| < \epsilon/3.$$

Let $m_0 = \max\{n_0, n_0(s_0)\}$.

Since $m_0 \geq n_0(s_0)$ we have $|f_{m_0}(s_0) - f(s_0)| < \epsilon/3$.

Now if $n \geq n_0$ then
$$|f(s_0) - f_n(s_0)| \leq |f(s_0) - f_{m_0}(s_0)| + |f_{m_0}(s_0) - f_n(s_0)|$$
$$< \epsilon/3 + \epsilon/3 = \frac{2\epsilon}{3}.$$

So, for any $s_0 \in S$ we have
$$|f(s_0) - f_n(s_0)| < 2\epsilon/3 \quad \text{if} \quad n \geq n_0.$$

Hence, $d_{sup}(f, f_n) \leq 2\epsilon/3$ if $n \geq n_0$.

So $\underline{d_{sup}(f, f_n) < \epsilon \text{ if } n \geq n_0}$.

Thus, $\lim f_n = f$ in $(B(S), d_{sup})$. ∎

Recall that a function $f: \mathbb{R} \to \mathbb{R}$ is called continuous at a point $x_0 \in \mathbb{R}$ if for any $\epsilon > 0$ there is some $\delta > 0$ so that
$$|x - x_0| < \delta \text{ implies } |f(x) - f(x_0)| < \epsilon.$$

If f is continuous at all points we simply say that f is a continuous function.

If $I \subseteq \mathbb{R}$ is a closed and bounded interval we know that each continuous function $f: I \to \mathbb{R}$ has a maximum and a minimum. In particular, f is bounded.

Hence, if $C(I) = \{ f: I \to \mathbb{R} \mid f \text{ is continuous} \}$ is the set of all continuous functions on I then $C(I) \subseteq B(I)$. ($I = [a,b] \subseteq \mathbb{R}$).

<u>Theorem:</u> $C(I)$ is a closed subset of the complete metric space $(B(I), d_{sup})$ and thus $(C(I), d_{sup})$ is also a complete metric space.

<u>Proof:</u> Take any convergent sequence (f_n) in $B(I)$ with $f_n \in C(I)$, for all n. We must show $\lim f_n \in C(I)$.

Set $f = \lim f_n$ (in $(C(I), d_{sup})$).
<u>must show:</u> f is continuous on I.

Pick any $x_0 \in I$. Given $\epsilon > 0$ choose some $n_0 \in \mathbb{N}$ so that if $n \geq n_0$ then

$$d_{sup}(f, f_n) < \epsilon/3.$$

Hence, $|f(x) - f_n(x)| < \epsilon/3$ for all $x \in I$ and $n \geq n_0$.

Consider the function $f_{n_0}: I \to \mathbb{R}$, which is continuous on I by the assumption. In particular, f_{n_0} is continuous at x_0 and thus there is some $\delta > 0$ so that

$$|x - x_0| < \delta \Rightarrow |f_{n_0}(x) - f_{n_0}(x_0)| < \epsilon/3.$$

Then if $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| = |f(x) - f_{n_0}(x) + f_{n_0}(x) - f_{n_0}(x_0) + f_{n_0}(x_0) - f(x_0)|$$

$$\leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)|$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

Hence, f is continuous at x_0. Since $x_0 \in I$ was arbitrary we conclude that f is continuous on I. This finishes the proof. ∎

Hence, for a closed and bounded interval $I = [a, b]$ the space of continuous functions $(C(I), d_{sup})$ is a complete metric space.

Continuity of Functions:

Recall that a function $f: \mathbb{R} \to \mathbb{R}$ is continuous at a point $x_0 \in \mathbb{R}$ if for any given $\epsilon > 0$ there is some $\delta > 0$ so that

$$|x - x_0| < \delta \text{ implies } |f(x) - f(x_0)| < \epsilon.$$

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Hence, if $x \in B(x_0, \delta)$ then $f(x) \in B(f(x_0), \epsilon)$.

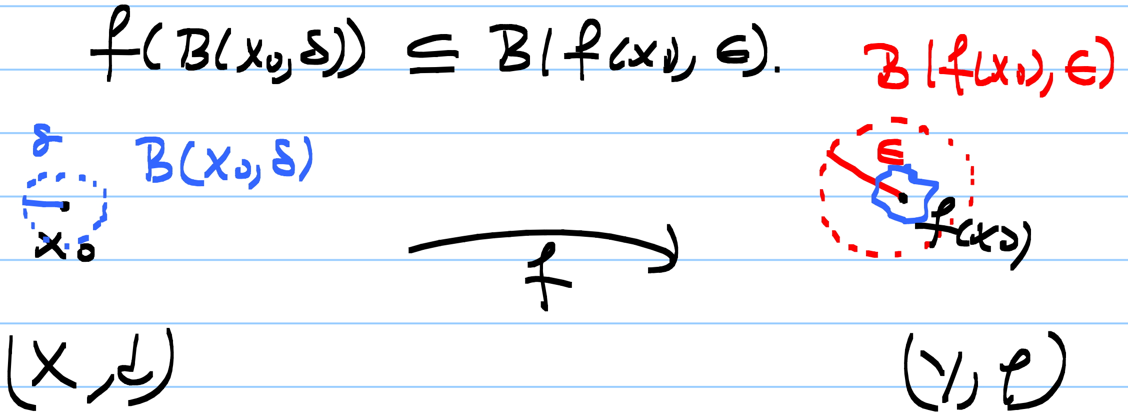
Or, equivalently, f is continuous at x_0 provided that for a given $\epsilon > 0$ there is some $\delta > 0$ so that

$$f(B(x_0,\delta)) \subseteq B(f(x_0), \epsilon).$$

This formulation suggests the following definition.

<u>Definition</u>: Let $f : (X,d) \longrightarrow (Y,\rho)$ be a function between two metric spaces. Let $x_0 \in X$ be any point. We say that f is continuous at x_0, if for any $\epsilon > 0$ there is some $\delta > 0$ so that

$$f(B(x_0,\delta)) \subseteq B(f(x_0), \epsilon).$$



Or, equivalently $d(x,x_0) < \delta$ implies $\rho(f(x), f(x_0)) < \epsilon$.

The function $f : (X,d) \to (Y,\rho)$ will be called continuous if it is continuous at all points of X.

<u>Proposition</u> A function $f : X \to Y$ is continuous at $x \in X$ if and only if for every sequence (x_n) with $\lim x_n = x$ we have $\lim f(x_n) = f(x)$.

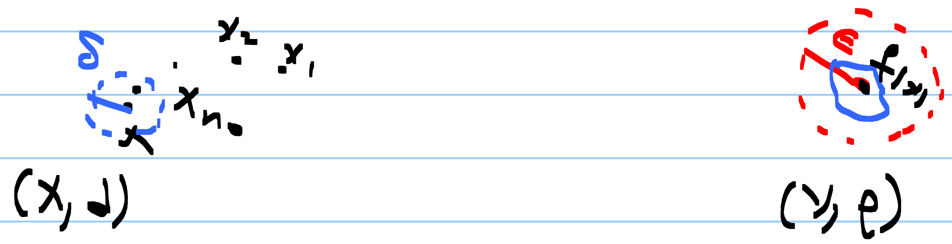
<u>Proof</u>: ($\Rightarrow$) Assume that f is continuous at some

$x \in X$ and let (x_n) be a sequence with $\lim x_n = x$.

must prove: $\lim f(x_n) = f(x)$.

Given $\epsilon > 0$. Since f is continuous at x, there is some $\delta > 0$ so that

$$d(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \epsilon.$$



Since $\lim x_n = x$ and $\delta > 0$ there is some $n_0 \in \mathbb{N}$ so that $n \geq n_0 \Rightarrow d(x, x_n) < \delta$. In particular, $\rho(f(x), f(x_n)) < \epsilon$, provided that $n \geq n_0$.

Hence, $\lim f(x_n) = f(x)$ in (Y, ρ).

($\Leftarrow$) Assume now that f is not continuous at x. Hence, we must construct a sequence (x_n) in (X, d) so that

$$\lim x_n = x \quad \text{but} \quad \lim f(x_n) \neq f(x).$$

Since we are given that f is not continuous at x there is some $\epsilon > 0$ so that for any $\delta > 0$ there is some $y \in X$ with

$$d(x,y) < \delta \quad \text{but} \quad \rho(f(x), f(y)) \geq \epsilon.$$

Now for any $n \in \mathbb{N}$ choose x_n with $d(x, x_n) < 1/n$ with $\rho(f(x), f(x_n)) \geq \epsilon$.

Hence, $\lim x_n = x$ and $\lim f(x_n) \neq f(x)$
This finishes the proof. ∎

<u>Remark:</u> This is called the formulation of continuity in terms of sequences.

<u>Proposition:</u> A function $f: X \to Y$ is continuous on X if and only if for every open subset O of Y the inverse image $f^{-1}(O)$ is open in X.

<u>Proof:</u> $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$.

($\Longrightarrow$) Assume that $f: X \to Y$ is continuous at all points. Take any open subset O of Y.

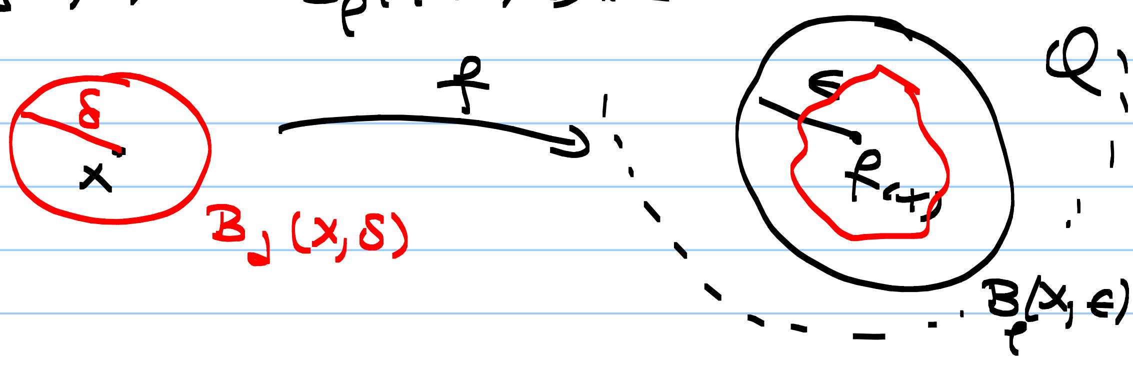
<u>must show:</u> $f^{-1}(O) \subseteq X$ is an open subset.

Take any $x \in f^{-1}(O)$. Then $f(x) \in O$, which is open. Thus there is some $\epsilon > 0$ so that

$$B_\rho(f(x), \epsilon) \subseteq O \text{ in the metric space } (Y, \rho).$$

Since, f is continuous at x there is some $\delta > 0$ so that

$$f(B_d(x, \delta)) \subseteq B_\rho(f(x), \epsilon) \subseteq O$$



In particular, $B_d(x,\delta) \subseteq f^{-1}(Q)$. Hence, $f^{-1}(Q)$ is an open subset.

($\Leftarrow$) For the other direction take any $x \in X$ and $\epsilon > 0$. The ball $B_\rho(f(x),\epsilon)$ is an open subset of Y. So by the assumption the subset

$$U = f^{-1}(B_\rho(f(x),\epsilon)) \text{ is open in } (X,d).$$

Since $f(x) \in B_\rho(f(x),\epsilon)$, $x \in U = f^{-1}(B_\rho(f(x),\epsilon))$.

Finally, since U is open there is some $\delta > 0$ so that $B_d(x,\delta) \subseteq U$.

Hence, $f(B_d(x,\delta)) \subseteq f(U) \subseteq B_\rho(f(x),\epsilon)$.

Thus f is continuous at x. Since $x \in X$ were arbitrary this finishes the proof. ∎

This immediately gives the following consequence:

Proposition: Let $f: X \to Y$ be a function between metric spaces. Then the following statements are equivalent.

1) f is continuous on X.
2) For any open subset $Q \subseteq Y$ the inverse image $f^{-1}(Q)$ is open in X.
3) For any $x \in X$ and sequence (x_n) in X with $\lim x_n = x$ we have $\lim f(x_n) = f(x)$

4) For any closed subset A of Y the inverse image $f^{-1}(A)$ is closed in X.

Proof: We've already proved that the first three statements are equivalent. To finish the prove we'll show that (3) is equivalent to (4).

(3) ⟹ (4): Let $A \subseteq Y$ be any closed subset.

$X \setminus f^{-1}(A) = f^{-1}(Y) \setminus f^{-1}(A) = f^{-1}(Y \setminus A)$, where

$Y \setminus A$ is an open subset of Y. Now by assumption $f^{-1}(Y \setminus A)$ is open. Hence, $f^{-1}(A)$ is closed in X.

(4) ⟹ (3) is left as an exercise. ∎

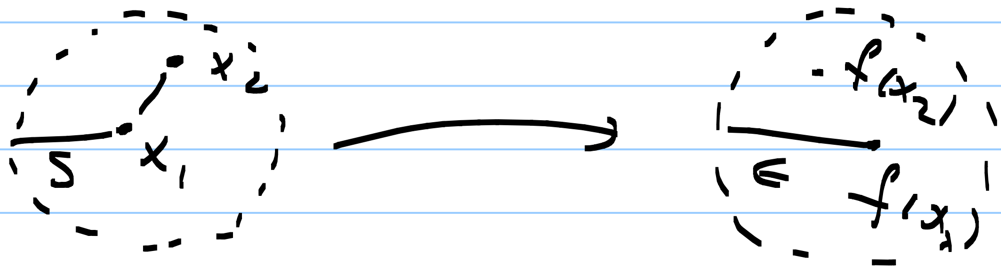
Definition: A function $f: X \longrightarrow Y$ is called uniformly continuous if for any $\epsilon > 0$ there is some $\delta > 0$ so that

$d(x_1, x_2) < \delta$ implies $\rho(f(x_1), f(x_2)) < \epsilon$.

Or equivalently, $f(B_d(x_1, \delta)) \subseteq B_\rho(f(x_1), \epsilon)$.

Here, δ is a function of ϵ only it can be choosen independently from x

$\epsilon > 0$ given, find some $\delta > 0$



This is called "uniform continuity since the $\delta > 0$ works for all $x \in X$, once $\epsilon > 0$ is given.

Example: Consider the function $f: (\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$

given by $f(x) = x^2$, $x \in \mathbb{R}$.

Since $f(x) = g(x) \cdot g(x)$, where $g(x) = x$, $\forall x \in X$ and g is continuous, $f(x)$ is continuous.

For the sake of completeness let's prove that f is continuous. Take any $x_0 \in \mathbb{R}$ and $\epsilon > 0$. Choose $\delta = \min\{1, \frac{\epsilon}{2|x_0|+1}\}$. (Note that our choice of δ depends on ϵ and x_0!)

Now, if $|x - x_0| < \delta$ then $|x - x_0| < 1$. Hence

$$|x| - |x_0| \leq |x - x_0| < 1 \Rightarrow |x| < 1 + |x_0|.$$

So, $|x + x_0| \leq |x| + |x_0| \leq 1 + 2|x_0|$.

Now, if $|x - x_0| < \delta$ then

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |x - x_0|\,|x + x_0|$$

$$\Rightarrow |f(x)-f(x_0)| < \delta \cdot (1+2|x_0|) \le \frac{\epsilon}{1+2|x_0|}(1+2|x_0|)$$

$$\Rightarrow \underline{|f(x)-f(x_0)|} < \underline{\epsilon}.$$

Hence, f is continuous at x_0.

Note that $\delta = \delta(x_0, \epsilon) = \min\left\{1, \frac{\epsilon}{1+2|x_0|}\right\}$.

<u>Claim</u>: f is <u>not</u> uniformly continuous on $\mathbb{R}$.

<u>Proof</u>: <u>Must prove</u>: There is some $\epsilon > 0$ so that for any $\delta > 0$ there is some $x_1, x_2 \in X$ so that

$$|x_1 - x_2| < \delta \quad \text{but} \quad |f(x_1) - f(x_2)| \ge \epsilon.$$

$f(x) = x^2$

Take $\epsilon = 1$.

δ

Let any $\delta > 0$ be given. Choose $x_1 = \frac{1}{\delta}$ and $x_2 = x_1 + \frac{\delta}{2}$. Then $|x_1 - x_2| = \frac{\delta}{2} < \delta$.

However, $|f(x_1) - f(x_2)| = |x_1^2 - x_2^2|$

$$\Rightarrow |f(x_1)-f(x_2)| = |x_1-x_2||x_1+x_2|$$
$$= \frac{\delta}{2} \cdot \left|\frac{1}{\delta}+\frac{1}{\delta}+\frac{\delta}{2}\right|$$
$$> \frac{\delta}{2} \cdot \left(\frac{1}{\delta}+\frac{1}{\delta}\right) = 1 = \epsilon.$$

This finishes the proof that f is not uniformly continuous.

Examples of Uniformly continuous functions:

1) Any linear function $f: \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

$f(x) = ax+b$, $a, b \in \mathbb{R}$, $a \neq 0$.

$$|f(x_1)-f(x_2)| = |(ax_1+b)-(ax_2+b)|$$
$$= |ax_1 - ax_2|$$
$$= |a|\,|x_1-x_2|.$$

If $\epsilon > 0$ is given, then choose $\delta = \frac{\epsilon}{|a|}$.

Then if $|x_1-x_2| < \delta$ then $|f(x_1)-f(x_2)| < |a| \cdot \delta = \epsilon$.

2) Indeed, any continuous function $f: I \to \mathbb{R}$, where $I = [a, b]$ is a closed and bounded interval, is uniformly continuous.

3) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$.

We know that $|\sin x - \sin y| \leq |x-y|$.

Hence, given $\epsilon > 0$ just choose $\delta = \epsilon$. Then if $|x_1 - y| < \delta$ then $|f(x) - f(y)| = |\sin x - \sin y| < \epsilon$.

Hence, f is uniformly continuous

4) Let (X, d) be a discrete metric space. Then any subset U of X is open. In particular, for any function

$$f: (X, d) \longrightarrow (Y, \rho)$$

the inverse image $f^{-1}(Q) \subseteq X$ will be open for any open Q in Y. In particular, f is continuous.

Indeed, any function $f: (X, d) \to (Y, \rho)$ is uniformly continuous.
Let $\epsilon > 0$ be given. Just take $\delta = 1/2 > 0$.

Then, if $d(x_1, x_2) < \delta = 1/2$ then $d(x_1, x_2) = 0$ and hence, $x_1 = x_2$. However, in this case

$$\rho(f(x_1), f(x_2)) = \rho(f(x_1), f(x_1)) = 0 < \epsilon.$$

Hence, f is uniformly continuous.

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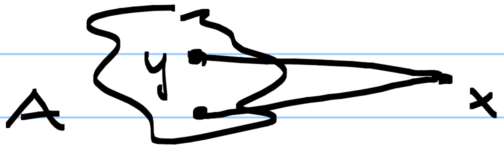
Let $A \subseteq X$ be any subset. Now consider the function

$\theta_A : X \longrightarrow \mathbb{R}$ defined by

$\theta_A(x) = \inf\{d(x,y) \mid y \in A\}$, the distance from A to x.



For any $x, y \in X$ and $z \in A$ we have

$$\theta_A(x) = \inf\{d(x,w) \mid w \in A\} \leq d(x,z) \leq d(x,y) + d(y,z)$$

$$\Rightarrow \theta_A(x) \leq d(x,y) + d(y,z)$$

$$\Rightarrow \theta_A(x) - d(x,y) \leq d(y,z) \text{ for all } z \in A.$$

Hence $\theta_A(x) - d(x,y)$ is a lower bound for $\{d(y,z) \mid z \in A\}$. Thus

$$\theta_A(y) = \inf\{d(y,z) \mid z \in A\} \geq \theta_A(x) - d(x,y)$$

$$\Rightarrow \theta_A(x) - \theta_A(y) \leq d(x,y), \text{ for any } x, y \in X.$$

Interchanging x and y we get

$$\theta_A(y) - \theta_A(x) \leq d(y,x) = d(x,y).$$

Hence, $d(x,y) \geq |\theta_A(x) - \theta_A(y)|$

It follows that the function

$\theta_A : X \longrightarrow \mathbb{R}$ is uniformly continuous (just take $\delta = \epsilon$)

<u>Some applications</u>:

1) The set $\{x \in X \mid \theta_A(x) = 0\} = \theta_A^{-1}(\{0\})$, where $\{0\}$ is closed in $(\mathbb{R}, |\cdot|)$. Hence $\{x \in X \mid \theta_A(x) = 0\}$ is a closed set containing A.

$(x \in A,\ \theta_A(x) = \inf\{d(x,y) \mid y \in A\} = 0)$

Hence, $\overline{A} \subseteq \{x \in X \mid \theta_A(x) = 0\}$.

Indeed, if $x \notin \overline{A}$ then there is some $r > 0$ so that $B(x,r) \subset X \setminus \overline{A}$. In particular, for any $y \in A$, $y \notin B(x,r)$ so that

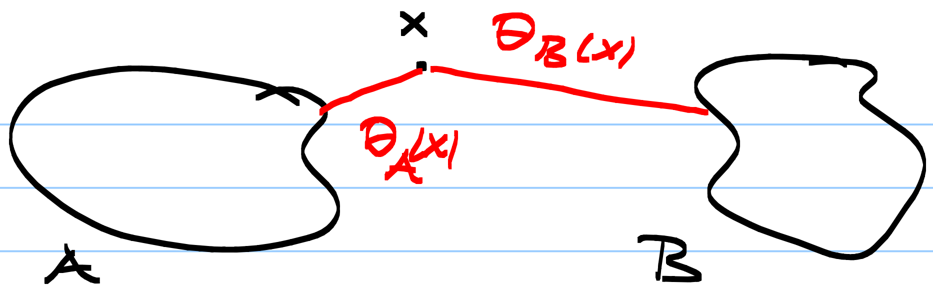
$d(x,y) \geq r$. Hence, $\theta_A(x) = \inf\{d(x,y) \mid y \in A\} \geq r$.

Hence, $\theta_A(x) > 0$ if $x \notin \overline{A}$. Thus

$\theta_A^{-1}(\{0\}) = \overline{A}$.

2) Let $A, B \subseteq X$ be closed and disjoint subsets and consider the function

$g : X \longrightarrow \mathbb{R},\quad g(x) \doteq \theta_A(x) - \theta_B(x)$



Now, for any $x, y \in X$, we have

$$|g(x) - g(y)| = |(\theta_A(x) - \theta_B(x)) - (\theta_A(y) - \theta_B(y))|$$
$$= |(\theta_A(x) - \theta_A(y)) + (\theta_B(y) - \theta_B(x))|$$
$$\leq |\theta_A(x) - \theta_A(y)| + |\theta_B(y) - \theta_B(x)|$$
$$\leq d(x,y) + d(x,y) = 2d(x,y).$$

In particular, $g : X \to \mathbb{R}$ is uniformly continuous (just take $\delta = \epsilon/2$).

If $x \in A = \overline{A}$, $x \notin B = \overline{B}$ then $\theta_A(x) = 0$ but $\theta_B(x) > 0$. Hence, $g(x) = \theta_A(x) - \theta_B(x) < 0$.

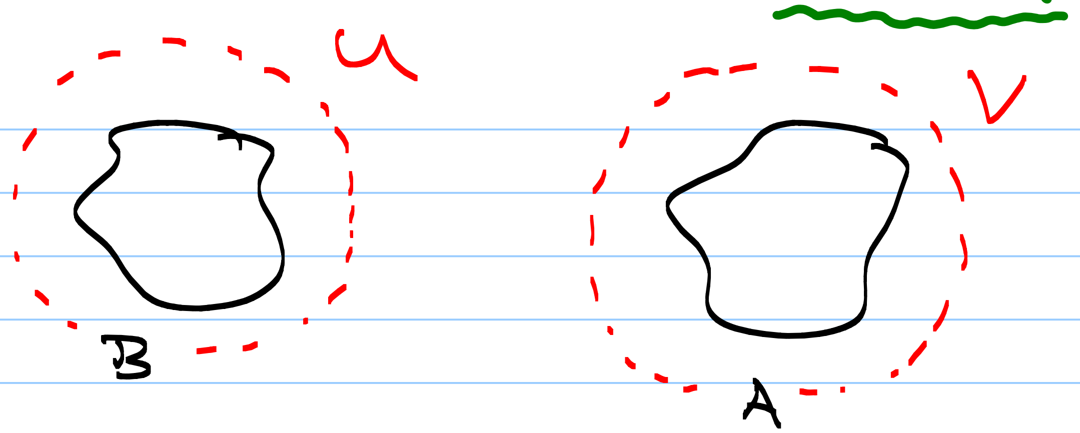
Similarly, if $x \in B = \overline{B}$, then $x \notin A = \overline{A}$ and thus $g(x) = \theta_A(x) - \theta_B(x) > 0$.

$g(x) > 0$ if $x \in B$ and $g(x) < 0$ if $x \in A$.

Let $U = g^{-1}((0, \infty))$ which is an open subset containing B. Similarly, $V = g^{-1}((-\infty, 0))$ is an open subset containing A.

$U \cap V = \phi$ because since $(0,\infty) \cap (-\infty, 0) = \phi$.

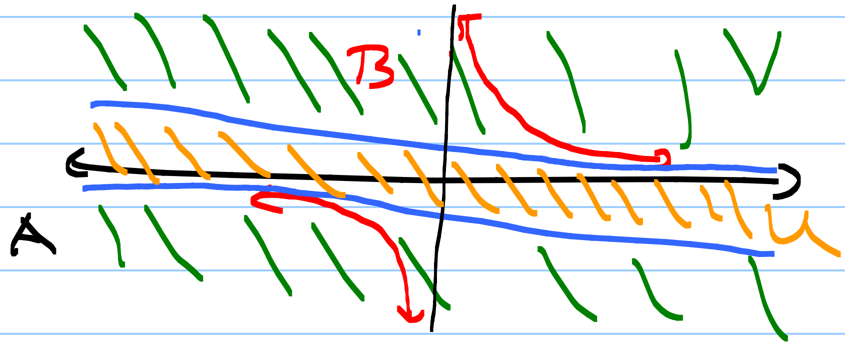
Hence, $B \subseteq U$, $A \subseteq V$ and $U \cap V = \phi$.



Example: $(X,d) = (\mathbb{R}^2, d_2)$

$A = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}$ the x-axis

$B = \{(x,y) \in \mathbb{R}^2 \mid xy = 1\}$



A, B closed and $A \cap B = \emptyset$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = y$, which is clearly continuous. Then

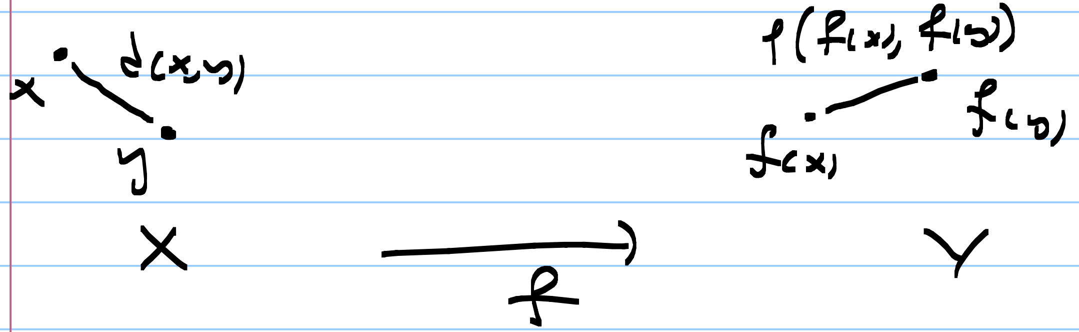
$A = f^{-1}(\{0\})$ and thus A is closed.

Similarly, if $h : \mathbb{R}^2 \to \mathbb{R}$, $h(x,y) = xy$, then $B = h^{-1}(\{1\})$, which is closed since h is continuous and $\{1\}$ is closed.

Proposition: Let A and B be closed and disjoint subsets of a metric space (X,d). Then there are open disjoint subsets U and V of X so that $A \subseteq V$ and $B \subseteq U$.

Definition: A function between two metric spaces $f: X \to Y$ is called a homeomorphism if f is a continuous bijection with continuous inverse $f^{-1}: Y \to X$.

Definition: A function $f:(X,d) \to (Y,\rho)$ is called an isometry if $d(x,y) = \rho(f(x), f(y))$ for all $x,y \in X$.



Remark: Taking $\delta = \epsilon$ we see that an isometry $f: X \to Y$ is uniformly continuous.

Moreover, if $f: X \to Y$ is an isometry and a bijection then its inverse
$f^{-1}: Y \to X$ is also an isometry:

[$y_1, y_2 \in Y \Rightarrow y_1 = f(x_1)$ and $y_2 = f(x_2)$ for some $x_1, x_2 \in X$.
Then $\rho(y_1, y_2) = d(x_1, x_2) = d(f^{-1}(y_1), f^{-1}(y_2))$
$\Rightarrow f^{-1}$ is an isometry]

As a conclusion, a isometry which is also a bijection is a homeomorphism.

Definition: A subset A of a metric space (X,d) is called dense if $\overline{A}=X$.

Examples: 1) $\mathbb{Q}$ is dense in $(\mathbb{R},|\cdot|)$.

2) $\mathbb{R}\setminus\{0\}$ is dense in $(\mathbb{R},|\cdot|)$.

Proposition: A continuous function $f:X\to Y$ is determined by its values on a dense subset of X.

Proof: Let $f:X\to Y$ and $g:X\to Y$ be two continuous functions and $A\subseteq X$ a dense subset. Assume that $f(x)=g(x)$, whenever $x\in A$. Then we must show

$$f(x)=g(x) \text{ for all } x\in X.$$

Let $x\in X=\overline{A}$. Then there is a sequence (x_n) in A with $\lim x_n=x$. Now, since f and g are continuous we get

$$f(x)=\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty} g(x_n)=g(x),$$ which

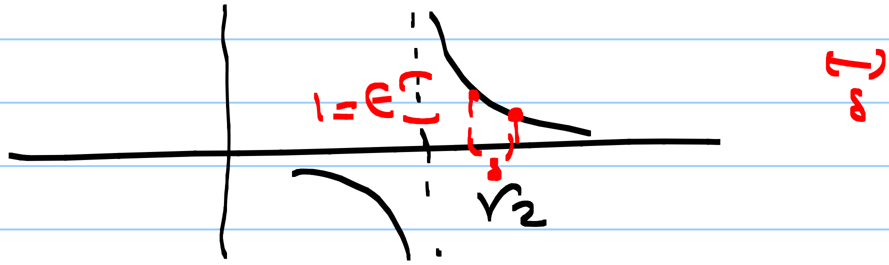
finishes the proof. $\blacksquare$

Proposition: Let (Y,ρ) be a complete metric space, A a dense subset in X and $f_0:A\to Y$ a uniformly continuous function. Then there is a unique uniformly continuous function $f:X\to Y$ such that $f(x)=f_0(x)$, for all $x\in A$. Further, if f_0 is an isometry then so is f.

<u>Example</u> $X=\mathbb{R}$, $Y=\mathbb{R}$, $A=\mathbb{R}\setminus\{\sqrt{2}\}$

$f_0 : A \to Y$, $f_0 : A \to \mathbb{R}$, $f_0(x) = \dfrac{1}{x-\sqrt{2}}$

f_0 is clearly continuous. However, this function cannot be extended to $\mathbb{R}$ as a continuous function. This is <u>not</u> a contradiction because f_0 is <u>not</u> uniformly continuous.



2) $X=\mathbb{R}$, $A=\mathbb{Q}$, $Y=\mathbb{Q}$, $f_0 : A \to Y$, $f_0 : \mathbb{Q} \to \mathbb{Q}$

by $f_0(x) = x$. Clearly, f_0 is uniformly continuous. However, f_0 cannot be extended to $\mathbb{R}$, because $Y=\mathbb{Q}$ is <u>not</u> a complete metric space.

<u>Proof</u>: 1) $f_0 : A \to Y$ is uniformly continuous.

2) (Y, ρ) is a complete metric space.

3) $\overline{A} = X$ (i.e., A is dense)

<u>Existence of the extension</u> <u>$f : X \to Y$</u>:

<u>Claim</u>: Whenever (x_n) is a Cauchy sequence in A then $(f_0(x_n))$ is a Cauchy sequence in Y.

Proof. Let $\epsilon > 0$ be given. Since $f_0 : A \to Y$ is uniformly continuous there is some $\delta > 0$ so that $d(x,y) < \delta$ implies $\rho(f_0(x), f_0(y)) < \epsilon$.

Now since (x_n) is a Cauchy sequence in A there is some $n_0 \in \mathbb{N}$ so that
$m, n \geq n_0 \Rightarrow d(x_n, x_m) < \delta$. Thus,

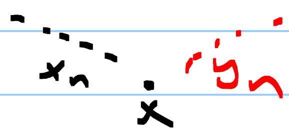
$\rho(f_0(x_n), f_0(x_m)) < \epsilon$, whenever $m, n \geq n_0$.

Hence, $(f_0(x_n))$ is a Cauchy sequence in (Y, ρ).

Definition of $f : X \to Y$: Take any $x \in X$. Since $X = \overline{A}$ there is a sequence (x_n) in A with $\lim x_n = x$. In particular (x_n) is a Cauchy sequence in (A, d). Hence, $(f_0(x_n))$ is a Cauchy sequence in (Y, ρ). Since (Y, ρ) is a complete metric space the Cauchy sequence $(f_0(x_n))$ must be convergent. Now we define $f(x)$ as

$$f(x) := \lim_{n \to \infty} f_0(x_n).$$

To show that f is well defined, we must prove the following: If $(y_n) \in A$ is another sequence with $\lim y_n = x$, then



$$\lim_{n \to \infty} f_0(y_n) = \lim_{n \to \infty} f_0(x_n).$$

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Aim: To show if $L_1 = \lim f_0(x_n)$ and $L_2 = \lim f_0(y_n)$ then $L_1 = L_2$.

Let $\epsilon > 0$ be given. Choose $\delta > 0$ so that whenever $x, y \in A$ with $d(x,y) < \delta$ then

$\rho(f(x), f(y)) = \rho(f_0(x), f_0(y)) < \epsilon/3$. This δ possible since $f_0 : A \to Y$ is uniformly continuous.

Since $\lim x_n = x$ and $\lim y_n = x$ we may choose $n_1 \in \mathbb{N}$ so that

$n \geq n_1$ implies $d(x_n, x) < \delta/2$ and $d(y_n, x) < \delta/2$.

Hence, $d(x_n, y_n) \leq d(x_n, x) + d(x, y_n) < \delta/2 + \delta/2 = \delta$

Moreover, since $\lim f(x_n) = L_1$ and $\lim f(y_n) = L_2$ there is some $n_2 \in \mathbb{N}$ so that

$n \geq n_2$ implies $\rho(f(x_n), L_1) < \epsilon/3$ and

$\rho(f(y_n), L_2) < \epsilon/3$.

Now let $n_0 = \max\{n_1, n_2\}$. So if $n \geq n_0$ then

$$0 \leq \rho(L_1, L_2) \leq \rho(L_1, f(x_n)) + \rho(f(x_n), f(y_n)) + \rho(f(y_n), L_2)$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

Thus $0 \leq \rho(L_1, L_2) < \epsilon$. Hence, by the ϵ-lemma $\rho(L_1, L_2) = 0$ and thus $L_1 = L_2$.

Therefore, $f : X \to Y$ is a well defined function: $x \in X$, choose (x_n) in A with $\lim x_n = x$. Then $f(x) = \lim f_0(x_n)$.

$f: X \to Y$ is uniformly continuous:

Let $\epsilon > 0$ be given. Since $f_0 : A \to Y$ is uniformly continuous there is some $\delta_0 > 0$ so that

$d(x,y) < \delta_0$ implies $\rho(f(x), f(y)) < \epsilon/3$.

$(x \in A, (x_n) = (x), f(x) = \lim f_0(x_n) = \lim f_0(x) = f_0(x))$

Choose sequences (x_n) and (y_n) in A with $\lim x_n = x$ and $\lim y_n = y$. We know that $\lim f(x_n) = f(x)$ and $\lim f(y_n) = f(x)$. Hence, there is some $n_0 \in \mathbb{N}$ so that

$n \geq n_0 \Rightarrow \rho(f(x), f(x_n)) < \epsilon/3$, $\rho(f(y), f(y_n)) < \epsilon/3$,

$d(x_n, x) < \delta_0/3$ and $d(y_n, y) < \delta_0/3$. In particular,

If $x, y \in X$ are with $d(x,y) < \delta_0/3$ then

$d(x_{n_0}, y_{n_0}) \leq d(x_{n_0}, x) + d(x,y) + d(y_{n_0}, y) < \frac{\delta_0}{3} + \frac{\delta_0}{3} + \frac{\delta_0}{3} = \delta_0$.

Choose $\delta = \frac{\delta_0}{3}$. Then if $d(x,y) < \delta$, then

$\rho(f(x), f(y)) \leq \rho(f(x), f(x_{n_0})) + \rho(f(x_{n_0}), f(y_{n_0})) +$
$+ \rho(f(y_{n_0}), f(y))$

$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$.

Hence, f is uniformly continuous.

Uniqueness of f: Assume that $\tilde{f}: X \to Y$ is another continuous extension of $f_0: A \to Y$. Then we must prove

$$\tilde{f}(x) = f(x) \text{ for any } x \in X.$$

Take any $x \in X$ and choose some (x_n) in A with $\lim x_n = x$.

Then $\tilde{f}(x) = \lim \tilde{f}(x_n) = \lim f_0(x_n) = \lim f(x_n) = f(x)$.

Hence, the extension f of f_0 to X is unique.

Finally, we'll prove that $f: X \to Y$ is an isometry provided that $f_0: A \to Y$ is an isometry.
Let $x, y \in X$ and choose sequence (x_n) and (y_n) in A with $\lim x_n = x$ and $\lim y_n = y$.

Note that

$$d(x,y) \leq d(x,x_n) + d(x_n,y_n) + d(y_n,y) \Rightarrow$$

$$d(x,y) - d(x_n,y_n) \leq d(x,x_n) + d(y_n,y).$$

Replace x with x_n and y with y_n to get

$$d(x_n,y_n) - d(x,y) \leq d(x,x_n) + d(y_n,y).$$

Hence, $|d(x,y) - d(x_n,y_n)| \leq d(x,x_n) + d(y_n,y)$.

Since $\lim x_n = x$ and $\lim y_n = y$ given $\varepsilon > 0$

there is some $n_0 \in \mathbb{N}$ so that $n \geq n_0 \Rightarrow$ $d(x, x_n) < \epsilon/2$ and $d(y, y_n) < \epsilon/2$. Hence

$|d(x,y) - d(x_n, y_n)| < \epsilon$.

Thus, $\lim d(x_n, y_n) = d(x,y)$.

Similarly, $f(x_n) \to f(x)$ and $f(y_n) \to f(y)$.
So, the same arguments imply that

$$\lim \rho(f(x_n), f(y_n)) = \rho(f(x), f(y)).$$

Hence, $\rho(f(x), f(y)) = \lim \rho(f(x_n), f(y_n))$, $x_n, y_n \in A$

$$= \lim \rho(f_0(x_n), f_0(y_n))$$

$$= \lim d(x_n, y_n) \quad (f_0 \text{ is an isometry})$$

$$= d(x,y).$$

Hence, f is an isometry.

This finishes the proof. ▬

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Hölder's and Minkowski's Inequalities:

$f:[a,b]\to\mathbb{R}$, $f(x)=x^{1-\alpha}$, where $0<\alpha<1$ is fixed number, $0<a<b$.

Apply the Mean Value Theorem to $f(x)$:

$$\frac{f(b)-f(a)}{b-a}=f'(c) \text{ for some } c\in(a,b).$$

$f'(x)=(1-\alpha)x^{-\alpha}$ and hence $\dfrac{b^{1-\alpha}-a^{1-\alpha}}{b-a}=(1-\alpha)c^{-\alpha}$

$$a<c<b \Rightarrow c^{-\alpha}=\frac{1}{c^{\alpha}}<\frac{1}{a^{\alpha}}=a^{-\alpha}$$

$$\Rightarrow b^{1-\alpha}-a^{1-\alpha}=(1-\alpha)c^{-\alpha}(b-a)<(1-\alpha)(b-a)a^{-\alpha}.$$

Divide by $a^{-\alpha}$ to get

$$b^{1-\alpha}a^{\alpha}-a<(1-\alpha)(b-a)$$

$$\Rightarrow \boxed{a^{\alpha}b^{1-\alpha}<b(1-\alpha)+\alpha a} \text{ for all } 0<\alpha<1$$

and $0<a<b$. If now $a>b>0$ then

$0<\frac{1}{a}<\frac{1}{b}$ and thus $\dfrac{1}{a^{\alpha}}\cdot\dfrac{1}{b^{1-\alpha}}<\dfrac{1}{b}(1-\alpha)+\alpha\cdot\dfrac{1}{a}$

Multiply this by ab to get

$a^{1-\alpha}b^{\alpha}<a(1-\alpha)+b\alpha$. However, these hold for all $0<\alpha<1$ and thus we may replace α with $1-\alpha$ in the inequality in the red box. Hence,

$$a^\alpha b^{1-\alpha} < \alpha a + (1-\alpha) b \quad \text{for all } a,b>0$$

and $1>\alpha>0$. Taking limits we see that

$$a^\alpha b^{1-\alpha} \le \alpha a + (1-\alpha) b, \quad \text{for all } a,b \ge 0 \text{ and } 1>\alpha>0.$$

Now fix some $p>1$ and let $\alpha = 1/p$. Then $1-\alpha = \frac{1}{q}$ for some $q>1$. So

$p,q>1$ and $1/p + 1/q = 1$.

Set $a = c_i^p$ and $b = d_i^q$. Then the above inequality becomes

$$c_i d_i \le \frac{1}{p} c_i^p + \frac{1}{q} d_i^q, \quad \text{for } i=1,\dots,k.$$

Take sum over i to get

$$\sum_{i=1}^k c_i d_i \le \sum_{i=1}^k \left(\frac{1}{p} c_i^p + \frac{1}{q} d_i^q\right)$$

Now finally let $c_i^p = \dfrac{|\xi_i|^p}{\left(\sum_{j=1}^k |\xi_j|^p\right)^{\cancel{1/p}}}$, $d_i = \dfrac{|\eta_i|^q}{\left(\sum_{j=1}^k |\eta_j|^q\right)^{\cancel{1/q}}}$

Plug them into the above inequality to get

$$\left(\sum_{i=1}^k |\xi_i||\eta_i|\right) \frac{1}{\left(\sum_{j=1}^k |\xi_j|^p\right)^{1/p} \left(\sum_{j=1}^k |\eta_j|^q\right)^{1/q}} \le \frac{1}{p}\cdot 1 + \frac{1}{q}\cdot 1 = 1.$$

$$\Rightarrow \boxed{\sum_{i=1}^{k} |\xi_i|\,|\eta_i| \le \left(\sum_{i=1}^{k} |\xi_i|^p\right)^{1/p} \left(\sum_{i=1}^{k} |\eta_i|^q\right)^{1/q}}$$

for all $\xi_i, \eta_i \in \mathbb{R}$.

This is called the Hölder's Inequality.

Minkowski's Inequality: Consider the inequality

$$|\xi_i+\eta_i|^p = |\xi_i+\eta_i|\,|\xi_i+\eta_i|^{p-1}$$

$$\le (|\xi_i|+|\eta_i|)\,|\xi_i+\eta_i|^{p-1}$$

$$\le |\xi_i|\,|\xi_i+\eta_i|^{p-1} + |\eta_i|\,|\xi_i+\eta_i|^{p-1}.$$

Take summation over $i=1,\dots,k$ and apply Hölder's Inequality to the products:

$$\sum_{i=1}^{k} |\xi_i|\,|\xi_i+\eta_i|^{p-1} \le \left(\sum_{i=1}^{k} |\xi_i|^p\right)^{1/p} \left(\sum_{i=1}^{k} |\xi_i+\eta_i|^{q(p-1)}\right)^{1/q}$$

Since $1/p+1/q=1 \Rightarrow q+p=pq \Rightarrow qp-q=p$

$$\Rightarrow \sum_{i=1}^{k} |\xi_i|\,|\xi_i+\eta_i|^{p-1} \le \left(\sum_{i=1}^{k} |\xi_i|^p\right)^{1/p} \cdot \left(\sum_{i=1}^{k} |\xi_i+\eta_i|^p\right)^{1/q}$$

Hence,

$$\sum_{i=1}^{k} |\xi_i+\eta_i|^p \le \left[\left(\sum_{i=1}^{k} |\xi_i|^p\right)^{1/p} + \left(\sum_{i=1}^{k} |\eta_i|^p\right)^{1/p}\right] \cdot \left(\sum_{i=1}^{k} |\xi_i+\eta_i|^p\right)^{1/q}$$

$$\left(\sum_{i=1}^{k}|\xi_i+\eta_i|^p\right)^{1-\frac{1}{q}} \leq \left(\sum_{i=1}^{k}|\xi_i|^p\right)^{1/p} + \left(\sum_{i=1}^{k}|\eta_i|^p\right)^{1/p}$$

$$\Rightarrow \boxed{\left(\sum_{i=1}^{k}|\xi_i+\eta_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{k}|\xi_i|^p\right)^{1/p} + \left(\sum_{i=1}^{k}|\eta_i|^p\right)^{1/p}}$$

for all $\xi_i, \eta_i \in \mathbb{R}$.

This is called Minkowski's Inequality.

<u>Corollary</u>: Let $(X_1, d_1), \ldots, (X_k, d_k)$ be metric spaces. Then for any $p > 1$ the function

$d_p : X \times X \longrightarrow \mathbb{R}$, where $X = X_1 \times \cdots \times X_k$ and

$$d_p(x,y) = \left(\sum_{i=1}^{k} d_i(x_i,y_i)^p\right)^{1/p}, \quad x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k).$$

defines a metric on $X = X_1 \times \cdots \times X_k$.

<u>Proof</u> (M1), (M2) trivially hold. For (M3) one needs to use the Minkowski's inequality, left as an exercise. ∎

<u>Proposition</u>: The metrics d_1, d_p and d_∞ are all equivalent, where

$$d_1(x,y) = \sum_{i=1}^{k} d_i(x_i,y_i) \quad \text{and}$$

$$d_\infty(x,y) = \max\{d_1(x_1,y_1), \ldots, d_k(x_k,y_k)\}.$$

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Recall that we prove this for

$(X_i, d_i) = (\mathbb{R}, |\cdot|)$, $d_i(x_i, y_i) = |x_i - y_i|$, before.

Proposition: If $(X_1, d_1), \dots, (X_k, d_k)$ are complete metric spaces then the Cartesian Product metric space (X, d), where $X = X_1 \times \dots \times X_k$ and $d = \tilde{d}_p$, $p \geq 1$ or $p = \infty$, are all complete.

Proof: First let's prove this for $k=2$ and for d_1.

$X = X_1 \times X_2$, $\tilde{d}_1((x_1, x_2), (y_1, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$

Must prove $(X, \tilde{d}_1)$ is complete. Take a Cauchy sequence (z_n) in X. So $z_n = (x_n, y_n)$ is Cauchy sequence implies that for any $\epsilon > 0$ there is some $n_0 \in \mathbb{N}$ so that
$m, n \geq n_0$, $\tilde{d}_1((x_n, y_n), (x_m, y_m)) < \epsilon$.

So, $d(x_n, x_m) + d(y_n, y_m) < \epsilon$ and hence $d(x_n, x_m) < \epsilon$ and $d(y_n, y_m) < \epsilon$, provided that $m, n \geq n_0$.

Hence (x_n) is Cauchy in (X_1, d_1) and (y_n) is Cauchy in (X_2, d_2). Since (X_i, d_i) is complete both sequences are convergent, say $\lim x_n = x_0$ and $\lim y_n = y_0$, for some $x_0 \in X_1$ and $y_0 \in X_2$.

Now we claim that $z_n = (x_n, y_n)$ is convergent.

Let $\epsilon > 0$ be given. Then $\epsilon/2 > 0$. Since $\lim x_n = x_0$ and $\lim y_n = y_0$ there are $n_1, n_2 \in \mathbb{N}$ so that

$n \geq n_1 \Rightarrow d_1(x_n, x_0) < \epsilon/2$ and

$n \geq n_2 \Rightarrow d_2(y_n, y_0) < \epsilon/2$.

So, let $n_0 = \max\{n_1, n_2\}$. Then if $n \geq n_0$ then

$$\tilde{d}_1(z_n, z_0) = \tilde{d}_1((x_n, y_n), (x_0, y_0))$$

$$= d_1(x_n, x_0) + d_2(y_n, y_0) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence, $(X_1 \times X_2, \tilde{d}_1)$ is complete. Since $\tilde{d}_p$ are all equivalent $(X_1 \times X_2, \tilde{d}_p)$ are all complete.

Finally, by induction $(X_1 \times \cdots \times X_k, \tilde{d}_p)$ is complete. ∎

Corollary $(\mathbb{R}^n, d_p)$ is complete.

Proof: $(\mathbb{R}, |\cdot|)$ is complete. Then

$$d_p(x, y) = d_p((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}.$$

Hence $(\mathbb{R}^n, d_p)$ is complete by the previous Proposition. ∎

Completion of a metric Space:

Theorem: Let (X,d) be a metric space. Then there is a complete complete metric space $(\tilde{X},\tilde{d})$ and an isometry

$$J:(X,d)\longrightarrow(\tilde{X},\tilde{d}) \quad \text{so that}$$

$J(X)$ is a dense subset of $\tilde{X}$: $\overline{J(X)}=\tilde{X}$.

Moreover, $(\tilde{X},\tilde{d})$ is unique upto isometry. More precisely, if $(\tilde{X}_1,d_1)$ and $(\tilde{X}_2,d_2)$ are two complete metric spaces and

$$J_1:(X,d)\longrightarrow(\tilde{X}_1,d_1) \quad \text{and} \quad J_2(X,d)\longrightarrow(\tilde{X}_2,d_2)$$

are isometric embeddings with $\overline{J_i(X)}=\tilde{X}_i$ for $i=1,2$, then there are isometries

$$\theta_1:(\tilde{X}_1,d_1)\longrightarrow(\tilde{X}_2,d_2) \quad \text{and}$$

$$\theta_2:(\tilde{X}_2,d_2)\longrightarrow(\tilde{X}_1,d_1)$$

so that the diagram below commutes:

$$\begin{tikzcd}
 & (\tilde{X}_1,d_1) \arrow[d, shift right, "\theta_1"'] \\
(X,d) \arrow[ur, "J_1"] \arrow[dr, "J_2"'] & \\
 & (\tilde{X}_2,d_2) \arrow[u, shift right, "\theta_2"']
\end{tikzcd}$$

$\theta_1\circ J_1=J_2$

$\theta_2\circ J_2=J_1$

$\theta_1\circ\theta_2=id_{\tilde{X}_2}$

$\theta_2\circ\theta_1=id_{\tilde{X}_1}$.

Remark: The book suggests two proofs for this. One of them mimics the construction of real numbers from rationals via Cauchy sequences in rationals:
$(\mathbb{Q}, |\cdot|)$ We just let $\mathbb{R}$ as the set of all Cauchy sequences in $(\mathbb{Q}, |\cdot|)$ up to an equivalence:
$(x_n), (y_n)$ Cauchy sequences in $(\mathbb{Q}, |\cdot|)$:

Then we say that $(x_n) \sim (y_n)$ if and only if $\lim |x_n - y_n| = 0$

Next we consider the set of equivalence classes $[x_n]$ of this relation.

$[x_n] = \{(y_n) \mid (y_n)$ Cauchy in $(\mathbb{Q}, |\cdot|), \lim (x_n - y_n) = 0\}$.

$X = \{[x_n] \mid (x_n)$ is Cauchy in $(\mathbb{Q}, |\cdot|)\}$

One can put a metric on X as follows:

$d([x_n], [y_n]) = \lim |x_n - y_n|$ $\qquad \sqrt{2} = \lim r_n$

$\sqrt{2} = \lim s_n$

$\sqrt{2} - \sqrt{3} = \lim r_n - t_n$ $\qquad \sqrt{3} = \lim t_n$

$= \lim s_n - t_n$

Finally, (X, d) is a completion of $(\mathbb{Q}, |\cdot|)$.

Instead, we will use another approach to construct the completion of a given metric space.

Idea: Given the metric space (X,d) try to embedd (X,d) into the complete metric space of bounded functions $(B(X), d_{sup})$.

We'll construct an isometry $J: X \hookrightarrow B(X)$.

Then $\overline{J(X)}$ is a closed subset of the $(B(X), d_{sup})$ and thus $\overline{J(X)}$ is also complete.

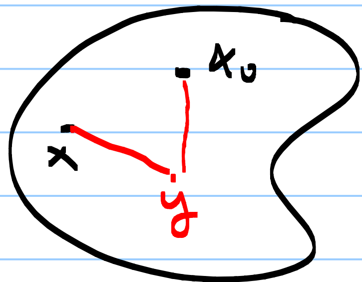
Theorem: Any metric space (X,d) has a unique completion upto isometry.

Proof: First consider the map

$J: X \longrightarrow B(X)$ defined as follows: Fix some point $x_0 \in X$. Then

$J(x) = f_x : X \longrightarrow \mathbb{R}$ defined by

$$f_x(y) = d(y,x) - d(y,x_0)$$



Note that $|f_x(y)| = |d(y,x) - d(y,x_0)| \leq d(x,x_0)$, for all $y \in X$. Hence f_x is a bounded function on X.

$$\left[\begin{array}{l} d(y,x) \leq d(y,x_0) + d(x,x_0) \Rightarrow \underline{d(y,x) - d(y,x_0)} \leq d(x,x_0) \\ \underline{d(y,x_0) - d(y,x)} \leq d(x,x_0). \end{array}\right]$$

Hence, $J : X \longrightarrow B(X)$

<u>Claim</u>: J is an isometry.

<u>Proof</u> Take any points x and x'. Then, for any $y \in X$ we have

$$f_x(y) - f_{x'}(y) = \big(d(x,y) - d(y,x_0)\big) - \big(d(x',y) - d(y,x_0)\big)$$
$$= d(x,y) - d(x',y)$$
$$\leq d(x,x'), \text{ for any } y \in X.$$

Hence, $|f_x(y) - f_{x'}(y)| \leq d(x,x')$ for all $y \in X$.

Thus, $d_{sup}(f_x, f_{x'}) = \sup\{|f_x(y) - f_{x'}(y)| \mid y \in X\}$
$$\leq d(x,x').$$

However, if we let $y = x$ then

$$|f_x(y) - f_{x'}(y)| = |d(x,y) - d(x',y)|$$
$$= |0 - d(x',x)| = d(x,x').$$

Hence, $d_{sup}(f_x, f_{x'}) = d(x,x')$.

In other words, the map $J : X \to B(X)$ sending $x \mapsto J(x) = f_x$ is an isometry.

Now let $\widehat{X} = \overline{J(X)}$ in $(B(X), d_{sup})$.

Since $\mathcal{B}(X)$ is complete and $\widetilde{X}$ is a closed subset $(\widetilde{X}, d_{sup})$ is a complete complete metric space and

$$\overline{J(X)} = \widetilde{X}, \text{ so that it contains}$$

an isometric copy $J(X)$ of X as a dense subset.

This completes the existence part of the proof. For the uniqueness we'll proceed as follows:

Let (X, d) be a metric space and assume that $(\widetilde{X}_1, d_1)$ and $(\widetilde{X}_2, d_2)$ are complete metric spaces,

$J_1 : X \longrightarrow \widetilde{X}_1$, $J_2 : X \longrightarrow \widetilde{X}_2$ are isometries

so that $\overline{J_1(X)} = \widetilde{X}_1$ and $\overline{J_2(X)} = \widetilde{X}_2$.

must construct $\theta_1 : \widetilde{X}_1 \longrightarrow \widetilde{X}_2$, $\theta_2 : \widetilde{X}_2 \longrightarrow \widetilde{X}_1$ isometries so that

$$\theta_1 \circ \theta_2 = id_{\widetilde{X}_2} \text{ and } \theta_2 \circ \theta_1 = id_{\widetilde{X}_1}, \text{ and}$$

finally

$$\begin{tikzcd} & \widetilde{X}_2 \arrow[d, shift right, "\theta_2"'] \\ X \arrow[ur, "J_2"] \arrow[r, "J_1"'] & \widetilde{X}_1 \arrow[u, shift right, "\theta_1"'] \end{tikzcd} \qquad \begin{array}{l} \theta_1 \circ J_1 = J_2 \\ \theta_2 \circ J_2 = J_1 \end{array}$$

<u>Construction of θ_1</u>: $A = J(X) \subseteq \widetilde{X}_1$. So $\overline{A} = \widetilde{X}_1$ so that A is a dense subset.

$$\begin{tikzcd}
X \arrow[r, shift left, "J_1"] \arrow[dr, "J_2"'] & A = J_1(X) \subseteq \widetilde{X}_1 \arrow[l, shift left, "J_1^{-1}"] \arrow[d, "J_2 \circ J_1^{-1}"] \arrow[dl, phantom, ""] \\
 & \widetilde{X}_2 & \arrow[l, bend left, "\theta_1"']
\end{tikzcd}$$

Since each J_i is an isometry so is $J_2 \circ J_1^{-1}$. In particular, $J_2 \circ J_1^{-1}$ is uniformly continuous.

So by the extension result we've proved earlier there is a unique map $\theta_1 : \widetilde{X}_1 \to \widetilde{X}_2$ so that $\theta_1(x) = J_2 \circ J_1^{-1}$, whenever $x \in A = J_1(X)$. Moreover, θ_1 is an isometry since $J_2 \circ J_1^{-1}$ is an isometry.

Similarly, there is an isometry $\theta_2 : \widetilde{X}_2 \to \widetilde{X}_1$ so that

$$\theta_2(x) = J_1 \circ J_2^{-1}, \text{ whenever } x \in B = J_2(X) \subseteq \widetilde{X}_2$$

$$\begin{tikzcd}
\widetilde{X}_1 \arrow[rr, shift left, "\theta_1"] & & \widetilde{X}_2 \arrow[ll, shift left, "\theta_2"] \\
 & X \arrow[ul, "J_1"] \arrow[ur, "J_2"'] &
\end{tikzcd}$$

If $x \in X$, then $\theta_1(J_1(x)) = J_2(x)$ and $\theta_2(J_2(x)) = J_1(x)$.

To finish the proof we need to show that θ_1 and θ_2 are inverses of each other.

If $x \in X$, then $\theta_2 \theta_1 (J_1(x)) = \theta_2(J_2(x)) = J_1(x)$. Hence, $\theta_2 \circ \theta_1 = \text{id}$ on A and $\theta_1 \circ \theta_2 = \text{id}$ on B.

$\theta_2 \circ \theta_1 : \widetilde{X}_1 \longrightarrow \widetilde{X}_1$, $(\theta_2 \circ \theta_1)(x) = x$, $\forall x \in A$.

So $\theta_2 \circ \theta_1$ is an extension of $id : A \to A$ to $\widetilde{X}_1 = \overline{A}$. On the other hand, $id_{\widetilde{X}_1}$ is also an extension of $id : A \to A$.

Hence, $\theta_2 \circ \theta_1$ must be $id_{\widetilde{X}_1}$.

Similarly, $\theta_1 \circ \theta_2$ must be $id_{\widetilde{X}_2}$.

This finishes the proof. $\blacksquare$

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COMPACTNESS

Definition: Let E be a subset of a metric space. A collection of open sets $\{O_i\}_{i\in\Lambda}$ in X is called an open cover for E if
$$E \subseteq \bigcup_{i\in\Lambda} O_i.$$

Definition: A subset E of X is called compact if any open cover $\{O_i\}_{i\in\Lambda}$ of E has a finite subcover:

$$E \subseteq \bigcup_{i\in\Lambda} O_i, \quad O_i \subseteq X \text{ is open for all } i\in\Lambda$$
$$\Rightarrow E \subseteq O_{i_1} \cup O_{i_2} \cup \dots \cup O_{i_k}, \text{ for some}$$

$i_1, \dots, i_k \in \Lambda$.

Examples: 1) Any finite subset E is compact.

Let $E = \{x_1, \dots, x_k\} \subseteq X$. Let $\{O_i\}_{i\in\Lambda}$ be any open cover for E.

Hence, $E = \{x_1, \dots, x_k\} \subseteq \bigcup_{i\in\Lambda} O_i$.

Since $x_1 \in \bigcup_{i\in\Lambda} O_i$, there is some $i_1 \in \Lambda$ so that $x_1 \in O_{i_1}$. Similarly, $x_2 \in O_{i_2}$ and $x_3 \in O_{i_3}$ for some $i_2, i_3 \in \Lambda$. This way we find

$i_1, \dots, i_k \in \Lambda$ so that $x_j \in O_{i_j}$, $j = 1, 2, \dots, k$.

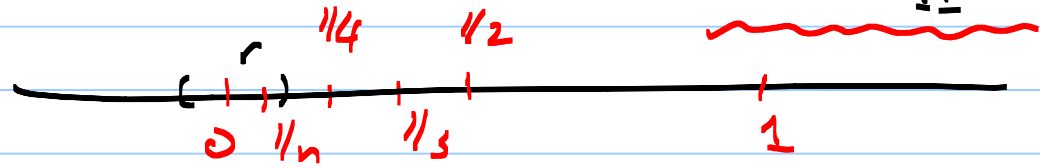
Now, $E = \{x_1, \dots, x_k\} \subseteq O_{i_1} \cup O_{i_2} \cup \dots \cup O_{i_k}$.

Hence, E is compact.

2) Let $E = \{1/n \mid n = 1, 2, \ldots\} \cup \{0\} \subseteq (\mathbb{R}, |\cdot|)$.

E is compact: Let $\{O_i\}_{i \in \Delta}$ be an open cover for E.

$$E \subseteq \bigcup_{i \in \Delta} O_i.$$



Since $0 \in E$ then $\exists$ some $i_0 \in \Delta$ so that $0 \in O_{i_0}$. Since O_{i_0} is open then $\exists$ some $r > 0$ so that

$$0 \in (-r, r) \subseteq O_{i_0}.$$

The sequence $1/n$ has limit 0, i.e., $\lim \frac{1}{n} = 0$ and hence there is some $n_0 \in \mathbb{N}$ so that
$n \geq n_0 \implies 1/n \in (-r, r)$.

$1/n_0, 1/n_{0+1}, 1/n_{0+2}, \ldots$ are in $(-r, r) \subseteq O_{i_0}$.

Now for the remaining terms $1, 1/2, \ldots, 1/n_{0-1}$ pick open sets $O_{i_1}, O_{i_2}, \ldots, O_{i_{n_0-1}}$ so that

$1 \in O_{i_1}$, $1/2 \in O_{i_2}, \ldots, 1/n_{0-1} \in O_{i_{n_0-1}}$.

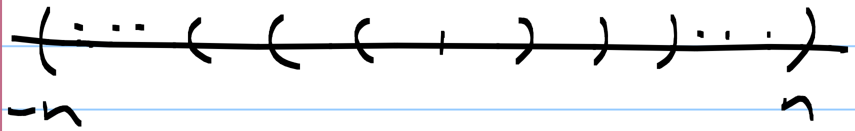
Now, $E = \{1/n \mid n = 1, \ldots \infty\} \cup \{0\} \subseteq O_{i_0} \cup O_{i_1} \cup O_{i_2} \cup \ldots \cup O_{i_{n_0-1}}$

Hence, E is a compact subset of $(\mathbb{R}, |\cdot|)$.

Exercise: Let (x_n) be a convergent sequence in any metric space (X,d) with $\lim x_n = x_0$.

Then the subset $E = \{x_n \mid n = 1, 2, \dots\} \cup \{x_0\}$ is a compact subset of (X,d).

3) $E = \mathbb{R}$ is not compact in $(\mathbb{R}, |\cdot|)$.



Let $O_n = (-n, n)$, $n = 1, 2, 3, \dots$. Clearly,

$$\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n) = \bigcup_{n=1}^{\infty} O_n \quad \text{and} \quad \mathbb{R} \not\subseteq O_{n_1} \cup \dots \cup O_{n_k}$$

for any $n_1, \dots, n_k$, because $O_{n_1} \cup \dots \cup O_{n_k} = (-n_0, n_0)$, when $n_0 = \max\{n_1, \dots, n_k\}$.

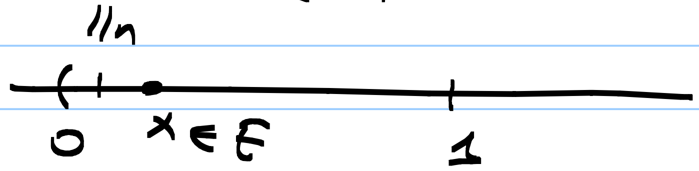
Hence, the open cover $\{O_n\}_{n \in \mathbb{N}}$ has no finite subcover.

So $\mathbb{R}$ is not a compact subset of $(\mathbb{R}, |\cdot|)$.

4) Let $E = (0, 1] \subseteq (\mathbb{R}, |\cdot|)$.

<u>Claim:</u> E is not compact.

<u>Proof</u>: Let $O_n = (1/n, 2)$.



Since for any $x > 0$ then $\exists$ some $n \in \mathbb{N}$ with $n > 1/x$ we see that $x > 1/n$ and thus $x \in (1/n, 2)$. So

$$E = (0,1] \subseteq \bigcup_{n=1}^{\infty} (1/n, 2) \quad , \quad (1/n, 1] \subseteq (0,2) \text{ are all open}$$

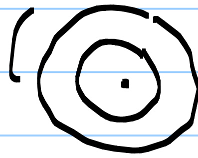
However, $E \not\subseteq \bigcup_{i=1}^{k} (1/n_i, 2)$ for any k and $n_1, \ldots, n_k$.

This is because, $\bigcup_{i=1}^{k} (1/n_i, 2) = (1/n_0, 2)$, where

$n_0 = \max\{n_1, \ldots, n_k\}$, and $(0,1) \not\subseteq (1/n_0, 2)$.

<u>Proposition</u>: A compact subset E of (X, d) is closed and bounded.

<u>Proof.</u> 1) Suppose E is <u>not</u> bounded. Choose any $x_0 \in X$ and consider the balls



$$O_i = B(x_0, i), \quad i = 1, 2, \ldots .$$

Since E is not bounded $E \not\subseteq O_i$ for any i.

Hence, E is <u>not</u> covered by finitely many O_i's.

$$E \subseteq X = \bigcup_{i=1}^{\infty} B(x_0, i), \text{ so that } \{B(x_0, i)\}_{i \in \mathbb{N}}$$

is an open cover for E which do <u>not</u> have

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a finite subcover. Hence, E is not compact. This is a contradiction to the assumption that E is compact, hence E must be bounded.

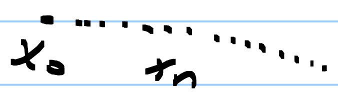
2) E is closed. Again assume on the contrary that E is not closed. Hence, there is a convergent sequence (x_n) so that each $x_n \in E$ but $x_0 = \lim x_n \notin E$.

Let $O_n = X \setminus B[x_0, 1/n]$, which is open in X, for each n.

$$\bigcup_{n=1}^{\infty} O_n = \bigcup_{n=1}^{\infty} \left(X \setminus B[x_0, 1/n]\right) = X \setminus \left(\bigcap_{n=1}^{\infty} B[x_0, 1/n]\right) = X \setminus \{x_0\}$$

Hence, $E \subseteq X \setminus \{x_0\} = \bigcup_{n=1}^{\infty} O_n$, so that $\{O_n\}_{n=1}^{\infty}$

is an open cover for E.



Note that any finite number $O_{n_1}, \dots, O_{n_k}$ of open subsets won't cover E, because

$$O_{n_1} \cup \dots \cup O_{n_k} = \bigcup_{i=1}^{k} X \setminus B[x_0, 1/n_i] = X \setminus \left(\bigcap_{i=1}^{k} B[x_0, 1/n_i]\right)$$

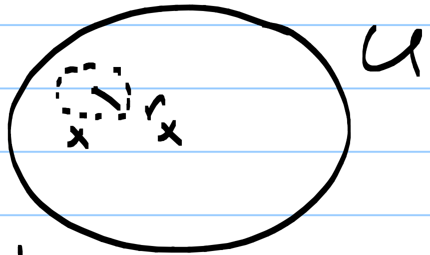
$$= X \setminus B[x_0, 1/n_0], \text{ where } n_0 = \max\{n_1, \dots, n_k\}.$$

Since $\lim x_n = x_0$ after some index all x_n's will lie in $B(x_0, 1/n_0)$ and thus $E \not\subseteq X \setminus B[x_0, 1/n_0]$. Hence, the open cover $\{O_n\}_{n=1}^{\infty}$ for E has no finite subcover.

Thus, E is not compact, which is a contradiction. Hence, E must be a closed subset. $\blacksquare$

<u>Remark:</u> Let (X, d) be a subspace of $(\tilde{X}, d)$.

Take any open set U in (X, d). Then for any $x \in U$ there is some $r_x > 0$ so that $B(x, r_x) \subseteq U$.



Hence $U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B(x, r_x) \subseteq U$

$$\Rightarrow U = \bigcup_{x \in U} B(x, r_x)$$

$B_X(x, \epsilon) = \{y \in X \mid d(x,y) < \epsilon\}$

$B_{\tilde{X}}(x, \epsilon) = \{y \in \tilde{X} \mid d(x,y) < \epsilon\}$

$$B_{\tilde{X}}(x, \epsilon) \cap X = B_X(x, \epsilon).$$

Hence, $U = \bigcup_{x \in U} B_X(x, r_x) \subseteq \bigcup_{x \in U} B_{\tilde{X}}(x, r_x) = \tilde{U}$, which is open in $\tilde{X}$ with $\tilde{U} \cap X = U$.

In other words, any open subset of the subspace X is the intersection of an open subset of $\tilde{X}$ with the subspace X.

Corollary If a metric space (X,d) is compact then it must be complete.

Proof: Assume on the contrary that X is not complete. Let $\tilde{X}$ be its unique completion (upto isometry). Since X is not complete and any closed subset of a complete metric space is complete we see that X cannot be a closed subset of $(\tilde{X},\tilde{d})$. Hence, X cannot be a compact subset of $\tilde{X}$.

<u>Claim</u>: X is <u>not</u> a compact metric space.

Proof Take any open cover $\{O_\alpha\}$ of X. Then by the above remark each $O_\alpha = X \cap U_\alpha$ for some open subset U_α of $\tilde{X}$.
Then $\{U_\alpha\}$ is an open cover for X.

Conversely, if $\{U_\alpha\}$ is a collection of open subsets in $\tilde{X}$ covering X then $\{O_\alpha\}$ is an open cover for X, where $O_\alpha = X \cap U_\alpha$.

Now since X is <u>not</u> a compact subset of $\tilde{X}$ then is an open cover $\{U_\alpha\}$ in $\tilde{X}$ so that

$X \subseteq \bigcup_\alpha U_\alpha$ but no finitely many U_α will

cover X. Finally, let $O_\alpha = X \cap U_\alpha$, all open in X. Moreover, $\{O_\alpha\}$ is an open cover for X which do <u>not</u> have a finite subcover. Hence, X is <u>not</u> a compact metric space. ∎

__Remark:__ This observation tells us that being a compact subset is an intrinsic property of the subspace E of X.

__Proposition:__ Every sequence in a compact set E has a subsequence which converges to an element of E.

__Proof:__ Assume that there is a sequence (x_n) in E which does not have any convergent subsequence. Hence, for any $x \in E$ there is some $r_x > 0$ so that $B(x, r_x)$ contains at most finitely many x_i's.

Hence, $\{i \mid x_i \in B(x, r_x)\}$ is finite.

$$E = \bigcup_{x \in E} \{x\} \subseteq \bigcup_{x \in E} B(x, r_x) \text{ so that } \{B(x, r_x)\}_{x \in E}$$

is an open cover for E. Since E is compact $E \subseteq B(y_1, r_{y_1}) \cup \dots \cup B(y_k, r_{y_k})$ for some $y_1, \dots, y_k \in E$.

Since (x_n) lies in E, we have

$$\mathbb{N} = \{i \mid x_i \in E\} = \{i \mid x_i \in B(y_1, r_{y_1}) \cup \dots \cup B(y_k, r_{y_k})\}$$

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This is a contradiction since $\mathbb{N}$ is infinite and each $\{i \mid x_i \in B(y_{j},r_{y_{j}})\}$ is finite.

This finishes the proof. ∎

Proposition: A closed subset E of a compact metric space X is compact.

Proof: Let $\{O_i\}_{i\in\Lambda}$ be an open cover for E.

So $E \subseteq \bigcup_{i\in\Lambda} O_i$. Since E is closed $X\setminus E = O$ is open.

Then $X = E \cup (X\setminus E) \subseteq (\bigcup_{i\in\Lambda} O_i) \cup O$ so that $\{O_i\}_{i\in\Lambda} \cup \{O\}$ is an open cover for X. Since X is compact $X \subseteq O_{i_1} \cup O_{i_2} \cup \dots \cup O_{i_k} \cup O$, for some $i_1, \dots, i_k \in \Lambda$.

Then $E \subseteq X$ and $E \cap O = \emptyset$ we have

$E \subseteq O_{i_1} \cup O_{i_2} \cup \dots \cup O_{i_k}$. Hence, E is compact. ∎

Compactness and Convergence of Sequences:

Definition: A subset E of X is called precompact if for every $\epsilon > 0$ there are finitely many $x_1, \dots, x_k$ in X so that

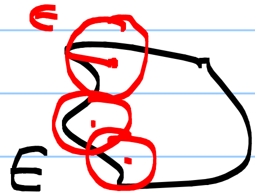
$$E \subseteq B(x_1,\epsilon) \cup B(x_2,\epsilon) \cup \dots \cup B(x_k,\epsilon).$$

Proposition: A compact set is precompact. A subset of a precompact is also precompact. A precompact set is bounded.

Proof: A subset $E \subseteq X$ is compact subset.

Claim: E is precompact.

proof of the claim: Given $\epsilon > 0$. For any $x \in E$ we have $x \in B(x, \epsilon)$.



Hence, $E = \bigcup_{x \in E} \{x\} \subseteq \bigcup_{x \in E} B(x, \epsilon)$.

In particular, $\{B(x, \epsilon)\}_{x \in E}$ is an open cover for E. Since E is compact we see that

$$E \subseteq B(x_1, \epsilon) \cup B(x_2, \epsilon) \cup \cdots \cup B(x_k, \epsilon), \text{ for}$$

some $x_1, \ldots, x_k \in E \subseteq X$. Hence, E is precompact.

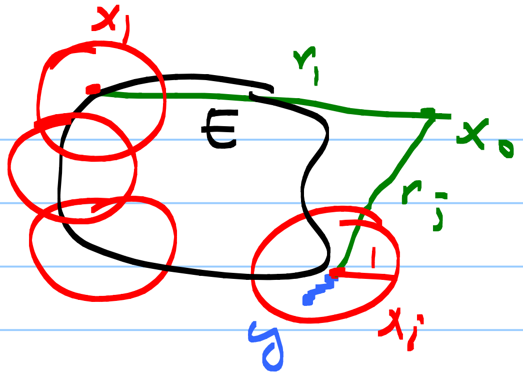
I leave as an exercise to show that a subset of a precompact subset is also precompact.

For the final statement let E be a precompact subset of X.

must show: E is bounded.

Let $\epsilon = 1$. Since E is precompact, then are point $x_1, \ldots, x_k \in X$ with

$$E \subseteq B(x_1, 1) \cup B(x_2, 1) \cup \cdots \cup B(x_k, 1).$$



Take any point $x_0 \in X$ and let $r_i = d(x_0, x_i)$, $i=1,\dots,k$. Let $r = \max\{r_1, \dots, r_k\}$.

Then clearly $B(x_i, 1) \subseteq B(x_0, r+1)$: To see this let $y \in B(x_i, 1)$. Then

$$d(y, x_0) \leq d(y, x_i) + d(x_i, x_0)$$

$$< 1 + r$$

so that $y \in B(x_0, 1+r)$.

Hence, $E \subseteq B(x_1, 1) \cup \dots \cup B(x_k, 1) \subseteq B(x_0, r+1)$ so that E is bounded. $\blacksquare$

Now we'll state and prove the sequential characterization of compact subsets:

<u>Theorem:</u> The following conditions are equivalent on a metric space (X, d).

a) (X, d) is compact.
b) Every sequence in X has a convergent subsequence.
c) (X, d) is complete and precompact.

<u>Proof:</u> We have already proved that (a) $\Rightarrow$ (b).

We'll prove (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a).

(b) $\Rightarrow$ (c): Assume that every sequence in X has a convergent subsequence.

must show: X is complete and precompact.

i) X is complete: Take any Cauchy sequence (x_n) in X. Then by the assumption (x_n) has a convergent subsequence say (x_{k_n}) to some x_0.

Let $\epsilon>0$ be given. Since (x_n) is Cauchy there is some n_0 so that $m,n \geq n_0$ implies $d(x_n,x_m)<\epsilon/2$. On the other hand

$\lim x_{k_n} = x_0$ and thus choosing n_0 bigger if necessary we see that

$$n \geq n_0 \Rightarrow d(x_{k_n}, x_0) < \epsilon/2.$$

So, if $n \geq n_0$ then $d(x_n,x_0) \leq d(x_n,x_{k_n}) + d(x_{k_n},x_0)$

$$< \epsilon/2 + \epsilon/2 = \epsilon,$$

because $n \geq n_0 \Rightarrow k_n \geq n_0$.

Hence, $\lim x_n = x_0$, so that (X,d) is a complete metric space.

ii) Precompact: Assume on the contrary

that (X,d) is not precompact. Then there is some $\epsilon>0$ so that for any $x_1,\dots,x_k$ in X, $X \not\subset B(x_1,\epsilon) \cup B(x_2,\epsilon) \cup \dots \cup B(x_k,\epsilon)$.

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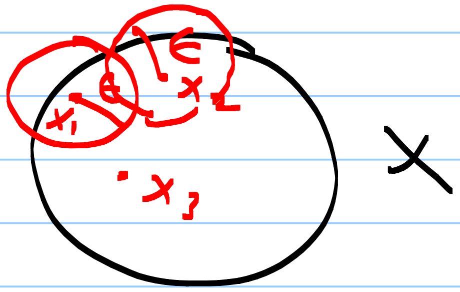
Now we form the following sequence.
Start with any $x_1 \in X$. Since $X \not\subset B(x_1, \epsilon)$ there is some $x_2 \in X \setminus B(x_1, \epsilon)$. In particular, $d(x_1, x_2) \geq \epsilon$.

Now, since $X \not\subset B(x_1,\epsilon) \cup B(x_2, \epsilon)$ then is some $x_3 \in X \setminus (B(x_1,\epsilon) \cup B(x_2,\epsilon))$. Hence, $d(x_3, x_1) \geq \epsilon$ and $d(x_3, x_2) \geq \epsilon$.



By induction we can form a sequence (x_n) so that $d(x_n, x_m) \geq \epsilon$ for any $n \neq m$.

This implies that (x_n) has no convergent subsequence, which contradicts to our assumption. Hence, (X,d) must be precompact. This finishes the proof of (b) $\Rightarrow$ (c).

Now, we'll prove (c) $\Rightarrow$ (a).

So we may assume (X,d) is complete and precompact.
must show: (X,d) is compact.

Assume on the contrary that (X,d) is not compact. Hence, there is some open cover, say $\{O_\alpha\}_{\alpha\in\Delta}$ so that this open cover has no finite subcover. So, for any $\alpha_1, \dots, \alpha_k$ $X \not\subset O_{\alpha_1} \cup \dots \cup O_{\alpha_k}$.

Let $\epsilon = 1$. Since (X,d) precompact X can be covered by finitely many balls of radius 1.

$X \subseteq B(x_1,1) \cup B(x_2,1) \cup \cdots \cup B(x_{k_1},1)$ for some

$x_1, \ldots, x_{k_1} \in X$. Since X cannot covered by finitely many O_α's at least one of the balls $B(x_i,1)$ cannot be covered by finitely many O_α's. Without loss of generality say $B(x_1,1)$ cannot be covered by finitely many O_α's.

$B(x_1,1) \not\subseteq O_{\alpha_1} \cup O_{\alpha_2} \cup \cdots \cup O_{\alpha_k}$ for any $\alpha_1, \ldots, \alpha_k$.

Now let $\epsilon = 1/2$. Since X is precompact its subset $B(x_1,1)$ is precompact and thus there are finitely many balls of radius $\epsilon = 1/2$ covering $B(x_1,1)$:

$B(x_1,1) \subseteq B(y_1,1/2) \cup B(y_2,1/2) \cup \cdots \cup B(y_{k_2},1/2)$

for some $y_1, \ldots, y_{k_2} \in X$. Here we may assume that $B(x_1,1) \cap B(y_i,k_i) \neq \emptyset$.

Since $B(x_1,1)$ cannot be covered by finitely many O_α's at least one of balls $B(y_i,1/2)$ cannot be covered by finitely many O_α's, say $B(y_1,1/2)$.

This way we may form a sequence of balls

$B(x_1,1), B(y_1,1/2), B(z_1,1/4), \ldots$ so that none of these is covered by finitely many O_α's and

$$B(x_1, 1) \cap B(y_1, 1/2) \neq \phi,\ B(y_1, 1/2) \cap B(z_1, 1/4) \neq \phi, \ldots$$

$x_1, y_1, z_1, \ldots\ldots$

Let's rename the centers as $x_1, x_2, x_3, \ldots$ (where $y_1 = x_2$, $z_1 = x_3$)

So

i) $B(x_i, \frac{1}{2^{i-1}})$ is <u>not</u> covered by finitely many O_α's, for all i.

ii) $B(x_i, 1/2^{i-1}) \cap B(x_{i+1}, \frac{1}{2^i}) \neq \phi$, for all i.

If $w \in B(x_i, 1/2^{i-1}) \cap B(x_{i+1}, 1/2^i)$ then

$$d(x_i, x_{i+1}) \leq d(x_i, w) + d(w, x_{i+1})$$

$$< \frac{1}{2^{i-1}} + \frac{1}{2^i} < \frac{1}{2^{i-2}}.$$

In particular, if $n \geq m$ then

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m).$$

$$< \frac{1}{2^{n-3}} + \frac{1}{2^{n-4}} + \cdots + \frac{1}{2^{m-2}}$$

$$= \frac{1}{2^{m-2}} \underbrace{\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-m-2}}\right)}_{< 2}$$

$$< \frac{2}{2^{m-2}} = \frac{1}{2^{m-3}}.$$

Hence, $d(x_n, x_m) < \frac{1}{2^{m-3}}$, if $m \le n$.

<u>Claim:</u> (x_n) is Cauchy.

<u>Proof</u>: Given $\epsilon > 0$. Choose $n_0 \in \mathbb{N}$ so that $2^{n_0-3} > 1/\epsilon$. Then $\epsilon > \frac{1}{2^{n_0-3}}$.

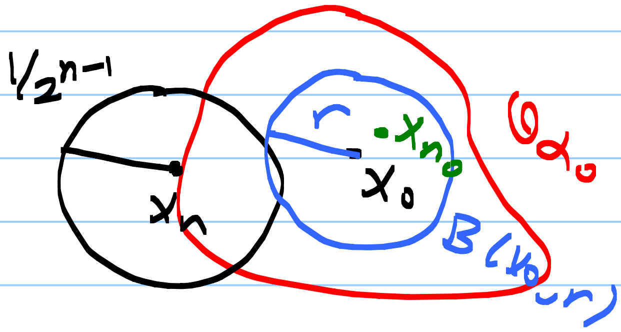
So, if $n \ge m \ge n_0$ then

$$d(x_n, x_m) \le \frac{1}{2^{m-3}} \le \frac{1}{2^{n_0-3}} < \epsilon.$$

Hence, the sequence (x_n) is Cauchy. ∎

Since (X, d) is a complete metric space this Cauchy sequence (x_n) is convergent to some element say $x_0 \in X$.

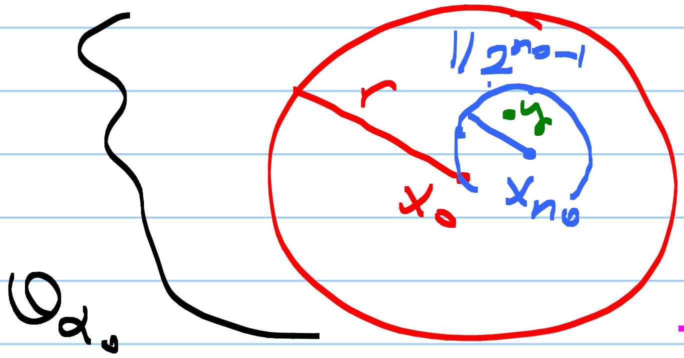
$\lim x_n = x_0$.



Since $X \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha$ and $x_0 \in X$ there is some O_{α_0} so that $x_0 \in O_{\alpha_0}$. So there is some $r > 0$ so that $B(x_0, r) \subseteq O_{\alpha_0}$, because O_{α_0} is an open subset.

Since $\lim x_n = x_0$ there is some n_0 so that if $n \ge n_0$ then $d(x_n, x_0) < r/2$. Choose n_0

big enough so that $\frac{1}{2^{n_0-1}} < r/2$.



$B(x_{n_0}, \frac{1}{2^{n_0-1}}) \subseteq B(x_0, r)$.

To see this let $y \in B(x_{n_0}, 1/2^{n_0-1})$.

Then $d(y, x_0) \leq d(y, x_{n_0}) + d(x_{n_0}, x_0)$
$< r/2 + r/2 = r$

$\Rightarrow y \in B(x_0, r)$.

Finally, since $B(x_0, r) \subseteq \mathcal{O}_{\alpha_{n_0}}$ we see that $B(x_{n_0}, \frac{1}{2^{n_0-1}}) \subseteq \mathcal{O}_{\alpha_{n_0}}$, which is a contradiction to the choice of $B(x_n, \frac{1}{2^{n-1}})$'s.

Hence, our assumption that (X, d) is not compact is false. So (X, d) must be compact.

This finishes the proof. ▬

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Proposition: The cartesian product of finitely many compact metric spaces is compact.

Proof $(X_1, d_1), \ldots, (X_k, d_k)$ compact metric spaces.

Let $X = X_1 \times \cdots \times X_k$ and ρ_r, $r \in [1, \infty]$.

$\rho_1 = d_1 + \cdots + d_k$, $\rho_r = \left(\sum d_i^r\right)^{1/r}$, $\rho_\infty = \max\{d_1, \ldots, d_k\}$

These metrics are equivalent and therefore a subset is open w.r.t. one metric ρ_r is open w.r.t. $\rho_{r'}$, for any $r' \in [1, \infty]$. Therefore, X is compact w.r.t. ρ_r if and only if it is compact w.r.t. $\rho_{r'}$, for any $r, r' \in [1, \infty]$.

Hence, we may work with any metric we wish on X. Let's work with d_∞.

To prove we use the sequential compactness. Let (x_n) be a sequence in (X, d_∞).

$x_n \in X = X_1 \times \cdots \times X_k$, $x_n = (a_n^1, a_n^2, \ldots, a_n^k)$, where

(a_n^i) is a sequence in (X_i, d_i).

<u>must show:</u> (x_n) has a convergent subsequence.

<u>Example:</u> $x_n = (\underset{\downarrow}{a_n}, b_n, c_n)$ with $a_n \in X_1$, $b_n \in X_2$, $c_n \in X_3$, $k = 3$

$(a_2, a_5, \underline{a_8}, \underline{a_{11}}, a_{20}, a_{109}, \underline{a_{250}}, \ldots) \longrightarrow \underline{a_0}$

$(b_2, b_5, b_8, b_{11}, b_{20}, \underline{b_{109}}, b_{250}, \ldots)$ may not converge.

$(\underline{b_8}, b_{11}, \underline{b_{109}}, b_{250}, \ldots) \longrightarrow \underline{b_0}$

$(c_8, c_{11}, c_{109}, c_{280}, \ldots)$ may not converge.

$(c_8, c_{109}, \ldots) \longrightarrow c_0$.

Now $x_8 = (a_8, b_8, c_8)$, $x_{109} = (a_{109}, b_{109}, c_{109}) \ldots$ and

$(x_8, x_{109}, \ldots) \longrightarrow (a_0, b_0, c_0)$.

I will leave the detail to you. ▬

Proposition: Let A be a precompact subset of a complete metric space. Then $\overline{A}$ is compact.

Proof: $A \subseteq X$, X complete.

Claim: $\overline{A}$ is also precompact.

Proof: Given $\epsilon > 0$. Since A is precompact we can cover A with finitely many $\epsilon/2$ balls:

$A \subseteq B(x_1, \epsilon/2) \cup B(x_2, \epsilon/2) \cup \ldots \cup B(x_n, \epsilon/2)$, for some $x_1, \ldots, x_n \in X$.

$A \subseteq B[x_1, \epsilon/2] \cup B[x_2, \epsilon/2] \cup \ldots \cup B[x_n, \epsilon/2]$, where the latter union is closed because it is the union finitely many closed subsets.

Hence, $\overline{A} \subseteq B[x_1, \epsilon/2] \cup \ldots \cup B[x_n, \epsilon/2]$.
Finally, each $B[x_i, \epsilon/2] \subseteq B(x_i, \epsilon)$ and thus

$\bar{A} \subseteq B(x_1, \epsilon) \cup B(x_2, \epsilon) \cup \cdots \cup B(x_n, \epsilon)$.

Hence, $\bar{A}$ is precompact.
On the other hand, $\bar{A}$ is a closed subset of the complete metric space (X, d) and thus $\bar{A}$ is a complete subspace.
Finally, $\bar{A}$ is compact because it is both complete and precompact, by the Sequential Characterization Theorem. ∎

<u>Theorem</u> (Heine–Borel Theorem) A subset of $\mathbb{R}^n$ is precompact if and only if it is bounded. A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

<u>Proof</u> $\mathbb{R}^n$, $d_r(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^r\right)^{1/r}$, $r \in [1, \infty)$

$d_\infty(x,y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$.

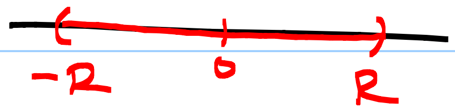
Let's work with d_∞ metric.
Let A be a precompact subset of $(\mathbb{R}^n, d_\infty)$.
We've proved earlier that A is bounded.
Now, assume that A is bounded.
<u>must show:</u> A is precompact.

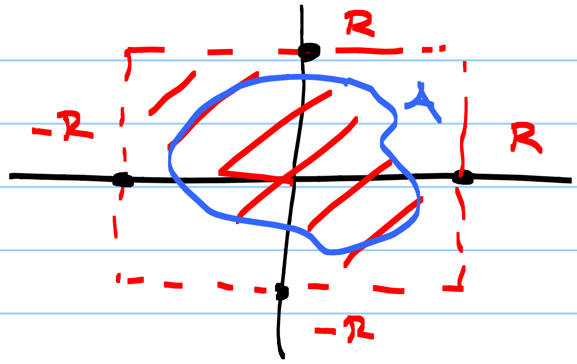
Since A is bounded there is some ball $B(0, R)$ so that $A \subseteq B(0, R)$.

$B(0, R) = \{x \in \mathbb{R}^n \mid d_\infty(x, 0) < R\}$.

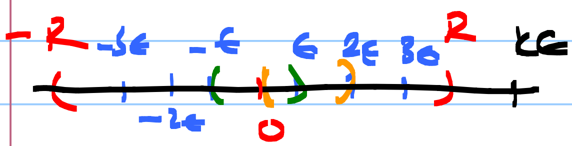
$d_\infty(x,0) = \max\{|x_1-0|, |x_2-0|, \dots, |x_n-0|\}$, $x=(x_1,\dots,x_n)$

$= \max\{|x_1|, |x_2|, \dots, |x_n|\}$.

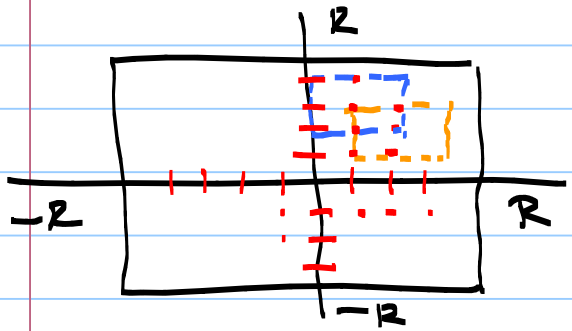




Now if $\epsilon > 0$ is given. Consider the squares of the form:



Clearly the balls $B(k\epsilon, \epsilon)$ covers $B(0,R)$, where $k \in \mathbb{N}$ so that $k\epsilon > R$.



Just take ϵ-balls around each red dot. They will cover $B(0,R)$

This way we conclude that $B(0,R)$ is covered by finitely many ϵ-balls so that A is precompact.

Here, for a subset A of $\mathbb{R}^n$ we have.

$\boxed{A \text{ is bounded}} \iff \boxed{A \text{ is precompact}}$

Also, since $\mathbb{R}^n$ is complete

$\boxed{A \text{ is closed}} \iff \boxed{A \text{ is complete.}}$

This finishes the proof. ∎

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Continuity and Compactness:

Proposition: Let $f: X \to Y$ be a continuous function of metric spaces. If $E \subseteq X$ is a compact subset then so is $f(E) \subseteq Y$.

Proof: Take any sequence (y_n) in $f(E)$.

Must show: (y_n) has a convergent subsequence.

Since $y_n \in f(E)$, there is some $x_n \in E$ so that $f(x_n) = y_n$. In particular, (x_n) is a sequence in E, which is a compact subspace. So (x_n) has a convergent subsequence, say (x_{k_n}) with

$\lim x_{k_n} = x_0$. Now, since f is continuous,

$$y_0 \doteq f(x_0) = f(\lim x_{k_n}) = \lim f(x_{k_n}) = \lim y_{k_n},$$ where

$y_0 \in f(E)$ since $x_0 \in E$. This finishes the proof. $\blacksquare$

Alternative Proof: To show that $f(E)$ is compact take any open cover $\{O_\alpha\}$ for $f(E)$:

$$f(E) \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha, \quad O_\alpha \subseteq Y \text{ open for all } \alpha \in \Lambda.$$

Then $E \subseteq f^{-1}(f(E)) \subseteq f^{-1}\left(\bigcup_{\alpha \in \Lambda} O_\alpha\right) = \bigcup_{\alpha \in \Lambda} f^{-1}(O_\alpha)$, where

each $f^{-1}(O_\alpha)$ is open in X since f is continuous, so that $\{f^{-1}(O_\alpha)\}_{\alpha \in \Lambda}$ is an open cover for E.

I'll leave the rest as an exercise!

Theorem (Homeomorphism Theorem)

Let $f: X \to Y$ be a continuous bijection. If X is compact then f is a homeomorphism.

Remark: $X = (\mathbb{R}, d)$, d: discrete metric.
$Y = (\mathbb{R}, |\cdot|)$

$f: X \to Y$, $f(x) = x$, for all $x \in \mathbb{R}$.

f is clearly a bijection. Since X has the discrete metric any subset of X is open and thus f is continuous.
However,
$$g = f^{-1}: Y \to X \text{ is } \underline{\text{not}} \text{ continuous.}$$

Because, $U = (0, 1]$ is open in X, but

$g^{-1}(U) = U = (0, 1]$ is not open in $Y = (\mathbb{R}, |\cdot|)$.

Proof of the Theorem: $f: X \to Y$ cont. bijection, X is compact. Let $g: Y \to X$ be the inverse function f^{-1}.

__must show:__ g is continuous.

Take any closed set $A \subseteq X$. Then

$g: Y \to X$

$g^{-1}(A) = \{y \in Y \mid \overset{x}{\overline{g^{-1}(y)}} \in A\}$

$= f(A)$

$x = g(y)$

$f(x) = f(g(y))$

$= y$

$A \subseteq X$ is a closed subset and thus A is compact. Since f is continuous $f(A)$ compact. Hence, $g^{-1}(A) = f(A)$ is a closed subset of Y,

so that g is continuous. Thus, f is a homeomorphism. ∎

Theorem (Minimum, Maximum Theorem)

Let (X, d) be a compact metric space and $f : (X, d) \to (\mathbb{R}, |\cdot|)$ be a continuous function.

Then there are points x_{max} and x_{min} in X so

that $f(x_{min}) \leq f(x) \leq f(x_{max})$, for all $x \in X$.

Proof: $f(X)$ is a compact subset of $(\mathbb{R}, |\cdot|)$.

In particular, $f(X)$ is closed and bounded.

Hence, $\sup f(X)$ and $\inf f(X)$ exist. Moreover, since $f(X)$ is closed both $\sup f(X)$ and $\inf f(X)$ belongs to $f(X)$

$M \doteq \sup f(X) \in f(X) \Rightarrow \exists\, x_{max} \in X$ so that $f(x_{max}) = M$.

$m \doteq \inf f(X) \in f(X) \Rightarrow \exists\, x_{min} \in X$ s.t. that $f(x_{min}) = m$.

Now, if $x \in X$ then $m \leq f(x) \leq M$. $\blacksquare$

<u>Corollary</u> If $f : [a,b] \to \mathbb{R}$ is a continuous function, where both have the absolute value metric, then f has a maximum and minimum.

<u>Proof.</u> $[a,b] \subseteq \mathbb{R}$ is a closed and bounded subset of $\mathbb{R}$. Hence, $[a,b]$ is compact. So we are done by the previous Max-Min Theorem. $\blacksquare$

<u>Corollary</u> If (X,d) is a compact metric space then $C(X)$ is a closed subset of $\mathcal{B}(X)$. In particular, $C(X)$ is complete.

<u>Proof</u>: $\mathcal{B}(X) = \{ f : X \to \mathbb{R} \mid f \text{ is bounded} \}$

$C(X) = \{ f : X \to \mathbb{R} \mid f \text{ is continuous} \}$

Note that since X is compact any continuous

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function $f: X \to \mathbb{R}$ is bounded (by its extreme values). So $C(X) \subseteq B(X)$.

For the second statement, recall that we have proved that if (f_n) is a sequence of continuous functions on X and $\lim f_n = f_0$ (d_∞-metric) then f_0 is also continuous.

In particular, $\overline{C(X)} = C(X)$, so that

$C(X)$ is a closed subset of $B(X)$.

<u>Lebesgue Number Of a Open Covering</u>:

Let $E \subseteq X$ be a compact subset and $\{O_\alpha\}_{\alpha \in \Lambda}$ an open cover for E. Then $E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha$.

For any $x \in E$ there is some O_{α_x} containing x. Since O_{α_x} is open there is some $\epsilon_x > 0$ so that

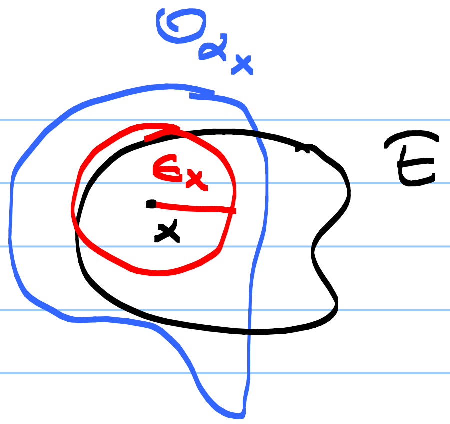
$$x \in B(x, \epsilon_x) \subseteq O_{\alpha_x}.$$

Then $$E = \bigcup_{x \in E} \{x\} \subseteq \bigcup_{x \in E} B(x, \epsilon_x/2).$$

Hence $\{B(x, \epsilon_x/2)\}_{x \in E}$ is also an open cover for E.

Since E is compact $E \subseteq B(x_1, \epsilon_{x_1}/2) \cup \dots \cup B(x_k, \epsilon_{x_k}/2)$, for some $x_1, \dots, x_k \in E$.

Let $\delta = \min\{\epsilon_{x_1}/2, \dots, \epsilon_{x_k}/2\}$.



Let $x \in E$, then $x \in B(x_i, \epsilon_{x_i}/2)$ for some $i \in \{1, \dots, k\}$.

Now if $y \in B(x, \delta)$ then

$$d(y, x_i) \le d(y, x) + d(x, x_i) < \delta + \epsilon_{x_i}/2 \le \epsilon_{x_i}/2 + \epsilon_{x_i}/2$$

$\Rightarrow d(y, x_i) < \epsilon_{x_i}$ so that $y \in B(x_i, \epsilon_{x_i})$.

Hence, $B(x, \delta) \subseteq B(x_i, \epsilon_{x_i}) \subseteq O_{\alpha_{x_i}}$.

This $\delta > 0$ is called a Lebesgue number of the covering.
For any $x \in E$ the ball $B(x, \delta) \subseteq O_\alpha$ for some $\alpha \in \Lambda$.

Theorem: Let $f : X \to Y$ be a continuous function. If X is a compact metric space then f is uniformly continuous.

Proof: Let $\epsilon > 0$ be given. For any $x \in X$ choose some $\delta_x > 0$ so that

$$f(B(x, \delta_x)) \subseteq B(f(x), \epsilon/2).$$

$$X = \bigcup_{x \in X} \{x\} \subseteq \bigcup_{x \in X} B(x, \delta_x) \subseteq X$$ and thus

$X = \bigcup_{x \in X} B(x, \delta_x)$ so that $\{B(x, \delta_x)\}_{x \in X}$ is an open cover for the compact space X. Let $\delta > 0$ be a Lebesgue number for this open cover.

Let $x, y \in X$ with $d(x,y) < \delta$. Choose some $x' \in X$ with $B(x,\delta) \subseteq B(x', \delta_{x'})$.

Then $x, y \in B(x,\delta) \subseteq B(x', \delta_{x'})$.

Hence, $f(x), f(y) \in B(f(x'), \epsilon/2)$. Finally,

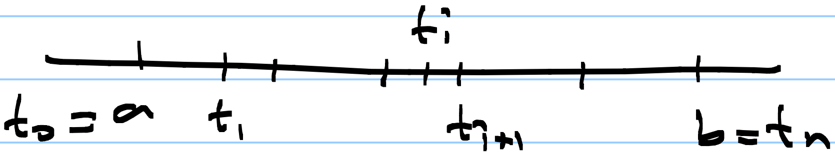
$$\rho(f(x), f(y)) \leq \rho(f(x'), f(x)) + \rho(f(x'), f(y))$$
$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence, f is uniformly continuous on X. ∎

Riemann Integrability of Functions:

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function. Take any partition

$$P = \{t_0 = a < t_1 < \cdots < t_n = b\} \text{ for } [a,b].$$



Norm of $P = \max\{|t_{i+1} - t_i| \mid i = 0, \ldots, n-1\}$ and we denote it as $|P|$.

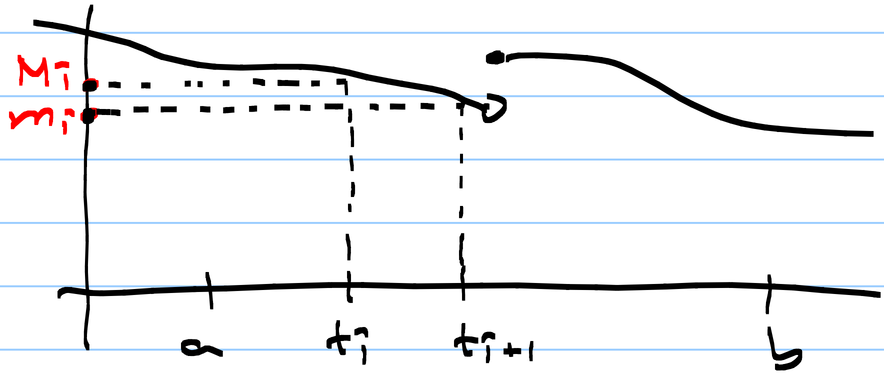
The upper and lower Riemann sums of f with respect to this partition are defined by

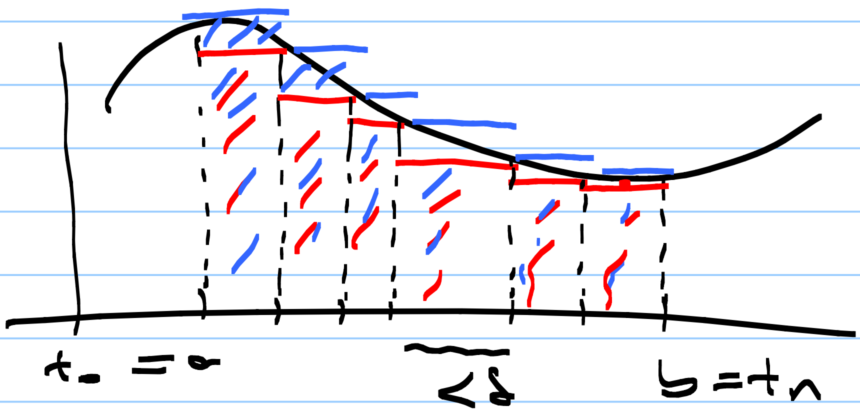
$$U(f,P)=\sum_{i=0}^{n-1}(t_{i+1}-t_i)\,M_i \quad \text{and}$$

$$L(f,P)=\sum_{i=0}^{n-1}(t_{i+1}-t_i)\,m_i, \quad \text{where}$$

$$M_i = \sup\{f(x) \mid x\in[t_i,t_{i+1}]\} \quad \text{and}$$

$$m_i = \inf\{f(x) \mid x\in[t_i,t_{i+1}]\},\ \text{for } i=0,\dots,n-1.$$





Definition: f is called Riemann integrable if for any $\epsilon>0$ there is some $\delta>0$ so that for any partition P of the interval $[a,b]$ with $|P|<\delta$ we have

$$0\le U(f,P)-L(f,P)<\epsilon.$$

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Theorem: Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of bounded Riemann integrable functions converging uniformly to some $f \in B([a,b])$. Then f is Riemann Integrable and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$$

Here, for a Riemann integrable function f the integral of f over $[a,b]$ is defined to be the limit

$$\int_a^b f(x)\,dx = \lim_{|P|\to 0} U(f,P) = \lim_{|P|\to 0} L(f,P)$$

Ex The function $f:[0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$$

For any partition P of $[0,1]$ we have

$U(f,P) = 1$ and $L(f,P) = 0$ so that f is not Riemann Integrable.

Proof: Let $\epsilon > 0$ be given. Choose $n_0 \in \mathbb{N}$ so that

$n \geq n_0$ implies $d_{sup}(f, f_n) < \epsilon/3(b-a)$.

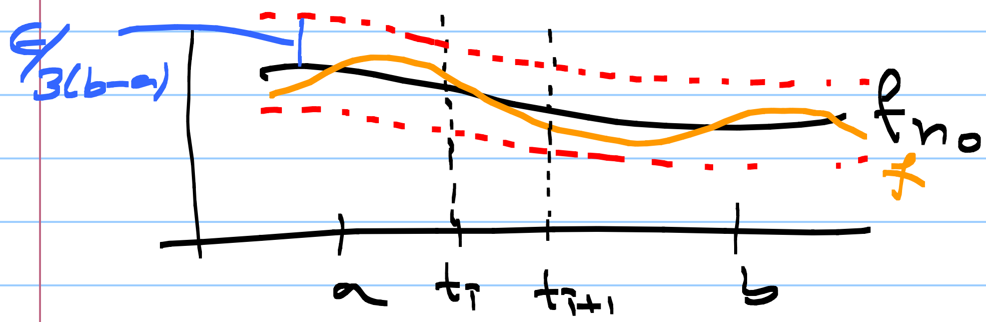
Since each f_n is Riemann integrable f_{n_0} is Riemann Integrable and thus there is some $\delta > 0$ so that for any partition P of $[a,b]$ with $|P| < \delta$

then $0 \le U(f_{n_0}, P) - L(f_{n_0}, P) < \epsilon/3$.

Since $|f_{n_0}(x) - f(x)| < \epsilon/3(b-a)$ for all $x \in [a,b]$

$|U(f_{n_0}, P) - U(f, P)| < \epsilon/3$ and

$|L(f_{n_0}, P) - L(f, P)| < \epsilon/3$.



$$|\sup\{f_{n_0}(t) \mid t \in [t_i, t_{i+1}]\} - \sup\{f(t) \mid t \in [t_i, t_{i+1}]\}| < \epsilon/3(b-a)$$

Then we get

$$\begin{aligned} |U(f,P) - L(f,P)| &\le |U(f,P) - U(f_{n_0}, P)| \\ &+ |U(f_{n_0}, P) - L(f_{n_0}, P)| \\ &+ |L(f_{n_0}, P) - L(f, P)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

Hence, f is Riemann integrable.

We have also $|U(f_n, P) - U(f, P)| < \epsilon/3$, for any $n \ge n_0$, and partition $|P| < \delta$.

Let $|P| \to 0$ to get

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| < \epsilon/3, \text{ for all } n \geq n_0.$$

Hence, $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$

This finishes the proof. ∎

Series of Functions

Let (f_n) be a sequence of functions

$f_n : X \to \mathbb{R}$ (or $\mathbb{C}$).

Define s_n as the $n^{\underline{th}}$ partial sum

$$s_n(x) = f_1(x) + \cdots + f_n(x).$$

If the sequence $(s_n(x))$ converges (in the supremum metric) in $B(X)$ then we say that the series

$\displaystyle\sum_{i=1}^{\infty} f_i(x)$ is convergent and

$$\sum_{i=1}^{\infty} f_i(x) = \lim_{n\to\infty} s_n(x).$$

We usually say in this case that the series $\displaystyle\sum_{i=1}^{\infty} f_i(x)$ converges uniformly.

Weierstrass M-Test:

Given a series of functions $\sum_{n=1}^{\infty} f_n(x)$, $f_n: X \to \mathbb{R}/\mathbb{C}$,

when each $f_n \in B(X)$. Assume that there is a sequence of positive real numbers (M_n) with

$$0 \leq \sup\{|f_n(x)| \mid x \in X\} \leq M_n, \text{ for all } n.$$

If the series $\sum_{n=1}^{\infty} M_n$ is convergent then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

Proof: We must show that the sequence (s_n) is convergent. Since $B(X)$ is complete it is enough to show that (s_n) is a Cauchy sequence.

Let $\epsilon > 0$ be given. Since $\sum_{n=1}^{\infty} M_n$ is convergent the sequence (t_n) of partial sums of $\sum_{n=1}^{\infty} M_n$ is convergent.

$$t_1 = M_1, \quad t_2 = M_1 + M_2, \ldots, t_n = M_1 + M_2 + \cdots + M_n, \ldots$$

In particular, (t_n) is Cauchy. So there is some $n_0 \in \mathbb{N}$ so that

$m, n \geq n_0$ implies $|t_n - t_m| < \epsilon$.

$$|M_n + M_{n-1} + \cdots + M_{m+2} + M_{m+1}| < \epsilon.$$

Note that $|f_n(x)| \leq M_n$ for all $x \in X$ and n.

$$\Rightarrow |f_n(x) + \cdots + f_{m+1}(x)| \leq M_n + \cdots + M_{m+1} < \epsilon.$$

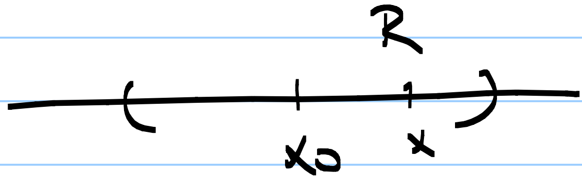
$$\Rightarrow |s_n(x) - s_m(x)| < \epsilon, \text{ for all } m, n \geq n_0.$$

Hence, $(s_n(x))$ converges in the supremum metric.

This finishes the proof. ∎

<u>Example:</u> Consider a power series $\sum_{n=0}^{\infty} \underbrace{a_n(x-x_0)^n}_{f_n(x)}$. Let $R > 0$ be the radius of convergence of this series

$$R = \limsup \left| \frac{a_n}{a_{n+1}} \right|.$$



Then $|x - x_0| < R$

$$\Rightarrow |x - x_0|^n < R^n \Rightarrow |a_n(x-x_0)^n| < |a_n| R^n$$
$$\forall x \in (x_0 - R, x_0 + R).$$

<u>Exercise:</u> Let $0 \leq R_1 < R$ and $M_n = |a_n| R_1^n$. Show that $\sum M_n$ is convergent.

So by the Weierstrass M-test the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges uniformly on any interval $[a,b] \subseteq (x_0-R, x_0+R)$.

<u>Remark</u>: Clearly, each $f_n(x)=a_n(x-x_0)^n$ is Riemann integrable on any $[a,b] \subseteq (x_0-R, x_0+R)$.
Thus
$$\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$ is Riemann integrable and if $f(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$, then

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \int_a^b \sum_{n=0}^{N} a_n(x-x_0)^n\,dx$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} \int_a^b a_n(x-x_0)^n\,dx \qquad \begin{pmatrix} a=x_0 \\ b=x \end{pmatrix}$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} a_n \left.\frac{(x-x_0)^{n+1}}{n+1}\right|_{x_0}^{x}$$

$$= \sum_{n=0}^{\infty} a_n \frac{(x-x_0)^{n+1}}{n+1}.$$

So, $$\int_{x_0}^{x} \sum_{n=0}^{\infty} a_n(t-x_0)^n\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1}.$$

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CONNECTEDNESS

A subset E of a metric space (X,d) is called disconnected if we can find open subsets A and B of (X,d) so that

i) $E \subseteq A \cup B$

ii) $E \cap A \neq \emptyset$, $E \cap B \neq \emptyset$

iii) $E \cap A \cap B = \emptyset$.

If E is not disconnected we say that E is connected.

If $E = X$ and X is connected then we say that (X,d) is a connected metric space.

In case E is disconnected then the pair of subsets A, B as above is called a separation for E.

<u>Remark</u>: Assume X is disconnected and $X = A \cup B$ for some open subsets with

i) $A \neq \emptyset$ and $B \neq \emptyset$

ii) $A \cap B = \emptyset$

Note that in this case $A = X \setminus B$ is both open and closed. Similarly, B is both open and closed.

Hence, if X is connected the only subsets of X which are both open and closed should be $\emptyset$ and X itself. Otherwise, i.e., if A is a subset of X which is both open and closed so that $A \neq \emptyset$ and $A \neq X$ then $A, B = X \setminus A$ would be a separation for X.

<u>Proposition</u>: The interval $[0,1]$ in $(\mathbb{R}, |\cdot|)$ is connected.

<u>Proof</u>: Assume on the contrary that $E=[0,1]$ is disconnected and $A, B \subseteq \mathbb{R}$ are open subsets so that

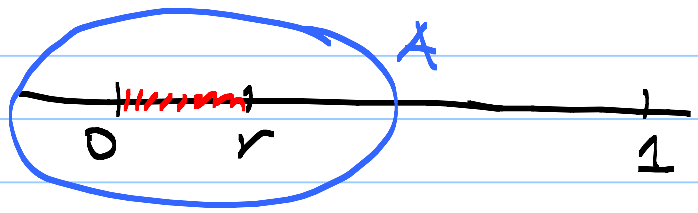
i) $E \subseteq A \cup B$,

ii) $E \cap A \neq \emptyset$ and $E \cap B \neq \emptyset$

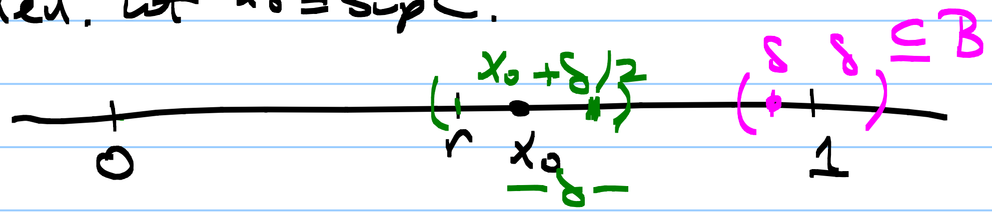
iii) $E \cap A \cap B = \emptyset$.

Since $0 \in E \subseteq A \cup B$ we may without loss of generality that $0 \in A$. Since A is open there is some $r_0 > 0$ so that

$(0 - r_0, 0 + r_0) \subseteq A$.



Let $C = \{ r > 0 \mid [0, r) \subseteq A \}$. Note that $r_0 \in C$ so that $C \neq \emptyset$. Note that $1 \notin C$ because otherwise $[0,1) \subseteq A$ and since $[0,1] \cap B \neq \emptyset$ Hence, $1 \in B$. Choose $\delta > 0$ so that $(1-\delta, 1+\delta) \subseteq B$ would lead to a contradiction. Hence, $1 \notin C$ and C is bounded. Let $x_0 = \sup C$.



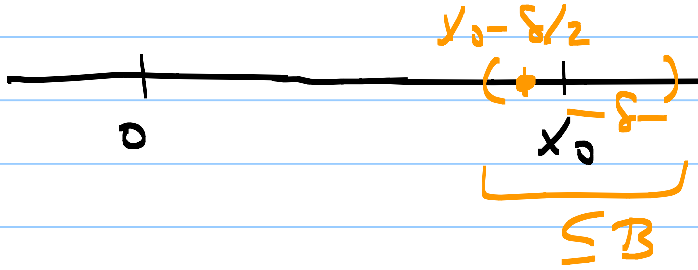
$x_0 \in [0,1] \subseteq A \cup B$ so that we have two cases:

<u>$x_0 \in A$</u> or <u>$x_0 \in B$</u>. If $x_0 \in A$ then choosing

$\delta > 0$ so that $(x_0 - \delta, x_0 + \delta) \subseteq A$ we see that

$[0, x_0 + \frac{\delta}{2}) \subseteq A$ so that $x_0 + \frac{\delta}{2} \in C$, contradicting

that $x_0 = \sup C$. Hence, $x_0 \notin A$ so that $x_0 \in B$.



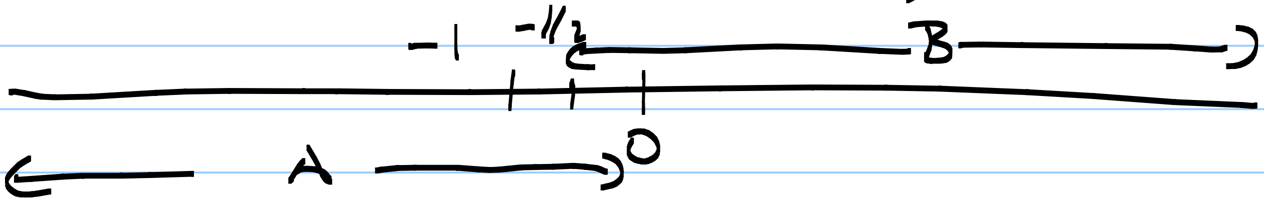
Since B is also open there is some $\delta > 0$ so that $(x_0 - \delta, x_0 + \delta) \subseteq B$. In particular, $x_0 - \frac{\delta}{2} \in B$

and $x_0 - \frac{\delta}{2} \in A$ so that $A \cap B \cap [0,1] \neq \emptyset$, a contradiction.

Hence, the interval $[0,1]$ is connected. ∎

<u>Example:</u> $\mathbb{Z} \subseteq \mathbb{R}$ is <u>not</u> connected.

Let $A = (-\infty, 0)$ and $B = (-1/2, \infty)$ then



i) $A \cap \mathbb{Z} \neq \emptyset$, $B \cap \mathbb{Z} \neq \emptyset$

ii) $\mathbb{Z} \subseteq A \cup B$

iii) $A \cap B \cap \mathbb{Z} = \emptyset$.

Hence, $\mathbb{Z}$ is disconnected.

Proposition: Let $f: X \to Y$ be a continuous map and $E \subseteq X$ a connected subset. Then $f(E) \subseteq Y$ is connected.

Proof: Assume on the contrary that $f(E)$ is not connected. Say $A, B \subseteq Y$ are open subsets with

i) $f(E) \cap A \neq \phi$, $f(E) \cap B \neq \phi$
ii) $f(E) \subseteq A \cup B$
iii) $f(E) \cap A \cap B = \phi$.

Let $A' = f^{-1}(A)$ and $B' = f^{-1}(B)$, which are both open subsets of X, because f is continuous.

Note that taking preimage via f we get

i) $f(E) \cap A \neq \phi \Rightarrow f^{-1}(f(E) \cap A) \neq \phi$
Let $y \in f(E) \cap A$. Then there is some $x \in E$ st. $y = f(x)$. Since $f(x) = y \in A$ we have $x \in f^{-1}(A) = A'$. Thus $x \in E \cap A'$ so that $E \cap A' \neq \phi$.

Similarly, $E \cap B' \neq \phi$.

ii) $f(E) \subseteq A \cup B \Rightarrow E \subseteq f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
$\Rightarrow E \subseteq A' \cup B'$.

iii) $f(E) \cap A \cap B = \phi \Rightarrow E \cap f^{-1}(A) \cap f^{-1}(B) = \phi$
$\Rightarrow E \cap A' \cap B' = \phi$.

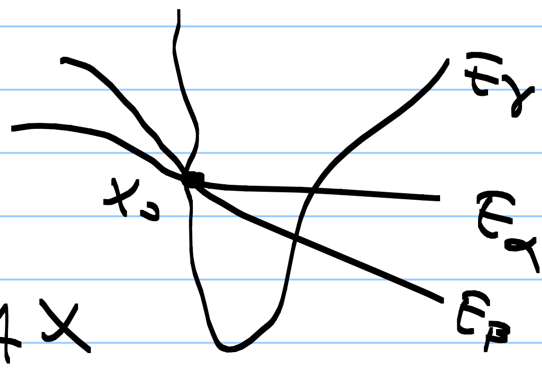
Hence, E is a disconnected subset of (X, d), which is a contradiction. Thus $f(E)$ must be connected. ∎

Proposition: Assume that $\{E_\alpha\}_{\alpha\in\Delta}$ is a collection of connected subsets of a metric space so that

$$\bigcap_{\alpha\in\Delta} E_\alpha \neq \emptyset.$$

Then $\bigcup_{\alpha\in\Delta} E_\alpha$ is connected.

Proof: Let $x_0 \in \bigcap_{\alpha\in\Delta} E_\alpha$.



Let A, B be open subsets of X giving a seperation for $\bigcup_\alpha E_\alpha$.

Then each $E_\alpha \subseteq \bigcup_\alpha E_\alpha \subseteq A \cup B$. Since E_α is connected so E_α should be completely inside A or B.

On the other hand $x_0 \in A \cup B$ and then $x_0 \in A$, without loss of generality. This implies that each E_α must lie in A.

So $\bigcup_{\alpha\in\Delta} E_\alpha \subseteq A$ and thus $\left(\bigcup_{\alpha\in\Delta} E_\alpha\right) \cap B = \emptyset$, a contradiction. ∎

Note that for any interval $[a,b]$ we have a homeomorphism

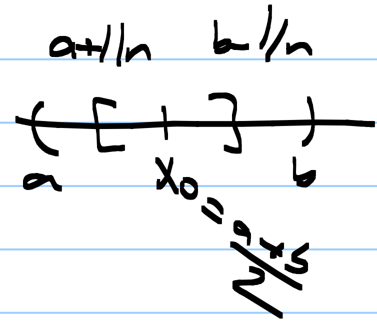
$$f : [0,1] \longrightarrow [a,b], \text{ given by}$$

$$f(t) = a + t(b-a),\ t \in [0,1].$$

Since $[0,1]$ is connected and $[a,b] = f([0,1])$ we see that $[a,b]$ is connected.

This implies that any interval I is connected.

i) $I = (a,b) = \bigcup_{n=n_0}^{\infty} \underbrace{[a+1/n,\ b-1/n]}_{\text{connected}}$

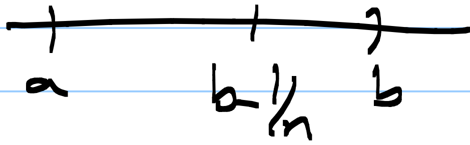


$\Rightarrow I$ is connected.

ii) $I = [a, \infty) = \bigcup_{n=1}^{\infty} \underbrace{[a, a+n]}_{\text{connected}}$ $\quad x_0 = a \in [a, a+n]$

$\Rightarrow I = [a, \infty)$ is connected.

iii) $I = [a, b) = \bigcup_{n=n_0}^{\infty} \underbrace{[a, b-\frac{1}{n}]}_{\text{connected}}$

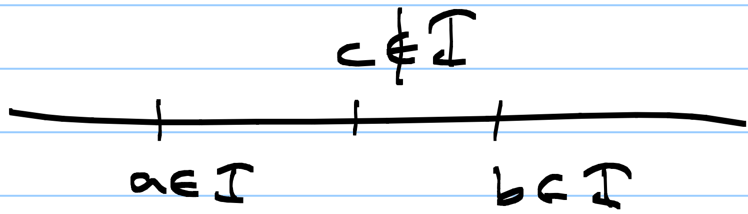


$x_0 = a \in [a, b-1/n]$. $\Rightarrow I = [a,b)$ is connected.

<u>Theorem</u> A subset I of $\mathbb{R}$ is connected if and only if it is an interval.

Proof. We've already seen that intervals are connected.

Conversely, $I \subseteq \mathbb{R}$ be a subset which is not an interval. Hence, there are points $a, b \in I$ so that $[a,b] \not\subseteq I$. Say $c \in [a,b]$ and $c \notin I$.



Let $A = (-\infty, c)$, $B = (c, \infty)$ which are both open.

i) $I \subseteq \mathbb{R} \setminus \{c\} = A \cup B$

ii) $a \in A \cap I \Rightarrow A \cap I \neq \emptyset$

$b \in B \cap I \Rightarrow B \cap I \neq \emptyset$

iii) $I \cap A \cap B = \emptyset$.

Hence, I is disconnected. This finishes the proof. ∎

Theorem (Intermediate Value Theorem)

Let $f: X \to \mathbb{R}$ be a continuous function, where X is a connected space. If $a = f(x_0)$ and $b = f(x_1)$ for some $x_0, x_1 \in X$, then for any $c \in [a,b]$ there $\exists$ some $x \in X$ with

$$f(x) = c.$$

Proof Since f is continuous and X is connected

$f(X)$ is a connected subset of $\mathbb{R}$. Hence, $f(X) = I$ is an interval.

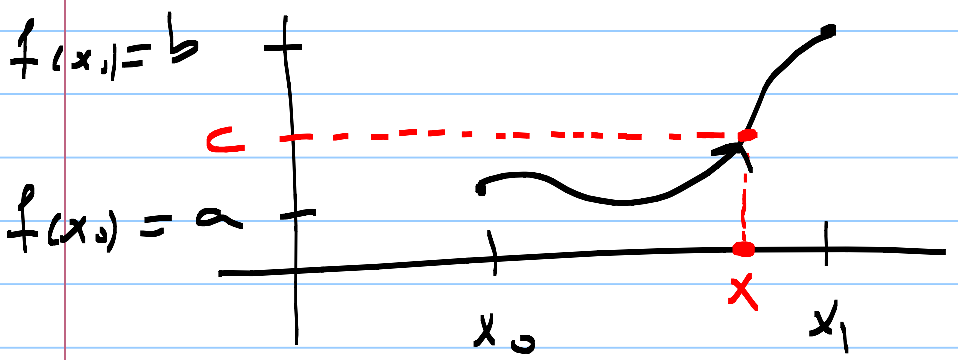
By assumption, $a = f(x_0) \in I$ and $b = f(x_1) \in I$. Since I is an interval,

$$[a,b] \subseteq I.$$

In particular, $c \in [a,b) \subseteq I = f(X)$.

So, there is some $x \in X$ with $f(x) = c$. ∎

Corollary: If $f: \mathbb{R} \to \mathbb{R}$ is a continuous function, $f(x_0) = a$ and $f(x_1) = b$ and $c \in [a,b)$, then there is some $x \in [x_0, x_1]$ with $f(x) = c$.



Proposition: Let E be a connected subset of a metric space X. If $E \subseteq F \subseteq \bar{E}$, then F is connected.

Proof: Suppose we have two open subsets A

and B of X with $F \subseteq A \cup B$ and $A \cap B \cap F = \emptyset$.

Since $E \subseteq F \subseteq A \cup B$, $E \cap A \cap B \subseteq F \cap A \cap B = \emptyset$ so that $E \cap A \cap B = \emptyset$. Since E is connected by assumption E must lie completely either in A or B. Without loss of generality, assume that $A \cap E = \emptyset$ (so $E \subseteq B$).

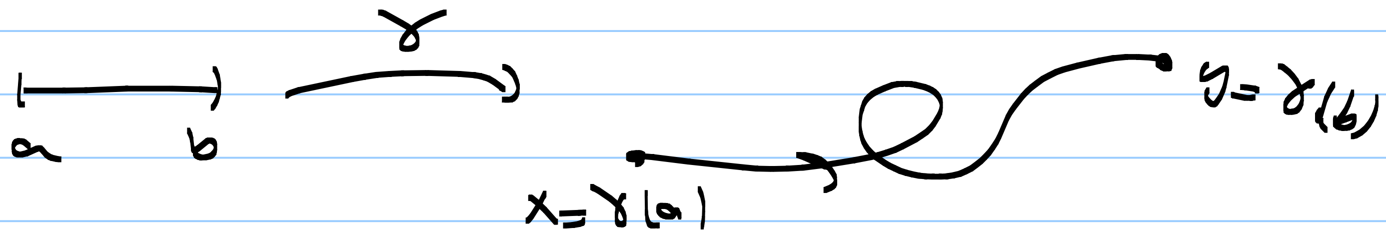
$E \subseteq X \setminus A$, where $X \setminus A$ is a closed subset. Hence, $\overline{E} \subseteq X \setminus A$. So $\overline{E} \cap A = \emptyset$. Hence,

$F \cap A = \emptyset$ because $F \subseteq \overline{E}$, so that $F \subseteq B$.

Hence, F must be connected. ∎

Connected Components:

Let $x, y \in X$ be points in some metric space. If there is some continuous function $\gamma : [a,b] \to X$ with $\gamma(a) = x$, $\gamma(b) = y$, we say that x and y are connected by the path γ.



If any two points of X are connected by a path, then we say that X is *path connected*.

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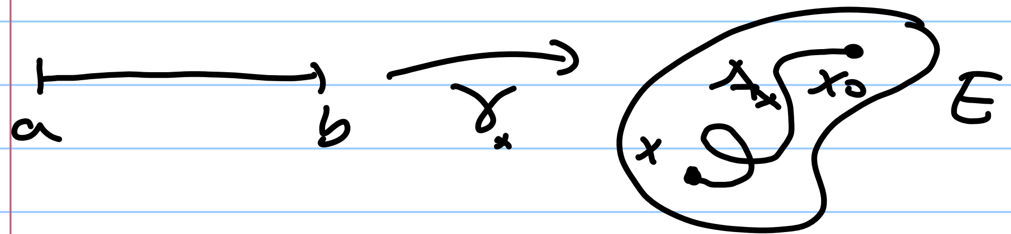
Proposition: Any path connected subset E is connected.

Proof: $E \subseteq X$ is path connected.

must show: E is connected.

Let $x_0 \in E$ be a fixed point and $x \in E$ any other point. Since E is path connected there is some continuous path $\gamma_x : [a,b] \to E \subseteq X$ so that $\gamma_x(a) = x_0$ and $\gamma_x(b) = x$.



$A_x \subseteq E$

Since $[a,b] \subseteq \mathbb{R}$ is connected and γ is continuous its image $\gamma_x([a,b])$ is connected.

$A_x \doteq \gamma_x([a,b])$. $x, x_0 \in A_x$

$$x_0 \in \bigcap_{x \in E} A_x \Rightarrow \bigcap_{x \in E} A_x \neq \emptyset.$$

Since each A_x is connected we have $\bigcup_{x \in E} A_x$ is connected.

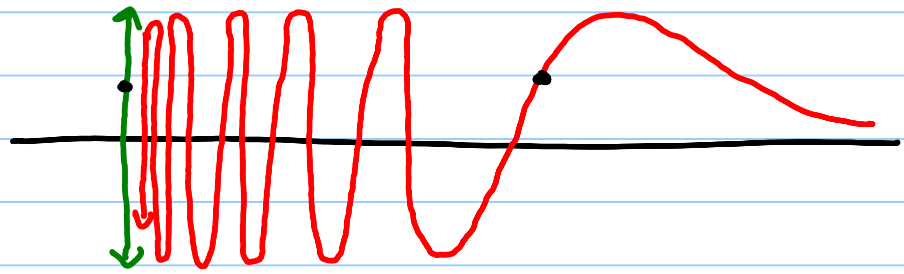
$$\bigcup_{x \in E} A_x \subseteq E = \bigcup_{x \in E} \{x\} \subseteq \bigcup_{x \in E} A_x.$$ It follows

that $E = \bigcup_{x \in E} A_x$. Since $\bigcup_{x \in E} A_x$ is connected we are done. $\blacksquare$

Remark: There are connected subsets of $(\mathbb{R}^2, d_2)$ which are not path connected.

Example: Topologist Sine Curve

$$E = \{(0,y) \mid y \in \mathbb{R}\} \cup \{(x, \sin\tfrac{1}{x}) \mid x > 0\}$$

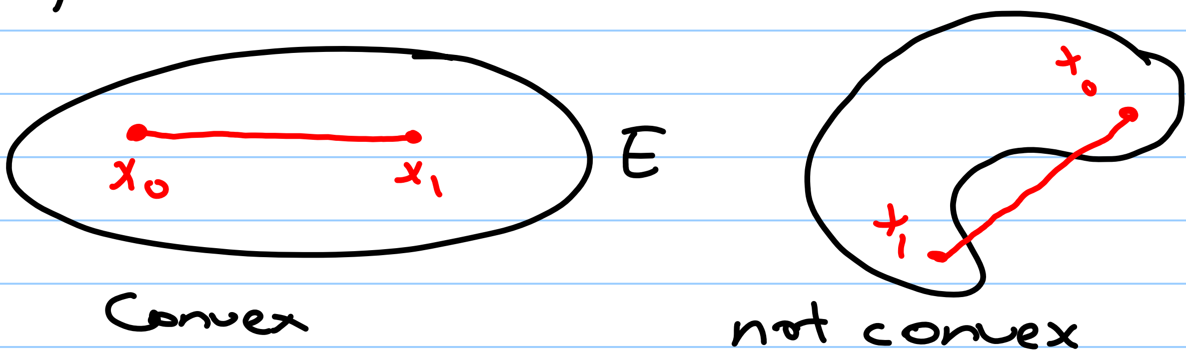


Claim: E is connected but <u>not</u> path connected.

Remark: Arcwise connected = Pathwise connected

Proposition: A convex subset of $\mathbb{R}^k$ is pathwise connected and hence connected. $\mathbb{R}^k$ is pathwise connected. A ball in $\mathbb{R}^k$ is pathwise connected.

Proof: Let E be a convex subset of $\mathbb{R}^k$.



Recall that a subset $E \subseteq \mathbb{R}^k$ is called convex if whenever we are given two points $x_0, x_1 \in E$ the line segment

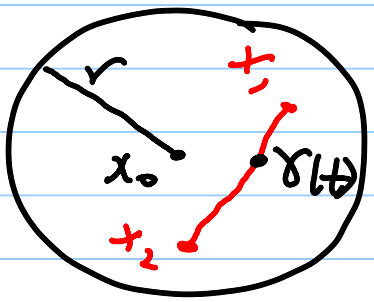
$\gamma : [0,1] \to \mathbb{R}^k$, $\gamma(t) = (1-t)x_0 + tx_1$ has image in E.

$$\gamma(0) = x_0, \quad \gamma(1) = x_1$$

Clearly, $\gamma(t)$ is continuous and it joins x_0 to x_1. Hence, E is path connected.

Clearly, $\mathbb{R}^k$ is convex and path conneced.

Finally, let $E = B(x_0, r)$ be a ball in $\mathbb{R}^k$.



Claim: If $x_1, x_2 \in B(x_0, r)$ then $\gamma(t) = (1-t)x_1 + tx_2 \in B(x_0, r)$ for all $t \in [0,1]$.

Note that the claim shows that $B(x_0, r)$ is convex and thus $B(x_0, r)$ is path connected.

$$d(x_0, \gamma(t)) = \sqrt{(a(t) - a_0)^2 + (b(t) - b_0)^2}$$

$x_0 = (a_0, b_0)$, $\gamma(t) = (a(t), b(t)) = (1-t)(a_1, b_1) + t(a_2, b_2)$

$x_1 = (a_1, b_1)$ $\quad \Rightarrow \gamma(t) = ((1-t)a_1 + ta_2, (1-t)b_1 + tb_2)$

$x_2 = (a_2, b_2)$

Exercise: Finish the proof (for any d_p, $p \in [1, \infty]$).

Connected Component:

Let $x, y \in X$ be arbitrary points. We say that x and y are related and write $x \sim y$ if there is a connected subset E of X containing both x and y.

$x \sim y$.

Claim: This is an equivalence relation on X.

Proof: i) (Reflexive) Let $x \in X$, then $E = \{x\}$ is clearly connected and $x, x \in E$. Hence, $x \sim x$.

ii) (Symmetric) Let $x, y \in X$ with $x \sim y$. Then there is a connected subset E of X with $x, y \in E$. Then, $y, x \in E$ and thus $y \sim x$.

iii) (Transitive) Let $x, y, z \in X$ with $x \sim y$ and $y \sim z$. So there are connected subsets E_1 and E_2 so that

$$x, y \in E_1 \quad \text{and} \quad y, z \in E_2.$$

Since $y \in E_1 \cap E_2$ and both subsets are connected we see that $E_1 \cup E_2$ is connected. $x, z \in E_1 \cup E_2$ and hence $x \sim z$.

This finishes the proof. ∎

Hence, $\sim$ is an equivalence relation on X.

For any $x \in X$ let E_x be the equivalence class of $\sim$ containing the point x.

$$E_x = \{y \in X \mid x \sim y\}$$

Claim E_x is the union of all connected subsets of X containing x. In particular, E_x is the largest connected subset containing x.

Proof: Let A be any connected subset of X containing x. Then for any $y \in A$ we have $x \sim y$. Hence, $y \in E_x$. So $A \subseteq E_x$ and thus

E_x contains all the connected subsets of X containing x.

So, $\displaystyle\bigcup_{\substack{x \in A \\ A \subseteq X \text{ connected}}} A \subseteq E_x$. Since the subsets in the union have a common point, namely x, their union is also connected.

Hence, $\displaystyle\bigcup_{\substack{x \in A \\ A \subseteq X \text{ connected}}} A$ is the largest connected subset containing E_x.

Finally, if $y \in E_x$ then there is a connected set A containing both x, y. Hence A is one of the subsets in the union $\displaystyle\bigcup_{\substack{x \in A \\ A \subseteq X \text{ connected}}} A$.

So, $y \in A \subseteq \bigcup_{\substack{x\in A \\ A\subseteq X \text{ connected}}} A \implies E_x \subseteq \bigcup_{\substack{x\in A \\ A\subseteq X \text{ conn.}}} A$

Thus, $E_x = \bigcup_{\substack{x\in A \\ A\subseteq X \text{ connected}}} A$.

<u>Claim:</u> E_x is closed.

<u>Proof:</u> Since E_x is connected $\overline{E_x}$ is also connected. Since E_x is the largest connected set containing x we see that $\overline{E_x} \subseteq E_x$. Hence $E_x = \overline{E_x}$.

As a summary we have: For any $x \in X$ the subset E_x, which is the largest connected subset of X containing x, is called the connected component of X containing x.

$X = \bigcup_{x\in X} E_x$, E_x is the equivalence class of x.

If $E_x \neq E_y$ then $E_x \cap E_y = \emptyset$.

Hence, X is written as the disjoint union of its (closed) connected components.

<u>Proposition:</u> Let X be a metric space and $A \subseteq X$ a connected subset, which is both open and closed. Then $A = E_x$ for some $x \in X$.

Proof: Note that $X = A \cup (X\setminus A)$, where both A and $X \setminus A$ are open. Take any $x \in A$.

Then $A \subseteq E_x$, because $x \in A$ and A is connected.

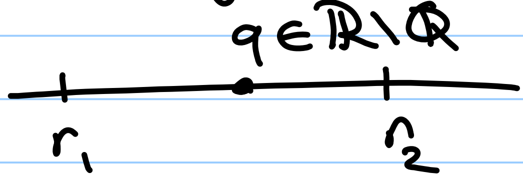
However, $E_x \subseteq A \cup (X \setminus A)$. Since E_x is connected and $x \in A \cap E_x$ then other subset $(X\setminus A)\cap E_x$ should be empty, because otherwise E_x would be disconnected.

So $E_x \cap (X \setminus A) = \emptyset \Rightarrow E_x \subseteq A$.

Thus, $A = E_x$. ∎

Example: Consider the metric space $(\mathbb{Q}, |\cdot|)$.

Let $E \subseteq \mathbb{Q}$ containing at least two points, say r_1, r_2.



Choose an irrational number $q \in (r_1, r_2)$.

Then $E \subseteq \mathbb{Q} \subseteq \mathbb{R}\setminus\{q\} = \underset{\substack{\uparrow \\ r_1}}{(-\infty, q)} \cup \underset{\substack{\uparrow \\ r_2}}{(q, \infty)}$

$E \cap (-\infty, q) \neq \emptyset$ and $E \cap (q, \infty) \neq \emptyset$.

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So, E is <u>not</u> connected.

Therefore, any connected subset of $(\mathbb{Q}, |\cdot|)$ must be a singleton. In particular, $E_r = \{r\}$, the connected component of $\mathbb{Q}$ containing r, for each $r \in \mathbb{Q}$.

$$\mathbb{Q} = \bigcup_{r\in\mathbb{Q}} E_r = \bigcup_{r\in\mathbb{Q}} \{r\}.$$

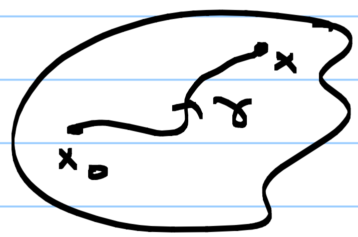
Clearly, each E_r is closed. However, $E_r = \{r\}$ is not open, because it does <u>not</u> contain any interval of rational numbers.

<u>Proposition</u>: An open connected subset of $\mathbb{R}^n$ is arcwise connected.

<u>Proof</u>: Pick a point $x_0 \in U$, where U is open and connected subset.

<u>Aim</u>: U is arcwise connected.

Let $A = \{x \in U \mid$ there is a path γ joining x_0 to $x\}$.



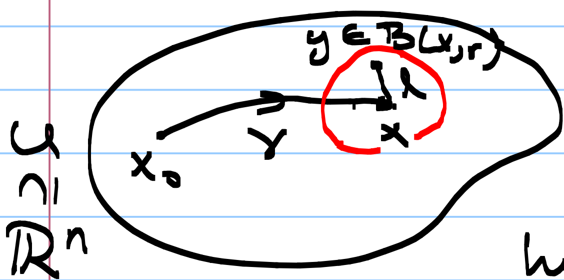
$\gamma : [a,b] \longrightarrow X$, $\gamma(a) = x_0$
$\gamma(b) = x$

Note that it is enough to show that $A = U$.

To prove that $A = U$ we'll show that A is a nonempty subset of U which is both open and closed. Since U is already connected this would imply $A = U$.

$x_0 \in A$ because the constant path $\gamma : [0,1] \to U$ by $\gamma(t) = x_0$ joins x_0 to x_0. In particular, $A \neq \emptyset$.

<u>A is open:</u> Let $x \in A$. Since U is open there is some ball $B(x,r)$ for some $r > 0$ so that



$$B(x,r) \subseteq U.$$

Let $\ell : [0,1] \to U$ be given by

$\ell(t) = (1-t)x + ty$, the line segment joining x to y. Clearly $\ell(t) \in B(x,r) \subseteq U$, because any ball is convex.

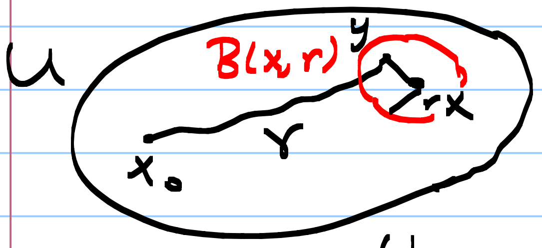
On the other hand, $x \in A$ so that there is path $\gamma : [0,1] \to U$ so that $\gamma(0) = x_0$ and $\gamma(1) = x$. Finally, define

$$\gamma * \ell : [0,1] \to U \text{ by } \gamma * \ell(t) = \begin{cases} \gamma(2t), & 0 \leq t \leq 1/2 \\ \ell(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

$\gamma * \ell$ is continuous because γ and ℓ are continuous and $\gamma(2 \cdot 1/2) = \gamma(1) = x = \ell(0) = \ell(2t-1)$ and $(\gamma * \ell)(0) = \gamma(2 \cdot 0) = \gamma(0) = x_0$ and $(\gamma * \ell)(1) = \ell(2 \cdot 1 - 1) = \ell(1) = y$. Hence, $\gamma * \ell$ joins x_0 to y and thus $y \in A$. Therefore, $B(x,r) \subseteq A$.

Hence, A is open.

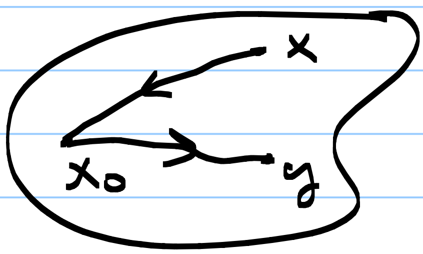
A is closed: It is enough to show that $U \setminus A$ is open.



Take $x \in U \setminus A$. Since U is open there is some $r > 0$ so that $B(x,r) \subseteq U$. Take any point $y \in B(x,r)$. If there were a path γ joining x_0 to y then composing this γ by the line segment ℓ in $B(x,r)$ joining y to x, we would obtain a path joining x_0 to x, a contradiction. Hence, $y \in U \setminus A$.

So $B(x,r) \subseteq U \setminus A$ and then $U \setminus A$ is open.

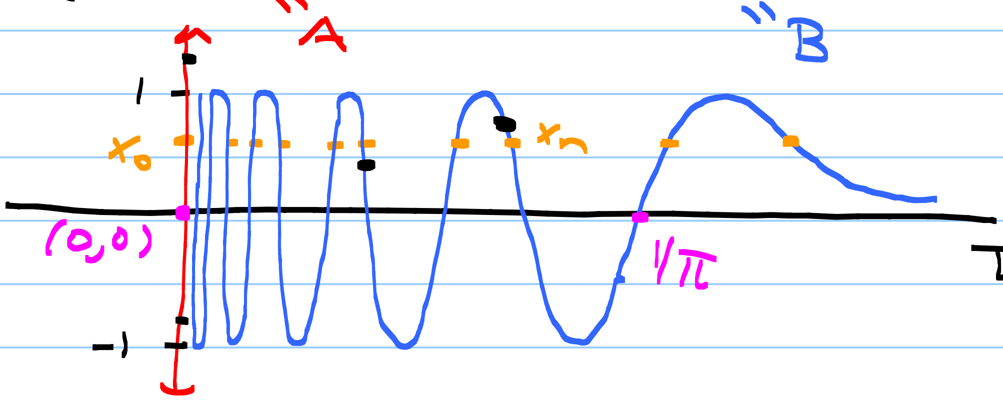
Now $U = A$ so that any point $x \in U$ is joined to x_0 by a path. Taking composition of paths as below we see that U is path connected.



This finishes the proof of the proposition. ∎

Back to the Topologist Sine Curve:

$$C = \underbrace{\{(0,y) \mid y \in \mathbb{R}\}}_{\text{"A"}} \cup \underbrace{\{(x, \sin 1/x) \mid x > 0\}}_{\text{"B"}}$$



$x_n \in B$

$\lim x_n = x_0$

Then $x_0 \in \overline{B}$.

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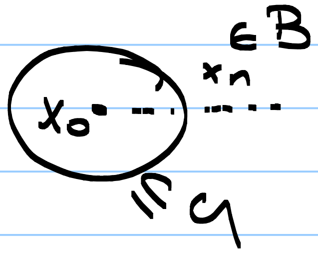
Aim: C is connected by not path connected.

<u>C is connected</u>: Clearly, A and B are path connected so that they are connected.

$C = A \cup B$, where $x_0 \in A \cap \overline{B}$. If $C \subseteq U \cup V$ where U, V open and $U \cap V = \emptyset$, then since A and B are connected we would have

$$A \subseteq U \text{ or } A \subseteq V$$
$$B \subseteq U \text{ or } B \subseteq V$$



If $A \subseteq U$ then $x_0 \in U$ and this would imply $U \cap B \neq \emptyset$ so that $B \subseteq U$. Hence, $C = A \cup B \subseteq U$ so that C is connected.

<u>C is not path connected</u>: Assume on the contrary that C is path connected and

$$\gamma : [0,1] \to C \text{ is a path with } \gamma(0) = (0,0)$$

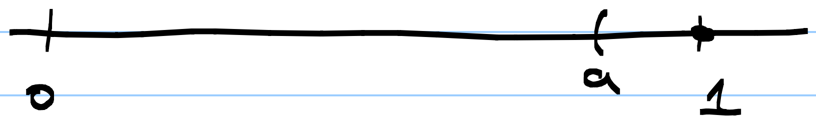
and $\gamma(1) = (1/\pi, 0)$.

Let $\gamma(t) = (x(t), y(t))$, where $x(t)$ and $y(t)$ are real valued continuous functions on $[0,1]$.

Since $\gamma(0) = (x(0), y(0)) = (0,0)$ and $\gamma(1) = (x(1), y(1)) = (1/\pi, 0)$ we see that

$$x(0) = 0 \text{ and } x(1) = 1/\pi.$$

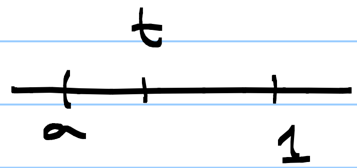
Since x is continuous the subset $x^{-1}((0,\infty))$ is open and contains 1 and $0 \notin x^{-1}((0,\infty))$.



$x^{-1}((0,\infty))$

Choose a minimal so that $(a,1] \subseteq x^{-1}((0,\infty))$.
This implies $x(a)=0$. Otherwise, $x(a)>0$ and this would contradict to the minimality of a, since $x(t)$ is continuous.

$0 \leq a < 1, \quad x(a)=0$



$t \in (a,1] \Rightarrow x(t) > 0.$

Finally, $y(a) = \lim\limits_{t \to a^+} y(t) = \lim\limits_{t \to a^+} \sin \dfrac{1}{x(t)}$, a

contradiction since the $\lim\limits_{x \to 0^+} \sin \dfrac{1}{x}$ does not exist.

Hence, $C = A \cup B$ cannot be path connected. ∎

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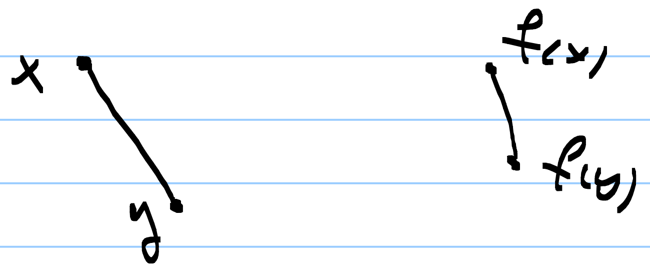
Some Applications

1) Banach Contraction Mapping Theorem:

Definition: A function $f: X \to X$ on a metric space (X, d) is called a contraction if there is a constant $0 \leq \lambda < 1$ so that

$$d(f(x), f(y)) \leq \lambda\, d(x,y), \text{ for all } x, y \in X.$$



Remark: Taking $\delta = \varepsilon/\lambda$ we see that f is uniformly continuous.

Theorem: Let $f: X \to X$ be a contraction mapping on a complete metric space (X, d). Then there is a unique point $x_0 \in X$ so that

$$f(x_0) = x_0.$$

A point $x \in X$ with $f(x) = x$ is called a fixed point of f.

Proof: Uniqueness: Assume that there are $x, y \in X$ with $f(x) = x$ and $f(y) = y$. Then

$$d(x,y) = d(f(x), f(y)) \leq \lambda\, d(x,y), \text{ where } 0 \leq \lambda < 1$$

is the contraction ratio of f.

$\Rightarrow$ $(1-\lambda)\, d(x,y) \leq 0$ which implies $d(x,y) = 0$.

Hence, $x=y$.

Existence: Start with any point $x_1 \in X$ and construct the sequence (x_n) defined by

$$x_n = f(x_{n-1}), \quad n \geq 2.$$

x_1 $\quad x_2 = f(x_1)$ $\quad x_3 = f(x_2)$ $\quad \cdots$

Claim (x_n) is a Cauchy sequence.

Since (X,d) is complete (x_n) has to be convergent, say $\lim x_n = x_0$.

Then $f(x_0) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x_0$ so that x_0 is a fixed point of f, which finishes the proof.

Proof of the Claim: For any $n \geq 2$ we have

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1}))$$
$$\leq \lambda d(x_n, x_{n-1}) \quad \text{and by induction}$$
$$\vdots$$
$$\leq \lambda^{n-1} d(x_2, x_1)$$

Now let $m \geq n \geq 1$, then

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n)$$
$$\leq \lambda^{m-2} d(x_2, x_1) + \lambda^{m-3} d(x_2, x_1) + \cdots + \lambda^{n-1} d(x_2, x_1)$$

$$d(x_m, x_n) \le d(x_2, x_1)(\lambda^{m-2} + \lambda^{m-3} + \cdots + \lambda^{n-1})$$

$$\le d(x_2, x_1)\lambda^{n-1}(1 + \lambda + \cdots + \lambda^{m-n-1}) \quad (0<\lambda<1)$$

$$\le d(x_2, x_1)\lambda^{n-1}\left(\sum_{n=0}^{\infty} \lambda^n\right)$$

$$\le d(x_2, x_1)\lambda^{n-1}\frac{1}{1-\lambda}$$

$$= d(x_2, x_1)\frac{\lambda^{n-1}}{1-\lambda}, \text{ for all } m \ge n \ge 1.$$

$\lim \lambda^n = 0$ since $0<\lambda<1$. So if $\epsilon > 0$, choose $n_0 \in \mathbb{N}$ so that

$$\lambda^{n_0-1} < \frac{\epsilon(1-\lambda)}{d(x_2, x_1)}.$$

Then, if $m \ge n \ge n_0$ then

$$d(x_m, x_n) \le d(x_2, x_1)\frac{\lambda^{n-1}}{1-\lambda} \le d(x_2, x_1)\frac{\lambda^{n_0-1}}{1-\lambda} < \epsilon.$$

Hence, (x_n) is Cauchy. This finishes the proof. ∎

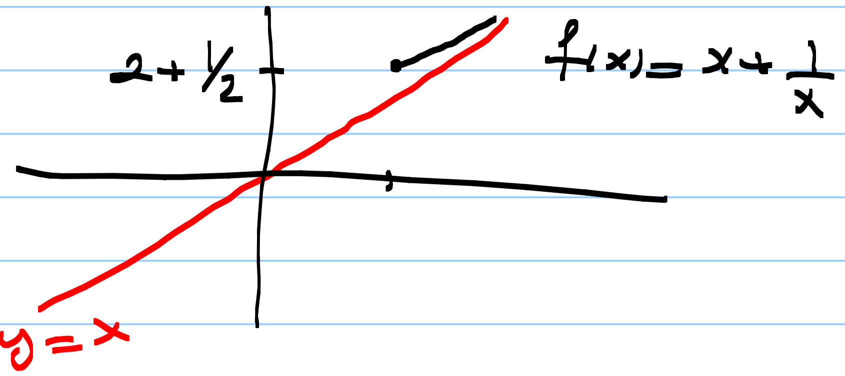
<u>Remark</u>: The assumption that $0<\lambda<1$ cannot be weakened to $\lambda = 1$. Here is a counter example.

Take $(X = [2, \infty), |\cdot|)$, which is a complete metric space and

$f : X \to X$ by $f(x) = x + \frac{1}{x}$. Clearly,

$f(x) = x + \frac{1}{x} > x$ so that f has no fixed points. However, for any $x, y \in X$, we have

$$|f(x) - f(y)| = \left| x + \frac{1}{x} - y - \frac{1}{y} \right|$$
$$= \left| (x-y) + \frac{y-x}{xy} \right|$$
$$= \left| (x-y)\left(1 - \frac{1}{xy}\right) \right|$$
$$= |x-y| \left| 1 - \frac{1}{xy} \right| , \quad x, y \geq 2$$
$$< |x-y|$$



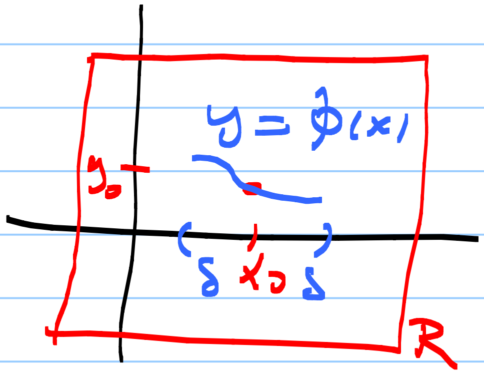
__Theorem:__ (Existence Uniqueness Theorem for O.D.E.'s)

Consider the initial value problem

$$\begin{cases} y' = f(x, y) \\ y'(x_0) = y_0 \end{cases}$$

, where $f: \mathbb{R} \to \mathbb{R}$ and $\frac{\partial f}{\partial y}$ are continuous on some rectangle $R: |x - x_0| < a, |y - y_0| < b$. Then there is a unique solution $y = \phi(x)$ of the I.V.P. defined on some interval $(x_0 - \delta, x_0 + \delta)$

for some $\delta > 0$.



$$\phi'(x) = f(x, \phi(x))$$
$$\phi(x_0) = y_0$$

Proof: First we convert the I.V.P. to an integral equation:

I.V.P. $\begin{cases} y' = f(x,y) \Rightarrow y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$

So $y(x) - y(x_0) = \int_{x_0}^{x} y'(t)\,dt = \int_{x_0}^{x} f(t, y(t))\,dt$.

$$y(x) = y(x_0) + \int_{x_0}^{x} f(t, y(t))dt.$$

$$\boxed{y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt}$$ Integral Equation.

Conversely, if $y = y(x)$ satisfies the above integral equation then taking derivative of both sides we get

$$y'(x) = 0 + f(x, y(x)) \Rightarrow y' = f(x,y).$$

Moreover, $y(x_0) = y_0 + \underbrace{\int_{x_0}^{x_0} f(t, y(t))dt}_{=0} = y_0$

Hence, $y(x)$ solves the I.V.P.

$$(*)\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}.$$

Thus the I.V.P (*) is equivalent to the Integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt.$$

From now on let's concentrate on this integral equation. To solve this integral equation we use so called Picard Iteration.

Start with any function $y_1(x)$ and let

$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\,dt$. Then define

$y_3(x) = y_0 + \int_{x_0}^{x} f(t, y_2(t))\,dt$ and similarly,

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))\,dt.$$

This way we obtain a sequence of functions

$(y_n(x))$.

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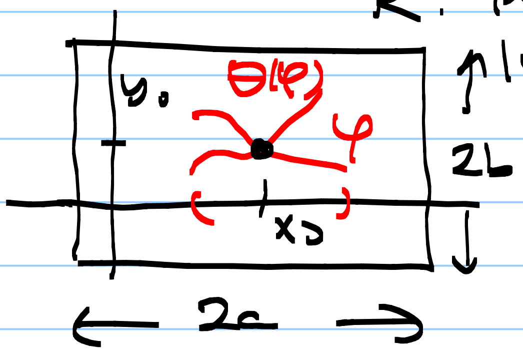
Definition: A function $\varphi : X \to X$ is called Lipschitz if there is some $L > 0$ so that

$$d(f(x), f(y)) \leq L\, d(x,y).$$

$R: |x - x_0| \leq a$ and $|y - y_0| \leq b$

Back to the proof:

$y' = f(x,y)$
$y(x_0) = y_0$



$f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous on the rectangle R, which is a compact subset of $\mathbb{R}^2$. Hence both $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ have maximum value on R.

Say there are $M > 0$ and $L > 0$ so that

$|f(x,y)| \leq M$ and $\left|\frac{\partial f}{\partial y}(x,y)\right| \leq L$ for all

$(x,y) \in R$. Fix some $0 < \lambda < 1$. Choose $\delta > 0$

so that $\delta < \frac{b}{M}$ and $\delta < \frac{\lambda}{L}$.

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

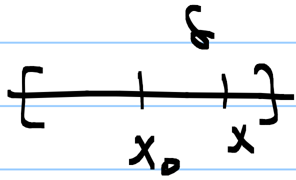
Let $X = (C[x_0 - \delta, x_0 + \delta])$ equipped with the supremum metric. We know that X is a complete metric space.

Define $\Theta : X \longrightarrow X$ as follows: If $\varphi \in X = C([x_0-\delta, x_0+\delta])$ let

$$\Theta(\varphi)(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))\,dt.$$

Claim: $\Theta(\varphi)(x) \in [y_0 - M, y_0 + M]$, whenever $\varphi \in X$.

$$\Theta(\varphi)(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))\,dt$$



$$|\Theta(\varphi)(x) - y_0| = \left| \int_{x_0}^{x} f(t, \varphi(t))\,dt \right|$$

$$\leq M \cdot |x - x_0|$$

$$\leq M \cdot \delta < b \quad \text{if}$$

$$\Rightarrow \Theta(\varphi)(x) \in (y_0 - b, y_0 + b)$$

$$\Theta : C([x_0-\delta, x_0+\delta]) \longrightarrow C([x_0-\delta, x_0+\delta])$$

Claim: Θ is contraction mapping.

Proof $\Theta(\varphi_1)(x) - \Theta(\varphi_2)(x)$

$$= \cancel{y_0} + \int_{x_0}^{x} f(t, \varphi_1(t))\,dt - \cancel{y_0} - \int_{x_0}^{x} f(t, \varphi_2(t))\,dt$$

$$= \int_{x_0}^{x} \big(\underbrace{f(t, \varphi_1(t)) - f(t, \varphi_2(t))}\big)\,dt.$$

$$|f(x,y_1)-f(x,y_2)| = \left|\frac{\partial f}{\partial y}(x,\tilde{y})\right| |(y_1-y_2)|$$

$$\leq L \cdot |y_1-y_2|$$

So $$|\Theta(\varphi_1)(x)-\Theta(\varphi_2)(x)| \leq \int_{x_0}^{x} L \cdot \underbrace{|\varphi_1(t)-\varphi_2(t)|}_{\wedge} \, dt$$

$$\leq \int_{x_0}^{x} L \cdot \overbrace{d(\varphi_1,\varphi_2)}\, dt$$

$$\leq d(\varphi_1,\varphi_2) \int_{x_0}^{x} L \, dt$$

$$\leq d(\varphi_1,\varphi_2) \underbrace{|x-x_0|}_{<\delta} \cdot L$$

$$\leq \lambda \, d(\varphi_1,\varphi_2)$$

for all $x \in [x_0-\delta, x_0+\delta]$

$d_{sup}(\Theta(\varphi_1), \Theta(\varphi_2)) \leq \lambda \, d_{sup}(\varphi_1,\varphi_2)$.

Since $0 \leq \lambda < 1$, we see that

$$\Theta : C([x_0-\delta, x_0+\delta]) \to C([x_0-\delta, x_0+\delta])$$

is a contraction mapping. Hence, Θ has a unique fixed point, say

$\varphi \in X = C([x_0-\delta, x_0+\delta])$.

$\varphi = \Theta(\varphi)$ implies that

$$\varphi(x) = \Theta(\varphi)(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))\, dt$$

Hence, $\varphi = \varphi(x)$ is the unique solution of the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

Thus, $\varphi = \varphi(x)$ is the unique solution of the I.V.P. $\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$

$\varphi : [x_0 - \delta, x_0 + \delta] \longrightarrow \mathbb{R}$.

This finishes the proof of the theorem. ■

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2. Application: Baire Category Theorem.

Theorem: Let (X, d) be a complete metric space and E_n is a sequence of closed subsets of X. If $E = \bigcup_{n=1}^{\infty} E_n$ has nonempty interior then one of E_n's must have non empty interior.

Example: $(X, d) = (\mathbb{R}, |\cdot|)$

$\mathbb{Q} = \{ p/q \mid p, q \in \mathbb{Z}, q \neq 0 \}$ the set of rationals.

$\mathbb{Q}$ is countable $\mathbb{Q} = \{r_1, r_2, r_3, \ldots, r_n, \ldots\}$.

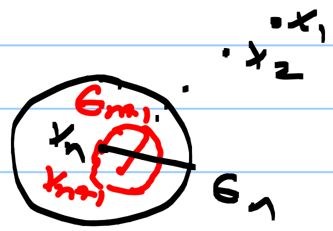
Let $E_n = \{r_n\} \subseteq \mathbb{R}$ a closed subset with $\operatorname{Int} E_n = \phi$. Since $\mathbb{Q} = \bigcup_{n=1}^{\infty} E_n$, we see that

$\operatorname{Int} \mathbb{Q} = \phi$.

To prove the theorem we need a lemma.

Lemma: Let $\epsilon_n > 0$ be a sequence of real numbers with $\lim \epsilon_n = 0$. If $B[x_n, \epsilon_n] \subseteq B[x_{n-1}, \epsilon_{n-1}]$ for all $n \geq 2$ and some sequence $(x_n) \in X$, then there is some $x \in X$ so that

$$\bigcap_{n=1}^{\infty} B[x_n, \epsilon_n] = \{x\}.$$



Proof of the lemma: Let $m, n \in \mathbb{N}$ with $m \geq n$.

Then $B[x_n, \epsilon_n] \supseteq B[x_{n+1}, \epsilon_{n+1}] \supseteq \cdots \supseteq B[x_m, \epsilon_m]$.

$x_m \in B[x_n, \epsilon_n]$ and thus $d(x_m, x_n) \leq \epsilon_n$.

Let $\epsilon > 0$ be given. Since $\lim \epsilon_n = 0$ there is some $n_0 \in \mathbb{N}$ so that if $n \geq n_0$ then

$$|\epsilon_n - 0| < \epsilon \implies 0 < \epsilon_n < \epsilon.$$

Hence, if $m, n \geq n_0$ then

$$d(x_m, x_n) \leq \epsilon_n < \epsilon.$$

Thus (x_n) is a Cauchy sequence in (X, d). Since (X, d) is complete (x_n) is convergent, say

$$\lim x_n = x, \text{ for some } x \in X.$$

$$\underbrace{x_1, \underbrace{x_2, \overbrace{x_3, \ldots, x_n, x_{n+1}, \ldots, x_m, \ldots}^{\subseteq B[x_3, \epsilon_3]}}_{\subseteq B[x_2, \epsilon_2]}}_{\subseteq B[x_1, \epsilon_1]} \to x$$

Since $B[x_1, \epsilon_1]$ is closed we see that $x \in B[x_1, \epsilon_1]$. Similarly, $x \in B[x_2, \epsilon_2]$ and so on.

In particular, $x \in B[x_n, \epsilon_n]$ for all $n \in \mathbb{N}$ so that

$$x \in \bigcap_{n=1}^{\infty} B[x_n, \epsilon_n].$$

If $y \in \bigcap_{n=1}^{\infty} B[x_n, \epsilon_n]$ then $y \in B[x_n, \epsilon_n]$ and

$d(x_n, y) \leq \epsilon_n$. Taking limit as $n \to \infty$ we see that

$$0 \leq d(x,y) = \lim d(x_n, y) \leq \lim \epsilon_n = 0$$

$\Rightarrow d(x,y) = 0$ and thus $y = x$.

Hence, $\bigcap_{n=1}^{\infty} B[x_n, \epsilon_n] = \{x\}$.

This finishes the proof of the lemma. ∎

<u>Proof of the Baire Category theorem:</u>

$E = \bigcup_{n=1}^{\infty} E_n$, $E_n \subseteq X$ closed, for all n.

Assume that $x_0 \in \mathrm{Int}(E)$.

<u>must show:</u> $\mathrm{Int}(E_n) \neq \emptyset$ for some n.

Assume on the contrary that each E_n has empty interior.

Since $x_0 \in \mathrm{Int}(E)$ there is some $\epsilon_0 > 0$ with $B(x_0, \epsilon_0) \subseteq E$. Now since E_1 has empty interior $B(x_0, \epsilon_0) \not\subseteq E_1$. Thus there is some

$$x_1 \in \underbrace{B(x_0, \epsilon_0)}_{\text{open}} \cap \underbrace{(X \setminus E_1)}_{\text{open}}.$$

Since $B(x_0, \epsilon_0) \cap (X \setminus E_1)$ there is some $\epsilon_1' > 0$ so that

$$B(x_1, \epsilon_1') \subset B[x_0, \epsilon_0] \cap (X \setminus E_1).$$

Now let $\epsilon_1 = \min\{\frac{\epsilon_1'}{2}, 1\}$. Then $\epsilon_1 \le \frac{\epsilon_1'}{2} < \epsilon_1'$.

Thus $B[x_1, \epsilon_1] \subset B[x_1, \epsilon_1') \subseteq B[x_0, \epsilon_0] \cap (X \setminus E_1)$.

Similarly, since E_2 has empty interior $B(x_1, \epsilon_1) \not\subset E_2$. Hence, there is some

$x_2 \in B(x_1, \epsilon_1) \cap (X \setminus E_2)$. Similarly, we find

$\epsilon_2 > 0$ so that $\epsilon_2 < 1/2$ and

$B[x_2, \epsilon_2] \subseteq (X \setminus E_2) \cap B(x_1, \epsilon_1)$ and thus

$B[x_2, \epsilon_2] \subseteq B[x_1, \epsilon_1] \subseteq B[x_0, \epsilon_0]$.

By induction we construct a sequence of

balls $B[x_n, \epsilon_n]$ so that $0 < \epsilon_n \le 1/n$

and $B[x_n, \epsilon_n] \supseteq B[x_{n+1}, \epsilon_{n+1}]$.

Thus, by the lemma $\bigcap_{n=1}^{\infty} B[x_n, \epsilon_n] = \{x\}$,

where $x = \lim x_n$.

Since $x \in B[x_n, \epsilon_n] \subseteq X \setminus E_n$, for all n, and thus $x \notin E_n$ for all n. Therefore,

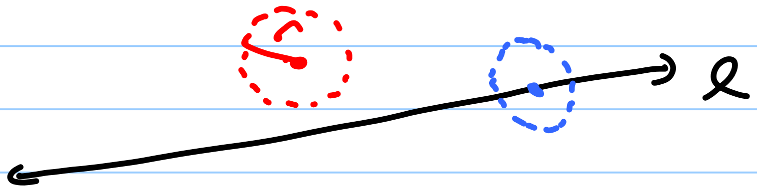
$$x \notin \bigcup_{n=1}^{\infty} E_n = E.$$

On the other hand, $x \in B[x_1, \epsilon_1] \subseteq B(x_0, \epsilon_0) \subseteq E$,

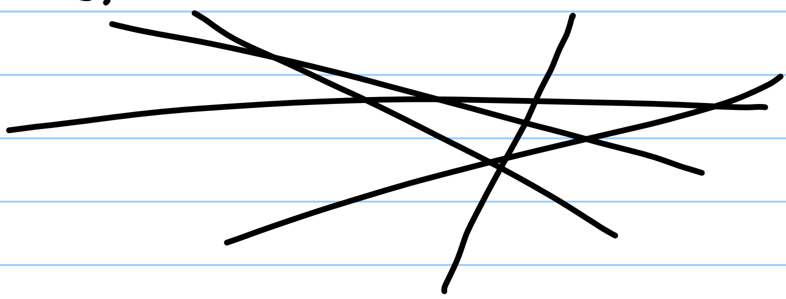
which is a contradiction. Therefore, at least one E_n must have non empty interior.
This finishes the proof. ∎

<u>Example</u>: $(\mathbb{R}^2, d_2)$ is a complete metric space.

Any line $\ell \subseteq \mathbb{R}^2$ has empty interior and is closed.



So, if $E = \bigcup_{n=1}^{\infty} \ell_n$ then $\mathrm{Int}(E) = \phi$.



<u>Example</u>. The set of irrational numbers $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$ is <u>not</u> contained in the union of countably many closed subsets with empty interiors.

<u>Remark</u>: $\mathbb{Q} = \{r_1, r_2, r_3, \ldots\}$ countable.

$\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\}$, $\{r_n\}$ is closed with empty interior.

Solution: Let $P \subseteq \bigcup_{n=1}^{\infty} E_n$, where each $E_n \subseteq \mathbb{R}$ is closed with empty interior.

Hence, $\mathbb{R} = P \cup \mathbb{Q} \subseteq \left(\bigcup_{n=1}^{\infty} E_n\right) \cup \left(\bigcup_{n=1}^{\infty} \{r_n\}\right) \subseteq \mathbb{R}$

so that $\mathbb{R} = \left(\bigcup_{n=1}^{\infty} E_n\right) \cup \left(\bigcup_{n=1}^{\infty} \{r_n\}\right)$, which is a countable union of closed subsets with empty interiors. This is a contradiction since $\mathbb{R}$ is complete by the Baire Category Theorem.

Thus, the set of irrational numbers is not contained in a union of countably many closed subsets of $\mathbb{R}$, with empty interior.

Arzela–Ascoli Theorem:

Recall that for any set $X \neq \emptyset$, the metric space of bounded functions on X, $(B(X), d_{sup})$ is a complete metric space.

If X is already a metric space then the subset $C(X) \cap B(X)$ of $B(X)$ consisting of continuous functions is a closed and thus $(C(X) \cap B(X), d_{sup})$ is also a complete metric space.

(X, d) metric space

$C(X) = \{ f : X \to \mathbb{R} \mid f \text{ is continuous} \}$

$C(X) \cap B(X)$ is complete.

If X is a compact metric space then any continuous function $f : X \to \mathbb{R}$ has extreme values. In particular, f is bounded. So $C(X) \subseteq B(X)$. Therefore, $(C(X), d_{sup})$ is a complete metric space.

Arzela–Ascoli Theorem describes compact subsets of $(C(X), d_{sup})$. First we need a definition.

Definition: A subset E of $C(X)$ is called equicontinuous

if for every $\epsilon > 0$ there is some $\delta > 0$ so that

$d(x,y) < \delta$ implies $|f(x) - f(y)| < \epsilon$, for all $x, y \in X$ and $f \in E$.

Example: Let $(X,d) = ([0,1], |\cdot|)$ a compact metric space.

$$C(X) = C([0,1]) = \{f:[0,1] \to \mathbb{R} \mid f \text{ is continuous}\}.$$

Let $E = \{f_n : [0,1] \to \mathbb{R} \mid n = 1,2,\dots\}$, where

$f_n : [0,1] \to \mathbb{R}$, $f_n(x) = nx$.

Is E equicontinuous? Answer is NO!

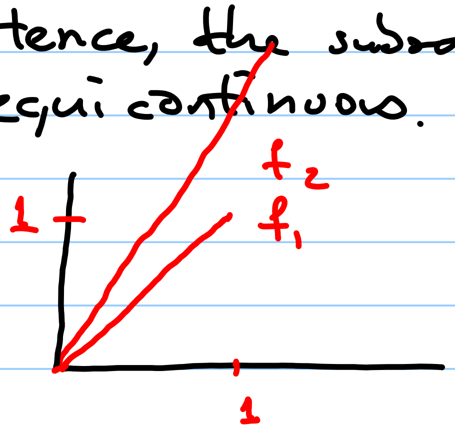
Let $\epsilon = 1$. Then if $\delta > 0$ then choose n_0 so that

$n_0 \delta > 2$. Now let $x = 0$, $y = \min\{1, \frac{\delta}{2}\}$. Then

$|x-y| \leq \frac{\delta}{2} < \delta$ but $|f_{n_0}(x) - f_{n_0}(y)| = |n_0 \cdot 0 - n_0 \cdot \frac{\delta}{2}|$

$$= n_0 \frac{\delta}{2} > 1 = \epsilon.$$

Hence, the subset $E \subseteq (C([0,1]), d_{sup})$ is not equicontinuous.



Theorem (Arzelà–Ascoli)

Let (X,d) be a compact metric space. A subset E of $C(X)$ is precompact if and only if it is equicontinuous and bounded. Consequently, E is compact if and only if E is closed, bounded and equicontinuous.

Proof: Note that in a complete metric space a subset is compact if and only if it is closed and precompact. Thus it is enough to prove the first assertion.

First assume that E is precompact. Then we must show:

1) E is bounded
2) E is equicontinuous.

(1) is proved earlier when we discuss sequential compactness.

E is equicontinuous: Given $\epsilon > 0$. Since E is compact and $\epsilon/3 > 0$ there are $f_1, \ldots, f_k$ so that

$$E \subseteq B(f_1, \epsilon/3) \cup \cdots \cup B(f_k, \epsilon/3).$$

Since (X,d) is compact each $f_i \in C(X,d)$ is indeed uniformly continuous. Thus there is $\delta_i > 0$ so that

$$d(x,y) < \delta_i \implies |f_i(x) - f_i(y)| < \epsilon/3.$$

Let $\delta = \min\{\delta_1, \ldots, \delta_k\}$. Then $0 < \delta \leq \delta_i$, for all $i = 1, \ldots, k$. Now if $f \in E$ and $x, y \in X$ with $d(x,y) < \delta$ then there is some $i \in \{1, \ldots, k\}$ so that $f \in B(f_i, \epsilon/3)$ and hence

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)|$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Hence, E is equicontinuous.

Now we assume $\mathcal{E}$ is bounded and equicontinuous.

<u>must show:</u> $\mathcal{E}$ is precompact.

Given $\epsilon>0$. Since $\epsilon/4>0$ and $\mathcal{E}$ is equicontinuous there is some $\delta>0$ so that for any $f\in\mathcal{E}$ and $x,y\in X$ with $d(x,y)<\delta$ we have

$$|f(x)-f(y)|<\epsilon/4.$$

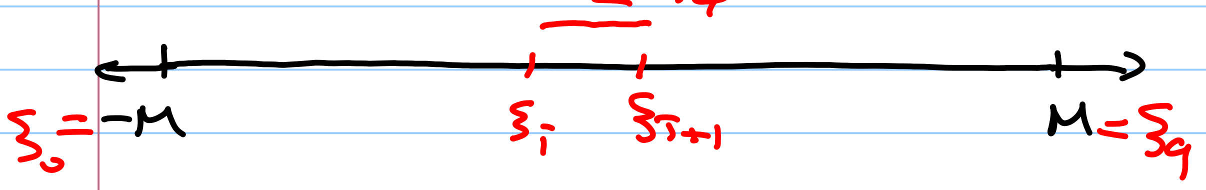
Since (X,d) is compact X is precompact and there are points $x_1,\dots,x_p$ in X with

$$X = B(x_1,\delta)\cup\dots\cup B(x_p,\delta).$$

Let $\bar{0}: X\to\mathbb{R}$ be the zero function: $\bar{0}(x)=0, \forall x\in X$. Clearly, $\bar{0}\in C(X)$. Since $\mathcal{E}$ is bounded, there is some $M>0$ so that $\mathcal{E}\subseteq B(\bar{0}, M)$.

So, for any $x\in X$ and $f\in\mathcal{E}$,

$$|f(x)-\underbrace{\bar{0}(x)}_{0}|<M \Rightarrow |f(x)|\le M, \text{ for all } x\in X \text{ and } f\in\mathcal{E}.$$



Choose a partition $P=\{\xi_0,\xi_1,\dots,\xi_q\}$ of $[M,M]$ so that

$$-M=\xi_0<\xi_1<\xi_2<\dots<\xi_q=M \text{ and } \xi_{i+1}-\xi_i<\epsilon/4$$

for all $i=0,1,\dots,q-1$.

For each $f \in E$, $f(x_i) \in [-M, M]$ and thus there is some ξ_{k_i} so that

$$|f(x_i) - \xi_{k_i}| < \epsilon/4, \quad i = 1, \dots, p.$$

Hence, for any $f \in E$ there is a p-tuple $(\xi_{k_1}, \xi_{k_2}, \dots, \xi_{k_p})$ so that $|f(x_i) - \xi_{k_i}| < \epsilon/4$.

Note that there are at most $(q+1)^p$ such p-tuples. Let $\mathcal{Z}$ be the set of all these p-tuples. For each p-tuple in $\mathcal{Z}$ choose a function g so that

$$|g(x_i) - \xi_{k_i}| < \epsilon/4 \quad \text{for all } i = 1, \dots, p.$$

In particular, the set A of such g's is at most $(q+1)^p$.

Now let $f \in E$. Then there is a p-tuple $(\xi_{k_1}, \dots, \xi_{k_p})$ and $g \in A$ so that

$$|f(x_i) - \xi_{k_i}| < \epsilon/4 \text{ and } |g(x_i) - \xi_{k_i}| < \epsilon/4.$$

Finally, for any $x \in X$, choose some $i = 1, \dots, p$ with $x \in B(x_i, \delta)$ and thus

$$|f(x) - g(x)| \le |f(x) - f(x_i)| + |f(x_i) - \xi_{k_i}| + |\xi_{k_i} - g(x_i)| + |g(x_i) - g(x)|$$

$$< \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4$$

$\Rightarrow |f(x) - g(x)| < \epsilon$ so that $f \in B(g, \epsilon)$.

Hence, $\mathcal{E} \subseteq \bigcup_{g \in A} B(g, \epsilon)$, where A is a finite subset of $\mathcal{E}$.

In other words, $\mathcal{E}$ is precompact.

This finishes the proof. ∎

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Theorem: Assume that (f_n) is a sequence of functions $f_n:[a,b]\to\mathbb{R}$. Further assume the following:

1) Each f_n is differentiable and f_n' is also continuous.
2) There is some $x_0\in[a,b]$ so that the sequence of real numbers $(f_n(x_0))$ is convergent.
3) The sequence of functions $(f_n'(x))$ is uniformly convergent to some $g\in C([a,b])$.

Then (f_n) converges to some differentiable function f so that $f'(x)=g(x)$ for all $x\in[a,b]$.

Proof: Note that each $f_n(x)$ satisfies

$$f_n(x)-f_n(x_0)=\int_{x_0}^{x} f_n'(t)\,dt \quad \text{and hence}$$

$$f_n(x)=f_n(x_0)+\int_{x_0}^{x} f_n'(t)\,dt.$$

Therefore, $$|f_n(x)-f_m(x)|=\left|f_n(x_0)+\int_{x_0}^{x} f_n'(t)\,dt-f_m(x_0)-\int_{x_0}^{x} f_m'(t)\,dt\right|$$

$$\Rightarrow |f_n(x)-f_m(x)|\le |f_n(x_0)-f_m(x_0)|+\left|\int_{x_0}^{x}(f_n'(t)-f_m'(t))\,dt\right|$$

Given $\epsilon>0$. Since $(f_n(x_0))$ is convergent there is some $n_1\in\mathbb{N}$ so that

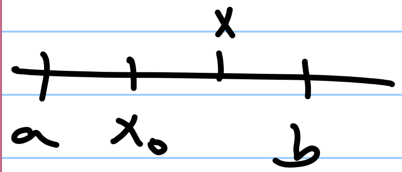
$$m,n\ge n_1 \Rightarrow |f_n(x_0)-f_m(x_0)|<\epsilon/2.$$

Similarly, $(f_n'(x))$ is Cauchy and thus there is some $n_2 \in \mathbb{N}$ so that

$$m, n \geq n_2 \Rightarrow |f_n'(x) - f_m'(x)| < \frac{\epsilon}{2(b-a)}.$$

In particular, $x, x_0 \in [a,b]$ and thus

$$\left| \int_{x_0}^{x} (f_n'(t) - f_m'(t))\,dt \right| \leq \left| \int_{x_0}^{x} |f_n'(t) - f_m'(t)|\,dt \right|$$



$$\leq \left| \int_{x_0}^{x} \frac{\epsilon}{2(b-a)}\,dt \right|$$

$$= \frac{\epsilon}{2(b-a)} \overbrace{|x - x_0|}^{< b-a} < \epsilon/2.$$

Let $n_0 = \max\{n_1, n_2\}$. Then if $m, n \geq n_0$ then

$$|f_n(x) - f_m(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence, the sequence $(f_n(x))$ is Cauchy in $(C([a,b]), d_{\sup})$. Since $(C([a,b]), d_{\sup})$ is complete (f_n) converges to some $f \in C([a,b])$.

Note that $f(x) = \lim_{n\to\infty} f_n(x)$ and

$$f(x) - f(x_0) = \lim_{n\to\infty} (f_n(x) - f_n(x_0))$$

$$= \lim_{n\to\infty} \int_{x_0}^{x} f_n'(t)\,dt \qquad (f_n'(t) \to g(t) \text{ uniformly on } [a,b])$$

$$= \int_{x_0}^{x} g(t)\,dt.$$

Hence, $f(x) = f(x_0) + \int_{x_0}^{x} g(t)\,dt$, where $g(x)$ is a continuous function. Thus, again by the Fundamental Theorem of Calculus $f(x)$ is differentiable and

$$f'(x) = \frac{d}{dx}(f(x_0)) + \frac{d}{dx}\left(\int_{x_0}^{x} g(t)\,dt\right)$$

$$= 0 + g(x)$$

$\Rightarrow f'(x) = g(x)$ so that $f(x)$ is differentiable with derivative $g(x)$. ∎

<u>Example:</u> Let $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ be a Power Series with radius of convergence $R>0$. Then for any $[a,b] \in (x_0-R, x_0+R)$ the series

$\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges uniformly.

The series $\sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1}$ has the radius of convergence and thus it converges uniformly.

Let $f_n(x) = \frac{a_n}{n+1}(x-x_0)^{n+1}$. Then $f_n(x)$ is continuously differentiable on $[a,b]$ with $f_n'(x) = a_n(x-x_0)^n$. Moreover, $\sum f_n'(x)$ converges uniformly on $[a,b]$. Finally, the series

$$\sum_{n=0}^{\infty} f_n(x_0) = \sum_{n=0}^{\infty} 0 = 0 \text{ is convergent.}$$

Hence, by the previous theorem the series

$$\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1}$$ is uniformly convergent and it is differentiable with derivative

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} f_n(x)\right) = \sum_{n=0}^{\infty} f_n'(x).$$

$$\Rightarrow \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1}\right) = \sum_{n=0}^{\infty} a_n(x-x_0)^n.$$

This theorem together with the one about integrals allows us to differentiate and integrate power series termwise.

Indeed, the same holds for complex valued functions and power series.