Video 1

CHAPTER 9: Sequences, Series and Power Series

Note Title 25.07.2022

§9.1. Sequences and Convergence:

A sequence in $\mathbb{R}$ is a function $a : \mathbb{N} \to \mathbb{R}$.

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$

$a(1) = -1/2, \quad a(2) = \frac{1}{4}, \quad a(3) = -\frac{1}{8}, \dots$

$a(n)$ is denoted simply as a_n.

$(a_n) = \{-1/2, 1/4, -1/8, 1/16, \dots\}$

If we know a formula for the function $a : \mathbb{N} \to \mathbb{R}$ we may represent using its formula also:

$$a(n) = \left(-\frac{1}{2}\right)^{n+1} \quad n = 0, 1, 2, \dots$$

$$a_0 = (-1/2)^{0+1} = -\frac{1}{2}, \quad a_1 = (-1/2)^{1+1} = (-1/2)^2 = 1/4, \quad a_2 = -\frac{1}{8}, \dots$$

$$(a_n) = \left(\left(-1/2\right)^{n+1}\right)$$

A formula for $a(n) = a_n$ is also called the general term of the sequence.

Examples: 1) $\{n\} = \{1, 2, 3, 4, 5, \dots\}$ $\quad n \in \mathbb{N} = \{1, 2, 3, \dots\}$

2) $\left\{\frac{n-1}{n}\right\} = \{0, 1/2, 2/3, \dots\}$

3) $\left\{\frac{n^2}{2^n}\right\} = \left\{\frac{1}{2}, 1, \frac{9}{8}, 1, \frac{25}{32}, \dots\right\}$

4) $\left\{\left(1 + \frac{1}{n}\right)^n\right\} = \left\{2, \left(\frac{3}{2}\right)^2, \left(\frac{4}{3}\right)^3, \dots\right\}$

5) $a_1 = 1$, $a_{n+1} = \sqrt{6+a_n}$, $n = 1, 2, \ldots$.
(Recursive Relation)

$a_2 = \sqrt{6+a_1} = \sqrt{7}$, $a_3 = \sqrt{6+a_2} = \sqrt{6+\sqrt{7}}$, ...

6) Fibonacci Sequence $\{a_n\}$, where $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}$, for $n \geq 3$.

$a_3 = a_2 + a_1 = 1+1 = 2$, $a_4 = a_3 + a_2 = 2+1 = 3$, ...

$\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$.

<u>Definition</u>: a) The sequence $\{a_n\}$ is <u>bounded below</u> by L, and L is a lower bound for $\{a_n\}$ if $a_n \geq L$, for every $n = 1, 2, 3$. The sequence $\{a_n\}$ is bounded above by M, and M is an upper bound for $\{a_n\}$ if $a_n \leq M$, for every $n = 1, 2, 3, \ldots$

The sequence $\{a_n\}$ is bounded if it is both bounded below and bounded above. In this case there is a constant K so that $|a_n| \leq K$ for every $n = 1, 2, 3, \ldots$

<u>Examples</u>: 1) $\{n\} = \{1, 2, 3, 4, \ldots\}$ is bound below by 1 and also by -7. It is not bounded above because for every n there is some M so that $M \leq a_n$.

2) $\left\{\frac{n-1}{n}\right\} = \{0, 1/2, 2/3, \ldots\}$ is bounded below by 0 and above by 1. This is a bounded sequence. In fact, $|a_n| \leq 1 = K$, for any $n = 1, 2, 3, \ldots$

b) The sequence $\{a_n\}$ is positive if it is bounded below by zero, that $a_n \geq 0$, for every n, and it is negative if it is bounded above by zero, that is $a_n \leq 0$ for every n.

$\{n\} = \{1, 2, 3, \dots\}$ is positive, the sequence $\{-n^2+1\} = \{0, -3, -8, -15, \dots\}$ is negative.

c) The sequence $\{a_n\}$ is increasing if $a_{n+1} \geq a_n$ for every n and it is decreasing if $a_{n+1} \leq a_n$ for every n. We say $\{a_n\}$ is monotonic if it is increasing or decreasing.

<u>Examples</u> 1) $\{n\} = \{1, 2, 3, \dots\}$ is increasing.

2) $\{-n\} = \{-1, -2, -3, \dots\}$ is decreasing.

3) $\{1\} = \{1, 1, 1, \dots\}$ is increasing and decreasing.

4) $\{(-1)^n\} = \{-1, 1, -1, 1, \dots\}$ is neither increasing nor decreasing. It is <u>not</u> monotonic.

d) The sequence $\{a_n\}$ is alternating if $a_n a_{n+1} < 0$ for every $n = 1, 2, 3, \dots$ that is any two consecutive terms of the sequence have opposite signs. Note that in this case no a_n is zero.

$\{10, 11, 10, 11, 10, 11, \dots\}$ is <u>not</u> monotonic and it is <u>not</u> alternating.

Subtract $10{,}5$ from each terms you get

$\{-0{,}5, 0{,}5, -0{,}5, 0{,}5, \dots\}$

Video 2

Sometimes a sequence is described by means of a function $f: \mathbb{R} \to \mathbb{R}$.

Example: If $a_n = \frac{n}{n^2+1}$, then $a_n = f(n)$, where

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{x}{x^2+1}$.

Note that $f'(x) = \frac{1\cdot(x^2+1) - x\cdot(2x)}{(1+x^2)^2} = \frac{\overset{\leq 0}{-x^2+1}}{\underset{>0}{(1+x^2)^2}}$

and $f'(x) \leq 0$ if $x \geq 1$. Hence, $f(x)$ is decreasing, in other words, $f(x) \leq f(y)$ when $x > y$, when $x, y \geq 1$.

In particular, since $n+1 > n$, for every $n \geq 1$,
$\underset{a_{n+1}}{\underset{\shortparallel}{f(n+1)}} \leq \underset{a_n}{\underset{\shortparallel}{f(n)}}$.

$$a_{n+1} = \frac{n+1}{(n+1)^2+1} \leq \frac{n}{n^2+1} = a_n \text{ for } n = 1, 2, 3, \ldots$$

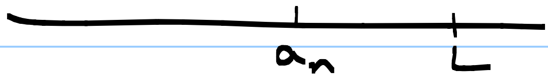
Definition: The adverb "ultimately" is used to describe any terminal property of a sequence that the terms have from some point on, but not necessarily at the beginning of the sequence.

Example: 1) $\{n-3\} = \{-2, -1, \underline{0, 1, 2, 3, \ldots}\}$ is not a positive sequence but it is positive after the third term. Therefore, we say that the sequence is ultimately positive.

2) $\{a_n\} = \{5, 0, -3, 0, 2, -1, 0, 1, 2, 3, 4, \ldots, n, \ldots\}$

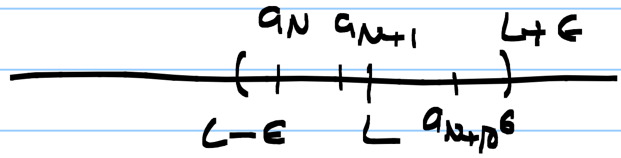
The sequence above is not increasing but it is increasing after some point on, and thus it is ultimately increasing.

<u>Convergence of Sequences</u>: We say that a sequence $\{a_n\}$ converges to the limit L, and write $\lim_{n\to\infty} a_n = L$, provided that the distance from a_n to L on the real line approaches to 0 as n increases toward ∞.



(Mathematical) <u>Definition</u>: We say that the sequence $\{a_n\}$ converges to the limit L, and we write $\lim_{n\to\infty} a_n = L$, if for every positive real number $\epsilon > 0$ there exists some integer N (which may depend on ϵ) such that if $n \geq N$ then $|a_n - L| < \epsilon$.

$\epsilon > 0$



<u>Examples</u>: $\{a_n\} = \{1/n\} = \{1, 1/2, 1/3, 1/4, \ldots\}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1/n = 0. \quad (L = 0)$

<u>Proof</u>: Given $\epsilon > 0$. Choose $N \in \mathbb{N}$ so that $N > 1/\epsilon$. Then if $n \geq N$, then

$|a_n - L| = |1/n - 0| = 1/n \leq 1/N < \epsilon$ since $N > 1/\epsilon$. ∎

Examples: a) $\left\{\frac{n-1}{n}\right\} = \left\{0, 1/2, 2/3, 3/4, 5/6, \ldots\right\}$

$$\lim_{n\to\infty} \frac{n-1}{n} = 1 \qquad \left(\frac{n-1}{n} = 1 - \frac{1}{n} \to 0 \to 1 \right)$$

b) $\{n\} = \{1, 2, 3, 4, \ldots\}$ diverges to ∞.

c) $\{-n\} = \{-1, -2, -3, -4, \ldots\}$ diverges to $-\infty$.

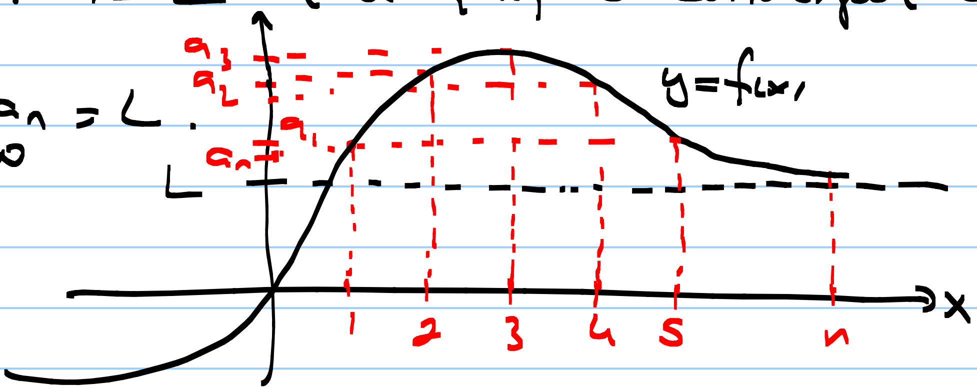
d) $\{(-1)^n\} = \{-1, 1, -1, 1, \ldots\}$ is divergent.

$-1 \qquad 1$

e) $\left\{\frac{(-1)^n}{n}\right\} = \left\{-1, 1/2, -1/3, 1/4, -1/5, \ldots\right\}$ is convergent with $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Remark: If a sequence is not convergent then we say that it is divergence.

Fact: If a sequence $\{a_n\}$ is given by a function $f: \mathbb{R} \to \mathbb{R}$, that $a_n = f(n)$ and if $\lim_{x\to\infty} f(x) = L$ then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.



Video 3

Rules for Limits: Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences. Then,

1) $\{a_n \mp b_n\}$ is convergent and
$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n.$$

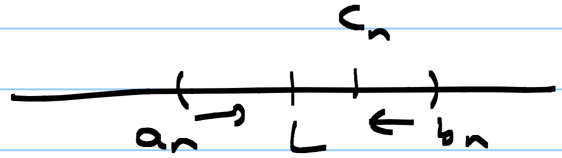
2) $\{a_n \cdot b_n\}$ is convergent and
$$\lim_{n\to\infty}(a_n \cdot b_n) = (\lim_{n\to\infty} a_n)\cdot(\lim_{n\to\infty} b_n)$$

3) $\lim_{n\to\infty}(c\,a_n) = c\,(\lim_{n\to\infty} a_n)$, where $c \in \mathbb{R}$ any real number.

4) If $b_n \neq 0$ for all n, and if $\lim b_n \neq 0$, then the sequence $\{a_n/b_n\}$ is convergent and
$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim a_n}{\lim b_n}$$

5) If $a_n \leq b_n$ ultimately then $\lim a_n \leq \lim b_n$.

Similarly, if $a_n \leq c_n \leq b_n$ ultimately and $\lim a_n = \lim b_n = L$ then $\{c_n\}$ is convergent and $\lim c_n = L$.



This is called the squeezing theorem.

Examples Calculate the limits of the sequences

a) $\left\{\dfrac{2n^2-n-1}{5n^2+n-3}\right\}$ b) $\left\{\dfrac{\cos n}{n}\right\}$ c) $\left\{\sqrt{n^2+2n}-n\right\}$.

Solutions: a) $\lim_{n\to\infty} \frac{2n^2-n-1}{5n^2+n-3} = \lim_{n\to\infty} \frac{\cancel{n^2}(2-\frac{1}{n}-\frac{1}{n^2})}{\cancel{n^2}(5+\frac{1}{n}-\frac{3}{n^2})}$

$$= \lim_{n\to\infty} \frac{2-\overset{\to 0}{1/n}-\overset{\to 0}{1/n^2}}{5+\underset{\to 0}{1/n}-\underset{\to 0}{3/n^2}} = \frac{2}{5}$$

$\lim 1/n = 0$ is known. $\frac{1}{n^2} = 1/n \cdot 1/n$ and

thus $\lim_{n\to\infty} 1/n^2 = (\lim_{n\to\infty} 1/n)(\lim_{n\to\infty} 1/n) = 0 \cdot 0 = 0$

$2/n = 2\ 1/n$ and then $\lim \frac{2}{n} = 2 \lim 1/n = 2.0 = 0.$

Hence, $\lim_{n\to\infty} 2 - 1/n - 1/n^2 = 2 - 0 - 0 = 2.$

Similarly, $\lim(5+\frac{1}{n}-\frac{3}{n^2}) = 5 - 0 - 3.0 = 5.$

Hence, $\lim \frac{2n^2-n-1}{5n^2+n-3} = 2/5.$

b) $\lim \frac{\cos n}{n}$

Note that $-1 \leq \cos n \leq 1$ and thus

$-1/n \leq \frac{\cos n}{n} \leq 1/n$. Also $\lim 1/n = \lim -1/n = 0$

and thus by the Squeezing Theorem $\lim \frac{\cos n}{n}$ exists and equals 0.

$$\underset{\downarrow 0}{-1/n} \leq \underset{\downarrow 0}{\frac{\cos n}{n}} \leq \underset{\downarrow 0}{1/n}$$

c) $\lim_{n\to\infty}(\sqrt{n^2+2n}-n) = \lim_{n\to\infty} \frac{(\sqrt{n^2+2n}-n)(\sqrt{n^2+2n}+n)}{\sqrt{n^2+2n}+n}$

$$= \lim_{n\to\infty} \frac{(n^2+2n)-n^2}{\sqrt{n^2+2n}+n}$$

$$= \lim_{n\to\infty} \frac{2n \quad /n}{\sqrt{n^2+2n}+n \quad /n}$$

$$= \lim_{n\to\infty} \frac{2}{\underbrace{\sqrt{1+2/n}}_{\to 1}+1} = \frac{2}{1+1} = 1.$$

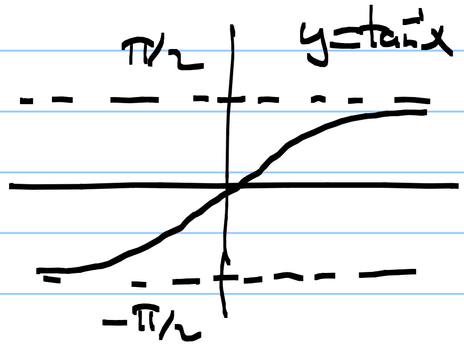
$\frac{2}{n} = 2\cdot\frac{1}{n} \to 2\cdot 0 = 0.$

$x = 1+\frac{2}{n} \to 1+0 = 1.$

$f(x) = \sqrt{x}$, $\lim_{x\to 1} f(x) = \lim_{x\to 1}\sqrt{x} = 1.$

$\sqrt{1+2/n} \to 1$ as $n\to\infty$. Here we use the fact that $f(x)=\sqrt{x}$ is continuous at $x=1$.

<u>Example</u> $\lim_{n\to\infty} n\tan^{-1}(1/n)$



$f(x) = x\tan^{-1} 1/x$, $x>0$.

So, $\lim_{n\to\infty} n\tan^{-1}(1/n) = \lim_{x\to\infty} x(\tan^{-1} 1/x)$ (if the R.H.S. exists)

$$= \lim_{x\to\infty} \frac{\tan^{-1} 1/x}{1/x} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to\infty} \frac{(\tan^{-1} 1/x)'}{(1/x)'}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+(1/x)^2} \cdot \cancel{\frac{-1}{x^2}}}{\cancel{-1/x^2}}$$

$$= \lim_{x\to\infty} \frac{x^2}{x^2+1}$$

$$= \lim_{x\to\infty} \frac{1}{1+1/x^2} \quad 1/x^2 \to 0$$

$$= \frac{1}{1} = 1.$$

<u>Theorem</u>: If $\{a_n\}$ converges then $\{a_n\}$ is bounded.

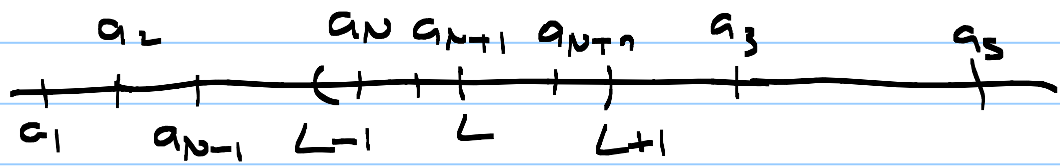
<u>Proof</u>: We are given that $\lim a_n = L$. So let $\epsilon = 1 > 0$.

Then by the assumption there is some $N \in \mathbb{N}$ so that if $n \geq N$ then $|a_n - L| < \epsilon = 1$.

$n \geq N$, $\quad -1 < a_n - L < 1$

$$L-1 < a_n < L+1 \Rightarrow \underline{\underline{|a_n| \leq \max\{|L+1|, |L-1|\}}}.$$

Let $K = \max\{|L-1|, |L+1|, |a_1|, |a_2|, \dots, |a_{N-1}|\}$
> 0



Moreover, $|a_n| \leq K$ for all $n = 1, 2, \ldots$

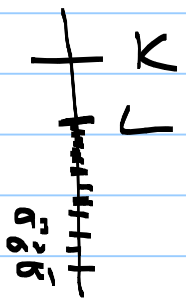
The above theorem implies that an unbounded sequence cannot be convergent.

For example, $\{(-1)^n n^2\}$ is <u>not</u> bounded and thus it is <u>not</u> convergent.

(not convergent $\equiv$ divergent)

<u>Remark:</u> A bounded sequence need not to be convergent. For example, the sequence $\{(-1)^n\} = \{-1, 1, -1, 1, -1, 1, \ldots\}$ is bounded by $K=1$ but it is <u>not</u> convergent.

<u>Theorem:</u> If a sequence is both bounded and monotonic then it is convergent.



$a_{n+1} \geq a_n$

<u>Example</u> Consider the sequence defined by the recursive relation

$$a_1 = 1, \quad a_{n+1} = \sqrt{6 + a_n}, \quad n = 1, 2, 3, \ldots$$

Show that $\lim a_n$ exists.

<u>Solution:</u> $1, \sqrt{7}, \sqrt{6+\sqrt{7}}, \sqrt{6+\sqrt{6+\sqrt{7}}}, \ldots$

<u>Claim</u> $\{a_n\}$ is increasing: $a_{n+1} \geq a_n$, $n=1,2,\ldots$

$n=1 \Rightarrow a_2 \geq a_1$, $\sqrt{7} \geq 1$ ✓

Let's use induction. Assume the result for n: $a_{n+1} \geq a_n$.

<u>must show:</u> $a_{n+2} \geq a_{n+1}$.

$$a_{n+2} = \sqrt{6+a_{n+1}} \geq \sqrt{6+a_n} = a_{n+1}. \checkmark$$

Hence, $\{a_n\}$ is increasing.

<u>Claim</u> $1 \leq a_n < 3$, for all n.

Again let's use induction

$n=1$, $1 \leq a_1 = 1 < 3$ ✓

Assume $1 \leq a_n < 3$. Then for a_{n+1} we have

$$1 \leq \sqrt{7} \leq a_{n+1} = \sqrt{6+a_n} < \sqrt{6+3} = \sqrt{9} = 3$$

So $\{a_n\}$ is a monotonic bounded sequence. Hence, it is convergent.

$\lim a_n = L \in \mathbb{R}$, $L = ?$

$$1 \leq a_{n+1} = \sqrt{6+a_n} \Rightarrow \underbrace{\lim a_{n+1}}_{= L} = \underbrace{\lim \sqrt{6+a_n}}_{= \sqrt{6+L}}$$

$L \geq 1$ since $a_n \geq 1$.

Video 4

$f(x)=\sqrt{6+x}$ continuous function.

$\lim_{x\to L} f(x)=\sqrt{6+L} \Rightarrow \lim_{n\to\infty} f(\underline{\underline{a_n}})=\lim_{x\to L} f(x)=\sqrt{6+L}$

$\Rightarrow L=\sqrt{6+L} \Rightarrow L^2=6+L$
$\Rightarrow L^2-L-6=0 \Rightarrow (L-3)(L+2)=0$

So, $L=3$ or $L=-2$. $L\neq -2$ since $L\geq 1$.

Hence, $\lim a_n = L = 3$.

Ex: $\{(1+\frac{1}{n})^n\}$ $\quad \lim_{n\to\infty}(1+\frac{1}{n})^n=\lim_{x\to\infty}(1+\frac{1}{x})^x=e.$

Because $f(x)=(1+\frac{1}{x})^x$ has limit as $x\to\infty$, which is equal to e.

Theorem: If $\{a_n\}$ is (ultimately) increasing then it is either bounded above and therefore convergent or it is not bounded and diverges to infinity.

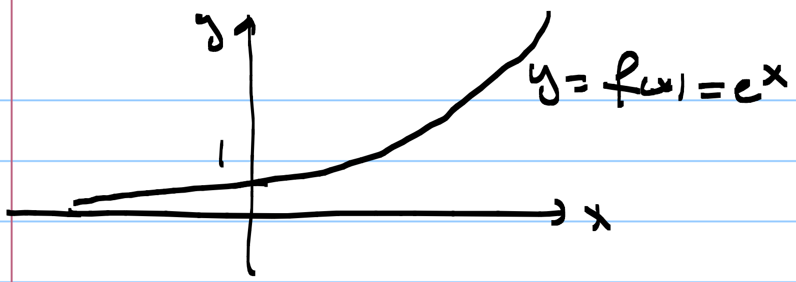
Theorem: a) If $|x|<1$, then $\lim_{n\to\infty} x^n=0$

b) If x is any real number then $\lim_{n\to\infty}\frac{x^n}{n!}=0$.

Proof: a) First observe that, if $x\neq 0$,

$$\lim_{n\to\infty} \ln|x|^n=\lim_{n\to\infty} n\underbrace{\ln\underbrace{|x|}_{<1}}_{<0}=-\infty.$$

Since, $f(x)=e^x$ is continuous and $\lim_{x\to-\infty} e^x=0$



Now, $\lim_{n\to\infty} |x|^n = \lim_{n\to\infty} e^{\ln|x|^n} = \lim_{n\to -\infty} e^n = 0.$

Finally,
$$\begin{array}{ccccc} -|x|^n & \le & x^n & \le & |x|^n \\ \downarrow {\scriptstyle n\to\infty} & & \downarrow & & \downarrow {\scriptstyle n\to\infty} \\ -0=0 & & 0 & & 0 \end{array}$$

So, $\lim x^n = 0$ by the Squeeze Theorem.

b) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any $x \in \mathbb{R}$.

Given $x \in \mathbb{R}$, choose some integer N so that $N > |x|$. Now if $n > N$ then

$$\left|\frac{x^n}{n!}\right| = \underbrace{\frac{|x|}{1}\cdot\frac{|x|}{2}\cdot\frac{|x|}{3}\cdots\frac{|x|}{N-1}}\cdot\frac{|x|}{N}\cdot\frac{|x|}{N+1}\cdots\frac{|x|}{n}$$

$$< \frac{|x|^{N-1}}{(N-1)!}\,\underbrace{\frac{|x|}{N}\cdot\frac{|x|}{N}\cdot\frac{|x|}{N}\cdot\ldots\cdot\frac{|x|}{N}}_{n-(N-1)=n+1-N \text{ terms}}$$

$$< \underbrace{\frac{|x|^{N-1}}{(N-1)!}}_{=K}\left(\frac{|x|}{N}\right)^{n+1-N}$$

$$< K\, r^{n+1-N}, \text{ where } K = \frac{|x|^{N-1}}{(N-1)!} \text{ and } r = \frac{|x|}{N}.$$

Since $N > |x|$, $r < 1$.

$$0 \le \lim \frac{|x|^n}{n!} < K\, r^{n+1-N} = \underbrace{K\, r^{-N+1}}\, r^n$$

$$= K' \cdot r^n \quad , 0<r<1$$

$$\downarrow \quad \downarrow \text{ as } n\to\infty.$$

$$= K' \cdot 0$$

$$= \underline{\underline{0}}$$

$$\Rightarrow \lim \frac{|x|^n}{n!} = 0. \quad ∎$$

__Example:__ $\displaystyle \lim_{n\to\infty} \frac{3^n+4^n+5^n}{5^n} = \lim_{n\to\infty} \left(\frac{3}{5}\right)^n + \left(\frac{4}{5}\right)^n + 1$

$0<\frac{3}{5}<1$ and $0<4/5<1$ $\quad = 0+0+1$

$= 1.$

__§9.2. Infinite Series:__ An infinite series, usually denoted as

$$\sum_{n=1}^{\infty} a_n \quad \text{is a formal sum of}$$

infinitely many terms.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

We'll mostly consider series starting from $n=0$.
But in general it can start from any integer.

$$\sum_{n=-2}^{\infty} a_n \quad \text{or} \quad \sum_{n=5}^{\infty} a_n.$$

Sequence of Partial Sums: If $\sum_{n=1}^{\infty} a_n$ is a series its sequence of partial sums is defined as follows (s_n)

$s_1 = a_1$
$s_2 = a_1 + a_2$
$s_3 = a_1 + a_2 + a_3$
$\vdots$
$s_n = a_1 + a_2 + \cdots + a_n$
$\vdots$

(s_n) is a sequence of real numbers.

Definition: (Convergence of a series)

We say that the series $\sum_{n=1}^{\infty} a_n$ converges to the sum s, and we write

$$\sum_{n=1}^{\infty} a_n = s$$

if $\lim_{n\to\infty} s_n = s$, where s_n is the n^{th} partial sum of the series:

$$s_n = a_1 + a_2 + \cdots + a_n.$$

Geometric Series:

Consider a series of the form $\sum_{n=1}^{\infty} ar^{n-1}$, where $a, r \in \mathbb{R}$.

Note that if we let $a_n = ar^{n-1}$, then

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} a_n.$$

Convergence? Sequence of partial sums:

$a_n = ar^{n-1}$

$s_1 = a_1 = a \cdot r^0 = a$

$s_2 = a_1 + a_2 = a + ar$

$s_3 = a_1 + a_2 + a_3 = a + ar + ar^2$

$\vdots$

$s_n = a_1 + a_2 + \dots + a_n = a + ar + ar^2 + \dots + ar^{n-1}$

$\lim s_n = ?$ $\qquad s_n = a(1 + r + r^2 + \dots + r^{n-1})$

$\Rightarrow s_n = a \dfrac{1-r^n}{1-r}$ since $(1-r)(1+r+\dots+r^{n-1}) = 1-r^n$

$= \dfrac{a}{1-r}(1-r^n) \qquad (r \neq 1)$

If $|r| < 1$, then $\lim\limits_{n\to\infty} r^n = 0$ so that

$\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \dfrac{a}{1-r}(1-r^n) = \dfrac{a}{1-r}(1-0) = \dfrac{a}{1-r}$

So, if $|r| < 1$ then the sequence (s_n) is convergent with limit $a/_{1-r}$ so that the series

$\sum\limits_{n=1}^{\infty} ar^n$ is convergent and $\sum\limits_{n=1}^{\infty} ar^n = \dfrac{a}{1-r}$.

If $r = 1$, then $\sum\limits_{n=1}^{\infty} a \cdot r^n = \sum\limits_{n=1}^{\infty} a = a + a + a + \dots + a + \dots$

$s_n = \underbrace{a + \dots + a}_{n \text{ terms}} = na$, $\lim s_n = \begin{cases} 0 & \text{if } a = 0 \\ +\infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0 \end{cases}$

Hence, $\sum\limits_{n=1}^{\infty} a \cdot 1^n$ is convergent only if $a = 0$.

If $r > 1$, $r^n \to +\infty$ as $n \to \infty$, so that $s_n = a + ar + \dots + ar^{n-1} \to +\infty$.

Video 5

If $r < -1$ then again r^n is divergent.

$r = -3$, $\{r^n\} = \{-3, 9, -27, 81, -243, \ldots\}$ is divergent.

As a result we have

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \text{converges to } 0 & \text{if } a=0 \\ \text{converges to } a/1-r & \text{if } |r|<1 \\ \text{diverges to } \infty & \text{if } r \geq 1,\ a>0 \\ \text{diverges to } -\infty & \text{if } r \geq 1,\ a<0 \\ \text{divergent} & \text{if } r \leq -1 \text{ and } a \neq 0. \end{cases}$$

Examples 1) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} (1/2)^{n-1} = \frac{a}{1-r}$

$a=1$ $\quad |r = \frac{1}{2}| = 1/2 < 1$ $\quad = \frac{1}{1-1/2} = 2.$

2) $\pi - e + \frac{e^2}{\pi} - \frac{e^3}{\pi^2} + \cdots = \sum_{n=1}^{\infty} \pi\left(-\frac{e}{\pi}\right)^{n-1} = \frac{a}{1-r}$

$a = \pi$, $r = -\frac{e}{\pi}$, $|r| = \frac{e}{\pi} < 1$

$$= \frac{\pi}{1-\left(-\frac{e}{\pi}\right)}$$

$$= \frac{\pi^2}{\pi + e}$$

Telescoping Series and Harmonic Series:

Ex: Show that the series below is convergent and find its sum: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solution $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} a_n$, $\quad a_n = \frac{1}{n(n+1)}$

The sequence of partial sums is

$s_n = a_1 + a_2 + \cdots + a_n$

$= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) \cancel{+} \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$

$= 1 - \frac{1}{n+1}$

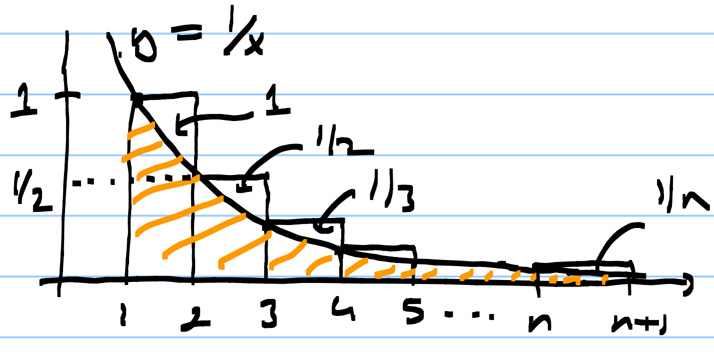
$s_n = 1 - \frac{1}{n+1}$, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1 - 0 = 1$

Hence, the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and equals to 1.

Series of this kind are called telescoping series.

<u>Harmonic Series:</u> $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

Sequence of partial sums $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.



$s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the sum areas of the rectangles.

Note that $s_n >$ the area under the curve $y = 1/x$ from $x=1$ to $x=n+1$.

Then, $s_n > \int_1^{n+1} \frac{1}{x}\,dx$

So, $s_n > \ln x \Big|_1^{n+1} = \ln(n+1) - \ln 1 = \ln(n+1)$

$\Rightarrow s_n > \ln(n+1)$.

Since $\lim_{x \to \infty} \ln x = +\infty$, we see that $\lim_{n \to \infty} s_n = +\infty$.

Hence, $\{s_n\}$ and thus $\sum_{n=1}^{\infty} 1/n$ is divergent.

<u>Some Theorem About Series:</u>

<u>Theorem</u>: If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n \to \infty} a_n = 0$.

Therefore if $\{a_n\}$ is not convergent to zero then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(This theorem is also known as the n^{th} term test for convergence of series)

<u>Proof</u>: $s_n = a_1 + a_2 + \cdots + a_n$. By assumption

$\lim s_n = L$ exists. Then $\lim s_{n-1} = L$.

Now, $a_n = s_n - s_{n-1}$ so that

$$\lim a_n = \lim (s_n - s_{n-1}) = \lim s_n - \lim s_{n-1} = L - L = 0. \blacksquare$$

<u>Example</u>: 1) $\sum_{n=1}^{\infty} \frac{n}{2n-1}$, $a_n = \frac{n}{2n-1}$. Note that

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n}{2n-1} = \lim_{n \to \infty} \frac{1}{2 - 1/n} = \frac{1}{2-0} = 1/2 \neq 0$$

Hence, by the n^{th} term test the series

$\sum_{n=1}^{\infty} \frac{n}{2n-1}$ is divergent.

Example $\sum_{n=1}^{\infty} (-1)^n n \sin \frac{1}{n}$, $a_n = (-1)^n n \sin \frac{1}{n}$

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} |(-1)^n n \sin 1/n| = \lim_{n\to\infty} n \sin 1/n$

$\lim_{x\to\infty} x \sin \frac{1}{x} = \lim_{x\to\infty} \frac{\sin 1/x}{1/x} \quad \left[\frac{0}{0}\right]$

$= \lim_{x\to\infty} \frac{(\sin 1/x)'}{(1/x)'}$

$= \lim_{x\to\infty} \frac{\cos 1/x \cdot \cancel{(-1/x^2)}}{\cancel{-1/x^2}}$

$= \lim_{x\to\infty} \cos 1/x$

$= \cos 0 = 1.$

Hence, $\lim_{n\to\infty} n \sin 1/n = \lim_{x\to\infty} x \sin 1/x = 1$ and

thus $\lim |a_n| = 1$ so that $\lim a_n \neq 0$.

Therefore, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n n \sin 1/n$

is divergent by the n^{th} term test.

Theorem: $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} a_n$

converges, for every integer $N \geq 1$.

Proof If $s_n = a_1 + \cdots + a_N + \cdots + a_n$ is the nth partial sum of $\sum_{n=1}^{\infty} a_n$ then

$s_n - s_{N-1} = a_N + \cdots + a_n$ is the nth partial sum of the series $\sum_{n=N}^{\infty} a_n$.

Hence, $\{s_n\}$ is convergent if and only if $\{s_n - s_N\}$ is convergent. $\blacksquare$

Theorem: If $\{a_n\}$ is ultimately positive, then the series $\sum_{n=1}^{\infty} a_n$ must either converge (if the sequence of partial sums is bounded) or diverge to infinity (if the sequence of partial sums is not bounded above).

Proof

$$s_n = \underbrace{a_1 + \cdots + a_N}_{} + \underset{\geq 0}{a_{N+1}} + \underset{\geq 0}{a_{N+2}} + \cdots + \underset{\geq 0}{a_n} + a_{n+1} + a_{n+2}$$

(s_n) is ultimately increasing. Hence the result follows. $\blacksquare$

Theorem: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge to A and B, respectively, then

a) $\sum_{n=1}^{\infty} (ca_n)$ converges to cA, where c is any constant

b) $\sum_{n=1}^{\infty} (a_n \pm b_n)$ converges to $A+B$ or $A-B$.

c) If $a_n \leq b_n$ for all n, then $A \leq B$.

Proof $A = \sum_{n=1}^{\infty} a_n$, $s_n = a_1 + \cdots + a_n$, $\lim s_n = A$.

$B = \sum_{n=1}^{\infty} b_n$, $\quad r_n = b_1 + \dots + b_n$, $\quad \lim r_n = B$.

$\sum_{n=1}^{\infty} (a_n + b_n)$, the n^{th} partial sum of this series is

$$(a_1 + b_1) + \dots + (a_n + b_n) = s_n + r_n$$

$$\lim (s_n + r_n) = \lim s_n + \lim r_n = A + B.$$

$\sum_{n=1}^{\infty} (a_n + b_n) = A + B$ is convergent $\blacksquare$

<u>Examples</u> 1) $$\sum_{n=1}^{\infty} \frac{1 + 2^{n+1}}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + 2\left(\frac{2}{3}\right)^n$$

Both are geometric series with $r = 1/3$ and $r = 2/3$.

$$= \sum_{n=1}^{\infty} (1/3)^n + \sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^n$$

$$= \sum_{n=1}^{\infty} (1/3)^n + 2\sum_{n=1}^{\infty} (2/3)^n$$

$$= \frac{1}{3} \sum_{n=1}^{\infty} (1/3)^{n-1} + 2 \cdot \frac{2}{3} \sum_{n=1}^{\infty} (2/3)^{n-1}$$

$$= \frac{1}{3} \cdot \frac{1}{1 - 1/3} + \frac{4}{3} \cdot \frac{1}{1 - 2/3}$$

$$= \frac{1}{3} \cdot \frac{3}{2} + \frac{4}{3} \cdot \frac{3}{1}$$

$$= 1/2 + 4 = 9/2.$$

<u>Remark</u> $\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r}$, $|r| < 1$.

$$\sum_{n=1}^{\infty} \frac{1}{4^n} = 1/4 + 1/16 + \frac{1}{64} + \dots = 1/4\left(1 + \frac{1}{4} + \frac{1}{16} + \dots\right)$$

$$= 1/4 \cdot \frac{1}{1-1/4}$$
$$= 1/4 \cdot \frac{4}{3} = 1/3.$$

§9.3. Convergence Tests for Positive Series:

The Integral Test:

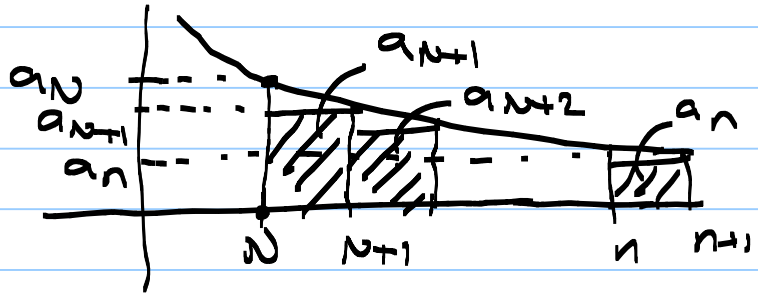
Theorem: Suppose that $a_n = f(n)$, where $f(x)$ is a positive, continuous, and non-increasing function on $[N, \infty)$, for some positive integer N. Then

$\sum_{n=1}^{\infty} a_n$ and $\int_N^{\infty} f(t)\,dt$ both converge or both diverge to infinity.

Proof: $s_n = a_1 + a_2 + \cdots + a_n$, $n > N$.

$$= s_N + a_{N+1} + \cdots + a_n$$
$$= s_N + f(N+1) + \cdots + f(n)$$

$= s_N +$ sum of the areas of rectangles shaded.



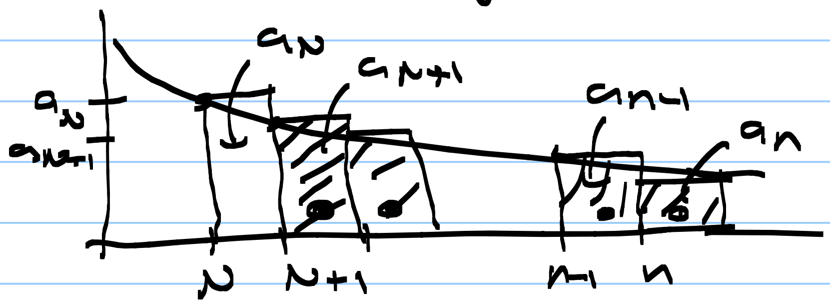
$$s_n \leq s_N + \int_N^{\infty} f(t)\,dt$$

So if $\int_N^{\infty} f(t)\,dt$ is convergent then $\{s_n\}$ is bounded above and thus convergent,

because $s_{n+1} \geq s_n$ ($\{s_n\}$ is increasing).
Hence, the series $\sum_{n=1}^{\infty} a_n$ is convergent.

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then

consider the diagram below



$$s_n = a_1 + \cdots + a_N + \underbrace{a_{N+1} + \cdots + a_n}$$
$= S_{N+1}$ + sum of areas of the rectangles shaded

$$s_n \geq S_{N+1} + \int_{N+1}^{\infty} f(t)\,dt \geq 0$$

convergent and is bounded. $\Rightarrow \int_{N+1}^{\infty} f(t)\,dt$ is

bounded and thus it is convergent, because $f(t)$ is a positive function. ■

Example (p-test) ($p > 0$)

$$\sum_{n=1}^{\infty} n^{-p} = \sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges if } p > 1 \\ \text{diverges to } \infty \text{ if } p \leq 1. \end{cases}$$

Proof: Use the integral test.
Let $f(x) = 1/x^p$, $x > 0$.

Video 7

Then $f(n) = \frac{1}{n^p} = a_n$. Clearly, $f(x)$ is continuous and it is decreasing $\left(f'(x) = \frac{-p}{x^{p+1}} < 0 \right)$

$$\int_N^\infty f(t)\,dt = \int_N^\infty \frac{1}{t^p}\,dt = \frac{1}{-(p-1)} \frac{1}{t^{p-1}} \Big|_N^\infty, \quad \text{if } p>1$$

$t^{p-1} \to \infty$, $p-1>0$

$$= 0 + \frac{1}{p-1} \cdot \frac{1}{N^{p-1}} < +\infty.$$

$\Rightarrow \int_N^\infty f(t)\,dt$ is convergent. Hence, by the integral test the series $\sum_{n=1}^\infty \frac{1}{n^p}$ is conv.

Case 2 $0 < p \leq 1$. $(p<1)$

$$\int_N^\infty f(t)\,dt = \int_N^\infty \frac{1}{t^p}\,dt = \int_N^\infty t^{-p}\,dt = \frac{t^{-p+1}}{-p+1} \Big|_N^\infty$$

$$= \frac{1}{1-p} \left(\lim_{t\to\infty} t^{-p+1} - N^{-p+1} \right)$$

$$= \frac{1}{1-p} \left(\infty - N^{-p+1} \right) = \infty \quad \text{divergent.}$$

$\Rightarrow \sum_{n=1}^\infty a_n = \sum_{n=1}^\infty \frac{1}{n^p}$ is divergent for $0<p<1$.

For $p=1$, $\int_N^\infty f(t)\,dt = \int_N^\infty \frac{1}{t}\,dt = (\ln t) \Big|_N^\infty = \infty$.

again divergent. Hence, $\sum a_n = \sum \frac{1}{n}$ is divergent.

Comparison Tests

Theorem (Comparison Test)
Let $\{a_n\}$ and $\{b_n\}$ be sequences for which there exists a positive constant K such that, ultimately, $0 \leq a_n \leq K b_n$.

a) If the series $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

b) If $\sum_{n=1}^{\infty} a_n$ diverges then so does $\sum_{n=1}^{\infty} b_n$.

Proof: Let $s_n = a_N + \cdots + a_n$ and $r_n = b_N + \cdots + b_n$.

By assumption there is some N so that $n \geq N$ implies $0 \leq a_n \leq K b_n$.

Then $s_n \leq K r_n$, for all $n \geq N$. Both s_n and r_n are positive and increasing sequences.

a) We are given that $\sum_{n=1}^{\infty} b_n$ is convergent. Then $\sum_{n=N}^{\infty} b_n$ is convergent. Then the sequence $\{r_n\}$ is convergent. $\{r_n\}$ is bounded. Then there is some $M > 0$ so that $0 \leq r_n \leq M$ for all n.
$\Rightarrow 0 \leq K r_n \leq KM, \ \forall n.$

$\Rightarrow \ 0 \leq s_n \leq K r_n \leq KM$, for all n so that the sequence $\{s_n\}$ is bounded. Hence, $\{s_n\}$ is convergent.
$\Rightarrow \sum_{n=N}^{\infty} a_n$ is convergent. $\Rightarrow \sum_{n=1}^{\infty} a_n$ is convergent.

(b) follows from part (a). ∎

Examples Check the series below for convergence.

a) $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ b) $\sum_{n=1}^{\infty} \frac{3n+1}{n^3+1}$ c) $\sum_{n=1}^{\infty} \frac{1}{\ln n}$

Solution a) $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ Let $a_n = \frac{1}{2^n+1}$ and $b_n = \frac{1}{2^n}$.

Then $2^n+1 > 2^n$ so that $a_n = \frac{1}{2^n+1} < \frac{1}{2^n} = b_n$.

Moreover, $\sum b_n = \sum \frac{1}{2^n}$ is convergent. So by the comparison test, where we take $K=1$, the series $\sum a_n = \sum \frac{1}{2^n+1}$ is convergent.

b) $\sum_{n=1}^{\infty} \frac{3n+1}{n^3+1}$. Let $a_n = \frac{3n+1}{n^3+1}$ and $b_n = \frac{1}{n^2}$.

Then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the

p-test ($p=2>1$). Also note that

$$0 \leq a_n = \frac{3n+1}{n^3+1} \leq \frac{3n+1}{n^3} = \frac{3}{n^2} + \frac{1}{n^3} \leq \frac{3}{n^2} + \frac{1}{n^2}$$

because $n^3 \geq n^2$ (for $n \geq 1$).

$0 \leq a_n \leq \frac{4}{n^2} = 4 b_n$. Since $\sum b_n = \sum \frac{1}{n^2}$ is

convergent and $0 \leq a_n \leq K b_n$ ($K=4$) for all $n \geq N=1$ by the comparison test the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3n+1}{n^3+1} \text{ is convergent.}$$

$\left(\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6 \right)$

c) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$. Let $a_n = \frac{1}{\ln n}$ and $b_n = 1/n$.

$n \geq \ln n$ if $n \geq 1$. ($f(x) = x - \ln x$ for $x \geq 1$.

Then $f'(x) = 1 - 1/x \geq 0 \Rightarrow f(x)$ is increasing on $[1, \infty)$. Thus, $f(x) \geq f(1) \Rightarrow$

$x - \ln x \geq 1 - \ln 1 = 1 \Rightarrow x \geq 1 + \ln x \geq \ln x$ for $x \geq 1$)

$\Rightarrow \frac{1}{\ln n} > \frac{1}{n}$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is the Harmonic series, which is divergent, the series $\sum_{n=2}^{\infty} 1/\ln n$ is also divergent.

Theorem (A Limit Comparison Test)

Suppose that $\{a_n\}$ and $\{b_n\}$ are positive sequences and that $\lim a_n/b_n = L$, where L is either a nonnegative finite number or $+\infty$.

a) If $L < \infty$ and $\sum b_n$ converges then $\sum a_n$ also converges.

b) If $L > 0$ and $\sum b_n$ diverges then so does $\sum a_n$.

Proof: a) $L < \infty$. So $L = \lim \frac{a_n}{b_n}$ exists and finite.

Take $\epsilon = 1$. Then there is some $N \in \mathbb{N}$ so that $n \geq N$ implies $a_n/b_n \in (L-\epsilon, L+\epsilon)$.

$\Rightarrow L - 1 \leq \frac{a_n}{b_n} \leq L + 1$.

$\Rightarrow \quad 0 < a_n \leq (L+1) b_n$ for $n \geq N$.

By assumption $\sum_{n=N}^{\infty} b_n$ is convergent. So by the comparison test $\sum_{n=N}^{\infty} a_n$ is convergent. Hence, $\sum_{n=1}^{\infty} a_n$ is convergent.

b) Assume $L > 0$. $0 < L < +\infty$ or $L = \infty$.

If $0 < L < +\infty$ then let $\epsilon = L/2$. Then there is some N so that $n \geq N \Rightarrow$

$$\frac{a_n}{b_n} \in (L-\epsilon, L+\epsilon) = (L/2, 3L/2).$$

$0 < L/2 < a_n/b_n < 3L/2 \Rightarrow \frac{L}{2} b_n < a_n < \frac{3L}{2} b_n$.

If $\sum b_n$ is divergent then $\sum \frac{L}{2} b_n$ is divergent and hence $\sum a_n$ is divergent by the comparison test.

$L = +\infty$. $\lim a_n/b_n = L = \infty$. Then again

$a_n/b_n \geq 1$ for all $n \geq N$, for some N.
This implies $a_n \geq b_n$ for all $n \geq N$.
Since $\sum b_n$ is divergent then so is $\sum a_n$. ∎

Examples Check for convergence.

a) $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ $\qquad$ b) $\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$.

<u>Solution:</u> a) $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$. Let $a_n = \frac{1}{1+\sqrt{n}}$ and $b_n = \frac{1}{\sqrt{n}}$. Note that $\sum b_n = \sum \frac{1}{\sqrt{n}}$ is divergent since $p = 1/2 < 1$ (p-test).

Also, $\lim \frac{a_n}{b_n} = \lim \frac{1/1+\sqrt{n}}{\sqrt{n}} = \lim \frac{\sqrt{n}}{1+\sqrt{n}}$

$$= \lim \frac{1}{1/\sqrt{n} + 1} = \frac{1}{0+1} = 1.$$

$L = 1 > 0$ and $\sum b_n$ is divergent. Thus by part (b) of the Limit Comparison Test the series $\sum a_n$ is divergent.

b) $\sum_{n=1}^{\infty} \frac{n+5}{n^3 - 2n + 3}$. Let $a_n = \frac{n+5}{n^3-2n+3}$

$b_n = \frac{1}{n^2}$. Then $L = \lim \frac{a_n}{b_n}$

$$= \lim \frac{n+5}{n^3-2n+3} \cdot \frac{n^2}{1}$$

$$= \lim \frac{n^3+5n^2}{n^3-2n+3}$$

$$= 1.$$

$L = 1 < \infty$ and $\sum b_n = \sum \frac{1}{n^2}$ is convergent. Hence, by part (a) of the Limit Comparison Test the series $\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$ is convergent.

Video 8

The Ratio and Root Tests

Theorem (The Ratio Test)

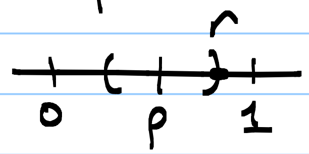
Suppose $a_n > 0$ (ultimately) and $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists or is $+\infty$. Then

a) If $0 \le \rho < 1$, then the series $\sum a_n$ converges.

b) If $1 < \rho \le \infty$, then the series $\sum a_n$ diverges.

c) If $\rho = 1$ then no conclusion.

Proof: a) $\rho = \lim \frac{a_{n+1}}{a_n}$ and $0 \le \rho < 1$.



$\epsilon = \frac{1-\rho}{2} > 0$. Since $\lim \frac{a_{n+1}}{a_n} = \rho$

there is some $N \in \mathbb{N}$ so that $n \ge N \Rightarrow$

$$\left| \frac{a_{n+1}}{a_n} - \rho \right| < \epsilon = \frac{1-\rho}{2}.$$

$$\frac{\rho-1}{2} < \frac{a_{n+1}}{a_n} - \rho < \frac{1-\rho}{2} \Rightarrow \frac{3\rho-1}{2} < \frac{a_{n+1}}{a_n} < \frac{1+\rho}{2}$$

$$0 < \frac{a_{n+1}}{a_n} < \frac{1+\rho}{2} \doteq r, \text{ for } n \ge N.$$

$a_{N+1} \le r a_N$.

$a_{N+2} \le r a_{N+1} \le r^2 a_N$

$a_{N+3} \le r a_{N+2} \le r^3 a_N$

$\vdots$

$a_{N+n} \le r^n a_N$, $n \ge N$. Let $K = a_N$.

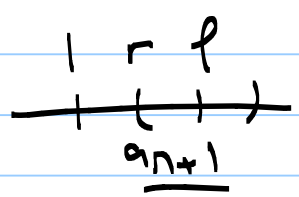
$0 \le b_n = a_{N+n} \le K r^n$, $\quad r = \frac{1+\rho}{2} < 1$.

Since $0 \le b_n \le K r^n$ and $\sum_{n=1}^{\infty} K r^n$ is convergent because it is a geometric series with $0 < r < 1$, we see that

$$\sum_{n=1}^{\infty} b_n = \sum_{n=0}^{\infty} a_{N+n} = \sum_{n=N}^{\infty} a_n \text{ is convergent}$$

(by the Comparison Test). Hence $\sum_{n=1}^{\infty} a_n$ is convergent.

b) $\rho = \lim \frac{a_{n+1}}{a_n} > 1$ or $+\infty$.



So there is some $r > 1$ so that

$$\frac{a_{n+1}}{a_n} \ge r \text{ for all } n \ge N.$$

Then
$$a_{N+1} \ge r a_N$$
$$a_{N+2} \ge r a_{N+1} \ge r^2 a_N$$
$$\vdots$$
$$b_n = a_{N+n} \ge r^n a_N = r^n K \quad (K = a_N)$$

The geometric series $\sum r^n$ is divergent since $r > 1$ and thus the series $\sum b_n$ is divergent by Comparison Test.

$$\Rightarrow \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} a_{N+n} = \sum_{n=N}^{\infty} a_n \text{ is divergent.}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

Remark: If $\rho = 1$ then the test indeed fails as the examples below show:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \rho = \lim \frac{a_{n+1}}{a_n} = \lim \frac{1/(n+1)^2}{1/n^2}$$
$$= \lim \frac{n^2}{(n+1)^2} = 1$$

and the series $\sum 1/n^2$ is convergent.

On the other hand, for the divergent series $\sum_{n=1}^{\infty} 1/\sqrt{n}$ (div. by the p-test $p=1/2$)

we still have, $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim \frac{\sqrt{n}}{\sqrt{n+1}} = 1.$

Examples a) $\sum_{n=1}^{\infty} \frac{99^n}{n!}$ b) $\sum_{n=1}^{\infty} \frac{n^5}{2^n}$ c) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

d) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$.

Solution: a) $\sum_{n=1}^{\infty} \frac{99^n}{n!}$. $\qquad a_n = \frac{99^n}{n!}$

$$\rho = \lim \frac{a_{n+1}}{a_n} = \lim \frac{99^{n+1}}{(n+1)!} \cdot \frac{n!}{99^n}$$
$$= \lim \frac{99}{n+1} = 0 < 1.$$

Thus, by the Ratio Test the series $\sum \frac{99^n}{n!}$ is convergent.

b) $\sum_{n=1}^{\infty} \frac{n^5}{2^n}$. $a_n = \frac{n^5}{2^n}$

$\rho = \lim \frac{a_{n+1}}{a_n} = \lim \frac{(n+1)^5}{2^{n+1}} \cdot \frac{2^n}{n^5}$

$= \lim \frac{1}{2}\left(\frac{n+1}{n}\right)^5$

$= \lim \frac{1}{2}\left(1+\underbrace{\frac{1}{n}}_{\to 0}\right)^5$

$= 1/2 \cdot 1^5 = 1/2 < 1$. It is convergent by the Ratio Test.

c) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$. $a_n = \frac{n!}{n^n}$.

$\rho = \lim \frac{a_{n+1}}{a_n} = \lim \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$

$= \lim \frac{n+1}{(n+1)^{n+1}} \cdot n^n$

$= \lim \frac{n^n}{(n+1)^n} = \lim \frac{1}{\left(\frac{n+1}{n}\right)^n}$

$= \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$.

Again by the Ratio Test the series $\sum \frac{n!}{n^n}$ is convergent.

d) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$. $a_n = \frac{(2n)!}{(n!)^2}$.

$$\rho = \lim \frac{a_{n+1}}{a_n} = \lim \frac{(2n+2)!}{(n+1)!\,(n+1)!} \cdot \frac{(n!)(n!)}{(2n)!}$$

$$= \lim \frac{(2n+2).(2n+1)}{(n+1)\ \ (n+1)}$$

$$= \lim \frac{4n^2+6n+2}{n^2+2n+1}$$

$$= 4 > 1$$

So the series is divergent by the Ratio Test.

The Root Test:

Theorem (The Root Test)

Suppose that $a_n > 0$ (ultimately) and that $\sigma = \lim_{n\to\infty} (a_n)^{1/n}$ exists or is ∞.

a) If $0 \le \sigma < 1$, then the series $\sum a_n$ converges.

b) If $1 < \sigma \le +\infty$ then the series $\sum a_n$ diverges.

c) If $\sigma = 1$ then no conclusion.

Example Use the Root Test to check the convergence of $\sum \left(\frac{n}{n+1}\right)^{n^2}$.

Solution: $a_n = \left(\frac{n}{n+1}\right)^{n^2}$. Then $a_n^{1/n} = \left(\frac{n}{n+1}\right)^n$.

So $\sigma = \lim a_n^{1/n} = \lim \left(\frac{n}{n+1}\right)^n = \lim \frac{1}{\left(1+\frac{1}{n}\right)^n}$

$$= \frac{1}{\lim \left(1+1/n\right)^n} = \frac{1}{e} < 1.$$

Hence, by the Root Test the series $\sum \left(\frac{n}{n+1}\right)^{n^2}$ is convergent.

Video 9

§9.4. Absolute and Conditional Convergence:

Definition: (Absolute Convergence)

A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Example: 1) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$. $a_n = \frac{(-1)^{n-1}}{n^2}$ and

hence, $|a_n| = 1/n^2$. $\sum 1/n^2$ is convergent by the p-test ($p = 1/2 < 1$) so then the series

$\sum a_n$ is absolutely convergent.

2) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, $a_n = \frac{(-1)^n}{n}$, $|a_n| = \frac{1}{n}$. We know that $\sum |a_n| = \sum \frac{1}{n}$ is divergent (since it is the Harmonic series) and the $\sum \frac{(-1)^n}{n}$ is not absolutely convergent.

Soon we'll see that both series above are indeed convergent.

Theorem: If a series converges absolutely then it is convergent.

Proof: Let $\sum a_n$ be a series which converges absolutely. Then $\sum |a_n|$ is convergent.

Let $b_n = a_n + |a_n| = \begin{cases} 2a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases}$.

Hence, $0 \leq b_n \leq 2|a_n|$.

Since $\sum |a_n|$ is convergent by the comparison test $\sum b_n$ is convergent.

Then $a_n = (a_n + |a_n|) - |a_n| = b_n - |a_n|$.

Since both $\sum b_n$ and $\sum |a_n|$ are convergent $\sum (b_n - |a_n|)$ is convergent.

So $\sum a_n = \sum (b_n - |a_n|)$ is convergent.

Video 10

Last time: A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Theorem: If a series absolutely convergent then it is convergent.

Example: We'll see that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent. However, it is not absolutely convergent:

$$a_n = \frac{(-1)^{n-1}}{n}, \quad |a_n| = \frac{1}{n}$$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

Definition: A convergent sequence which is not absolutely convergent is called conditionally convergent.

So the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ in the above example is conditionally convergent.

The Alternating Series Test:

Theorem: Suppose $\{a_n\}$ is a sequence which satisfy the following conditions:

i) $a_n a_{n-1} < 0$, for all $n \geq N$, for some N.
ii) $|a_{n+1}| \leq |a_n|$, for all $n \geq N$.
iii) $\lim_{n\to\infty} a_n = 0$.

Then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Examples: 1) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ $\qquad a_n = \frac{(-1)^{n-1}}{n}$

i) $a_n \cdot a_{n+1} = \frac{(-1)^{n-1}}{n} \cdot \frac{(-1)^n}{n+1} = \frac{(-1)^{2n-1}}{n(n+1)} < 0$, for all $n \geq 1$. $(N=1)$

ii) $|a_n| = \left|\frac{(-1)^{n-1}}{n}\right| = \frac{1}{n} > \frac{1}{n+1} = \left|\frac{(-1)^n}{n+1}\right| = |a_{n+1}|$

for all $n \geq 1$.

iii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n-1}}{n} = 0$ since

$$-\frac{1}{n} \leq \frac{(-1)^{n-1}}{n} \leq 1/n.$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$0 \qquad 0 \qquad 0$$

So by the A.S.T. the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.

Examples: 1) $\sum_{n=1}^{\infty} \frac{n\cos n\pi}{2^n}$, $a_n = \frac{n\cos n\pi}{2^n} = \frac{n(-1)^n}{2^n}$

$|a_n| = \frac{n}{2^n}$, $\lim \frac{|a_{n+1}|}{|a_n|} = \lim \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}$

$$= \lim \frac{n+1}{n} \cdot \frac{1}{2}$$
$$= 1/2$$

So, $\rho = \lim \frac{|a_{n+1}|}{|a_n|} = 1/2 < 1$ and thus by the Ratio Test the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Hence the series $\sum_{n=1}^{\infty} a_n$ is (absolutely) convergent.

2) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$, $a_n = \frac{(-1)^{n-1}}{2n-1}$

Let's apply A.S.T.

i) $a_n \cdot a_{n+1} = \frac{(-1)^{n-1}}{2n-1} \cdot \frac{(-1)^n}{2n+1} = \frac{(-1)^{2n-1}}{(2n-1)(2n+1)} < 0$, for all $n \geq N = 1$.

ii) $|a_{n+1}| = \frac{1}{2n+1} < \frac{1}{2n-1} = |a_n|$, for all $n \geq N = 1$.

iii) $\lim a_n = \lim \frac{(-1)^{n-1}}{2n-1} = 0.$

Hence, by the A.S.T. the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ is convergent.

Is it absolutely convergent?

$\sum |a_n| = \sum_{n=1}^{\infty} \frac{1}{2n-1}$

We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Let $b_n = \frac{1}{2n-1}$ and $c_n = 1/n$. Then

$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2} = L > 0.$

So by the Limit Comparison Test the series $\sum \frac{1}{2n-1}$ is also divergent.

Hence, $\sum \frac{(-1)^{n-1}}{2n-1}$ is conditionally convergent.

__Proof:__ $\sum_{n=1}^{\infty} a_n$ i) $a_n \cdot a_{n+1} < 0$, for all $n \geq N$

ii) $|a_n| \geq |a_{n+1}| \quad \forall n \geq N$

iii) $\lim a_n = 0.$

We must show the sequence $\{s_n\}$ of partial sums of the series $\sum a_n$ is convergent.

$s_1 = a_1, \quad s_2 = a_1 + \underline{\underline{a_2}}, \quad s_3 = a_1 + a_2 + a_3, \ldots$

For simplicity assume that $N = 1$ and $a_1 > 0$.

$a_1 \cdot a_2 < 0 \Rightarrow a_2 < 0.$

$s_1 = a_1 > a_1 + a_2 = s_2$

$s_3 = a_1 + a_2 + a_3 = a_1 + \underbrace{(\overset{0}{\overset{\vee}{a_2}} + \overset{0}{\overset{\wedge}{a_3}})}_{<0} < a_1 = s_1$

$s_3 = a_1 + a_2 + a_3 = s_2 + \underset{0}{\underset{\vee}{a_3}} > s_2$

$s_1 > s_3 > s_2$

$s_4 = \underbrace{(a_1 + a_2)}_{>0} + (a_3 + a_4) = s_2 + \underbrace{(a_3 + a_4)}_{\overset{\vee}{0}} > s_2.$

$s_4 = (a_1 + a_2 + a_3) + \overset{0}{\overset{\vee}{a_4}} = s_3 + a_4 < s_3.$

$\underline{\underline{s_1}} > \underline{\underline{s_3}} > \underline{\underline{s_4}} > \underline{s_2}$

$\underline{\underline{s_{2n+1}}} = (a_1 + \cdots + a_{2n-1}) + a_{2n} + a_{2n+1}$

$= s_{2n-1} + \underbrace{\overset{\wedge}{\underset{0}{a_{2n}}} + \overset{\vee}{\underset{0}{a_{2n+1}}}}_{<0} < s_{2n-1}$

$s_{2n+1} = s_{2n} + \overset{0}{\overset{\vee}{a_{2n+1}}} > s_{2n}$

Video 11

$s_1 > s_3 > \cdots > s_{2n-1} > s_{2n+1} > s_{2n} > s_{2n-2} > \cdots > s_2$

$$s_{2n} = \overbrace{a_1 + \cdots + a_{2n-2}} + \underbrace{\overset{>0}{a_{2n-1}} + \overset{<0}{a_{2n}}}_{>0}$$
$$= s_{2n-2} + \underbrace{a_{2n-1} + a_{2n}}_{>0}$$
$$> s_{2n-2}$$

$$s_{2n+1} = s_{2n} + \overset{>0}{a_{2n+1}} > s_{2n}$$

The sequence $\{s_{2n-1}\}$ is decreasing and bounded from below by s_2 and $\{s_{2n}\}$ is increasing and bounded from above by s_1. Hence, they are both convergent.

Say $\lim s_{2n-1} = L_1$ and $\lim s_{2n} = L_2$.

Now $L_2 - L_1 = \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} s_{2n-1}$
$$= \lim_{n\to\infty} (s_{2n} - s_{2n-1})$$
$$= \lim a_{2n}$$
$$= 0. \qquad \Rightarrow L_1 = L_2.$$

So, $\lim s_n = L_1 = L_2$. ■

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$ Test for absolute convergence.

Solution $a_n = \frac{(-1)^n}{\ln n}$, $|a_n| = \frac{1}{\ln n} > \frac{1}{n}$, for all $n \geq 2$, since $\ln n < n$.

Since, $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the Comparison Test the series $\sum \frac{1}{\ln n}$ is divergent.

However, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is convergent by the A.S.T.

i) $a_{n+1} \cdot a_n = \frac{(-1)^n}{\ln n} \cdot \frac{(-1)^{n+1}}{\ln n+1} = \frac{(-1)^{2n+1}}{\ln n(\ln n+1)} < 0.$

ii) $|a_n| = \frac{1}{\ln n} > \frac{1}{\ln n+1} = |a_{n+1}|$ for $n \geq 2$.

iii) $\lim a_n = \lim \frac{1}{\ln n} = 0.$

So by the A.S.T. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is convergent.

Hence, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is conditionally convergent.

<u>Rearranging the terms in a series:</u>

Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, which is conditionally convergent.

$$L = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \cdots$$

$$\infty = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

This implies that both sum $1 + \frac{1}{3} + \frac{1}{5} + \cdots = \infty$

and $\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots - \infty$ are divergent.

We can arrange the terms of $\sum \frac{(-1)^n}{n}$ so that the new sequence will converge to any number you like, say 2022.

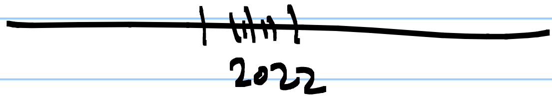
$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n_1+1} > 2022 \text{ but}$$

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n_1-1} \leq 2022$$

Then add negative terms so that it becomes less than 2022.

$$1+\frac{1}{3}+\cdots+\frac{1}{2n_1+1}-\frac{1}{2}-\frac{1}{4}-\cdots-\frac{1}{2n_2} < 2022$$

and $$1+\frac{1}{3}+\cdots+\frac{1}{2n_1+1}-\frac{1}{2}-\frac{1}{4}-\cdots-\frac{1}{2n_2-2} \geq 2022.$$



This way we may arrange the series so that the series converge to 2022.

__Remark.__ If a series $\sum a_n$ is absolutely converges then any rearrangement will converge to the same number, $\sum a_n$.

§9.5. Power Series:

A series of the form $\sum_{n=0}^{\infty} a_n(x-c)^n = a_0+a_1(x-c)$ $+a_2(x-c)^2+\cdots$ is called a power series in

powers of $(x-c)$ or a power series about c.
The constants $a_0, a_1, \dots, a_n, \dots$ are called the coefficients of the power series.

Ex For any real number $|x|<1$ the geometric series
$$\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\dots = \frac{1}{1-x}$$
So, $\sum_{n=0}^{\infty} x^n$ can be regarded as a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, where $c=0$

and $a_n=1$, for all n.

The set of real numbers $x \in \mathbb{R}$ for which the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ is convergent is

called the interval of convergence of the series.

Example: $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} a_n(x-c)^n$, $a_n=1, \forall n, c=0$.

Consider the series $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} |x|^n$

Apply Ratio Test to $\sum_{n=0}^{\infty} b_n$.

$$\rho = \lim \frac{b_{n+1}}{b_n} = \lim \frac{|x|^{n+1}}{|x|^n} = \lim |x| = |x|.$$

So, if $\rho<1$ then $\sum b_n$ is convergent.
If $\rho>1$ then $\sum b_n$ is divergent.
If $\rho=1$ then no conclusion.

So if $\rho<1$, $\sum |x^n|$ is convergent and hence $\sum x^n$ is absolutely convergent.

Video 12

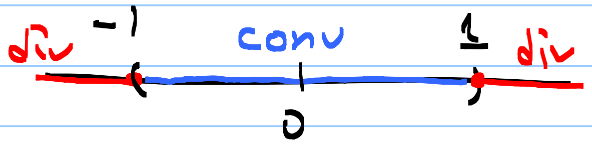
If $\rho > 1$ then $\sum |x|^n$ is divergent.

$$\sum x^n = \begin{cases} \text{conv. to } \frac{1}{1-x} & \text{if } |x|<1 \\ \text{div. if } & |x|>1 \\ ? & \text{if } |x|=1. \end{cases}$$

$|x|=1 \Rightarrow x=1$ or $x=-1$.

$x=1$, then $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 1$ is divergent by the n^{th} term test

$x=-1$ then $\sum x^n = \sum (-1)^n$ is divergent "



The interval of convergence of $\sum_{n=0}^{\infty} x^n$ is $(-1,1)$.

Theorem: For any power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ one of the following alternatives must hold:

i) the series may converge only at $x=c$.
ii) the series may converge at any real number x,
iii) there exists a positive real number R such that the series converges at every x satisfying $|x-c|<R$ and diverges at every x satisfying $|x-c|>R$.
In this case, the series may or may not converge at either of the end points $x=c-R$ and $x=c+R$.

Proof: Clearly $\sum_{n=0}^{\infty} a_n (x-c)^n$ converges for $x=c$ to the real number a_0. Let's apply Ratio Test to the series $\sum_{n=0}^{\infty} |a_n (x-c)^n|$.

$b_n = |a_n(x-c)^n|$, $\rho = \lim_{n\to\infty} \frac{b_{n+1}}{b_n}$

$$\Rightarrow \rho = \lim_{n\to\infty} \frac{|a_{n+1}(x-c)^{\cancel{n+1}}|}{|a_n \cancel{(x-c)^n}|} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x-c|$$

$$= |x-c| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

We have some cases:

Case 1 $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$. Then $\rho = \infty$ if $x \neq c$

Then $\rho > 1$ so that $\sum b_n = \sum |a_n(x-c)^n|$ diverges for $x \neq c$.

Case 2 $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists and is positive.

$$\rho = |x-c| \underbrace{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|}_{= L} \Rightarrow \rho = |x-c| L$$

By the Ratio Test $\sum |a_n(x-c)^n|$ is convergent if $|\rho| < 1 \Rightarrow |x-c| L < 1 \Rightarrow |x-c| < 1/L = R$.

If $|x-c| > R$ then it is divergent.
If $|x-c| = R$ then $\rho = 1$ then the test fails
$(x = c+R$ or $x = c-R)$

Case 3 $\lim \left|\frac{a_{n+1}}{a_n}\right| = 0$, $\rho = |x-c| \lim \left|\frac{a_{n+1}}{a_n}\right| = 0$.

$\rho < 1$. Thus the series $\sum |a_n(x-c)^n|$ converges for all $x \in \mathbb{R}$.

This proof works under the assumption that

$\lim \left|\frac{a_{n+1}}{a_n}\right|$ exists or $= \infty$. In general, $\lim \left|\frac{a_{n+1}}{a_n}\right|$ may <u>not</u> exist. In that one must replace

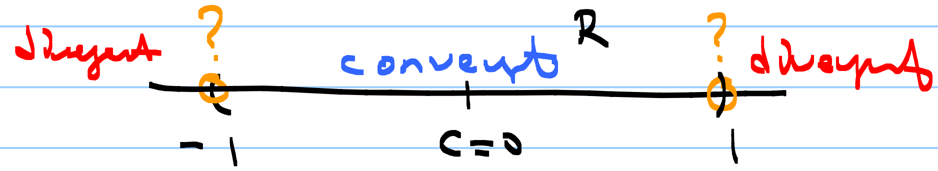
$\lim \left|\frac{a_{n+1}}{a_n}\right|$ by $\limsup \left|\frac{a_{n+1}}{a_n}\right|$.

<u>Examples:</u> 1) $\sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} a_n (x-c)^n$

$a_n = (-1)^n$, $c = 0$. $\quad L = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim 1 = 1$

$R = 1/L = 1/1 = 1$

So the series converges for $(c-R, c+R) = (-1, 1)$.
It is divergent for x with $|x-c| = |x| > R = 1$.



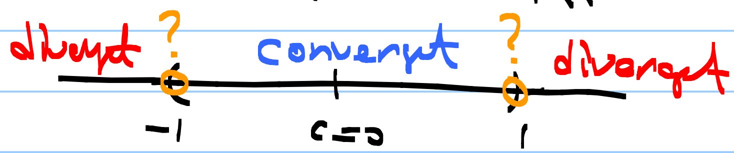
<u>End points:</u> $x = -1$, $\sum a_n (x-c)^n = \sum (-1)^n (-1)^n$
$= \sum 1$ divergent.
$x = 1 \Rightarrow \sum a_n (x-c)^n = \sum (-1)^n (1-0)^n = \sum (-1)^n$ divergent

So the interval of convergence is $(-1, 1)$.

2) $\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} a_n (x-c)^n \qquad a_n = 1/n,\ c = 0$

$L = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{1/n+1}{1/n}\right| = \lim \frac{n}{n+1} = 1.$

So $R = 1/L = 1$



<u>End points:</u> $x = 1$, $\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

since it is the Harmonic series.

$\underline{x=-1}$ $\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\cdots$

Use A.S.T.

i) $a_n \cdot a_{n+1} = \frac{(-1)^n}{n} \cdot \frac{(-1)^{n+1}}{n+1} < 0, \ \forall n \geq 1.$

ii) $|a_n| = \frac{1}{n} > \frac{1}{n+1} = |a_{n+1}|, \ \forall n \geq 1.$

iii) $\lim a_n = \lim \frac{(-1)^n}{n} = 0.$

So by A.S.T. the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conv.

So the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ is $[-1, 1)$.

3) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$= \sum_{n=0}^{\infty} a_n (x-c)^n \qquad c=0, \ a_n = \frac{1}{n!}$

$L = \lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{1/(n+1)!}{1/n!} \right| = \lim \frac{1}{n+1} = 0.$

So, $R = 1/L = \text{"}1/0\text{"} = +\infty.$

Hence, the series converges for all $x \in \mathbb{R}$.
The interval of convergence is $(-\infty, +\infty)$.

4) $\sum_{n=0}^{\infty} n! \, (x-4)^n = \sum_{n=0}^{\infty} a_n (x-c)^n, \quad a_n = n!, \ c=4$

Video 13

$L = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(n+1)!}{n!}\right| = \lim (n+1) = +\infty.$

$R = 1/L = 1/\infty = 0$. Hence, then the series converges only for $x = c$.

In other words, the interval of convergence is $\{4\} = [4, 4]$.

Algebraic Operations on Power Series:

Theorem: Let $\sum_{n=0}^{\infty} a_n (x-c)^n$ and $\sum_{n=0}^{\infty} b_n (x-c)^n$ be two power series with radii of convergence R_a and R_b, respectively. Let $\lambda \in \mathbb{R}$ be a constant. Then

i) $\sum_{n=0}^{\infty} \lambda a_n (x-c)^n$ has radius of convergence R_a

and $\sum_{n=0}^{\infty} \lambda a_n (x-c)^n = \lambda \sum_{n=0}^{\infty} a_n (x-c)^n$, whenever

the right hand side converges.

ii) $\sum_{n=0}^{\infty} (a_n + b_n)(x-c)^n$ has radius of convergence R at least as large as the the smaller of R_a and R_b ($R \geq \min\{R_a, R_b\}$) and

$$\sum_{n=0}^{\infty} (a_n+b_n)(x-c)^n = \sum_{n=0}^{\infty} a_n (x-c)^n + \sum_{n=0}^{\infty} b_n (x-c)^n,$$

whenever both series converge.

Proof: b) Let $x \in \mathbb{R}$ s.t. that both series are convergent at x.

$\sum_{n=0}^{\infty} a_n (x-c)^n$ conv. Let $s_n = \sum_{k=0}^{n} a_k (x-c)^k$

and if $r_n = \sum_{k=0}^{n} b_k (x-c)^k$. Then both sequences $\{s_n\}$ and $\{r_n\}$ are convergent.

Then $\{s_n + r_n\}$ is convergent.

$$s_n + r_n = \sum_{k=0}^{n} a_k (x-c)^k + \sum_{k=0}^{n} b_k (x-c)^k$$
$$= \sum_{k=0}^{n} \left(a_k (x-c)^k + b_k (x-c)^k\right)$$
$$= \sum_{k=0}^{n} (a_k + b_k)(x-c)^k$$

Note that $s_n + r_n$ is the n^{th} partial sum of the series $\sum_{k=0}^{\infty} (a_k + b_k)(x-c)^k$.

Hence, $\sum_{k=0}^{\infty} (a_k + b_k)(x-c)^k$ is convergent and equals $\lim (s_n + r_n) = \lim s_n + \lim r_n$

$$= \sum_{k=0}^{\infty} a_k (x-c)^k + \sum_{k=0}^{\infty} b_k (x-c)^k.$$

What about product?

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = (a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots)$$

$$= \underbrace{(a_0 b_0)}_{c_0} + \underbrace{(a_0 b_1 + a_1 b_0)}_{c_1} x + \underbrace{(a_0 b_2 + a_1 b_1 + a_2 b_0)}_{c_2} x^2 + \underbrace{(a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0)}_{c_3} x^3 + \cdots$$
$$+ \underbrace{(a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0)}_{c_n} x^n + \cdots$$

This is called the Cauchy product of two series.

Theorem: Let $\sum a_n x^n$, $\sum b_n x^n$ and $\sum c_n x^n$ be as above, where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$, for all n. If the radii of convergence of $\sum a_n x^n$ and $\sum b_n x^n$ are R_a and R_b, respectively, then the Cauchy product $\sum c_n x^n$ has radius of convergence at least $\min\{R_a, R_b\}$ and

$$\sum_{n=0}^{\infty} c_n x^n = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right).$$

Differentiation and Integration of Power Series

Theorem: If the series $\sum_{n=0}^{\infty} a_n (x-c)^n$ converges to the sum $f(x)$ on the interval $(c-R, c+R)$, that is

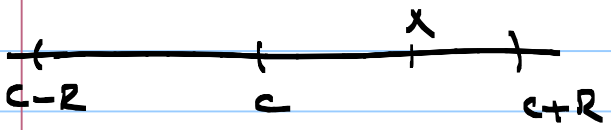
$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n + \cdots$$

Then $f(x)$ is differentiable on $(c-R, c+R)$ and

$$f'(x) = 0 + a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots + na_n(x-c)^{n-1} + \cdots$$

Also, f is integrable over any subinterval of $(c-R, c+R)$ and if $|x-c| < R$ then

$$\int_c^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1} = a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \frac{a_2}{3}(x-c)^3 + \cdots + \frac{a_n}{n+1}(x-c)^{n+1} + \cdots$$



$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

$$\int_c^x f(t)\,dt = \int_c^x \left(\underline{a_0} + a_1(t-c) + a_2(t-c)^2 + \cdots + a_n(t-c)^n + \cdots\right)dt$$

$$= \left. a_0 t + \frac{a_1}{2}(t-c)^2 + \frac{a_2}{3}(t-c)^3 + \cdots + \frac{a_n(t-c)^{n+1}}{n+1} + \cdots \right|_c^x$$

$$= \underline{a_0 x} + \frac{a_1}{2}(x-c)^2 + \frac{a_2}{3}(x-c)^3 + \cdots + \frac{a_n}{n+1}(x-c)^{n+1} + \cdots$$

$$- (\underline{a_0 c} + 0 + 0 + \cdots \quad)$$

$$= a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \frac{a_2}{3}(x-c)^3 + \cdots + \frac{a_n}{n+1}(x-c)^{n+1} + \cdots$$

We have one last theorem:

<u>Theorem</u> (Abel's Theorem)

The sum of a power series is a continuous function everywhere on the interval of convergence of the series. In particular, if $\sum_{n=0}^{\infty} a_n R^n$ converges for some $R>0$ then

$$\lim_{x \to R_-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n$$ and if $\sum_{n=0}^{\infty} a_n (-R)^n$

converges then $\lim_{x \to -R_+} \sum_{n=0}^{\infty} a_n(-x^n) = \sum_{n=0}^{\infty} a_n(-R)^n$.

Video 14

Examples

1) Find power series representations for the functions

a) $\dfrac{1}{(1-x)^2}$ b) $\dfrac{1}{(1-x)^3}$ c) $\ln(1+x)$

by starting from the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots, \quad (-1<x<1).$$

Solution: Just take the derivative of the above power series:

a) $$\left(\frac{1}{1-x}\right)' = \left(\sum_{n=0}^{\infty} x^n\right)' = 0 + 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots$$
for $-1<x<1$.

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots, \quad -1<x<1.$$
$4x^3$

b) $$\left(\frac{1}{(1-x)^2}\right)' = 0 + 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots$$
$-1<x<1$.

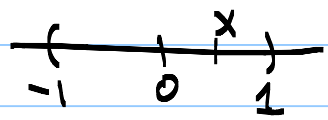
$$\frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots$$

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + \cdots + \frac{n(n-1)}{2}x^{n-2} + \cdots$$

c) $$\frac{1}{1-x} = 1 + x + x^2 + \cdots, \quad -1<x<1.$$

Just replace x by $-x$. Then we get

$$\frac{1}{1+x} = 1 - x + (-x)^2 + (-x)^3 + \cdots, \quad -1<-x<1$$
$$= 1 - x + x^2 - x^3 + x^4 - \cdots, \quad -1<x<1.$$



$$\int_0^x \frac{dt}{1+t} = \int_0^x (1-t+t^2-t^3+\cdots)\,dt \qquad -1<x<1$$

$$\ln(1+t)\Big|_0^x = \left(t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}-\cdots\right)\Big|_0^x$$

$$\ln(1+x)-\ln 1 = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots$$

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}, \qquad -1<x<1.$$

Let's compute $\ln 2$.
Let's check first if the series is convergent at the end points.

$x=-1$ $\quad \displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-1)^n}{n} = -\sum_{n=1}^{\infty}\frac{1}{n}$ divergent

since it is the Harmonic series.

$x=1 \Rightarrow \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}-\cdots$ conv.

by the A.S.T.

Finally, by Abel's theorem we

$$\ln 2 = \lim_{x\to 1}\ln(1+x) = \lim_{x\to 1}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}1^n$$

since $\ln x$ is continuous

since $x\to 1$ we have $-1<x<1$

Abel's Thm.

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

$$\ln(1+x) = \sum\frac{(-1)^{n-1}}{n}x^n, \text{ for } -1<x\leq 1.$$

So $\ln 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$

Example: Find a similar power series expansion for $\tan^{-1}x$.

Solution: $\frac{1}{1-x} = 1+x+x^2+\cdots+x^n+\cdots, \quad -1<x<1.$

Let $x=-t^2$ then we have

$$\frac{1}{1+t^2} = 1-t^2+t^4-t^6+t^8-t^{10}+\cdots, \quad -1<t^2<1$$

$$\int_0^x \frac{dt}{1+t^2} = t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}+\frac{t^9}{9}-\frac{t^{11}}{11}+\cdots\Big|_0^x \quad (-1<t<1)$$

$$\tan^{-1}t\Big|_0^x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\frac{x^{11}}{11}+$$

$$\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\frac{x^{11}}{11}+\cdots, \quad -1<x<1.$$

$$\pi/4 = \tan^{-1}1 \overset{(*)}{=} 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}\cdots$$

which is convergent by A.S.T. $a_n=\frac{(-1)^{n-1}}{2n-1}$

i) $a_n \cdot a_{n+1} < 0$ ✓
ii) $|a_n| \geq |a_{n+1}|$ ✓
iii) $\lim a_n = 0$ ✓

Finally the equality $(*)$ holds by the Abel's Theorem.

Example: Find the sum of the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ by first find the sum of the power series $\sum_{n=1}^{\infty} n^2x^n$

Solution: $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \; -1 < x < 1$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} n x^{n-1}$$

$$x \cdot \frac{1}{(1-x)^2} = (x + 0 \cdot x^2 + 0 \cdot x^3 + \cdots)(1 + 2x + 3x^2 + \cdots)$$

$$= x(1 + 2x + 3x^2 + \cdots)$$

$$= x + 2x^2 + 3x^3 + \cdots$$

$$= \sum_{n=1}^{\infty} n x^n$$

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n \Rightarrow \left[\frac{x}{(1-x)^2}\right]' = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

$$x \cdot \left[\frac{x}{(1-x)^2}\right]' = x \sum_{n=1}^{\infty} n^2 x^{n-1} = \sum_{n=1}^{\infty} n^2 x^n,$$

$$x \cdot \frac{1 \cdot (1-x)^2 + x \cdot 2(1-x)}{(1-x)^4} = \sum_{n=1}^{\infty} n^2 x^n, \quad -1 < x < 1.$$

$$x \cdot \frac{(1-x) + 2x}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^n$$

$$\Rightarrow \sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}, \quad -1 < x < 1.$$

Let $x = 1/2 \in (-1, 1)$. Then

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{1/2 \cdot (1 + 1/2)}{(1 - 1/2)^3} = \frac{\cancel{1/2} \cdot 3/2}{(1/2)^{\cancel{3}}} = 6.$$

Video 15

§9.6. Taylor and Maclaurin Series:

Theorem. Suppose the series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots + a_n(x-c)^n + \dots$$

converges to $f(x)$, for $c-R < x < c+R$, where $R>0$. Then

$$a_k = \frac{f^{(k)}(c)}{k!}, \text{ for } k = 0, 1, 2, \dots .$$

Proof: $f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots + a_n(x-c)^n + \dots$, $c-R < x < c+R$

$f'(x) = a_1 + 2a_2(x-c) + \dots + na_n(x-c)^{n-1} + \dots$, $c-R < x < c+R$

$f''(x) = 2a_2 + 6a_3(x-c) + \dots + n(n-1)(x-c)^{n-2} + \dots$, $c-R < x < c+R$.

$\vdots$

$f^{(k)}(x) = k!\, a_k + (k+1)(k)(k-1)\cdots 2 \cdot (x-c)^1 + \dots + n(n-1)(n-2)\cdots(n-k+1)(x-c)^{n-k} + \dots$

Plug $x=c$ in all lines:

$f(c) = a_0$, $f'(c) = a_1$, $f''(c) = 2a_2$, ...

$f^{(k)}(c) = k!\, a_k \Rightarrow a_k = \dfrac{f^{(k)}(c)}{k!}$ ∎

Definition: If $f(x)$ has derivatives of all orders at $x=c$, then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots + \frac{f^{(k)}(c)}{k!}(x-c)^k + \dots$$

is called the Taylor Series of f about $x=c$. If $c=0$ then the term Maclaurin series is usually used in place of Taylor series.

Definition: (Analytic Functions)

A function f is analytic at c if f has a Taylor series at c and that series converges to $f(x)$ in an open interval containing c. If f is analytic at each point on an open interval, then we say it is analytic on that interval.

$$f(x) = \sum \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Maclaurin Series of Some Elementary Functions

Example: Find the Taylor series of $f(x)=e^x$.

$f^{(k)}(x)=e^x$, $f^{(k)}(c)=e^c$. So the Taylor series of e^x about $x=c$ is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(c)}{k!}(x-c)^k=\sum_{k=0}^{\infty}\frac{e^c}{k!}(x-c)^k=e^c\sum_{k=0}^{\infty}\frac{(x-c)^k}{k!}$$

Let's determine if it is convergent? Where it is analytic?

$$\sum_{n=0}^{\infty}\frac{e^c}{n!}(x-c)^n \qquad a_n=\frac{e^c}{n!}$$

$$L=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\cancel{e^c}}{(n+1)\cancel{n!}}\cdot\frac{\cancel{n!}}{\cancel{e^c}}=\lim\frac{1}{n+1}=0$$

Hence, $R = 1/L = +\infty$. Hence the series converges for all $x \in \mathbb{R}$.

Suppose that $g(x) = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n, \quad x \in \mathbb{R}$.

Then $g(x) = e^c + \frac{e^c}{1!}(x-c) + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + \dots$

$\Rightarrow g'(x) = 0 + e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + \dots$

$= g(x)$

Thus $g(x)$ satisfies the "differential equation"

$y' = y$. Then the theory of differential equations tells us that $g(x) = e^x$.

So, $e^x = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n, \quad x \in \mathbb{R}$.

In particular, e^x is analytic on $\mathbb{R}$.

The Maclaurin Series for e^x is $(c=0)$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

——— o ———

<u>Error Estimation for Alternating Series</u>

Suppose that $\sum_{n=1}^{\infty} a_n$ is an alternating series which is convergent by A.S.T.. So

i) $a_n \cdot a_{n+1} < 0, \ \forall n$

ii) $|a_n| \geq |a_{n+1}|, \ \forall n$

iii) $\lim a_n = 0$.

Let $s = \lim s_n = \sum_{n=1}^{\infty} a_n$.

$s_n = a_1 + \cdots + a_n$. We may think of s_n as an approximation for the real number s. The error in this approximation is $|s - s_n|$.

<u>Fact:</u> $|s - s_n| \leq |a_{n+1}|$

<u>Proof:</u> $s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots\cdots$

$$|s - s_n| = |a_{n+1} + a_{n+2} + a_{n+3} + \cdots|$$

$$= |\underset{\vee\atop 0}{a_{n+1}} + \underbrace{\overbrace{(\underset{\wedge\atop 0}{a_{n+2}} + \underset{\vee\atop 0}{a_{n+3}})}^{<0} + \overbrace{(a_{n+4} + a_{n+5})}^{<0} + \cdots}|$$

$$= |\overbrace{(\underset{\vee\atop 0}{a_{n+1}} + \underset{\wedge\atop 0}{a_{n+2}})}^{>0} + \overbrace{(a_{n+3} + a_{n+4})}^{<0 \atop >0} + \cdots|$$

$$\Rightarrow |a_{n+1}| \geq \underset{\displaystyle |s - s_n|}{\underset{\parallel}{|(a_{n+1} + a_{n+2}) + (a_{n+3} + a_{n+4}) + \cdots|}}$$

$\blacksquare$

Video 16

Examples Find the Maclaurin Series for the functions $\sin x$ and $\cos x$. Determine the intervals of convergence of these series.

a) $f(x) = \sin x \quad c = 0 \qquad a_n = \dfrac{f^{(n)}(c)}{n!}$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(IV)}(x) = \sin x$

$$a_0 = \frac{f(0)}{0!} = \frac{\sin 0}{1} = 0$$

$$a_1 = \frac{f'(0)}{1!} = \frac{\cos 0}{1} = 1$$

$$a_2 = \frac{f''(0)}{2!} = \frac{-\sin 0}{2} = 0, \quad a_3 = \frac{f'''(0)}{3!} = \frac{-\cos 0}{3!} = \frac{-1}{3!}$$

$$a_4 = \frac{f^{(IV)}(0)}{4!} = \frac{\sin 0}{4!} = 0$$

Hence, the Maclaurin Series of $\sin x$ is

$$\sum_{n=0}^{\infty} a_n (x-c)^n = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$L = \lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \frac{1}{(2n+3)!} \cdot (2n+1)!$$

$$= \lim \frac{1}{(2n+3)(2n+2)} = 0. \quad R = 1/L = +\infty.$$

Hence, $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ converges for all $x \in \mathbb{R}$.

We'll see that indeed, $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

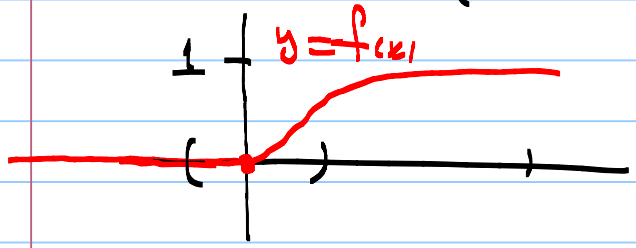
If we assume this taking derivative of both sides we get

$$(\sin x)' = \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right)'$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)\, x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

for all $x \in \mathbb{R}$.

Remark: $f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$



$f^{(n)}(0) = 0.$

$$f'(x) = \begin{cases} \frac{1}{x^2} e^{-1/x} & \text{if } x > 0 \\ ? & \text{if } x = 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$f'(0) = \lim_{h \to 0} \frac{f(0+h) - \overset{0}{f(0)}}{h}$ Need to compute left and right limits.

$$\lim_{h \to 0^+} \frac{f(h)}{h} = \lim_{h \to 0^+} \frac{e^{-1/h}}{h}$$

$$= \lim_{h \to 0^+} \frac{1/h}{e^{1/h}} \quad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{h \to 0^+} \frac{-1/h^2}{-1/h^2 \, e^{1/h}} = \lim_{h \to 0^+} \frac{1}{e^{1/h}} = 0.$$

$$\lim_{h\to 0^-} \frac{f(h)}{h} = \lim_{h\to 0^-} \frac{0}{h} = 0.$$

Hence, $\lim_{h\to 0} \frac{f(h)}{h}$ exists and equals 0.

So, $f'(0)$ exists and equals 0.

Similarly one can show that $f^{(n)}(0) = 0$.
Hence, the Maclaurin series for $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0.$$

Since $f(x) > 0$ for all $x > 0$, $f(x)$ cannot be analytic at $x = 0$. Hence, $f(x)$ not an analytic function.

<u>More examples</u> of <u>Analytic Functions</u>

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \forall x \in \mathbb{R}.$$

Just replace x by $-x$ to get

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}, \quad x \in \mathbb{R}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(\sum \frac{x^n}{n!} + \sum \frac{(-1)^n x^n}{n!} \right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(1 + (-1)^n\right) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \quad x \in \mathbb{R}$$

$$\sinh x = (\cosh x)' = \left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right)' = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right)'$$

$$= \sum_{n=0}^{\infty} \frac{2n\, x^{2n-1}}{(2n)!} = 0 + x + \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

Example Find Maclaurin Series for

a) $e^{-x^2/3}$ b) $\frac{\sin x^2}{x}$ c) $\sin^2 x$

Solution : a) $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$, $t \in \mathbb{R}$

Let $t = -x^2/3$. Then

$$e^{-x^2/3} = \sum_{n=0}^{\infty} \frac{(-x^2/3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{3^n \cdot n!}, \quad x \in \mathbb{R}$$

b) $\frac{\sin x^2}{x}$, $\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} -$

Just let $t = x^2$. $t \in \mathbb{R}$

Then $\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$, $x \in \mathbb{R}$

So $\frac{\sin x^2}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n+1)!}$ $x \neq 0$

c) $\sin^2 x = \frac{1 - \cos 2x}{2}$

$\cos t = \sum_{n=0}^{\infty} (-1)^n t^{2n}/(2n)!$, $t \in \mathbb{R}$.

Just let $t = 2x$. Then

$$\cos 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!}, \quad x \in \mathbb{R}$$

Hence, $\sin^2 x = \dfrac{1-\cos 2x}{2}$

$$= \frac{1}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{2(2n)!}, \quad x \in \mathbb{R}.$$

$$= 1/2 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 4^n x^{2n}}{2(2n)!}, \quad x \in \mathbb{R}$$

Example: Find the Taylor series for $\ln x$ in powers of $x-2$.

We know that $\ln 1+t = t - \dfrac{t^2}{2} + \dfrac{t^3}{3} - \dfrac{t^4}{4} + \cdots$ (*)

$|t|<1$

Then $\ln x = \ln(2+(x-2))$

$$= \ln 2\left(1 + \frac{x-2}{2}\right)$$

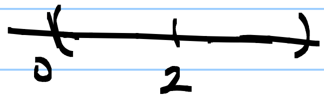
$$= \ln 2 + \ln\left(1 + \frac{x-2}{2}\right)$$

Now let $t = \dfrac{x-2}{2}$ in (*)

$$\ln x = \ln 2 + t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \cdots$$

$$= \ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{2 \cdot 2^2} + \frac{(x-2)^3}{3 \cdot 2^3} - \frac{(x-2)^4}{4 \cdot 2^4} \cdots$$

for $\left|\dfrac{x-2}{2}\right| < 1 \iff |x-2| < 2$.



Video 17

Example: Find the Taylor Series for $\cos x$ about $x = \pi/3$.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \cos(x - \pi/3 + \pi/3) = \cos(x-\pi/3)\cos\pi/3 - \sin(x-\pi/3)\sin\pi/3$$

$$\Rightarrow \cos x = \frac{1}{2}\cos(x-\pi/3) - \frac{\sqrt{3}}{2}\sin(x-\pi/3)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi/3)^{2n}}{(2n)!} - \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi/3)^{2n+1}}{(2n+1)!}$$

for all $x \in \mathbb{R}$.

Taylor's Formula:

Theorem: If the $(n+1)$st derivative of f exists on an interval containing c and x, and if

$P_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$ is the Taylor polynomial of $f(x)$ about $x=c$, then

$f(x) = P_n(x) + E_n(x)$ holds, where $E_n(x)$ is given by either of the following formulas:

Lagrange Remainder: $E_n(x) = \frac{f^{(n+1)}(s)}{(n+1)!}(x-c)^{n+1}$

for some s between c and x.

Integral Remainder:

$$E_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt.$$

Example: Let's show that $\sin x$ equals its Taylor expansion.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+1}$$

Let $f(x) = \sin x$. $P_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!} x^{2k+1}$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \quad + \frac{(-1)^n}{n!} x^n$$

Then $f(x) - P_n(x) = E_n(x)$, where

$$E_n(x) = \frac{f^{(n+1)}(s)}{(n+1)!}(x-c)^{n+1}$$

$c = 0$, for some s between 0 and x.

$$= \frac{f^{(n+1)}(s)}{(n+1)!} x^{n+1}$$

$f(x) = \sin x$
$f^{(n)}(x) = \pm \sin x$ or $\cos x$

$$|E_n(x)| = \left|\frac{f^{(n+1)}(s)}{(n+1)!} x^{n+1}\right|$$

$\Rightarrow \left|f^{(n+1)}(s)\right| \leq 1.$

$$\leq \underbrace{\frac{1}{(n+1)!}|x|^{n+1}}_{\to 0 \text{ as } n \to \infty}$$

$x = 10^6 \quad x^{n+1} = 10^{6(n+1)}$

$$\Rightarrow \sin x = P_n(x) + \underbrace{E_n(x)}_{\to 0 \text{ as } n \to \infty}$$

$$\text{So, } \sin x = \lim_{n\to\infty} P_n(x) + \overbrace{\lim_{n\to\infty} E_n(x)}^{=0}$$

$$= \lim_{n\to\infty} P_n(x)$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \text{ for all } x \in \mathbb{R}.$$

§ 9.7. Applications of Taylor and Maclaurin Series

Ex: Compute $\cos 43^\circ$ with error less that $1/10{,}000$.

Solution: $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$s = \cos 43^\circ = \cos \frac{43\pi}{180} = 1 - \frac{1}{2!}\left(\frac{43\pi}{180}\right)^2 + \frac{1}{4!}\left(\frac{43\pi}{180}\right)^4 - \frac{1}{6!}\left(\quad\right)^6$$

$s = a_0 + a_1 + a_2 + a_3 - \cdots + a_n + \cdots$, which is an alternating series.

So if $s_n = a_0 + a_1 + \cdots + a_n$, Then

$|s - s_n| \leq a_{n+1}$. Hence, we need find n so that $a_{n+1} < \frac{1}{10^4}$.

$$a_n = \frac{(-1)^n}{(2n)!}\left(\frac{43\pi}{180}\right)^{2n} \qquad a_0 = 1,\ a_1 = -\frac{1}{2}\left(\frac{43\pi}{180}\right)^2$$

$$a_{n+1} = \frac{(-1)^{n+1}}{(2n+2)!}\left(\frac{43\pi}{180}\right)^{2n+2}$$

$$|a_{n+1}| < \frac{1}{10^4} \Rightarrow \left| \frac{(-1)^{n+1}}{(2n+2)!} \left(\frac{43\pi}{180}\right)^{2n+2} \right| < 1/10^4$$

$$\Rightarrow (2n+2)! \left(\frac{180}{43\pi}\right)^{2n+2} > 10^4.$$

$n = 4$ works! $10! \left(\frac{180}{43\pi}\right)^{10} \approx 40{,}320 > 10^4.$

So $$\cos 43^\circ \approx 1 - \frac{1}{2!}\left(\frac{43\pi}{180}\right)^2 + \frac{1}{4!}\left(\frac{43\pi}{180}\right)^4 - \frac{1}{6!}\left(\frac{43\pi}{180}\right)^6 + \frac{1}{8!}\left(\frac{43\pi}{180}\right)^8$$

with error $< 1/10^4$.

Functions Defined by Integrals:

$$E(x) = \int_0^x e^{-t^2}\,dt$$

$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$, for all $t \in \mathbb{R}$.

So, $e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$, $t \in \mathbb{R}$.

$$E(x) = \int_0^x e^{-t^2}\,dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}\,dt$$

$$= \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^{2n}}{n!}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{t^{2n+1}}{2n+1} \Big|_0^x \right)$$

$$E(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R}$$

<u>Example</u> (Computing Limits using Taylor Series)

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{\cancel{x} - \left(\cancel{x} - \frac{x^3}{3!} + x^5/5! - x^7/7! + \cdots\right)}{x^3}$$

$$= \lim_{x \to 0} \left(\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \frac{x^6}{9!} \cdots \right)$$

$$= 1/3! - 0 + 0 - 0$$

$$= 1/3! = 1/6$$

Video 17

Recitation Hour:

1) Is the series $\sum_{n=1}^{\infty} \frac{1}{3^{\ln n}}$ convergent?

Solution:

$$3 = e^{\ln 3} \Rightarrow 3^{\ln n} = \left(e^{\ln 3}\right)^{\ln n}$$

$$= e^{(\ln 3)(\ln n)}$$

$$= \left(e^{\ln n}\right)^{\ln 3}$$

$$= n^{\ln 3}, \quad 3 > e \Rightarrow \ln 3 > 1.$$

Hence, the series $\sum_{n=1}^{\infty} \frac{1}{3^{\ln n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\ln 3}} = \sum_{n=1}^{\infty} \frac{1}{n^p}$

where $p = \ln 3 > 1$, is convergent by the p-test.

2) Check for convergence:

a) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{3^n \ln n}$ b) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^3}$ c) $\sum_{n=1}^{\infty} \frac{1+n^{4/3}}{2+n^{5/3}}$

Solution: a) $a_n = \frac{\sqrt{n}}{3^n \ln n}$ Let's use Ratio Test

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{\sqrt{n+1}}{3^{n+1} \ln(n+1)} \cdot \frac{3^n \ln n}{\sqrt{n}}$$

$$= \frac{1}{3} \lim \frac{\sqrt{n+1}}{\sqrt{n}} \frac{\ln n}{\ln(n+1)}$$

$$= \frac{1}{3} \lim \underbrace{\sqrt{1+1/n}}_{\downarrow \, 1} \, \underbrace{\frac{\ln n}{\ln(n+1)}}_{\downarrow \, 1} = 1/3 < 1$$

$$\lim_{x\to\infty} \frac{\ln x}{\ln x+1} \quad \left[\frac{\infty}{\infty}\right]. \text{ So use L'Hopital's Rule}$$

$$\lim_{x\to\infty} \frac{\ln x}{\ln x+1} = \lim_{x\to\infty} \frac{1/x}{1/x+1} = \lim_{x\to\infty} \frac{x+1}{x} = 1.$$

Hence, it is convergent by the Ratio Test.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^3}$ $\qquad a_n = \dfrac{(2n)!}{(n!)^3}$

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{(2n+2)!}{(n+1)!} \cdot \frac{(n!)^3}{(2n)!}$$

$$= \lim \frac{(2n+2)(2n+1)}{(n+1)^3}$$

$$= \lim \frac{4n^2+6n+2}{n^3+3n^2+3n+1}$$

$$= 0 < 1.$$

Hence, the series is convergent by the Ratio Test.

c) $\displaystyle\sum_{n=1}^{\infty} \frac{1+n^{4/3}}{2+n^{5/3}}$ $\qquad a_n = \dfrac{1+n^{4/3}}{2+n^{5/3}}$, let $b_n = \dfrac{1}{n^{1/3}}$

$$\lim \frac{a_n}{b_n} = \lim \frac{1+n^{4/3}}{2+n^{5/3}} \cdot n^{1/3}$$

$$= \lim \frac{n^{5/3}\,(1+n^{-4/3})}{n^{5/3}\,(1+2n^{-5/3})} = \lim \frac{1+\overset{\to 0}{n^{-4/3}}}{1+\underset{\to 0}{2n^{-5/3}}}$$

$$= 1/1 = 1$$

Since $L = 1$ then $\sum a_n$ and $\sum b_n$ will converge or diverge together. However,

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ is divergent by the p-test ($p = 1/3 < 1$).

$$\frac{1+n^{4/3}}{2+n^{5/3}} \geq \frac{1+n^{4/3}}{2n^{5/3}+n^{5/3}} \quad \text{because } n^{5/3} \geq 1.$$

$$= \frac{1}{3n^{5/3}} + \frac{n^{4/3}}{3n^{5/3}}$$

$$= \frac{1}{3n^{5/3}} + \frac{1}{3n^{1/3}}$$

$$\sum_{n=1}^{\infty} \frac{1+n^{4/3}}{2+n^{5/3}} \geq \underbrace{\sum_{n=1}^{\infty} \frac{1}{3n^{5/3}}}_{\text{conv.}} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{3n^{1/3}}}_{\text{div.}} = \infty$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1+n^{4/3}}{2+n^{5/3}}$ is div.

3) Use the Root Test to show that $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$ converges.

Solution $a_n = \frac{2^{n+1}}{n^n}$, $b_n = \frac{2^n}{n^n}$.

$\lim b_n^{1/n} = \lim \left(\frac{2^n}{n^n}\right)^{1/n} = \lim \frac{2}{n} = 0 < 1$ so that

$\sum b_n$ is convergent by the Root Test.

Since $a_n = 2b_n$, $\sum a_n = 2\sum b_n$ is also convergent.

4) $\cos x = \sum \frac{(-1)^n}{(2n)!} x^{2n}$

Let $f(x) = x\cos x^3$

a) Calculate $f^{(96)}(0)$, $f^{(97)}(0)$.

$$f(x) = x\cos x^3 = x \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} (x^3)^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!} = \sum_{n=1}^{\infty} a_n x^n$$

Since $f(x)$ is analytic $a_n = \frac{f^{(n)}(0)}{(2n)!}$

$f^{(n)}(0) = n!\, a_n$, So $f^{(96)}(0) = 96!\, a_{96} \overset{0}{=} 0$.

$f^{(97)}(0) = 16!\, a_{16}$

$= 16! \frac{(-1)^{16}}{(32)!}$

$97 = 6 \cdot n + 1$
$n = 16$

5) $a_1 = 3$, $a_{n+1} = \sqrt{15 + 2a_n}$, $n \geq 1$.

$3, \sqrt{21}, \sqrt{15 + 2\sqrt{21}}, \ldots$

Show that $\{a_n\}$ is convergent and find its limit.

$a_{n+1} = \sqrt{15+2a_n}$ $\qquad 3 \leq a_n \leq 5$

<u>Claim</u> $3 \leq a_n \leq 5$.

<u>Proof</u> Use induction $n=1$ $\quad 3 \leq a_1 = 3 \leq 5$ ✓

Assume that $3 \leq a_n \leq 5$. Then

$$9 \leq 15+6 \leq 15+2a_n \leq 15+10 = 25$$

$$\Rightarrow \sqrt{9} \leq \sqrt{15+2a_n} \leq \sqrt{25}$$

$$3 \leq a_{n+1} \leq 5$$

<u>Claim</u> $\{a_n\}$ is increasing. $\qquad 5 \geq a_n \geq 3$

<u>Proof</u> $a_{n+1} = \sqrt{15+2a_n}$ $\qquad a_{n+1}^2 = 15+2a_n$

$$a_{n+1}^2 - a_n^2 = 15+2a_n - a_n^2$$

$$= \underbrace{(3+a_n)}_{\geq 0}\underbrace{(5-a_n)}_{\geq 0}$$

$$\geq 0$$

$$\Rightarrow a_{n+1}^2 \geq a_n^2 \Rightarrow a_{n+1} \geq a_n$$

Since $\{a_n\}$ is bounded and increasing it is convergent. Say
$L = \lim a_n$.

Now $a_{n+1} = \sqrt{15+2a_n}$.

$\lim a_{n+1} = \lim \sqrt{15+2a_n}$

$L = \sqrt{15+2L}$ since $f(x) = \sqrt{15+2x}$ is continuous.

$L^2 = 15 + 2L \Rightarrow (L-5)(L+3) = 0$

$L = 5$ or $L = -3$.

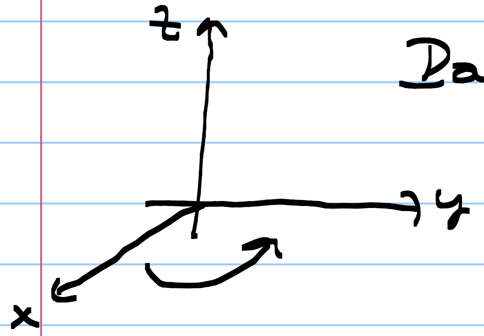
Since $3 \le a_n \le 5$, we must have $3 \le L \le 5$.

Hence, $L = 5$. ∎

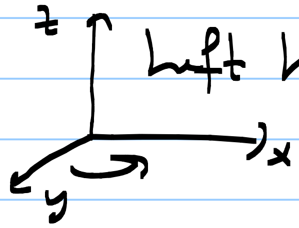
Video 19

CHAPTER 10. Vectors and Coordinate Geometry in 3-Space

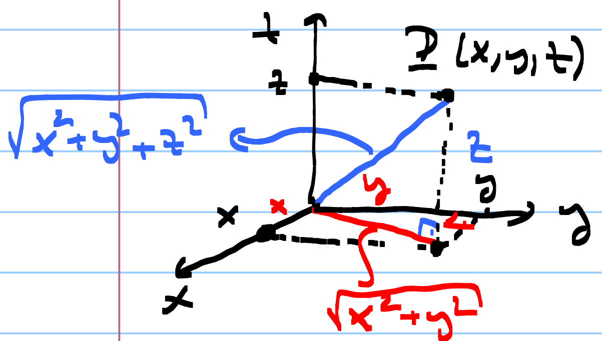
§10.1. Analytic Geometry in 3-dimensions:



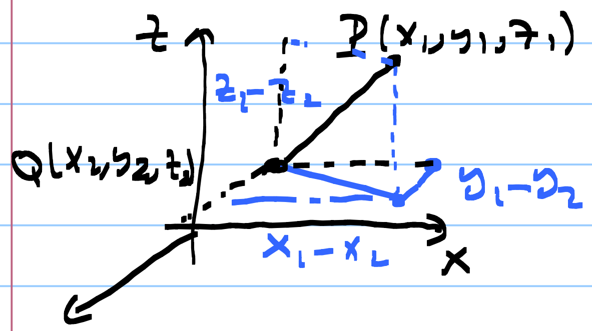
Positively oriented (right handed) coordinate system



Left handed coor. system



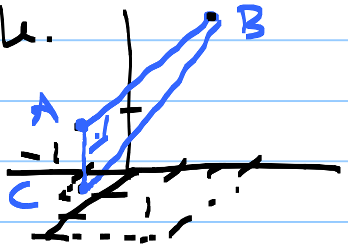
The distance from the point $P(x,y,z)$ to the origin $O(0,0,0)$ is given as $(x^2+y^2+z^2)^{1/2}$.



The distance from Q to P is

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$$

Example: Show that the triangle with vertices $A=(1,-1,2)$, $B=(3,3,8)$ and $C=(2,0,1)$ is a right triangle.



$$|AB| = \sqrt{(3-1)^2+(3+1)^2+(8-2)^2} = \sqrt{4+16+36} = \sqrt{56}$$

$$|AC| = \sqrt{(2-1)^2+(0+1)^2+(1-2)^2} = \sqrt{1+1+1} = \sqrt{3}$$

$$|BC| = \sqrt{(2-3)^2+(0-3)^2+(1-8)^2} = \sqrt{1+9+49} = \sqrt{59}$$

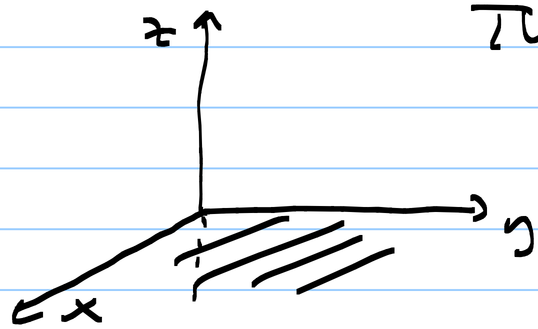
$|AB|^2 + |AC|^2 = 56 + 3 = 59 = |BC|^2$ and hence ABC is a right triangle, where A is the right angle.

<u>Example</u> (Some Equations and the surfaces they represent)

a) The equation $z = 0$ represent the set of all points in the 3-space whose z coordinates are zero, namely the set

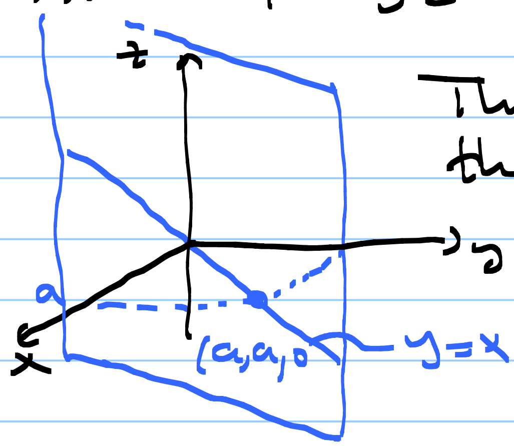
$$\{(x,y,z) \mid z = 0\} = \{(x,y,0) \mid x, y \in \mathbb{R}\}$$

This is just the xy-plane



b) The equation $x = y$ represent the set

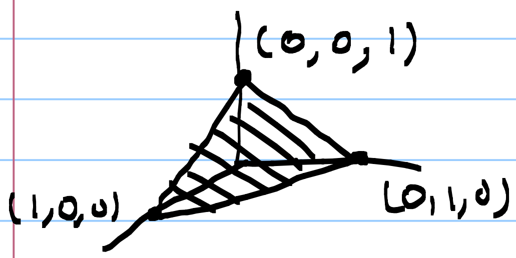
$$\{(x,y,z) \in \mathbb{R}^3 \mid x = y\} = \{(x,x,z) \mid x, z \in \mathbb{R}\}$$

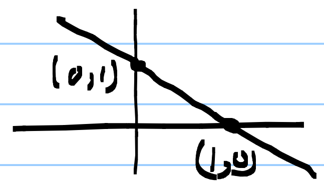


This is the plane containing the line $y = x$ in the xy-plane and the z-axis.

c) The equation $x+y+z=1$ represents all points whose sum of coordinates is 1.

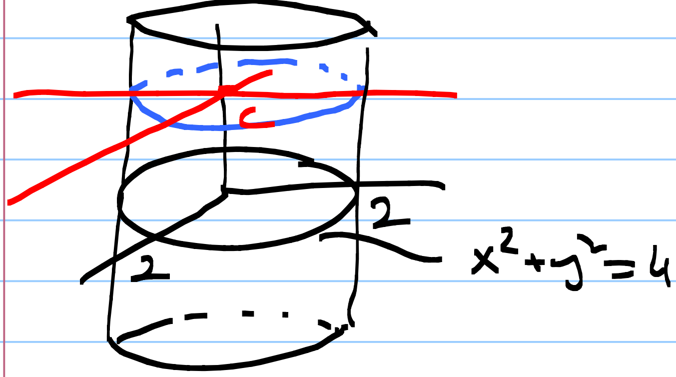
$$\{(x,y,z)\in\mathbb{R}^3 \mid x+y+z=1\}$$

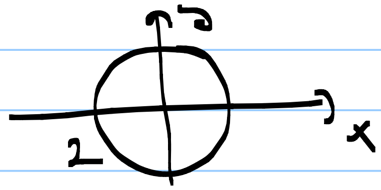




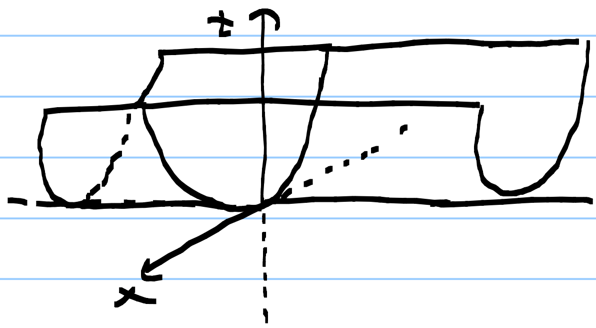
d) The equation $x^2+y^2=4$ represents the cylinder with base the circle $x^2+y^2=4$ in the xy-plane parallel to the z-axis.

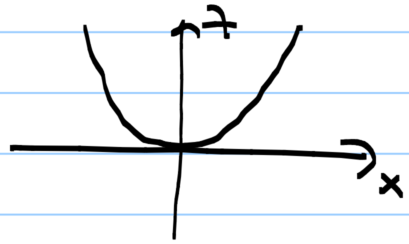
$$\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2=4\}\cap\{(x,y,z)\mid z=c\}$$



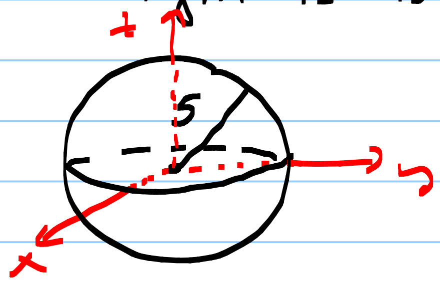


e) $z=x^2$ represent the surface obtained by translating the parabola given by $z=x^2$ in the xz-plane along the y-axis.





f) $x^2+y^2+z^2=25$ represents the set of all points (x,y,z) in 3-space whose distance to the origin is $\sqrt{25}=5$ units.



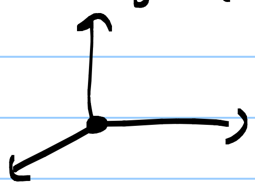
<u>Example</u> Identify the graphs of the equations

a) $y^2+(z-1)^2=4$ b) $y^2+(z-1)^2=0$

c) $x^2+y^2+z^2=0$ d) $x^2+y^2+z^2=-1$.

<u>Solution</u> d) $x^2+y^2+z^2=-1$ represent the empty set $\emptyset$.

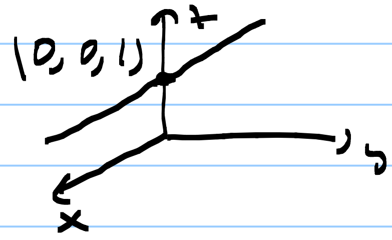
c) $x^2+y^2+z^2=0 \Rightarrow x=y=z=0$. So it represents the single point the origin $(0,0,0)$.



b) $y^2+(z-1)^2=0 \Rightarrow y^2=0$ and $(z-1)^2=0$

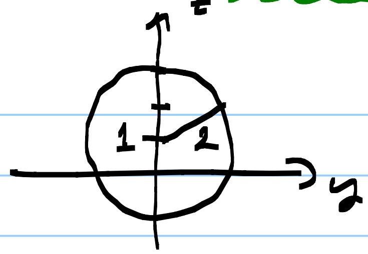
$\Rightarrow y=0$ and $z=1$.

There is no condition on the x-coordinate.
So it is the set $\{(x,0,1) \mid x\in\mathbb{R}\}$

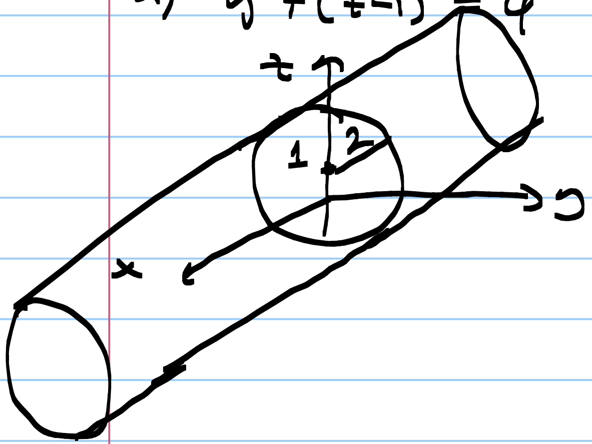


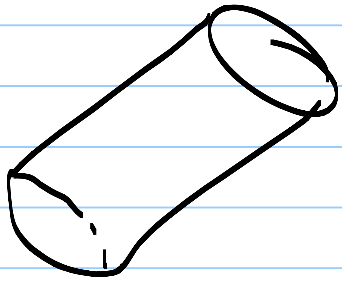
It is the line parallel to the x-axis passing through the point $(0,0,1)$.

Video 26



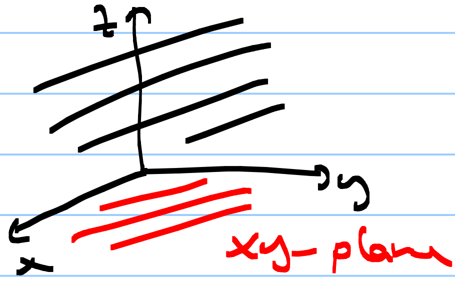
a) $y^2+(z-1)^2=4$

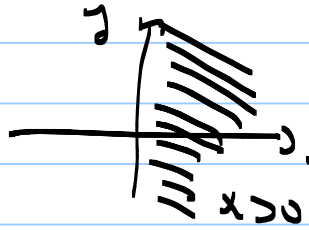




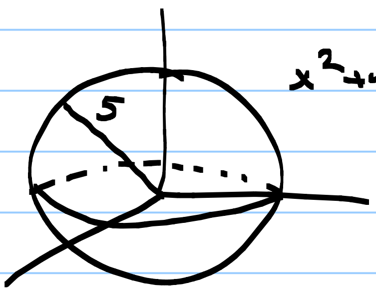
It is the cylinder obtained by translating the circle $y^2+(z-1)^2=4$ in the yz-plane along the x-axis.

Example 1) The inequality $z>0$ represents all points (x,y,z) in 3-space whose z-coordinate are positive.



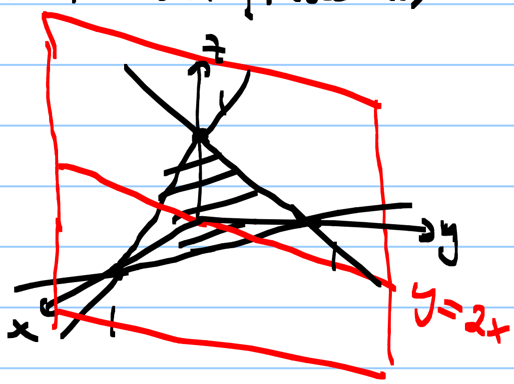


2) The inequality $x^2+y^2+z^2\le 25$ represents all points whose distance to the origin ≤ 5.

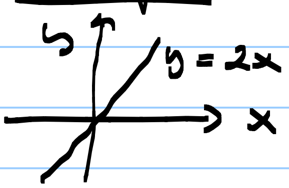


$x^2+y^2+z^2=25$

(Hilbert's 16th problem and related problems)

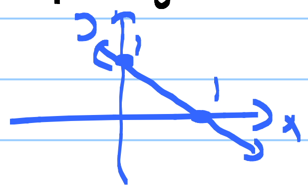


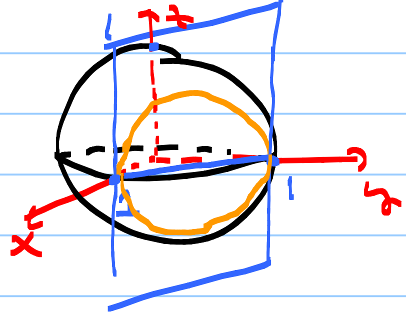
Example! a) $\begin{cases} x+y+z=1 \\ y-2x=0 \end{cases}$



This system of equations represent all the points in 3-space which lie on both planes: $x+y+z=1$ and $y-2x=0$. Therefore, it is just the intersection of these two planes. So it is a line.

b) $\begin{cases} x^2+y^2+z^2=1 \\ x+y=1 \end{cases}$



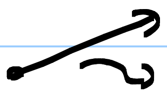


<u>Euclidean n-space</u>: $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n, \ldots$

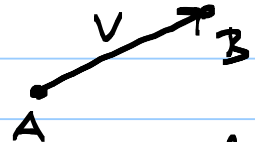
$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R},\ i=1,\ldots,n\}$.

<u>§ 10.2. Vectors</u>:

A vector is a quantity that involves both magnitude (size or length) and direction. For example the velocity vector contains both the information how fast the object moves, which is the speed and in which direction the object moves.



length 5 units = speed

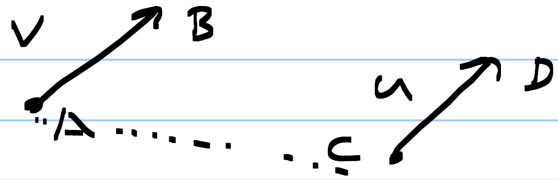


$v = \overrightarrow{AB}$ The magnitude of the the vector $v = \overrightarrow{AB}$ is the distance from A to B. The length of v is denoted as $|v| = |\overrightarrow{AB}|$.

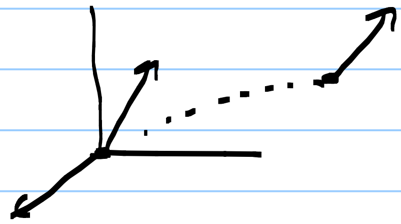
Two vectors $v = \overrightarrow{AB}$ and $u = \overrightarrow{CD}$ so that

i) $|v| = |u|$ and
ii) v and u are parallel

will be considered the same.

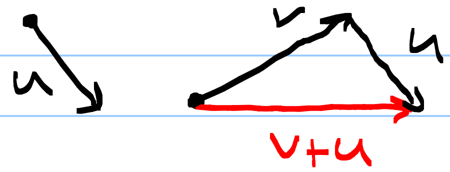


$\overrightarrow{AB} \neq \overrightarrow{BA}$

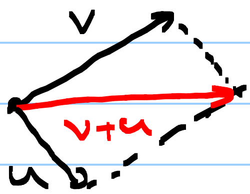


Arithmetic of Vectors:

Addition:



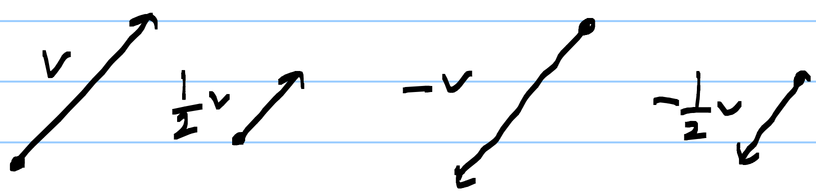
OR:



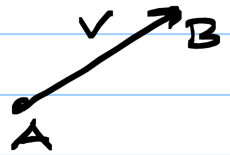
Note that $u+v = v+u$

Scalar multiplication: $\lambda \in \mathbb{R}$, v a vector

λv: this is the vector parallel to v with size $\lambda |v|$. If $\lambda < 0$ then you take the vector in opposite direction.



Mathematical Representation of vectors

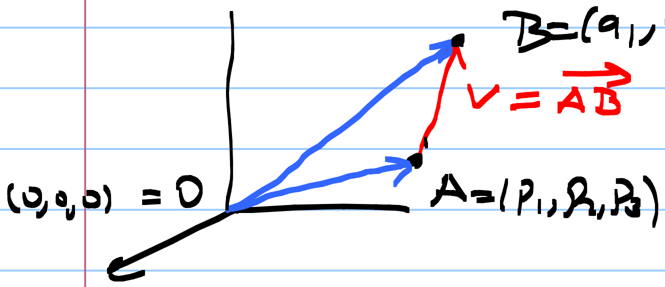


$v = \overrightarrow{AB}$

$A = (p_1, p_2, p_3)$
$B = (q_1, q_2, q_3)$

Then v is represented by the triple

$$v = \overrightarrow{AB} = (q_1 - p_1, q_2 - p_2, q_3 - p_3)$$

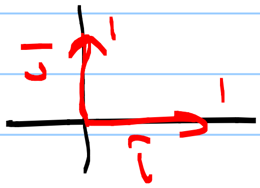


$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

or

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (q_1, q_2, q_3) - (p_1, p_2, p_3) \\ &= (q_1 - p_1, q_2 - p_2, q_3 - p_3). \end{aligned}$$

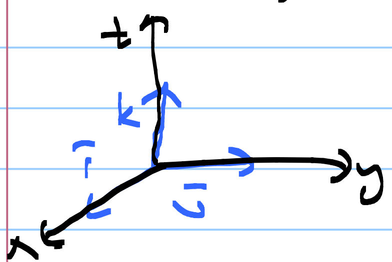
Another notation: In the plane the vectors $(1,0)$ and $(0,1)$ are denoted as $\hat{i}$ and $\hat{j}$.



$$\begin{aligned} v &= (a, b) \\ &= (a, 0) + (b, 0) \\ &= a(1,0) + b(0,1) \\ &= a\hat{i} + b\hat{j} \end{aligned}$$

In 3-space we have also k:

$\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$, $k = (0,0,1)$



$$v = (a, b, c) = a\hat{i} + b\hat{j} + ck$$

Example: $u = 3\hat{i} - 4\hat{j} + 5k$ and $v = \hat{j} + 7k$.

Then $2u + 4v = 2(3i - 4j + 5k) + 4(j + 7k)$

$$= 6i - 8j + 10k + 4j + 28k$$

$$= 6i - 4j + 38k \text{ or}$$
$$= (6, -4, 38)$$

Definition: A vector of length 1 unit is called a unit vector.

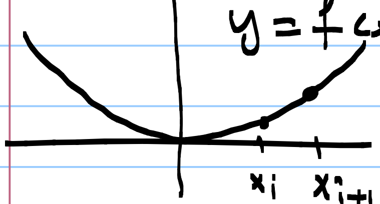
If v is any non-zero vector $(v \neq (0,0,0))$ then the vector

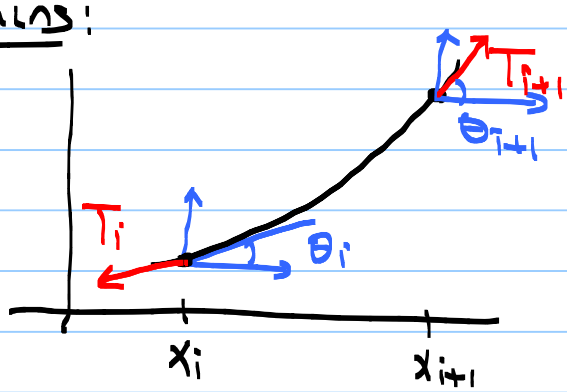
$$\frac{1}{|v|} v \text{ is a unit vector.}$$

$$\left|\underbrace{\frac{1}{|v|}}_{\lambda} v\right| = \lambda |v| = \frac{1}{|v|} |v| = 1.$$

Remark: $|\lambda v| = |\lambda|\,|v|$, for any scalar $\lambda \in \mathbb{R}$ and vector v.

Hanging Cables and Chains:





$$f'(x_i) = \tan\theta_i$$

Observations

(*) 1) $T_i \cos\theta_i = T_{i+1} \cos\theta_{i+1} = T$ because there is no horizontal motion.

(**) 2) $T_{i+1} \sin\theta_{i+1} = T_i \sin\theta_i + w_i$, where

w_i is the weight of the cable from the point x_i to x_{i+1}.

So, $w_i = \rho g \int_{x_i}^{x_i} \sqrt{1+(f'(t))^2}\, dt$, where

ρ is the density of cable in kg/m and g is the gravitational constant.

The Mean Value Theorem for integrals implies that

$$w_i = \rho g \sqrt{1+(f'(\xi_i))^2}\,(x_{i+1}-x_i), \text{ for}$$

some point ξ_i in $[x_i, x_{i+1}]$.

Dividing the equation (**) by T in equation (*) we get

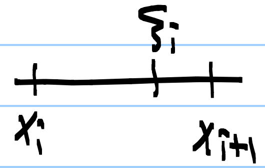
$$\tan\theta_{i+1} = \tan\theta_i + \frac{\rho g}{T}(x_{i+1}-x_i)\sqrt{1+(f'(\xi_i))^2}$$

Let k denote the constant $\rho g/T$. Then

$$\frac{f'(x_{i+1})-f'(x_i)}{x_{i+1}-x_i} = \frac{\rho g}{T}\sqrt{1+(f'(\xi_i))^2} \text{ is obtained.}$$

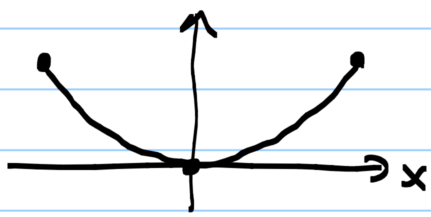
Letting $x_{i+1} - x_i \to 0$ we obtain

$$f''(x) = K\sqrt{1+(f'(x))^2}$$



Note that the graph of f below implies

$f(0) = f'(0) = 0.$



So $f(x)$ is a function satisfying the initial value problem

$$\begin{cases} f''(x) = K\sqrt{1+(f'(x))^2} \\ f(0) = 0,\ f'(0) = 0. \end{cases}$$

Finally, the theory of Ordinary Differential Equations implies that

$f(x) = \frac{1}{K}(-1 + \cosh Kx)$ is the unique solution of the initial value problem.

The Dot Product and Projections

Definition: Dot product of two vector in $\mathbb{R}^n$ is defined by the formula

$u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$

$$u \cdot v = u_1 v_1 + \dots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

In $\mathbb{R}^2$ it becomes $u \cdot v = (u_1, u_2) \cdot (v_1, v_2)$
$= u_1 v_1 + u_2 v_2$

and in $\mathbb{R}^3$ it is given by

$$u \cdot v = (u_1, u_2, u_3) \cdot (v_1, v_2, v_3)$$
$$= u_1 v_1 + u_2 v_2 + u_3 v_3.$$

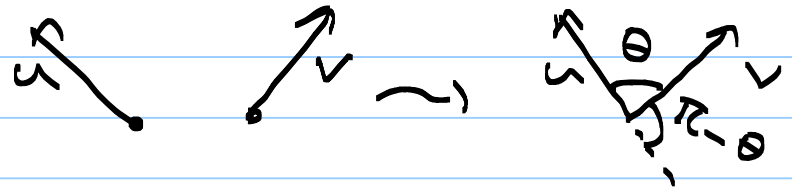
Example $(3, 2, -1) \cdot (4, 0, 5) = 3 \cdot 4 + 2 \cdot 0 + (-1) 5$
$= 12 - 5 = 7.$

Remark: If $u = (u_1, u_2, u_3)$ then $u \cdot u = u_1^2 + u_2^2 + u_3^2$ is just the square of its length, $|u| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

So, $u \cdot u = |u|^2$.

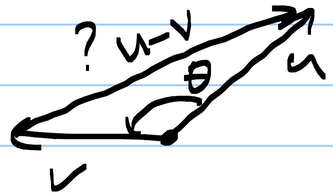
Theorem: For any two vectors u, v in $\mathbb{R}^3$ we have $u \cdot v = |u| |v| \cos \theta$, where θ is the angle between the vectors.

Video 22



$\cos \pi - \theta = \cos\theta$

Proof:



$v + ? = u$

$? = u - v$

Apply Cosine law to this triangle

$$|u-v|^2 = |v|^2 + |u|^2 - 2|u||v|\cos\theta$$

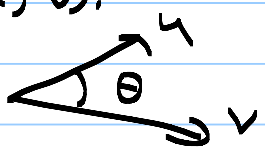
$$(u-v)\cdot(u-v) = |v|^2 + |u|^2 - 2|u||v|\cos\theta$$

$$u\cdot u - u\cdot v - v\cdot u + v\cdot v = |v|^2 + |u|^2 - 2|u||v|\cos\theta$$

$$\cancel{|u|^2} - 2u\cdot v + \cancel{|v|^2} = \cancel{|v|^2} + \cancel{|u|^2} - 2|u||v|\cos\theta$$

$$u\cdot v = |u||v|\cos\theta.$$

Example: Find the angle between the vectors $u = (2,-1,3)$ and $v = (-5,-2,0)$.



$$u\cdot v = |u||v|\cos\theta$$

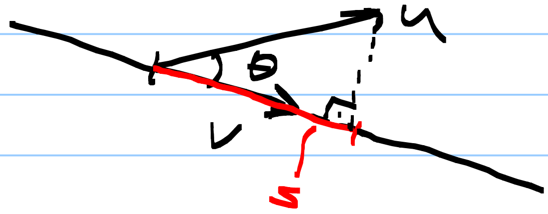
$$\cos\theta = \frac{u\cdot v}{|u||v|} = \frac{(2,-1,3)\cdot(-5,-2,0)}{|(2,-1,3)|\,|(-5,-2,0)|}$$

$$= \frac{-10+2+0}{\sqrt{4+1+9}\,\sqrt{25+1}} = \frac{-8}{\sqrt{14}\sqrt{26}}$$

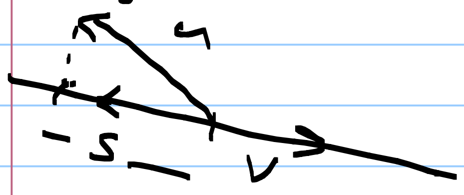
$$\theta = \cos^{-1}\left(-8/\sqrt{14\times 26}\right)$$

Projections The scalar projection of a vector v in the direction of a nonzero vector u is defined by the formula

$$s = \frac{u\cdot v}{|v|} = \frac{|u|\,|v|\cos\theta}{|v|} = |u|\cos\theta$$



Example: Find the projection of $u=(2,3,-5)$ along the vector $v=(-1,1,2)$.



$$s = \frac{u\cdot v}{|v|} = \frac{(2,3,-5)\cdot(-1,1,2)}{\sqrt{1+1+4}}$$

$$s = \frac{-2+3-10}{\sqrt{6}} = \frac{-9}{\sqrt{6}}$$

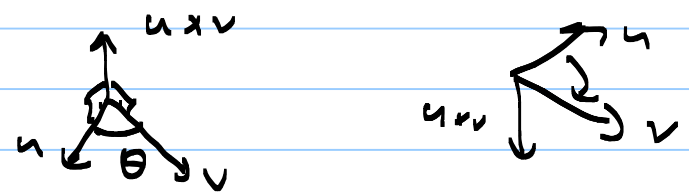
§10.3. The Cross Product in 3-space:

Definition: For any vectors u and v in $\mathbb{R}^3$, the cross product $u\times v$ is the unique vector satisfying the following three conditions:

i) $(u\times v)\cdot u = 0$ and $(u\times v)\cdot v = 0$

ii) $|u\times v| = |u|\,|v|\sin\theta$, where θ is the angle between u and v.

iii) u, v and $u\times v$ form a right-handed triad.



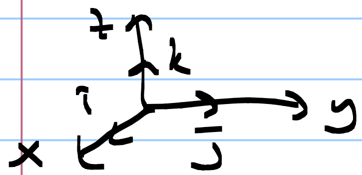
Remark: Note that indeed there is a unique vector satisfying these three conditions.

Theorem: If $u=(u_1,u_2,u_3)=u_1 i+u_2 j+u_3 k$ and $v=(v_1,v_2,v_3)=v_1 \hat{i}+v_2 \hat{j}+v_3 k$, then

$$u\times v=(u_2v_3-u_3v_2)\hat{i}+(u_3v_1-u_1v_3)\hat{j}+(u_1v_2-u_2v_1)k.$$

Remark: $\hat{i}\times \hat{j}=k$, $\hat{j}\times \hat{i}=-k$

$v\times v=|v||v| \cdot \sin 0$ $\Downarrow$ $v\times v=0$



$\hat{j}\times k=\hat{i}$, $k\times \hat{j}=-\hat{i}$

$k\times \hat{i}=\hat{j}$, $\hat{i}\times k=-\hat{j}$.

Proof: Assuming the cross product distributes over addition and scalar multiplication we can prove the theorem as follows:

$$(u+v)\times w=u\times w+v\times w$$
$$(\lambda u)\times w=\lambda(u\times w)$$

Note that in this case,

$$u\times v=(u_1\hat{i}+u_2\hat{j}+u_3k)\times(v_1\hat{i}+v_2\hat{j}+v_3k)$$

$$\begin{aligned}
&= u_1v_1\overbrace{\hat{i}\times\hat{i}}^{0}+u_1v_2\overbrace{\hat{i}\times\hat{j}}^{k}+u_1v_3\overbrace{\hat{i}\times k}^{-\hat{j}}\\
&+u_2v_1\overbrace{\hat{j}\times\hat{i}}^{-k}+u_2v_2\overbrace{\hat{j}\times\hat{j}}^{0}+u_2v_3\overbrace{\hat{j}\times k}^{\hat{i}}\\
&+u_3v_1\underbrace{k\times\hat{i}}_{\hat{j}}+u_3v_2\underbrace{k\times\hat{j}}_{-\hat{i}}+u_3v_3\underbrace{k\times k}_{0}
\end{aligned}$$

$$=(u_2v_3-u_3v_2)\hat{i}+(u_3v_1-u_1v_3)\hat{j}+(u_1v_2-u_2v_1)k$$

This finishes the proof $\blacksquare$

Remark: The book suggests another proof. Read it.

Video 23

Example $(-2,3,1)\times(4,2,-3) = (-2\hat{i}+3\hat{j}+k)\times(4\hat{i}+2\hat{j}-3k)$

$$= -4k + 6(-\hat{j}) + 12(-k) - 9\hat{i} + 4\hat{j} + 2(-\hat{i})$$

$$= -11\hat{i} - 2\hat{j} - 16k$$

Properties of the cross product:

If u, v and w are any vectors in $\mathbb{R}^3$, and t is any real numbers then

i) $u \times u = 0$
ii) $u \times v = -v \times u$ (the cross product is anti commutative)
iii) $(u+v)\times w = u\times w + v\times w$
iv) $u\times(v+w) = u\times v + u\times w$
v) $(tu)\times v = t\, u\times v$
vi) $u\cdot(u\times v) = 0 = v\cdot(u\times v)$

Remark: $\times$ is **not** an associative operation on vectors in $\mathbb{R}^3$.

$$(u\times v)\times w \neq u\times(v\times w) \text{ in general.}$$

Counter example:
$$(\hat{i}\times\hat{i})\times\hat{j} \neq \hat{i}\times(\hat{i}\times\hat{j})$$
$$0\times\hat{j} \neq \hat{i}\times k$$
$$0 \neq -\hat{j}$$

Determinants: An $m\times n$-matrix over $\mathbb{R}$ is an array of real numbers of the form

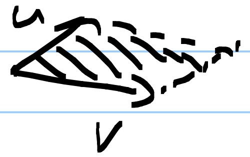
columns

$$\text{rows}\quad\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}_{m\times n}$$

m rows and n columns

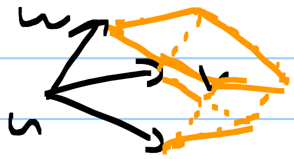
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3\times 3}$$

$u = (a, b)$
$v = (c, d)$



$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc = \pm$ Area of the parallelogram determined by the vector $u = (a,b)$, $v = (c,d)$.

In $\mathbb{R}^3$ $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \pm$ the volume of the parallelepiped formed by the vectors $u = (a,b,c)$, $v = (d,e,f)$ and $w = (g,h,i)$.



If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are two vectors in $\mathbb{R}^3$ then we have

$$u \times v = \det \begin{bmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = i(u_2v_3 - u_3v_2) - j(u_1v_3 - u_3v_1) + k(u_1v_2 - v_1u_2)$$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - hf) - b(di - gf) + c(dh - eg)$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = (-1)^{1+2} a_{12}(a_{21}a_{33} - a_{23}a_{31})$$

$$+(-1)^{2+2} a_{22}(a_{11}a_{33} - a_{13}a_{31}) + (-1)^{3+2} a_{32}(a_{11}a_{23} - a_{13}a_{21})$$

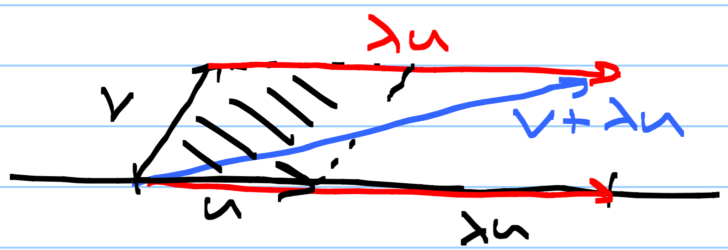
Examples $\det\begin{bmatrix} & \end{bmatrix} = \begin{vmatrix} & \end{vmatrix}$

$$\begin{vmatrix} 1 & 4 & -2 \\ -3 & 1 & 0 \\ 2 & 2 & -3 \end{vmatrix} = 1\cdot(-3) - 4(9-0) - 2(-6-2)$$
$$= -3 - 36 + 16 = -23.$$

Properties of Determinants: If two rows or colums are the same then the determinant is zero.

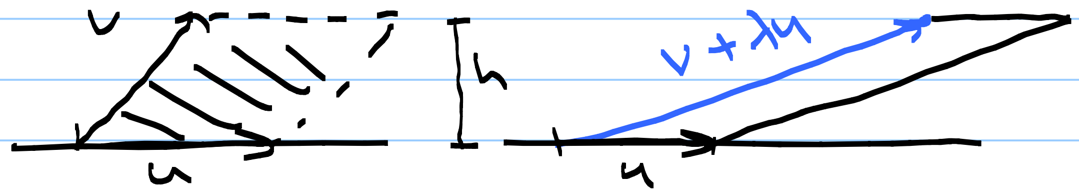
If we added a multiple of a row or a colum to another row or column, respectively, the the volume does not change.

$$\begin{vmatrix} u \\ v \\ w \end{vmatrix} = \begin{vmatrix} u \\ \lambda u + v \\ w \end{vmatrix} \qquad \begin{vmatrix} u & v & w \end{vmatrix} = \begin{vmatrix} u & v & \lambda u + w \end{vmatrix}$$



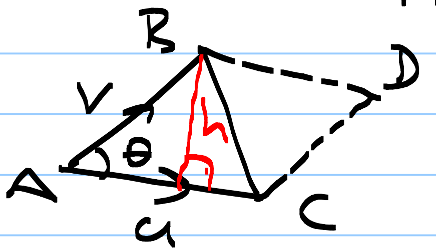
$$\begin{vmatrix} u \\ v \end{vmatrix} = \begin{vmatrix} u \\ \lambda u + v \end{vmatrix}$$

$\Downarrow$



Video 24

Example: Find the area of the triangle with vertices $A=(1,1,0)$, $B=(3,0,2)$ and $C=(0,-1,1)$.



$$\text{Area}(\overset{\triangle}{ABC}) = \frac{1}{2}\text{Area}(ACDB) = \frac{1}{2}|u\times v| = \frac{\sqrt{50}}{2}$$

$$|u\times v| = |u||v|\sin\theta = |u|(|v|\sin\theta) = |u|\cdot h = \text{Area}(ACDB)$$

$u = AC = (-1,-2,1)$

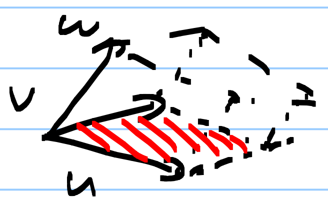
$v = AB = (2,-1,2)$

$$\underline{u\times v} = \begin{vmatrix} i & j & k \\ -1 & -2 & 1 \\ 2 & -1 & 2 \end{vmatrix} = i(-3) - j(-4) + k(5) = -3i + 4j + 5k$$

$$|u\times v| = \sqrt{9+16+25} = \sqrt{50}$$

Example: Find the volume of the parallelepiped spanned by the vectors u, v and w, where

$u=(3,0,1)$, $v=(-2,5,4)$, $w=(1,1,1)$.



$$\text{Volume} = \begin{vmatrix} u \\ v \\ w \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v\times w)\cdot(u_1,u_2,u_3) = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\cdot(u_1,u_2,u_3)$$

$$\text{Volume} = \begin{vmatrix} 3 & 0 & 1 \\ -2 & 5 & 4 \\ 1 & 1 & 1 \end{vmatrix} = 3\cdot(1) - 0\cdot ? + 1\cdot(-7) = 3 - 7 = -4$$

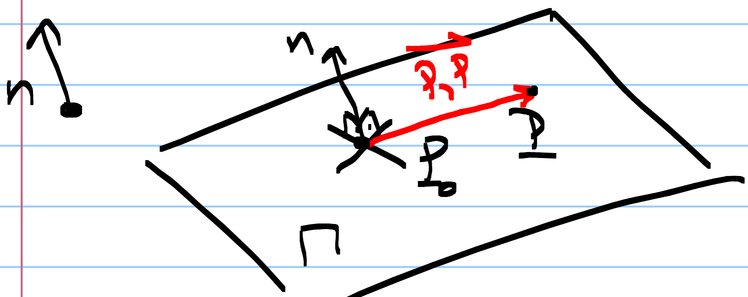
So, volume is $|-4| = 4$.

Definition: The quantity $u\cdot(v\times w)$ is called the scalar triple product of the vectors u, v and w.

$$u\cdot(v\times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$ – signed volume of the parallelepiped spanned by u, v, w.

§10.4. Planes and Lines:

A plane Γ in $\mathbb{R}^3$ is determined by a point contained in the plane and a normal vector to the plane.



$P_0 = (x_0, y_0, z_0)$
$P = (x, y, z)$
$u = \vec{P_0P}$
$= (x-x_0, y-y_0, z-z_0)$

Suppose $P_0 = (x_0, y_0, z_0)$ and $n = (A, B, C)$
$= A\hat{i} + B\hat{j} + C\hat{k}$.

How can we describe the points of Γ in terms of their coordinates?

Observation: P is on the plane Γ if and only if the vector $u = \vec{P_0P}$ lies on Γ.

Hence, P is on Γ if and only if $u \perp n$.

So $P = (x,y,z) \in \Gamma \iff u\cdot n = 0$

$$u\cdot n = (x-x_0, y-y_0, z-z_0)\cdot(A, B, C)$$
$$= A(x-x_0) + B(y-y_0) + C(z-z_0).$$

Therefore, a point $P=(x,y,z)$ is on Γ if and only if

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0.$$

OR, $Ax+By+Cz=D$, where $D=Ax_0+By_0+Cz_0$

This is called the equation of the plane Γ with normal vector $n=(A,B,C)$ passing through the point $P_0=(x_0,y_0,z_0)$.

Examples 1) Determine an equation for the plane with normal vector $n=(2,3,-5)$ passing through the origin.

Solution: $n=(2,3,-5)$, $P_0=(x_0,y_0,z_0)=(0,0,0)$.

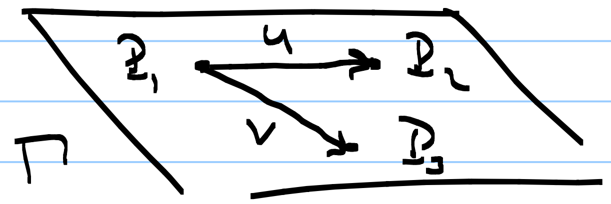
$$n\cdot\overrightarrow{PP_0}=0 \quad , \quad P=(x,y,z)$$

$$(2,3,-5)\cdot(x-0,y-0,z-0)=0$$

$$2x+3y-5z=0.$$

2) Find an equation for the plane passing through the points $P_1=(1,1,1)$, $P_2=(2,0,5)$ and $P_3=(4,2,-1)$.

Solution:



Let $u=\overrightarrow{P_1P_2}$ and $v=\overrightarrow{P_1P_3}$. The u,v lie on the plane Γ. Hence, their cross product $u\times v$ is perpendicular to the plane Γ. Hence, we may take $n=u\times v$ as the normal vector. Now, $u = \overrightarrow{P_1P_2} = (1,-1,4)$ and $v = \overrightarrow{P_1P_3} = (3,1,-2)$.

So $n = u \times v = \begin{vmatrix} i & j & k \\ 1 & -1 & 4 \\ 3 & 1 & -2 \end{vmatrix} = i(-2) - j(-14) + k(4)$

$n = -2i + 14j + 4k.$

Take $P_0 = P_1 = (1,1,1)$. Then an equation for the plane is

$n \cdot \overrightarrow{PP_0} = 0 \Rightarrow (-2,14,4)\cdot(x-1, y-1, z-1) = 0$

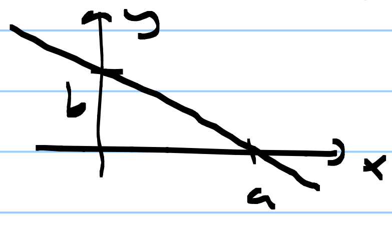
$-2x + 2 + 14y - 14 + 4z - 4 = 0$

Or, $-2x + 14y + 4z = -2 + 14 + 4$,

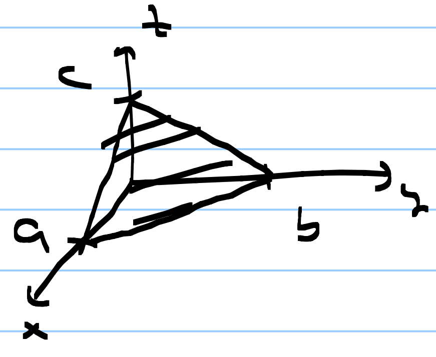
$$\boxed{-2x + 14y + 4z = 16}$$

Intercept form of a plane:

$\frac{x}{a} + \frac{y}{b} = 1$

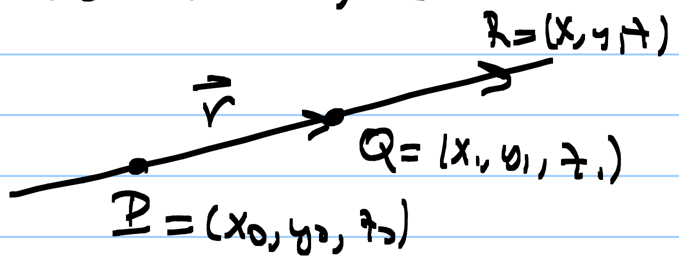


$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$



Video 25

Lines in 3-space.



$\vec{r}=\vec{PQ}$
$=(x_1-x_0,y_1-y_0,z_1-z_0)$

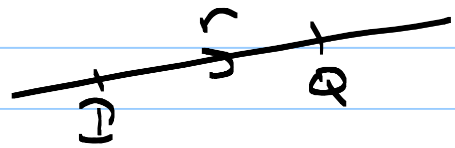
$PR \parallel \vec{r}=\vec{PQ} \Rightarrow \vec{PR}=t\vec{r}$
$\Rightarrow R-P=t\vec{r}$
$\Rightarrow (x,y,z)-(x_0,y_0,z_0)=t\vec{r}$

$\Rightarrow (x,y,z)=(x_0,y_0,z_0)+t\vec{r}=P+t\vec{r},\ t\in\mathbb{R}.$

This is called a parametric equation for the line passing through $P=(x_0,y_0,z_0)$ and parallel to $\vec{r}$.

<u>Example:</u> Write down a parametric equation for the line passing through the points $P=(2,-3,0)$ and $Q=(4,5,-7)$.

<u>Solution:</u>

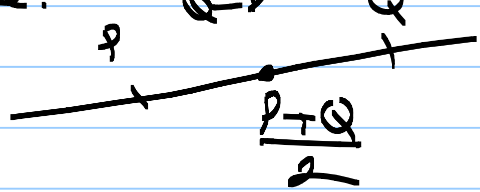


$\vec{r}=\vec{PQ}=(2,8,-7)$

$(x,y,z)=P+tr=(2,-3,0)+t(2,8,-7)$
$=(2+2t,-3+8t,-7t),\quad t\in\mathbb{R}$

$x=2+2t,\ y=-3+8t,\ z=-7t,\quad t\in\mathbb{R}.$

<u>Example:</u> Find the mid point of P and Q in the above line.

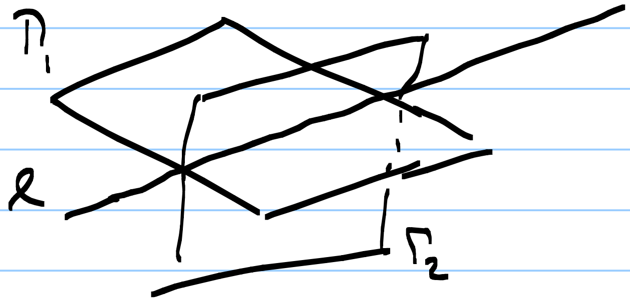


$P+\frac{Q-P}{2}=\frac{P+Q}{2}$

Remark: Any two generic planes intersect along a line and any line can be written as the intersection of two planes.

Example: Find a parametric equation of the line ℓ determined by the planes Π_1 and Π_2 given by the equations

$$\Pi_1: 2x - y + 3z = 0, \quad \Pi_2: x + y + z = 5$$



Solution: First we find a point on the line ℓ of intersection.

$$2x - y + 3z = 0$$
$$x + y + z = 5$$

Let's plug $z=0$ in both equations.

$$2x - y = 0$$
$$x + y = 5$$
$$\overline{3x + 0\cdot y = 5} \Rightarrow x = 5/3,\ y = 5 - \frac{5}{3} = 10/3.$$

$P = (5/3, 10/3, 0).$

For a second point let $z=1$.

$$\begin{array}{l} 2x - y + 3 = 0 \\ x + y + 1 = 5 \end{array} \Rightarrow \begin{array}{l} 2x - y = -3 \\ x + y = 4 \end{array}$$
$$+ \overline{\qquad\qquad}$$
$$3x = 1 \Rightarrow x = 1/3,\ y = 4 - 1/3 = 11/3$$

So, $Q = (1/3,\ 11/3,\ 1)$.

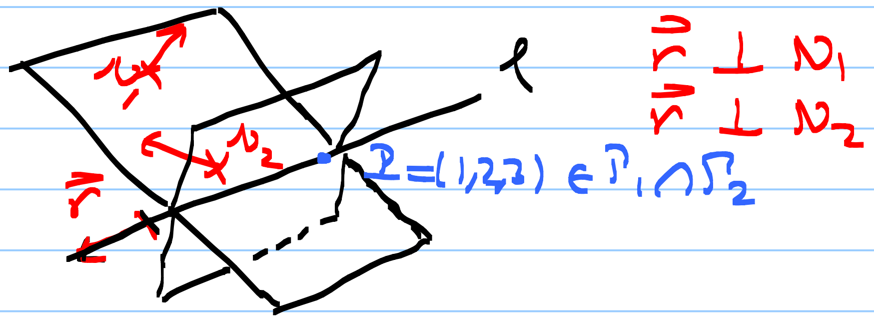
Hence, $r = PQ = (1/3,\ 11/3,\ 1) - (5/3,\ 10/3,\ 0)$
$= (-\frac{4}{3},\ 1/3,\ 1)$.

Parametric equation for the line l

$(x,y,z) = P + t r = (5/3,\ 10/3,\ 0) + t(-4/3,\ 1/3,\ 1),$
$t \in \mathbb{R}$.

<u>Example</u>: Find two planes Π_1 and Π_2 whose intersection is the line given by

$(x,y,z) = (1,2,3) + t(-1,0,4), \quad t \in \mathbb{R}$.



$\vec{r} = (-1,0,4)$. $\quad N_1 \cdot \vec{r} = 0$ and $N_2 \cdot \vec{r} = 0$

$(0,1,0)\cdot(-1,0,4) = 0 = (4,0,1)\cdot(-1,0,4)$

So we may just choose $N_1 = (0,1,0)$ and $N_2 = (4,0,1)$.

Π_1: $\big((x,y,z) - (1,2,3)\big)\cdot N_1 = 0$
$(x-1,\ y-2,\ z-3)\cdot(0,1,0) = 0 \Rightarrow y - 2 = 0$
$\Rightarrow y = 2$.

$P_2 : ((x,y,z)-(1,2,3)) \cdot N_2 = 0$

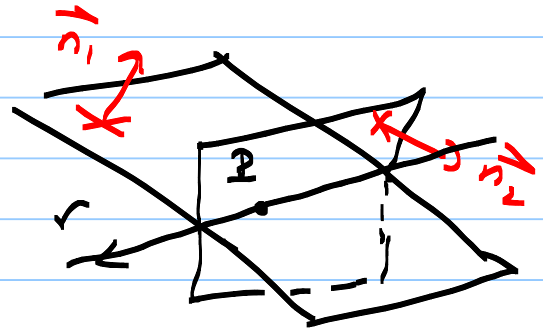
$(x-1,\, y-2,\, z-3)\cdot(4,0,1) = 0$

$\Rightarrow 4(x-1) + 0\cdot(y-2) + 1\cdot(z-3) = 0$

$4x + z = 7$

<u>Example:</u> Find a parametric equation for the line l which is the intersection of the planes passing through the point $P=(-3,4,5)$ with normal vectors $\vec{n}_1=(1,-2,0)$ and $\vec{n}_2=(3,7,-5)$.

<u>Solution:</u>



$r \perp \vec{n}_1$
$r \perp \vec{n}_2$

We may just take $\vec{r} = \vec{n}_1 \times \vec{n}_2$.

So, $\vec{r} = \begin{vmatrix} i & j & k \\ 1 & -2 & 0 \\ 3 & 7 & -5 \end{vmatrix} = (10, 5, 13)$

$l: (x,y,z) = P + tr = (-3,4,5) + t(10,5,13)$

$x = -3+10t,\quad y = 4+5t,\quad z = 5+13t,\quad t \in \mathbb{R}.$

$\Downarrow \qquad \Downarrow$

$\frac{x+3}{10} = t, \quad \frac{y-4}{5} = t, \quad \frac{z-5}{13} = t$

$$\Rightarrow \boxed{\frac{x+3}{10} = \frac{y-4}{5} = \frac{z-5}{13}}$$

Another way of representing a line

If $P=(x_0,y_0,z_0)\in \ell$ and $r=(a,b,c)\parallel \ell$ then

$$(x,y,z)=P+tr=(x_0,y_0,z_0)+t(a,b,c)$$

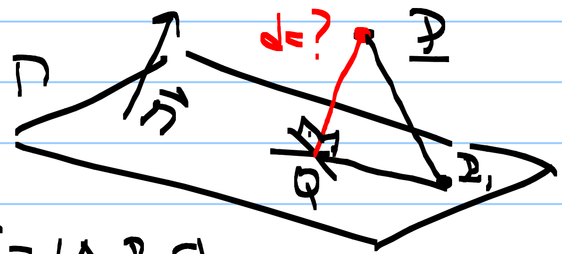
$$x=x_0+ta,\quad y=y_0+tb,\quad z=z_0+tc$$

OR $$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}\;(=t).$$

Distances to a line or a plane:

Distance from a point to a plane:

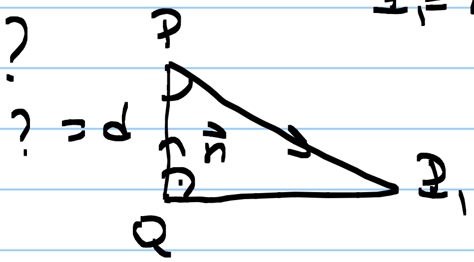
Find the distance from a point $P=(x_0,y_0,z_0)$ to the plane Π given by $Ax+By+Cz=D$.



$\vec{n}=(A,B,C)$

$d=|PQ|$, where Q is the closest point on Π to P. Take any other point $P_1=(x_1,y_1,z_1)$ on Π.

$d=|PQ|=?$



Note that d is just the scalar projection of the vector $\vec{PP_1}$ to the vector $\vec{n}=(A,B,C)$.

$$d=\left|\frac{\vec{PP_1}\cdot\vec{n}}{|\vec{n}|}\right|=\left|\frac{(P_1-P)\cdot\vec{n}}{|\vec{n}|}\right|=\left|\frac{P_1\cdot\vec{n}-P\cdot\vec{n}}{|\vec{n}|}\right|$$

Video 26

If $P_1=(x,y,z)\in P$ then $P_1\cdot n = Ax+By+Cz=D$.

So, $d=\left|\dfrac{D-P\cdot\vec{n}}{|\vec{n}|}\right|$ $\qquad (x,y,z)=P+tr$

As an example let's find the distance of the point $P=(2,-1,3)$ to the plane $2x-2y-z=9$.

$\vec{n}=(A,B,C)=(2,-2,-1)$, $D=9$, $P=(2,-1,3)$

$$d=\left|\frac{D-P\cdot\vec{n}}{|\vec{n}|}\right| = \left|\frac{9-(2,-1,3)\cdot(2,-2,-1)}{|(2,-2,-1)|}\right|$$

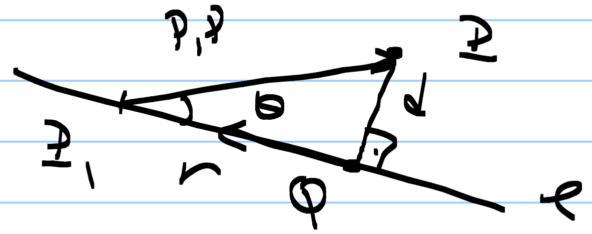
$$=\left|\frac{9-(4+2-3)}{\sqrt{4+4+1}}\right| = \left|\frac{9-3}{\sqrt{9}}\right| = 2.$$

If $P=(x_0,y_0,z_0)$ and $\vec{n}=(A,B,C)$, the normal of the plane $Ax+By+Cz-D=0$, then the distance is

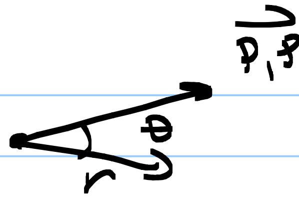
$$d=\left|\frac{D-P\cdot\vec{n}}{|\vec{n}|}\right| = \left|\frac{P\cdot\vec{n}-D}{|\vec{n}|}\right|$$

$$=\left|\frac{Ax_0+By_0+Cz_0-D}{\sqrt{A^2+B^2+C^2}}\right|$$

Distance from a point to a line:

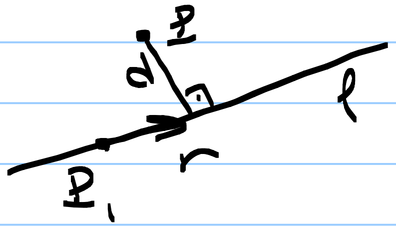


$$d=|PQ|$$
$$=\sin\theta\,|P_1P|$$



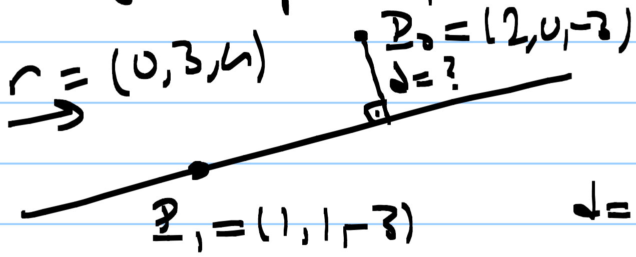
$$r \times \overrightarrow{P_1P} = |r|\,|\overrightarrow{P_1P}|\sin\theta$$

So, $|\overrightarrow{P_1P}|.\sin\theta = \dfrac{r\times \overrightarrow{P_1P}}{|r|}$



Hence, $d = \dfrac{|r \times \overrightarrow{P_1P}|}{|r|}$

Example: Find the distance from the point $P_0=(2,0,-3)$ to the line ℓ, with direction vector $r=(0,3,4)$ passing through the point $P_1=(1,1,-3)$.

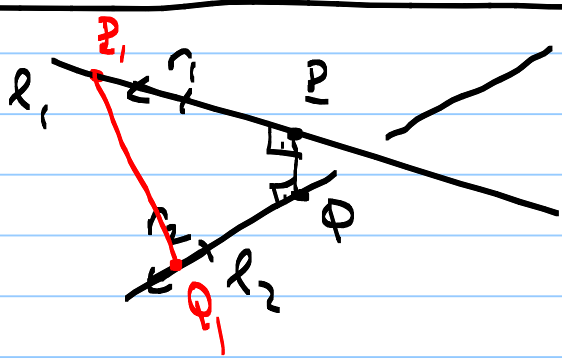


$$d=\frac{|r\times \overrightarrow{P_1P_0}|}{|r|}$$

$$r\times \overrightarrow{P_1P} = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 0 & 3 & 4 \end{vmatrix} = (-4,-4,3)$$

So, $d = \dfrac{|(-4,-4,3)|}{|(0,3,4)|} = \dfrac{\sqrt{16+16+9}}{\sqrt{9+16}} = \dfrac{\sqrt{41}}{5}.$

Distance between two lines:



If $P\in \ell_1$ and $Q \in \ell_2$ are the closest points on the lines to each other then $\overrightarrow{PQ}\perp r_1$ and $\overrightarrow{PQ}\perp r_2$

$\overrightarrow{PQ} \,//\, r_1 \times r_2$. If P_1 and Q_1 are ~~just~~ two points on ℓ_1 and ℓ_2, respectively, then $\overrightarrow{PQ}$ is the vector projection of $\overrightarrow{P_1Q_1}$ along $\overrightarrow{PQ}$ or $r_1 \times r_2$.

So, $d = |\overrightarrow{PQ}| = \dfrac{|\overrightarrow{P_1Q_1} \cdot (r_1 \times r_2)|}{|r_1 \times r_2|}$

<u>Example:</u> Find the distance between the lines l_1 : x-axis and l_2: $\dfrac{x-1}{-3} = \dfrac{y+2}{1} = \dfrac{z-1}{5}$.

<u>Solution:</u>

l_1 : x-axis, $r_1 = (1, 0, 0)$, $P_1 = (0, 0, 0)$

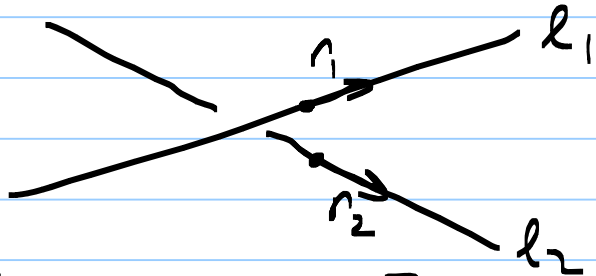
l_2: $\dfrac{x-1}{-3} = \dfrac{y+2}{1} = \dfrac{z-1}{5}$, $r_2 = (-3, 1, 5)$, $Q_1 = (1, -2, 1)$

$$r_1 \times r_2 = i \times (-3i + j + 5k) = -3\overset{0}{(i \times i)} + \overset{k}{i \times j} + 5\,\overset{-j}{i \times k}$$
$$= k + 5j = (0, 5, 1).$$

$P_1Q_1 = (1, -2, 1)$. So $d = \dfrac{|PQ \cdot (r_1 \times r_2)|}{|r_1 \times r_2|}$

$$\Rightarrow d = \frac{|(1,-2,1) \cdot (0,5,1)|}{|(0,5,1)|} = \frac{|-10+1|}{\sqrt{25+1}} = \frac{9}{\sqrt{26}}$$

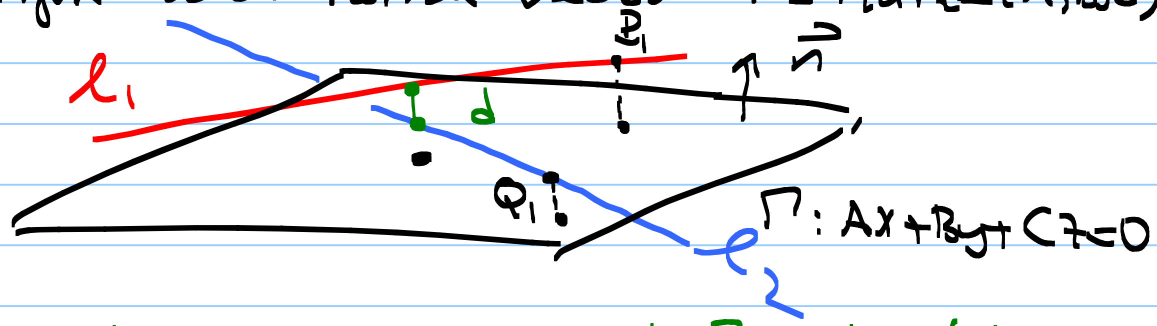
<u>Alternative Solution:</u>



$\vec{n} = r_1 \times r_2$

Note that if l_1 and l_2 do not intersect there is a plane Γ which is parallel to both lines. A normal vector to Γ is $n = \vec{r_1} \times \vec{r_2}$.

Let Γ be the plane passing through the origin with normal vector $\vec{n} = \vec{r}_1 \times \vec{r}_2 = (A, B, C)$



d = the distance from l_1 to Γ − the distance from l_2 to Γ.

$$= \left| \text{distance from } P_1 \text{ to } \Gamma - \text{distance from } Q_1 \text{ to } \Gamma \right|$$

$$= \frac{\left| P_1 \cdot (A,B,C) - Q_1 \cdot (A,B,C) \right|}{\sqrt{A^2+B^2+C^2}}$$

$$= \frac{\left| \overrightarrow{P_1Q_1} \cdot (r_1 \times r_2) \right|}{|r_1 \times r_2|}$$

Video 27

§10.5. Quadratic Surfaces:

A quadratic surface is the solution set of a quadratic polynomial in x, y, z:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz = J$$

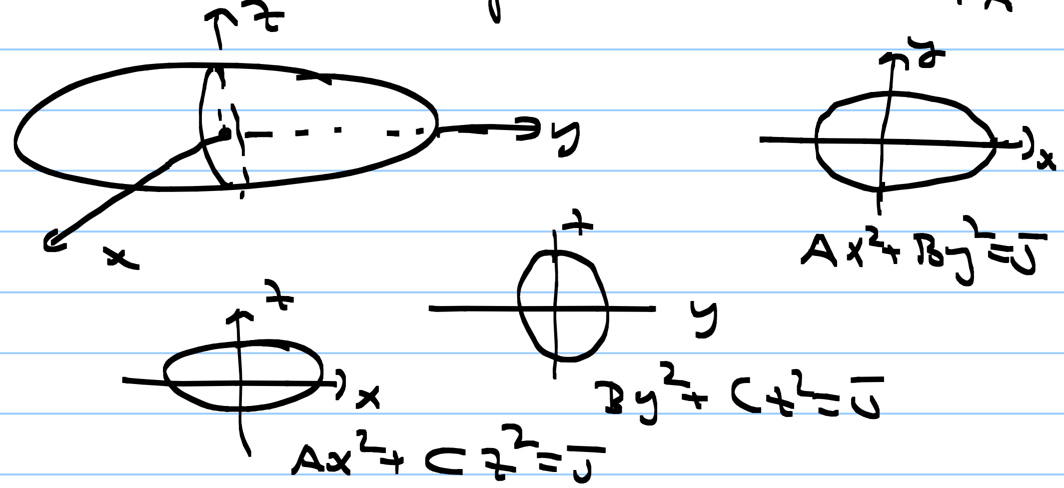
for some constants $A, B, \ldots, J \in \mathbb{R}$.

If $A = B = C = D = E = F = 0$ then the equation is a degenerate quadric, which is
$Gx + Hy + Iz = J$, a linear equation.

That is an equation for a plane.
We'll consider the case when the equation is <u>not</u> linear.

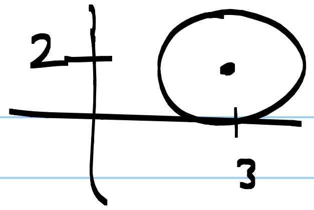
<u>Ellipsoids</u> $Ax^2 + By^2 + Cz^2 = J$

If $A = B = C$ then this is the sphere with center at the origin and radius $\sqrt{J/A}$



This is called an ellipsoid.

The center may be shifted to the point (x_0, y_0, z_0). Then the equation becomes:

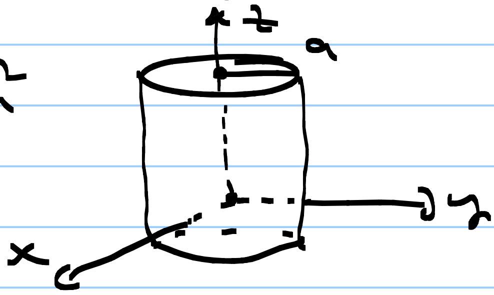


$(x-3)^2+(y-2)^2=R^2$

So the equation of a shifted ellipsoid is

$$A(x-x_0)^2+B(y-y_0)^2+C(z-z_0)^2=J.$$

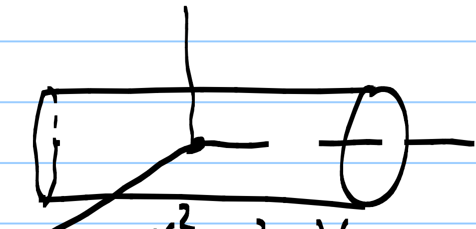
$$Ax^2+By^2+Cz^2-2Ax_0x-2By_0y-2Cz_0z$$
$$=J-Ax_0^2-By_0^2-Cz_0^2.$$

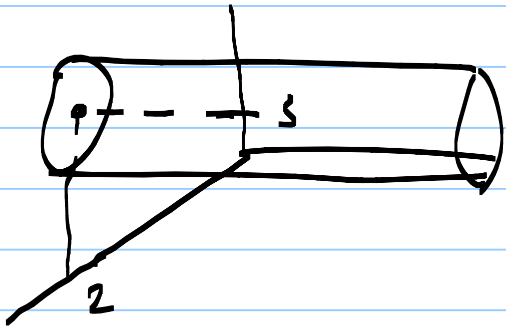


__Cylinders:__ $x^2+y^2=a^2$

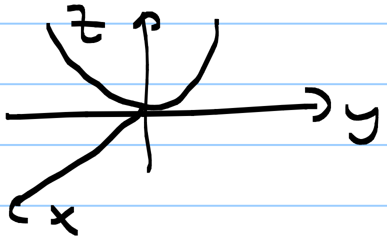
$(x-2)^2+(z-3)^2=16$

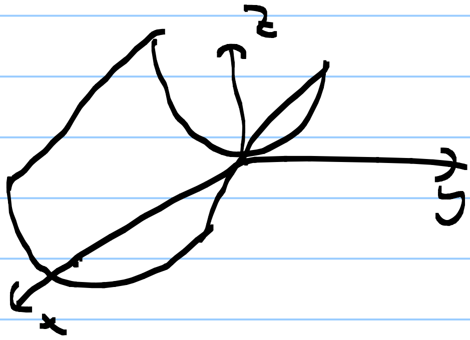
$r=4$





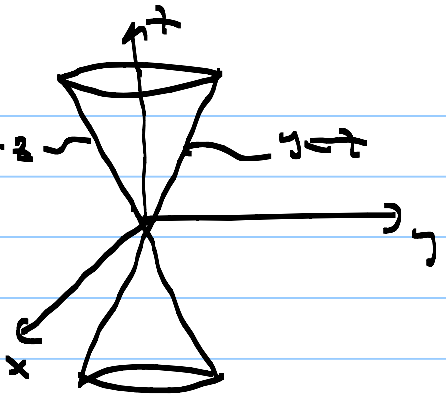
__Parabolic cylinders:__ $z=y^2$





Cones:

$$x^2+y^2=z^2$$



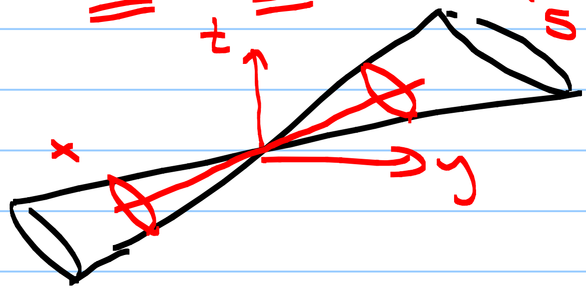
$x=0 \Rightarrow y^2=z^2$

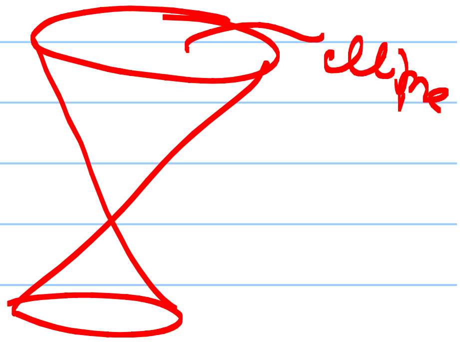
$y=\pm z$

yz-plane

The intersection of the surface $x^2+y^2=z^2$ with the yz-plane, which is given by the equation $x=0$ is obtained by plugging $x=0$ in the equation $x^2+y^2=z^2$.

Ex $\quad 4y^2+9z^2=x^2$





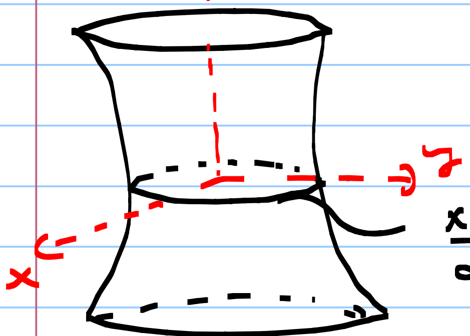
Elliptic Cone

Hyperboloids:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\underbrace{\frac{z^2}{c^2}}_{\geq 0}$$

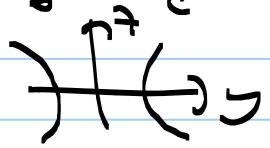
Hyperboloid of one sheet.



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

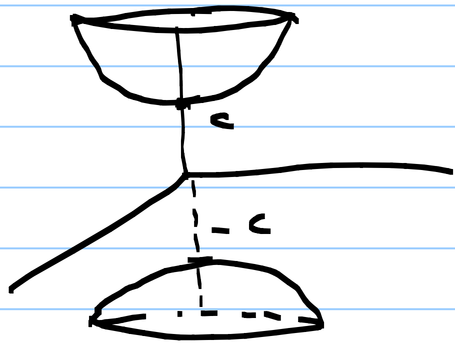
$x=0$

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

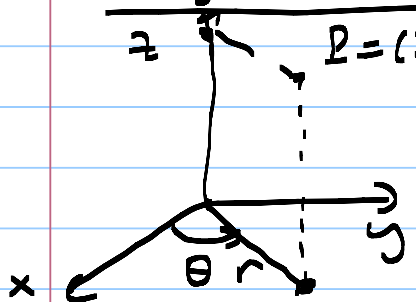


Hyperboloid of two sheets:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1 \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}-1$$



10.6. Cylindrical and Spherical Coordinates



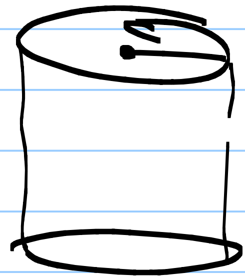
$P=(x,y,z) \Longleftrightarrow (r,\theta,z)$

$Q=(x,y,0)$

$r=\sqrt{x^2+y^2}$

$\tan\theta = y/x$

$r=c$

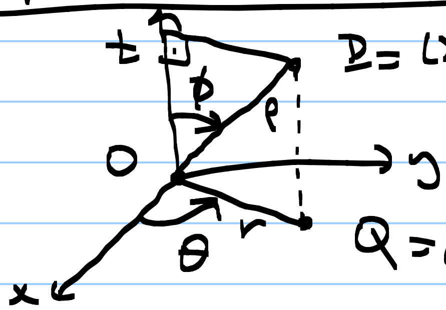


(r,θ,z) is called the cylindrical coordinate of P.

$x=r\cos\theta,\ y=r\sin\theta,\ z=z$

Spherical Coordinates:

(ρ,ϕ,θ) spherical coor.



$P=(x,y,z)$

$Q=(x,y,0)$

$\rho=|PO|$
$=\sqrt{x^2+y^2+z^2}$

$z=\rho\cos\phi$
$r=\rho\sin\phi$

$x=r\cos\theta=\rho\sin\phi\cos\theta$
$y=r\sin\theta=\rho\sin\phi\sin\theta$

$z=\rho\cos\phi$

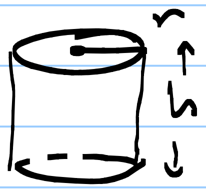
Video 2P

CHAPTER 12. Partial Differentiation

§12.1. Functions of Several Variables

$f: \mathbb{R} \to \mathbb{R}$, $f: \mathbb{R}^n \to \mathbb{R}$, $f: \mathbb{R}^2 \to \mathbb{R}$, $f: \mathbb{R}^3 \to \mathbb{R}$

Ex



Volume of the cylinder

$V = V(h,r) = \pi r^2 h$

$V: \mathbb{R}^2 \to \mathbb{R}$

Definition: A function f of n variables is a rule that assigns a unique number $f(x_1, x_2, \dots, x_n)$ to each point $(x_1, x_2, \dots, x_n)$ in some subset $D(f)$ of $\mathbb{R}^n$. $D(f)$ is called the domain of f. The set of all $f(x_1, x_2, \dots, x_n)$, where $(x_1, x_2, \dots, x_n) \in D(f)$, is called the range of f.

$f: D(f) \to \mathbb{R}$, $f(D(f))$ the range of f.

Example For the above volume function V we may choose $D(V) = \{(h,r) \mid h, r \in [0, \infty)\}$.

$V: D(V) \to \mathbb{R}$, $V(h,r) = \pi r^2 h$, Range of $f = [0, \infty)$.

Domain of Convention of a function is the set of all points so that the function is defined at those points.

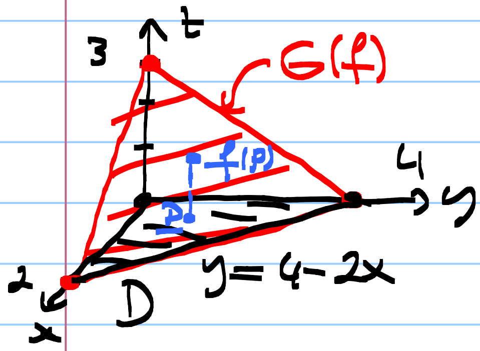
For example, for the rule $f(x,y) = \dfrac{1}{x^2+y^2}$

the domain of convention is $\mathbb{R}^2 \setminus \{(0,0)\}$.

Graphs: If $f: \mathbb{R}^n \to \mathbb{R}$ is a function the subset

$\{(x, f(x)) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n\}$ is called the graph of f.

Ex: let $f: D \to \mathbb{R}$, where $f(x,y) = 3(1 - \frac{x}{2} - \frac{y}{4})$, and $D = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 4-2x\}$. Draw a picture of its graph.



$G(f) = \{(x,y,z) \mid (x,y) \in D,\ z = f(x,y)\}$.

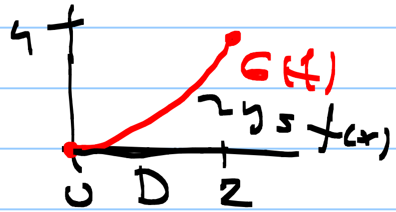
$z = 3(1 - \frac{x}{2} - \frac{y}{4})$

$f(2,0) = 3(1 - \frac{2}{2} - \frac{0}{4}) = 0$, $f(0,0) = 3$, $f(0,4) = 0$

Ex

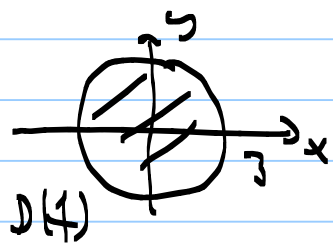
$f: \underset{D}{\underset{\shortparallel}{[0,2]}} \to \mathbb{R}$, $f(x) = x^2$

$y = f(x) = x^2$



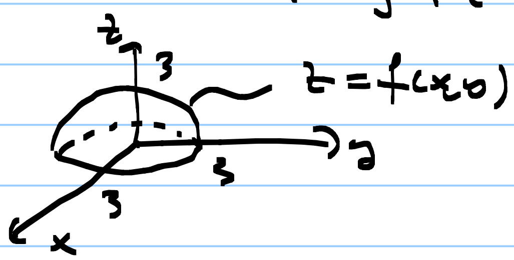
Example $f(x,y) = \sqrt{9 - x^2 - y^2}$

$D(f) = \{(x,y) \in \mathbb{R}^2 \mid 9 - x^2 - y^2 \ge 0\}$
$= \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 9\}$

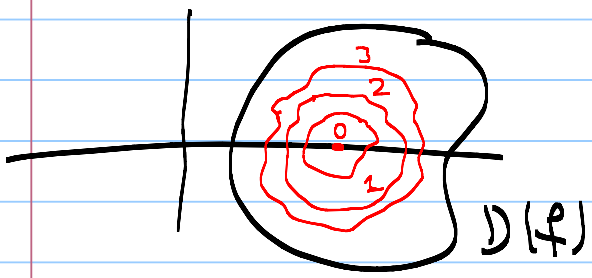


$z = f(x,y) = \sqrt{9 - x^2 - y^2} \Rightarrow z^2 = 9 - x^2 - y^2$
$\Rightarrow x^2 + y^2 + z^2 = 9 = 3^2$

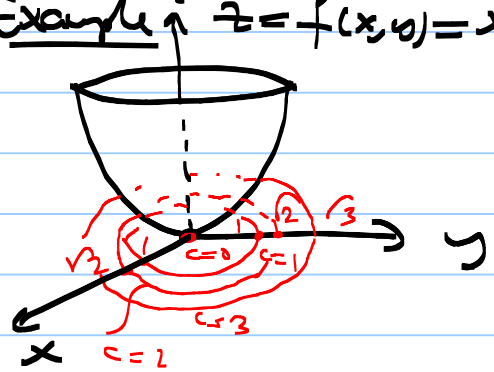
$z = f(x,y) \ge 0$

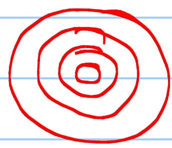


Level Curves: If $z=f(x,y)$ and $c\in\mathbb{R}$, the set of all pairs $(x,y)\in D(f)$ for which $f(x,y)=c$ is the level curve of f for $f=c$.



Example: $z=f(x,y)=x^2+y^2$

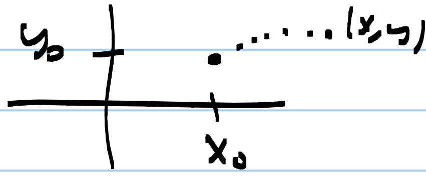




§12.2. Limits and Continuity:

A function $f:D\to\mathbb{R}$, $D\subseteq\mathbb{R}^n$ (for time being we'll take $D\subseteq\mathbb{R}^2$) is called continuous at a point $(x_0,y_0)\in D$, if f takes its expected value (limit value) at (x_0,y_0):

$$f(x_0,y_0)=\lim_{(x,y)\to(x_0,y_0)} f(x,y).$$



Definition: We say that $\lim\limits_{(x,y)\to(a,b)} f(x,y)=L$, provided that

i) every neighborhood of (a,b) contains points of domain of f different from (a,b), and

ii) for every positive number $\epsilon > 0$ there exists a positive number $\delta = \delta(\epsilon)$ such that $|f(x,y) - L| < \epsilon$ holds whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta = \delta(\epsilon), \text{ and } (x,y) \in D(f).$$

<u>Example:</u> $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 3x - 2y$.

Show that f is continuous at any point.

<u>Solution:</u> must show. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Fix some $(a,b) \in \mathbb{R}^2$. Let $L = f(a,b) = 3a - 2b$.
Given $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{6} > 0$.

Now $\exists$ $(x,y) \in \mathbb{R}^2$ is so that
$\sqrt{(x-a)^2 + (y-b)^2} < \epsilon/6 = \delta$, then

$$(x-a)^2 + (y-b)^2 < \frac{\epsilon^2}{36}$$

$$\Rightarrow (x-a)^2 < \frac{\epsilon^2}{36} \text{ and } (y-b)^2 < \frac{\epsilon^2}{36}.$$

$$|x-a| < \frac{\epsilon}{6} \text{ and } |y-b| < \frac{\epsilon}{6}.$$

Hence
$$\begin{aligned} |f(x,y) - f(a,b)| &= |3x - 2y - (3a - 2b)| \\ &= |3(x-a) - 2(y-b)| \\ &\leq 3|x-a| + 2|y-b| \\ &< 3\,\epsilon/6 + 2\cdot \epsilon/6 \end{aligned}$$

Video 29

$$\Rightarrow |f(x,y)-f(a,b)| < \frac{\epsilon}{2}+\frac{\epsilon}{3} < \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

Hence, $\lim_{(x,y)\to(a,b)} f(x,y) = L = 3a-2b = f(a,b)$.

In particular, f is continuous at (a,b).
Hence f is continuous at all points.

Some Useful Rules for evaluating limits:

Let f and g be functions so that

$\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{(x,y)\to(a,b)} g(x,y) = M$ both exist.

Then

1) $\lim_{(x,y)\to(a,b)} f(x,y) \mp g(x,y)$ exists and equals $L \mp M$.

$$\left(\lim_{(x,y)\to(a,b)} f(x,y)+g(x,y) = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y) = L+M\right)$$

2) $\lim_{(x,y)\to(a,b)} f(x,y)\cdot g(x,y) = \lim_{(x,y)\to(a,b)} f(x,y) \cdot \lim_{(x,y)\to(a,b)} g(x,y) = L\cdot M$

3) $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)} = L/M$, provided that the expressions in the denominator are non zero.

4) If $F:\mathbb{R}\to\mathbb{R}$ is continuous at $x=L$ then

$$\lim_{(x,y)\to(a,b)} F(f(x,y)) = F\left(\lim_{(x,y)\to(a,b)} f(x,y)\right) = F(L).$$

<u>Examples:</u> $\lim_{(x,y)\to(a,b)} x = a$, $\lim_{(x,y)\to(a,b)} y = b$.

$$\lim_{(x,y)\to(a,b)} \lambda = \lambda.$$

$\lambda \in \mathbb{R}$ fixed real number

$$\lim_{(x,y)\to(a,b)} x.y = \left(\lim_{(x,y)\to(a,b)} x\right)\left(\lim_{(x,y)\to(a,b)} y\right) = a.b$$

$$\lim_{(x,y)\to(3,-1)} \left(3x^2y - 5x^3y^2 + \frac{x}{y^2}\right) = 3.3^2(-1) - 5\,3^3(-1)^2 + \frac{3}{(-1)^2}$$

$$= -27 - 135 + 3$$

$$= -159.$$

$\lim_{(x,y)\to(3,-1)} x = 3$, $\lim_{(x,y)\to(3,-1)} y = -1$

<u>Example:</u> $\lim_{(x,y)\to(\pi/3,2)} y\sin\left(\frac{x}{y}\right) = 2.\sin\left(\frac{\pi/3}{2}\right)$

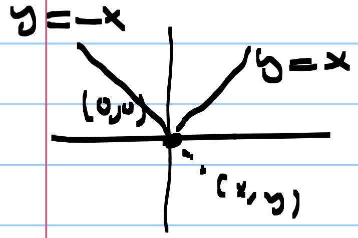
$$= 2\sin(\pi/6)$$

$= 2.\frac{1}{2} = 1$, since $\sin x$ is continuous.

<u>Example</u> Investigate if the function $f(x,y) = \frac{2xy}{x^2+y^2}$ has limit as (x,y) approaches to $(0,0)$.

<u>Solution:</u> Note that since $x^2+y^2=0$ at $(0,0)$ the above rule for ratios does not apply.

$$f(x,y) = \frac{2xy}{x^2+y^2}$$



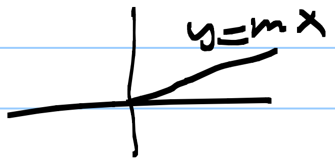
$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{2x^2}{2x^2} = 1.$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=-x}} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{-2x^2}{2x^2} = -1$$

So the limit $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ does <u>not</u> exist.

In fact if we approach to $(0,0)$ along the line $y=mx$ then we get

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{2mx^2}{x^2+m^2x^2}$$



$$= \lim_{x\to 0} \frac{2m}{1+m^2}$$

$$= 2m/1+m^2.$$

$$z = \frac{2xy}{x^2+y^2}$$

<u>Example:</u> Determine if the function $f(x,y) = \frac{2x^2y}{x^4+y^2}$ has limit as $(x,y)\to(0,0)$.

<u>Solution:</u> $y=mx$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{2x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{2x^3m}{x^4+m^2x^2}$$

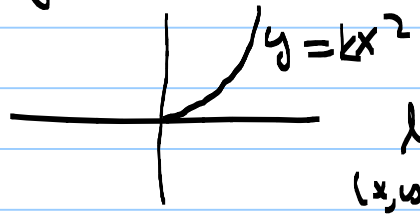
Video 30

$$= \lim_{x \to 0} \frac{2xm}{x^2 + m^2}$$

$$= 0 \quad \text{if } m \neq 0$$

Indeed, it is zero if $m = 0$ also.

Hence, limit exist as we approach to $(0,0)$ along any line $y = mx$.



$$\lim_{\substack{(x,y) \to (0,0) \\ y = kx^2}} \frac{2x^2y}{x^4 + y^2} = \lim_{x \to 0} \frac{2x^4k}{x^4 + k^2x^4}$$

$$= \lim_{x \to 0} \frac{2k}{1 + k^2}$$

$$= \frac{2k}{1 + k^2}$$

Hence, different k values give different limit values so that $\lim_{(x,y) \to (0,0)} f(x,y)$ does not exist.

<u>Example</u> Show that the function $f(x,y) = \frac{x^2y}{x^2 + y^2}$ does have a limit at the origin. (Note that $f(x,y)$ is continuous at any point $(x,y) \neq (0,0)$.)

<u>Solution:</u> $|f(x,y)| = \left| \frac{x^2y}{x^2 + y^2} \right| = \underbrace{\left| \frac{x^2}{x^2 + y^2} \right|}_{\leq 1} |y| \leq |y|$

$$-|y| \leq f(x,y) \leq |y| \quad \text{for any } (x,y) \neq (0,0).$$

$(x,y) \to (0,0)$: $-|y| \to 0$, $f(x,y) \to 0$, $|y| \to 0$

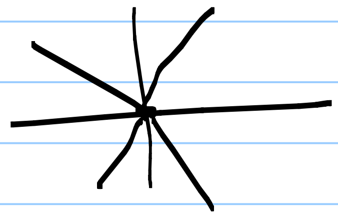
So by the Squeezing Theorem $\lim_{(x,y)\to(0,0)} f(x,y)=0$. In particular, f has limit 0 at $(0,0)$.

If we let $\tilde{f}(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$.

Then $\tilde{f}:\mathbb{R}^2\to\mathbb{R}^2$ is continuous at all points.

__Ex__ Show that $\lim_{(x,y)\to(0,0)} \dfrac{\sin xy}{x^2+y^2}$ does not exist.

__Solution__ $\lim\limits_{\substack{(x,y)\to(0,0)\\ y=mx}} \dfrac{\sin xy}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{\sin mx^2}{x^2+m^2x^2}$



$$= \lim_{x\to 0} \frac{\sin mx^2}{mx^2}\,\frac{m}{(1+m^2)}$$

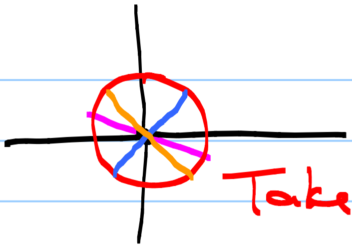
$$= 1\cdot\frac{m}{1+m^2} = \frac{m}{1+m^2}$$

Hence, $\lim_{(x,y)\to(0,0)} \dfrac{\sin xy}{x^2+y^2}$ does __not__ exist.

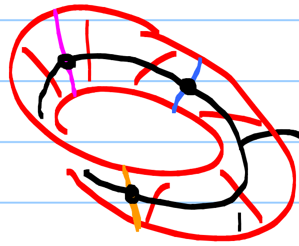
__Remark.__ Consider the limit of $f(x,y)=\dfrac{xy}{x^2+y^2}$

$\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$. If we approach to $(0,0)$ along $y=mx$ we get

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$$



Take out this disc and replace it by a Möbius Band



In algebraic geometry this process is called blowing up the plane.

$f(x,y)$ has singularity at $(0,0)$.

§12.3 Partial Derivatives:

Definition. The first partial derivatives of the function $f(x,y)$ with respect to the variables x and y are the functions $f_1(x,y)$ and $f_2(x,y)$ given by

$$f_1(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}, \text{ and}$$

$$f_2(x,y) = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}, \text{ provided that}$$

these limits exist.

Remark. Some books prefer f_x for f_1 and f_y for f_2.

Example Let $f(x,y) = x^2 \sin y$

$$f_1(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$. Let $\varphi(x) = x^2 \sin y$ for a fixed y.

$$= \lim_{h\to 0} \frac{\varphi(x+h) - \varphi(x)}{h}$$

$$= \varphi'(x)$$

So, $f_1(x,y) = 2x \sin y$ and similarly,

$f_2(x,y) = x^2 \cos y$.

Examples $f(x,y) = \underline{x^2 y} - \underline{\underline{e^x}} + \frac{xy}{1+y}$

$$f_1(x,y) = 2xy - e^x + \frac{y}{1+y}$$

$$f_2(x,y) = x^2 - 0 + x\left(\frac{y}{1+y}\right)'$$

$$= x^2 + x\,\frac{1\cdot(1+y) - y\cdot 1}{(1+y)^2}$$

$$= x^2 + x\,\frac{1}{(1+y)^2} = x^2 + \frac{x}{(1+y)^2}$$

More Notation: $z = f(x,y)$

$$\frac{\partial z}{\partial x} = f_x = f_1 = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f = D_1 f(x,y)$$

$$\frac{\partial z}{\partial y} = f_y = f_2 = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f = D_2 f(x,y).$$

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function and $g: \mathbb{R}^2 \to \mathbb{R}$ is any function. Then

$$f \circ g = f(g) : \mathbb{R}^2 \to \mathbb{R}, \quad (f \circ g)(x,y) = f(g(x,y))$$

Chain Rule $\quad h(x,y) = f(g(x,y))$

$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x} f(g(x,y)) = f'(g(x,y)) \cdot \frac{\partial g}{\partial x}$$

$$\frac{\partial h}{\partial y} = \frac{\partial}{\partial y} f(g(x,y)) = f'(g(x,y)) \cdot \frac{\partial g}{\partial y}$$

Example: $f(x) = x e^x$, $g(x,y) = x + 3y^2$

$$\frac{\partial}{\partial x}(f(g(x,y)) = \frac{\partial}{\partial x}\left((x+3y^2)\, e^{x+3y^2}\right)$$

or $$= f'(g(x,y)) \frac{\partial g}{\partial x}$$

$f(x) = x e^x$, $f'(x) = e^x + x e^x$, $\frac{\partial g}{\partial x} = 1$.

So $$\frac{\partial}{\partial x} f(g(x,y)) = f'(g(x,y)) \cdot 1$$
$$= e^{g(x,y)} + g(x,y)\, e^{g(x,y)}$$
$$= e^{x+3y^2} + (x+3y^2)\, e^{x+3y^2}$$

Exercise: Compute $\frac{\partial}{\partial y} f(g(x,y))$.

Video 31

Example Find $f_1(0,\pi)$ if $f(x,y) = e^{xy}\cos(x+y)$.

Solution: $f_1(x,y) = y e^{xy}\cos(x+y) + e^{xy}\cdot(-\sin(x+y)\cdot 1)$
$= y e^{xy}\cos(x+y) - e^{xy}\sin(x+y)$.

$$f_1(0,\pi) = \pi e^{0\cdot\pi}\cos(0+\pi) - e^{0\cdot\pi}\sin(0+\pi)$$
$$= \pi(-1) - 1\cdot 0$$
$$= -\pi.$$

Example: Suppose that f is a everywhere differentiable function of a single variable. Show that $z = f(x/y)$ satisfies the following "partial differential equation"

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0. \qquad z = g(x,y)\ ?$$

Solution: $z = f(x/y)$, $\dfrac{\partial z}{\partial x} = \dfrac{\partial}{\partial x}\left(f\left(\dfrac{x}{y}\right)\right)$

$$= f'\left(\frac{x}{y}\right)\cdot\frac{1}{y} = \frac{1}{y}f'(x/y).$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(f\left(\frac{x}{y}\right)\right)$$
$$= f'\left(\frac{x}{y}\right)\cdot\frac{-x}{y^2}$$
$$= \frac{-x}{y^2}f'\left(\frac{x}{y}\right)$$

Now, $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = x\cdot\dfrac{1}{y}f'\left(\dfrac{x}{y}\right) + y\cdot\left(\dfrac{-x}{y^2}f'\left(\dfrac{x}{y}\right)\right)$

$$= \frac{x}{y}f'\left(\frac{x}{y}\right) - \frac{x}{y}f'\left(\frac{x}{y}\right) = 0.$$

Clearly, we may talk about partial derivatives of functions of 3 or more variables.

Example: Let $f(x,y,z) = \dfrac{2xy}{1+xz+yz}$

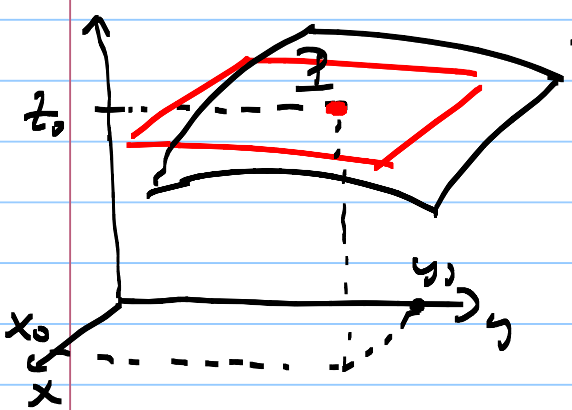
$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(\frac{2xy}{1+xz+yz}\right) = 2xy \cdot \frac{\partial}{\partial z}\left(\frac{1}{1+xz+yz}\right)$$

$$= 2xy\left(\frac{0\cdot(1+xz+yz) - 1\cdot(x+y)}{(1+xz+yz)^2}\right)$$

$$= \frac{-2xy(x+y)}{(1+xz+yz)^2}.$$

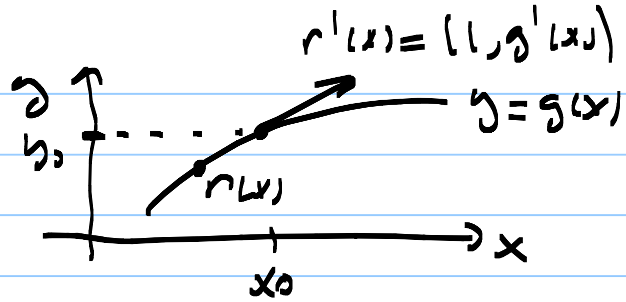
Tangent Planes and Normal Lines

Consider the graph of a function $z=f(x,y)$. We would like to find an equation for the tangent plane to the graph of $z=f(x,y)$ at a point say (x_0,y_0,z_0).



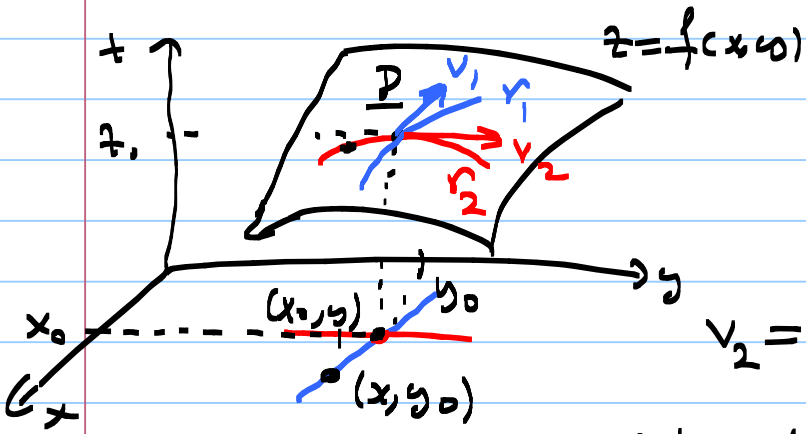
$z=f(x,y)$ $\quad P=(x_0,y_0,z_0)$

First recall that a tangent vector to the graph of a function $y=g(x)$ is given by $v=(1,g'(x))$.



$r(x) = (x, g(x))$

$r'(x) = (1, g'(x))$



$r_1(x) = (x, y_0, f(x, y_0))$

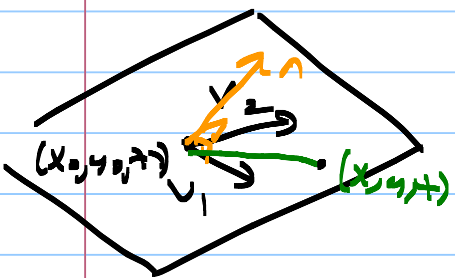
$r_1'(x) = (1, 0, f_1(x, y_0))$

$v_2 = r_1'(x_0) = (1, 0, f_1(x_0, y_0)).$

$r_2(y) = (x_0, y, f(x_0, y))$

$v_1 = r_2'(y) = (0, 1, f_2(x_0, y))$

$r_2'(y_0) = (0, 1, f_2(x_0, y_0))$



Let $n = v_1 \times v_2$, then $n \perp v_1$ and $n \perp v_2$ and, hence n is a normal vector to the tangent plane

$$n = \begin{vmatrix} i & j & k \\ 0 & 1 & f_2(x_0,y_0) \\ 1 & 0 & f_1(x_0,y_0) \end{vmatrix} = (f_1(x_0,y_0), f_2(x_0,y_0), -1)$$

Equation of the tangent plane:

$$(x-x_0, y-y_0, z-z_0) \cdot n = 0$$

$$(x-x_0, y-y_0, z-z_0) \cdot (f_1(x_0,y_0), f_2(x_0,y_0), -1) = 0$$

$$(x-x_0) f_1(x_0,y_0) + (y-y_0) f_2(x_0,y_0) - (z-z_0) = 0.$$

$$\Rightarrow \boxed{x f_1(x_0,y_0) + y f_2(x_0,y_0) - z = x_0 f_1(x_0,y_0) + y_0 f_2(x_0,y_0) - z_0.}$$

Example Determine an equation for the tangent plane to the graph of $z = \sin(xy)$ at the point $x = \pi/3$, $y = -1$, $z = \sin(-\pi/3) = -\sqrt{3}/2$.

Solution $P = (x_0, y_0, z_0) = (\pi/3, -1, -\sqrt{3}/2)$

$$z = f(x,y) = \sin(xy).$$

$$f_1(x,y) = y\cos(xy) \text{ and } f_2(x,y) = x\cos(xy).$$

$$f_1(x_0,y_0) = -1 \cdot \cos(-\pi/3) = -1/2, \quad f_2(x_0,y_0) = \pi/3 \cos(-\pi/3) = \pi/6.$$

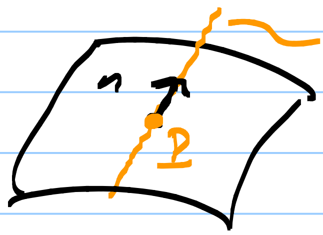
$$x f_1(x_0,y_0) + y f_2(x_0,y_0) - z = x_0 f_1(x_0,y_0) + y_0 f_2(x_0,y_0) - z_0$$

$$-\frac{x}{2} + \frac{\pi}{6} y - z = \frac{\pi}{3}\left(-\frac{1}{2}\right) + (-1)\cdot\left(\frac{\pi}{6}\right) - \left(-\frac{\sqrt{3}}{2}\right)$$

$$-\frac{x}{2} + \frac{\pi}{6} y - z = -\pi/6 - \pi/6 + \frac{\sqrt{3}}{2}$$

$$\frac{x}{2} - \frac{\pi}{6} y + z = \pi/3 - \frac{\sqrt{3}}{2}.$$

Normal Line to the Graph:



normal line to the graph at the point $P = (x_0, y_0, z_0)$.

$$n = (f_1(x_0,y_0), f_2(x_0,y_0), -1) = (A, B, C)$$

$$\frac{x-x_0}{A} = \frac{y-y_0}{B} = \frac{z-z_0}{C} \Rightarrow \frac{x-x_0}{f_1(x_0,y_0)} = \frac{y-y_0}{f_2(x_0,y_0)} = \frac{z-z_0}{-1} = t$$

Video 32

Or: $x - x_0 = t f_1(x_0,y_0),\ y - y_0 = t f_2(x_0,y_0),\ z - z_0 = -t$

$$\Rightarrow (x,y,z) = (x_0, y_0, z_0) + t(f_1(x_0,y_0), f_2(x_0,y_0), -1).$$

is a parametric equation for the normal line to the graph of $z = f(x,y)$ at the point (x_0,y_0,z_0).

Example: Let's write down an equation for the normal line to the graph of $z = \sin(xy)$ at the point $(\pi/3, -1, -\sqrt{3}/2)$.

Solution: $n = (-1/2, \pi/6, -1)$.

$$\frac{x-x_0}{-1/2} = \frac{y-y_0}{\pi/6} = \frac{z-z_0}{-1} \quad \text{or} \quad \frac{x-\pi/3}{-1/2} = \frac{y+1}{\pi/6} = \frac{z+\sqrt{3}/2}{-1}$$

Or, $(x,y,z) = (x_0,y_0,z_0) + tn$
$= (\pi/3, -1, -\sqrt{3}/2) + t(-1/2, \pi/6, -1)$.

Example: Which horizontal plane is tangent to the surface $z = x^2 - 4xy - 2y^2 + 12x - 12y - 1$.

Solution $z = f(x,y) = x^2 - 4xy - 2y^2 + 12x - 12y - 1$.
$16 + 16 - 2 - 48 - 12 - 1$

$n = (f_1(x,y), f_2(x,y), -1) \parallel z\text{-axis} \parallel (0,0,1)$

So we need to find (x_0,y_0,z_0) on the surface so that

$$f_1(x_0,y_0) = f_2(x_0,y_0) = 0.$$

$f_1(x,y) = 2x - 4y + 12$, $f_2(x,y) = -4x - 4y - 12$.

So, we have

$$\begin{array}{r} 2x - 4y + 12 = 0 \\ - \quad -4x - 4y - 12 = 0 \\ \hline 6x + 0 \cdot y + 24 = 0 \end{array} \Rightarrow 6x = -24, \Rightarrow x = -4$$

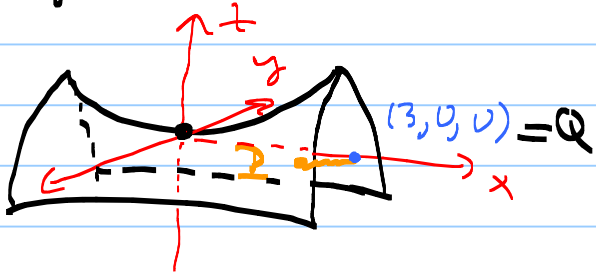
$$\Rightarrow 2x - 4y + 12 = 0 \Rightarrow y = \frac{2x+12}{4} = 1.$$

So $(x_0, y_0) = (-4, 1)$, $z_0 = f(x_0, y_0) = -31$.

Hence, the only point on the graph of $z = f(x,y)$ at which the tangent plane is horizontal is $(x_0, y_0, z_0) = (-4, 1, -31)$.

<u>Example:</u> Find the distance from the point $(3,0,0)$ to the hyperbolic paraboloid with equation $z = x^2 - y^2$.

<u>Solution</u>



$P = (x_0, y_0, z_0)$.

If $P = (x_0, y_0, z_0)$ is the closest point on the graph of $z = x^2 - y^2$ to the point $(3,0,0)$ then the line segment PQ would be normal to the surface.

So $\overrightarrow{PQ}$ should be parallel to the normal vector at P.

The normal vector to the surface at P is $n = (f_1(x_0, y_0), f_2(x_0, y_0), -1)$, $f(x,y) = x^2 - y^2$.

$n = (2x, -2y, -1)\Big|_{(x_0, y_0, z_0)} = (2x_0, -2y_0, -1).$

$\overrightarrow{PQ} = (x_0 - 3, y_0, z_0).$

$(2x_0, -2y_0, -1) \,//\, (x_0 - 3, y_0, z_0).$

Also we have $z_0 = x_0^2 - y_0^2$.

Assume that $(x_0 - 3, y_0, z_0) = \lambda(2x_0, -2y_0, -1)$.

$\Rightarrow x_0 - 3 = 2\lambda x_0, \quad y_0 = -2\lambda y_0, \quad z_0 = -\lambda.$

Assume $y_0 \neq 0$, then $-2\lambda = 1 \Rightarrow \lambda = -1/2$.

Then $x_0 - 3 = 2(-1/2)x_0 \Rightarrow 2x_0 = 3 \Rightarrow x_0 = 3/2$.

$z_0 = -\lambda = 1/2$. Finally, $z_0 = x_0^2 - y_0^2 \Rightarrow$

$1/2 = \frac{9}{4} - y_0^2 \Rightarrow y_0^2 = \frac{9}{4} - \frac{1}{2} = 7/4$

$\Rightarrow y_0 = \pm\sqrt{7}/2.$

So the candidates are $(x_0, y_0, z_0) = \left(\frac{3}{2}, \pm\frac{\sqrt{7}}{2}, 1/2\right)$

and the points when $y_0 = 0$ should be also considered.

$y_0 = 0, \quad z_0 = x_0^2, \quad (x_0 - 3, 0, z_0) \,//\, (2x_0, 0, -1).$

$x_0 - 3 = \lambda 2x_0, \quad z_0 = -\lambda$

$\Rightarrow (1 - 2\lambda)x_0 = 3, \quad z_0 = -\lambda$

$x_0 = 3/1-2\lambda$

$$z_0 = x_0^2 \Rightarrow -\lambda = \frac{9}{(1-2\lambda)^2}$$

$$\lambda(1-2\lambda)^2 = -9 \Rightarrow (4\lambda^2 - 4\lambda + 1)\lambda = -9$$

$$\Rightarrow 4\lambda^3 - 4\lambda^2 + \lambda + 9 = 0$$

Note that $\underline{\lambda = -1}$ is a solution.

$$0 = 4\lambda^3 - 4\lambda^2 + \lambda + 9 = (\lambda+1)\overbrace{(4\lambda^2 - 8\lambda + 9)}^{>0}$$

$$\underbrace{(2\lambda - 2)^2 + 1}_{} \neq 0$$

Moreover, $\lambda = -1$ is the only real solution to the equations.

$\lambda = -1$ gives us the point $(1, 0, 1)$.

So we have 3 candidates $\left(\frac{3}{2}, \pm\frac{\sqrt{7}}{2}, 1/2\right)$ and $(1, 0, 1)$. $Q = (3, 0, 0)$

$P = \left(3/2, \frac{\sqrt{7}}{2}, 1/2\right)$, $|PQ| = \left|\left(3/2, -\frac{\sqrt{7}}{2}, 1/2\right)\right| = \frac{\sqrt{17}}{2}$

$P = \left(3/2, -\frac{\sqrt{7}}{2}, 1/2\right)$, $|PQ| = \left|\left(3/2, \sqrt{7}/2, 1/2\right)\right| = \frac{\sqrt{17}}{2}$.

$P = (1, 0, 1)$, $|PQ| = |(2, 0, -1)| = \sqrt{5}$

So the closest points on the surface to the point $Q = (3, 0, 0)$ are $\left(3/2, \pm\frac{\sqrt{7}}{2}, 1/2\right)$.

Video33

§12.4. Higher-Order Derivatives:

If $z = f(x,y)$ is a function we may take its partial derivatives as many times as we want, provided that the derivatives exist.

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_1(x,y), \quad \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_2(x,y).$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(f_1(x,y)\right)$$

$$\frac{\partial z}{\partial x} = z_x$$

$$\frac{\partial^2 z}{\partial x^2} = f_{11}(x,y) \qquad \frac{\partial^2 z}{\partial x^2} = z_{xx}$$

OR, $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(f_1(x,y)\right)$ $\qquad \frac{\partial^2 z}{\partial x \partial y} = z_{xy}$

$$\frac{\partial^2 z}{\partial y \partial x} = f_{12}(x,y)$$

Note that we may consider derivatives such as

$$\frac{\partial^5 z}{\partial x^2 \partial y^3} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right)\right)\right) = z_{xxyyy}$$

Example: $f(x,y) = x^3 y^4$

$f_1(x,y) = 3x^2y^4$, $f_{11}(x,y) = 6xy^4$, $f_{21}(x,y) = 12x^2y^3$

$f_2(x,y) = 4x^3y^3$, $f_{22}(x,y) = 12x^3y^2$, $f_{12}(x,y) = 24xy^3$

$f_{12}(x,y) = 12x^2y^3$. Note that $f_{12} = f_{21} = 12x^2y^3$

Moreover, one can check that for this function $f_{1211221} = f_{1111222}$.

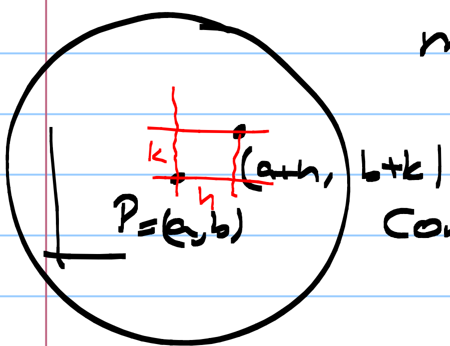
<u>Remark</u>: In general mixed partial derivatives may not be equal.

<u>Theorem</u>: (Equality of Mixed Partials)

Suppose that two mixed n^{th} order partial derivatives of a function f involve the same differentiations but in different orders. If those partial derivatives are continuous at a point P, and if f and all partials of f of order less than n are continuous in a neighborhood of P, then the two mixed partials are equal at the point P.

<u>Proof</u>: Will prove this in the special case, where we show $f_{12} = f_{21}$ under the assumption that $f, f_1, f_2, f_{12}, f_{21}, f_{11}, f_{22}$ are all continuous at the point P and f, f_1 and f_2 are continuous on a disc around P.

must show $f_{12}(P) = f_{21}(P)$.



Consider the expression

$Q = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b)$ and define then the single variable functions

$u(x) = f(x, b+k) - f(x,b)$ and $v(y) = f(a+h, y) - f(a,y)$.

Then $Q = u(a+h) - u(a) = v(b+k) - v(b)$.

Apply the Mean Value Theorem to $u(x)$.

Then there is some $\theta_1 \in (0,1)$ so that

$$\frac{u(a+h) - u(a)}{a+h-a} = u'(a+\theta_1 h)$$

$$Q = u(a+h) - u(a) = h u'(a+\theta_1 h)$$
$$= h\left[f_1(a+\theta_1 h, b+k) - f_1(a+\theta_1 h, b)\right]$$

Now apply M.V.T. to f_1 considered as function of its second variable and obtain $\theta_2 \in (0,1)$ so that

$$f_1(a+\theta_1 h, b+k) - f_1(a+\theta_1 h, b) = f_{21}(a+\theta_1 h, b+\theta_2 k)\, k$$

Then, $Q = h \cdot k \cdot f_{21}(a+\theta_1 h, b+\theta_2 k)$.

A similar consideration would give us

$$Q = kh \cdot f_{12}(a+\theta_1' h, b+\theta_2' k)$$

So we have

$$\cancel{kh}\, f_{12}(a+\theta_1' h, b+\theta_2' k) = \cancel{hk}\, f_{21}(a+\theta_1 h, b+\theta_2 k)$$

for all small enough $|h|, |k| > 0$. Then

$$f_{12}(a+\theta_1' h, b+\theta_2' k) = f_{21}(a+\theta_1 h, b+\theta_2 k)$$

Video 34

for all small enough $|h|, |k| > 0$.
Using the continuity assumption take limits as $h, k \to 0$ to obtain

$$f_{12}(a,b) = f_{21}(a,b).$$

§12.5. Chain Rule:

We already know that $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$,

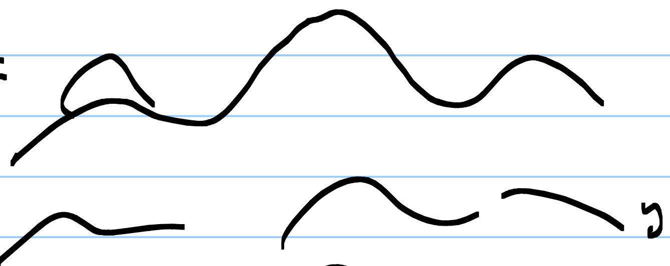
if both f and g differentiable single variable functions.

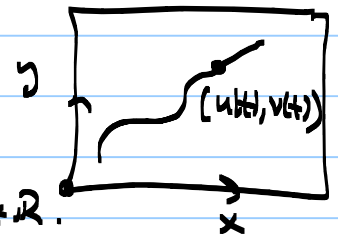
Example: Suppose you are hiking in a mountainous region for which you have a map. Let (x,y) be the coordinates of your position on the map. Let $z = f(x,y)$ denote the height of land (above the sea level) at position (x,y). Suppose you are walking on a trail so that your position at time t is given by $x = u(t)$ and $y = v(t)$, for some functions. Then at time t your altitude above sea level is given by

$$z = f(x(t), y(t)) = g(t),$$ a function of one variable t only. How fast is your altitude changing with respect to time t?

Solution.

$z = f(x,y)$





$x = u(t),\ y = v(t),\ t \in \mathbb{R}$
$z = f(x,y) = f(u(t), v(t)) = g(t),\ t \in \mathbb{R}.$

The answer is $g'(t)$.

$$g'(t) = \lim_{h\to 0} \frac{g(t+h)-g(t)}{h}$$

$$= \lim_{h\to 0} \frac{f(u(t+h),v(t+h)) - f(u(t),v(t))}{h}$$

$$= \lim_{h\to 0} \frac{f(u(t+h),v(t+h)) - f(u(t),v(t+h))}{h} + \frac{f(u(t),v(t+h)) - f(u(t),v(t))}{h}$$

$$= \lim_{\substack{h\to 0 \\ x_1 \to x_2}} \frac{f(\overbrace{u(t+h)}^{x_1},\overbrace{\underline{v(t+h)}}^{y_1}) - f(\overbrace{u(t)}^{x_2},\overbrace{\underline{v(t+h)}}^{y_1})}{\underbrace{u(t+h)}_{x_1} - \underbrace{u(t)}_{x_2}} \cdot \frac{u(t+h)-u(t)}{h}$$

$$+ \lim_{h\to 0} \frac{f(u(t),v(t+h)) - f(u(t),v(t))}{v(t+h)-v(t)} \quad \frac{v(t+h)-v(t)}{h}$$

$$= \lim_{x_1\to x_2} \frac{f(x_1,y_1) - f(x_2,y_1)}{x_1-x_2} \cdot \frac{u(t+h)-u(t)}{h}$$

$$= f_1(u(t),v(t)) \cdot u'(t) + f_2(u(t),v(t)) \cdot v'(t)$$

So, $\frac{d}{dt}\left(f(u(t),v(t))\right) = f_1(u(t),v(t))\,\underline{u'(t)} + f_2(u(t),v(t))\,\underline{v'(t)}$.

2) $z = f(x(t),y(t))$, $x = x(t)$, $y = y(t)$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= z_x\, x'(t) + z_y\, y'(t)$$

$\underline{\underline{Ex}}$ $z = f(x,y) = 3xy + e^{-x+y}$, $x(t) = t^2$
$y(t) = \cos t$

$$\frac{dz}{dt} = f_1(x,y)\, x'(t) + f_2(x,y)\, y'(t)$$
$$= (3y - e^{-x+y})\, x'(t) + (3x + e^{-x+y})\, y'(t)$$
$$= (3\cos t - e^{-t^2+\cos t}) \cdot (2t)$$
$$+ (3t^2 + e^{-t^2+\cos t})\,(-\sin t)$$

$$= 6t\cos t - 2t\, e^{-t^2+\cos t} - 3t^2\sin t - \sin t\, e^{-t^2+\cos t}.$$

Another Version: $z = f(x,y)$, $x = u(s,t)$
$y = v(s,t)$

$$z = z(s,t) = f(u(s,t), v(s,t))$$

$$\frac{\partial z}{\partial t} = f_1(u(s,t), v(s,t)) \cdot \frac{\partial u}{\partial t} + f_2(u(s,t), v(s,t)) \cdot \frac{\partial v}{\partial t}$$

$$\frac{\partial z}{\partial s} = f_1(u(s,t), v(s,t))\frac{\partial u}{\partial s} + f_2(u(s,t), v(s,t))\,\frac{\partial v}{\partial s}.$$

Example: $f(x,y) = x^2 + 3y$, $u(s,t) = st^2$, $v(s,t) = s-t$

$$g(s,t) = f(u(s,t), v(s,t)) = u^2(s,t) + 3v(s,t)$$
$$= s^2t^4 + 3(s-t)^2$$

$\frac{\partial g}{\partial s}$, $\frac{\partial g}{\partial t}$ = ?

$$\frac{\partial g}{\partial s} = f_1(u(s,t), v(s,t)) \cdot \frac{\partial u}{\partial s} + f_2(u(s,t), v(s,t)) \cdot \frac{\partial v}{\partial s}$$

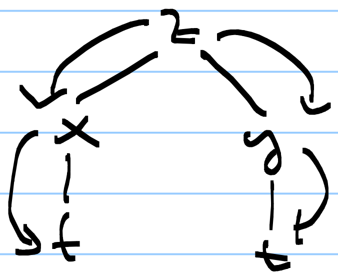
$$f(x,y) = x^2+3y,\ f_1(x,y) = 2x,\ f_2(x,y) = 3$$

$$\frac{\partial g}{\partial s} = 2u(s,t)\cdot\frac{\partial u}{\partial s} + 3\cdot\frac{\partial v}{\partial s}$$
$$= 2st^2\cdot t^2 + 3\cdot(1)$$
$$= 2st^4+3.$$

$$\frac{\partial g}{\partial t} = f_1(u(s,t),v(s,t))\cdot\frac{\partial u}{\partial t} + f_2(u(s,t),v(s,t))\frac{\partial v}{\partial t}$$
$$= 2u(s,t)\cdot 2st + 3\cdot(-1)$$
$$= 2st^2\cdot 2st - 3 = 4s^2t^3-3.$$

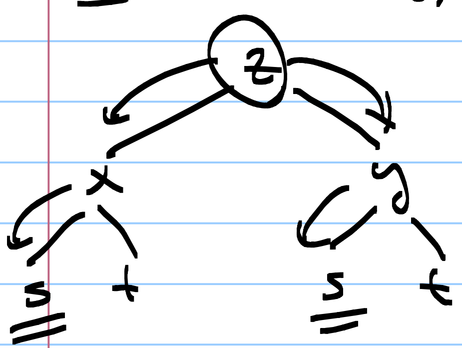
In computing these derivatives it is useful to write down a tree of variables appearing in the functions:

Ex $z=f(x,y)$, $x=u(t)$, $y=v(t)$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Ex $z=f(x,y)$, $x=u(s,t)$, $y=v(s,t)$

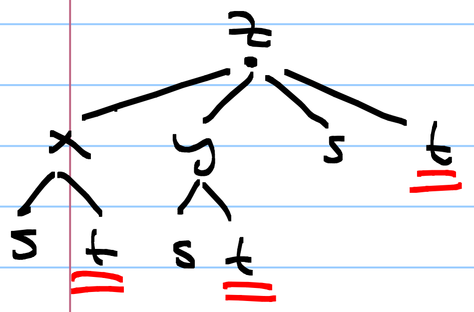


$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Ex: $z = f(x,y,s,t)$, $x = g(s,t)$, $y = h(s,t)$

$= \varphi(s,t)$

$$\frac{\partial z}{\partial t} = ?$$



Sol,
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial t}$$
$$= f_1 \frac{\partial x}{\partial t} + f_2 \frac{\partial y}{\partial t} + f_4$$
$$= f_1(x,y,s,t)\frac{\partial g}{\partial t} + f_2(x,y,s,t)\frac{\partial h}{\partial t} + f_4(x,y,s,t).$$

Remark: $z = z(x,y)$, $x = x(s,t)$, $y = y(s,t)$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\begin{bmatrix} \frac{\partial z}{\partial s} \\ \frac{\partial z}{\partial t} \end{bmatrix} = \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}$$

$F : \mathbb{R}^n \to \mathbb{R}^m$, $G : \mathbb{R}^m \to \mathbb{R}^k$

$G \circ F : \mathbb{R}^n \to \mathbb{R}^k$ matrix multiplication

$$D(G \circ F) = DG \circ DF$$

$z = F(x,y)$ $\quad F:\mathbb{R}^2 \to \mathbb{R}$

$x = x(s,t),\ y = y(s,t)$ $\quad G:\mathbb{R}^2 \to \mathbb{R}^2$

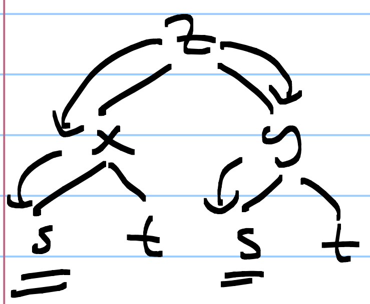
$G(s,t) = (x(s,t), y(s,t))$

$F \circ G = F(x(s,t), y(s,t))$

$$D(F\circ G) = DF \circ DG = \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}$$

<u>Examples</u>: 1) $z = \sin(x^2y)$, $x = st^2$, $y = s^2 + 1/t$

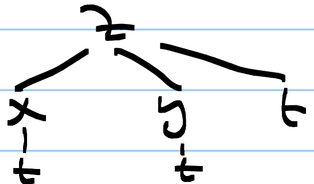
Find $\partial z/\partial s$ and $\partial z/\partial t$.



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$= 2xy\cos(x^2y)\cdot t^2 + x^2\cdot\cos(x^2y)\cdot 2s$$
$$= 2st^2(s^2 + 1/t)\cos\left(s^2t^4(s^2+1/t)\right) + s^2t^4\cos\left(s^2t^4(s^2+1/t)\right)\cdot 2s$$

$\frac{\partial z}{\partial t}$ is similar!

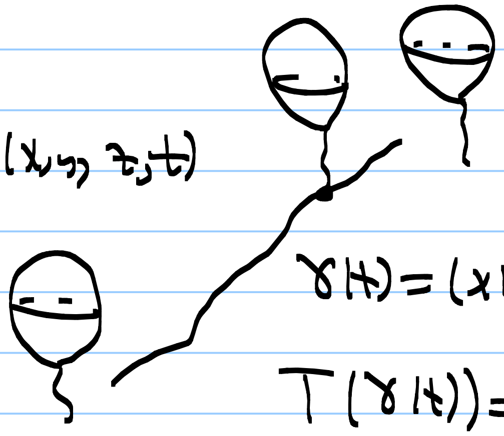
2) Find $\frac{dz}{dt}$, where $z = f(x,y,t)$, $x = g(t)$ and $y = h(t)$.



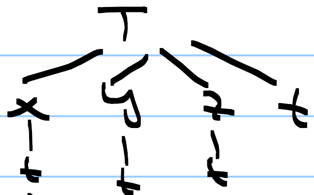
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}$$

$$= \frac{\partial f}{\partial x} g'(t) + \frac{\partial f}{\partial y} h'(t) + \frac{\partial f}{\partial t}.$$

OR $= f_1(x,y,t)\, g'(t) + f_2(x,y,t)\, h'(t) + f_3(x,y,t).$



3) $T(x,y,z,t)$



$\gamma(t) = (x(t), y(t), z(t))$

$T(\gamma(t)) = T(x(t), y(t), z(t), t)$

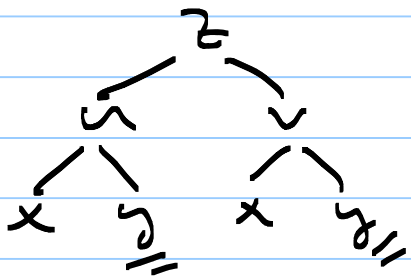
$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} + \frac{\partial T}{\partial t}\overset{1}{\overset{\shortparallel}{\frac{dt}{dt}}}$$

$$= T_1\, x'(t) + T_2\, y'(t) + T_3\, z'(t) + T_4$$

Higher-Order Derivatives:

Ex Calculate $\frac{\partial^2}{\partial x \partial y} f(x^2-y^2, xy)$ in terms of partial derivatives of f.

$z = f(u,v)$, $u = u(x,y) = x^2 - y^2$, $v = v(x,y) = xy$



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= f_1(x^2-y^2, xy)\cdot 2x + f_2(x^2-y^2, xy)\cdot x$$

So, $\frac{\partial z}{\partial y} = \underbrace{2x\, f_1(x^2-y^2, xy)}_{} + \underbrace{x \cdot f_2(x^2-y^2, xy)}_{}$.

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= 2 \cdot f_1(x^2-y^2, xy) + 2x \frac{\partial}{\partial x}\left(f_1(x^2-y^2, xy)\right)$$

$$+ 1 \cdot f_2(x^2-y^2, xy) + x \cdot \frac{\partial}{\partial x}\left(f_2(x^2-y^2, xy)\right)$$

$$= 2f_1(x^2-y^2, xy) + 2x\left(f_{11}(x^2-y^2, xy) \cdot 2x + f_{21}(x^2-y^2, xy) \cdot y\right)$$

$$+ f_2(x^2-y^2, xy) + x \cdot \left(f_{12}(x^2-y^2, xy) \cdot 2x + f_{22}(x^2-y^2, xy)\, y\right)$$

So $\frac{\partial^2 z}{\partial x \partial y} = 2f_1 + f_2 + 4x^2 f_{11} + 2xy f_{21} + 2x^2 f_{12} + xy f_{22}$

Ex A function $f(x,y)$ is called Harmonic if $f_{xx} + f_{yy} = 0$. Show that if f is harmonic then so is $f(x^2-y^2, 2xy)$.

Solution: $f(u,v)$, $u = x^2-y^2$, $v = 2xy$.

Since f is Harmonic we have $f_{uu} + f_{vv} = 0$.

must show: $g = g(x,y) = f(x^2-y^2, 2xy)$ is Harmonic.

In other words, we must show
$g_{xx} + g_{yy} = 0$.

$g(x,y) = f(\underset{\sim}{u(x,y)}, \underset{\sim}{v(x,y)})$

$g_x = f_u \cdot u_x + f_v \cdot v_x$, $g_y = f_u u_y + f_v v_y$.

$g_{xx} = \frac{\partial}{\partial x}\left(f_u \cdot u_x + f_v v_x\right)$ $u = u(x,y)$

$= \left[\left(f_{uu} u_x + f_{vu} v_x\right) \cdot u_x + f_u u_{xx}\right]$

$+ \left[\left(f_{uv} u_x + f_{vv} v_x\right) \cdot v_x + f_v \cdot v_{xx}\right]$

$u = x^2 - y^2$,
$u_x = 2x$, $u_y = -2y$
$u_{xx} = 2$, $u_{xy} = u_{yx} = 0$, $u_{yy} = -2$

$V = 2xy$
$v_x = 2y$, $v_y = 2x$
$v_{xx} = 0$, $v_{yy} = 0$
$v_{xy} = v_{yx} = 2$.

So, $g_{xx} = \left[(2x f_{uu} + 2y f_{vu})2x + f_u \cdot 2\right]$
$+ \left[(2x f_{uv} + 2y f_{vv}) \cdot 2y + f_v \cdot 0\right]$

$\Rightarrow g_{xx} = 4x^2 f_{uu} + 4xy(f_{vu} + f_{uv}) + 4y^2 f_{vv}$
$+ 2f_u$

Similarly, $g_{yy} = -2f_u + 4y^2 f_{uu} - 4xy(f_{uv} + f_{vu})$
$+ 4x^2 f_{vv}$

Hence, $g_{xx} + g_{yy} = 4x^2(\underbrace{f_{uu} + f_{vv}}_{=0}) + 4y^2(\underbrace{f_{uu} + f_{vv}}_{=0})$

$+ 0 \cdot (f_{uv} + f_{vu})$ since f is Harmonic

$= 0$.

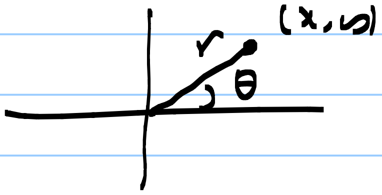
Video 36

Example: Let's write the Laplace's equation in Polar coordinates.

$u = u(x,y)$, $u_{xx} + u_{yy} = 0.$

$x = r\cos\theta$

$y = r\sin\theta$



$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}(y/x)$

In other words, we want to write the equation in the variables r and θ.

$$u = u(x,y) = u(\overbrace{r\cos\theta}^{x}, \overbrace{r\sin\theta}^{y})$$

$x = r\cos\theta$

$y = r\sin\theta$

$x_r = \cos\theta$

$y_r = \sin\theta$

$x_\theta = -r\sin\theta$

$y_\theta = r\cos\theta$

$$u_r = u_x x_r + u_y y_r$$
$$= u_x \cos\theta + u_y \sin\theta$$

$$u_\theta = u_x \cdot x_\theta + u_y y_\theta$$
$$= -r\sin\theta\, u_x + r\cos\theta\, u_y$$

$$u_{rr} = \frac{\partial}{\partial r}(u_x \cos\theta + u_y \sin\theta) \qquad x = r\cos\theta$$

$$= \cos\theta \frac{\partial}{\partial r}(u_x) + \sin\theta \frac{\partial}{\partial r}(u_y)$$

$$= \cos\theta\left(u_{xx} \overbrace{\frac{\partial x}{\partial r}}^{=\cos\theta} + u_{yx} \overbrace{\frac{\partial y}{\partial r}}^{=\sin\theta}\right)$$
$$+ \sin\theta\left(u_{xy} \overbrace{\frac{\partial x}{\partial r}}^{=\cos\theta} + u_{yy} \overbrace{\frac{\partial y}{\partial r}}^{=\sin\theta}\right)$$

$$= \cos^2\theta\, u_{xx} + 2\cos\theta\sin\theta\, u_{yx} + \sin^2\theta\, u_{yy}$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial\theta}(-r\sin\theta\, u_x + r\cos\theta\, u_y)$$

$$= -r\frac{\partial}{\partial\theta}(\sin\theta\, u_x - \cos\theta\, u_y)$$

$$= -r\left(\cos\theta\, u_x + \sin\theta \frac{\partial}{\partial\theta}(u_x) + \sin\theta\, u_y - \cos\theta \frac{\partial}{\partial\theta}(u_y) \right)$$

$x = r\cos\theta$
$y = r\sin\theta$

$$= -r\Big[\cos\theta\, u_x + \sin\theta\, u_y + \sin\theta \Big(u_{xx} \underbrace{\frac{\partial x}{\partial\theta}}_{= -r\sin\theta} + u_{yx} \underbrace{\frac{\partial y}{\partial\theta}}_{= r\cos\theta} \Big) - \cos\theta \Big(u_{xy} \underbrace{\frac{\partial x}{\partial\theta}}_{= -r\sin\theta} + u_{yy} \underbrace{\frac{\partial y}{\partial\theta}}_{= r\cos\theta} \Big) \Big]$$

$$= -r\Big[\cos\theta\, u_x - \sin\theta\, u_y - r\sin^2\theta\, u_{xx} + r\sin\theta\cos\theta\, u_{yx} + r\sin\theta\cos\theta\, u_{xy} - r\cos^2\theta\, u_{yy} \Big] + r\cos\theta\, u_x + r\sin\theta\, u_y$$

$$u_{rr} + \frac{1}{r^2} u_{\theta\theta} = (\cos^2\theta + \sin^2\theta)\, u_{xx} + (\cos^2\theta + \sin^2\theta)\, u_{yy} - \frac{1}{r}\underbrace{(\cos\theta\, u_x + \sin\theta\, u_y)}_{u_r}$$

$$\Rightarrow u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = u_{xx} + u_{yy} = 0$$

In polar coordinates, the Laplace equation becomes

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0.$$

Video 37

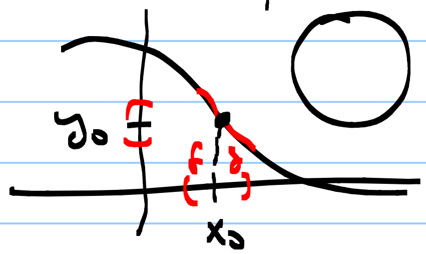
§12.8 Implicit Functions:

In the study of one variable functions we studied Implicit Differentiation: Suppose $y=f(x)$ is given implicitly by an equation of the form

$$F(x,y)=0.$$

Ex $\quad x^2y+e^{xy}-\cos y+5=0.$

Theory implies that if $\frac{\partial F}{\partial y}(x_0,y_0)\neq 0$ for some (x_0,y_0) with $F(x_0,y_0)=0$, then $y=f(x)$ for some function $f(x)$ around (x_0,y_0).



$F(x,y)=0$

$y=f(x) \quad x\in(x_0-\delta,x_0+\delta)$

Moreover, $f(x)$ is differentiable and thus taking derivative of $F(x,y)=\underline{F(x,f(x))}=0$

$$F_1(x,f(x))+F_2(x,f(x))\,f'(x)=0.$$

$$\Rightarrow f'(x)=-\frac{F_1(x,f(x))}{F_2(x,f(x))}=-\frac{F_1(x,y)}{F_2(x,y)}$$

Indeed, this holds for functions of several variables also.

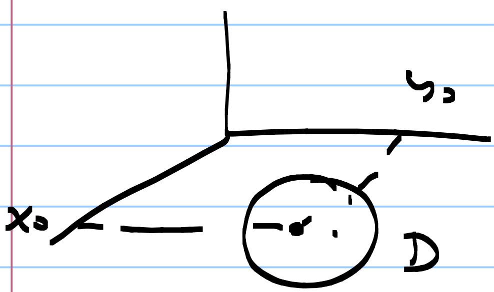
Suppose we are given an equation in the variables x,y and z as follows:

$F(x,y,z)=0.$

Fact: If F is differentiable and $\frac{\partial F}{\partial z}(x_0,y_0,z_0)\neq 0$ at some point then there is a disc D around (x_0,y_0) and a function $f(x,y)$ defined on D so that

$$F(x,y,f(x,y))=0 \text{ for all } (x,y)\in D.$$

In other words, $z=f(x,y)$ is solution of $F(x,y,z)=0$ on D.



$F(x,y,f(x,y))=0$

$F(x,y,z)=0,\ z=f(x,y).$

Moreover, taking $\partial/\partial x$ and $\partial/\partial y$ of the above equation we obtain: for all $(x,y)\in D$,

$F(x,y,f(x,y))=0.$ So $\frac{\partial}{\partial x}\left(F(x,y,f(x,y))\right)=0.$

$$F_1(x,y,f(x,y))+F_2(x,y,f(x,y))\underbrace{\frac{\partial y}{\partial x}}_{=0}+\underbrace{F_3(x,y,f(x,y))}_{\neq 0 \text{ by assumption}}\frac{\partial f}{\partial x}=0$$

$$\Rightarrow \frac{\partial f}{\partial x}=\frac{-F_1(x,y,f(x,y))}{F_3(x,y,f(x,y))}=\frac{-F_1(x,y,z)}{F_3(x,y,z)}$$

$$\frac{\partial f}{\partial x}(x_0,y_0)=\frac{-F_1(x_0,y_0,z_0)}{F_3(x_0,y_0,z_0)}.$$

Similarly, $\dfrac{\partial f}{\partial y}(x,y,z) = \dfrac{-F_2(x,y,z)}{F_3(x,y,z)}$.

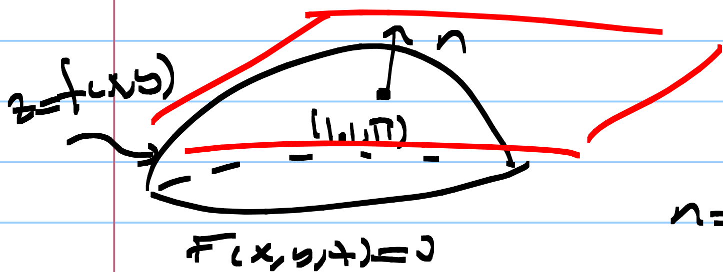
__Example:__ Suppose a function $z=f(x,y)$ is given implicitly as

$$e^{yz} - x^2 z \ln y = \pi.$$

Find the equation of the tangent plane to the surface given by above equation at the point $(1,1,\ln\pi)$.

__Solution:__ $F(x,y,z) = e^{yz} - x^2 z \ln y - \pi = 0$

$$F(1,1,\ln\pi) = e^{1\cdot 1\ln\pi} - 1^2\cdot \ln\pi \underbrace{\ln 1}_{=0} - \pi = e^{\ln\pi} - \pi = \pi - \pi = 0.$$



$z = f(x,y)$

$n = (f_x, f_y, -1)$

Note that $\dfrac{\partial F}{\partial z} = y e^{yz} - x^2 \ln y - 0$ and

$$\frac{\partial F}{\partial z}(x_0,y_0,z_0) = \frac{\partial F}{\partial z}(1,1,\ln\pi) = 1\cdot e^{\ln\pi} - 1^2\cdot \underbrace{\ln 1}_{=0} = \pi \neq 0$$

Hence, there is a function $f(x,y)$ defined near (x_0,y_0) so that

$$F(x,y,f(x,y)) = 0 \text{ for all } (x,y) \text{ near } (x_0,y_0).$$

$F(x,y,z) = 0, \quad z = f(x,y)$

$$\frac{\partial}{\partial x}(F(x,y,f(x,y))) = 0 \Rightarrow F_1 \overbrace{\frac{\partial x}{\partial x}}^{1} + F_2 \overbrace{\frac{\partial y}{\partial x}}^{0} + F_3 \frac{\partial f}{\partial x} = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{-F_1}{F_3}.$$

Similarly, $\frac{\partial}{\partial y}\left(F(x,y,f(x,y))\right)=0$

$$\Rightarrow F_1 \underbrace{\frac{\partial x}{\partial y}}_{\overset{\shortparallel}{0}} + F_2 \underbrace{\frac{\partial y}{\partial y}}_{\overset{\shortparallel}{1}} + F_3 \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{-F_2}{F_3}.$$

$$F(x,y,z) = e^{yz} - x^2 z \ln y - \pi$$

$$\frac{\partial f}{\partial x} = \frac{-F_1}{F_3} = \frac{-(-2xz\ln y)}{ye^{yz} - x^2\ln y} = \frac{2xz\ln y}{ye^{yz} - x^2\ln y}$$

$$\frac{\partial f}{\partial x}(1,1,\ln\pi) = \frac{2\cdot 1\cdot \ln\pi \ln 1}{- \;\; --} = 0$$

$$\frac{\partial f}{\partial y} = \frac{-F_2}{F_3} = -\frac{(ze^{yz} - x^2 z\,1/y)}{ye^{yz} - x^2\ln y}$$

$$\frac{\partial f}{\partial y}(1,1,\ln\pi) = -\frac{(\ln\pi\, e^{\ln\pi} - 1.\ln\pi)}{e^{\ln\pi} - 1.0} = -\ln\pi + \frac{\ln\pi}{\pi}$$

$$= \ln\pi\left(\frac{1}{\pi} - 1\right) = \frac{1-\pi}{\pi}\ln\pi$$

$$\vec{n} = (f_x, f_y, -1) = \left(0, \frac{1-\pi}{\pi}\ln\pi, -1\right)$$

Hence, an equation for the tangent plane is

$$(x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0) - (z-z_0) = 0$$

$$z = xf_x(x_0,y_0) + yf_y(x_0,y_0) - x_0 f_x(x_0,y_0) - y_0 f_y(x_0,y_0) + z_0$$

Video 38

$$z = x\cdot 0 + y\,\frac{1-\pi}{\pi}\ln\pi - x_0\cdot 0 - 1\cdot\frac{1-\pi}{\pi}\ln\pi + \ln\pi$$

$$z = \left(\frac{1-\pi}{\pi}\ln\pi\right)\cdot y + 2\ln\pi - \frac{\ln\pi}{\pi}.$$

If we are given an equation of the form $F(x,y,z,u)=0$ and if $\frac{\partial F}{\partial u}(x_0,y_0,z_0,u_0)\neq 0$ then

the equation can be solved for u in terms of the other variables as

$$u = f(x,y,z).$$

If F is differentiable the derivatives of u with respect to x, y and z can be computed as before:

$$F(x,y,z,u) = F(x,y,z,f(x,y,z)) = 0$$

$$F_1\underbrace{\frac{\partial x}{\partial x}}_{1} + F_2\underbrace{\frac{\partial y}{\partial x}}_{0} + F_3\underbrace{\frac{\partial z}{\partial x}}_{0} + F_4\frac{\partial f}{\partial x} = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} = -\frac{F_1}{F_4}.$$ Similarly, $\frac{\partial f}{\partial y} = -\frac{F_2}{F_4}$ and so on.

§12.7. Gradients and Directional Derivatives:

Definition: At any point (x,y) where the first partial derivatives of $f(x,y)$ exist, we define the gradient vector $\nabla f(x,y) = \text{grad}\, f(x,y)$ by

$$\nabla f(x,y) = \text{grad}\, f(x,y) = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}$$
$$= (f_x(x,y), f_y(x,y))$$

Ex $f(x,y) = 3xy + e^x - \cos xy$

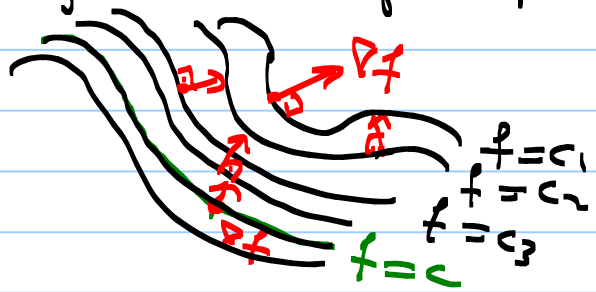
$f_1 = f_x = 3y + e^x + y\sin xy$ and
$f_2 = f_y = 3x + x\sin xy$

$\nabla f(x,y) = (f_x, f_y) = (3y + e^x + y\sin xy,\ 3x + x\sin xy)$
$= (3y + e^x + y\sin xy)\,\vec{i} + (3x + x\sin xy)\,\vec{j}.$

Notation: $\nabla = \vec{i}\,\dfrac{\partial}{\partial x} + \vec{j}\,\dfrac{\partial}{\partial y}$

$\nabla f = (\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y) f = \vec{i}\,\dfrac{\partial f}{\partial x} + \vec{j}\,\dfrac{\partial f}{\partial y}.$

Theorem: If $f(x,y)$ is differentiable at the point (a,b) and $\nabla f(a,b) \neq (0,0)$, $\nabla f(x_0,y_0)$ is a normal vector to the level curve of f that passes through (a,b).



Proof: Let $r(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$ be a parametrization for a level curve say $f(x,y) = c$

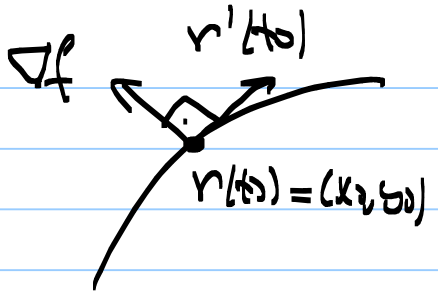
Then $c = f(r(t)) = f(x(t), y(t))$, for all t.
Take derivative w.r.t. t to get

$$0 = \frac{\partial c}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \quad (*)$$

$$0 = \nabla f(x_0,y_0)\cdot r'(t_0)$$

$$\Rightarrow \nabla f(x_0,y_0) \perp r'(t_0)$$

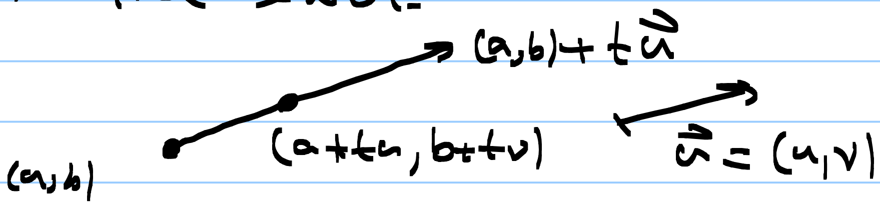


Directional Derivatives:

Definition: Let $\vec{u} = u\hat{i} + v\hat{j}$ be a unit vector, so that $u^2+v^2=1$. The directional derivative of $f(x,y)$ at (a,b) in the direction of $\vec{u}$ is the rate of change of $f(x,y)$ with respect to distance measured at (a,b) along a ray in the direction of $\vec{u}$ in the xy-plane. This directional derivative is given by

$$D_{\vec{u}} f(a,b) = \lim_{h\to 0^+} \frac{f(a+hu, b+hv) - f(a,b)}{h}$$

or $= \frac{d}{dt}\left(f(a+tu, b+tv)\right)\Big|_{t=0}$, if the derivative exist.



$$\boxed{\frac{d}{dt} f(a+t)\Big|_{t=0} = \lim_{h\to 0} \frac{f(a+h) - f(0)}{h}}$$

Theorem If f is "differentiable" at (a,b) and $u = u\hat{i} + v\hat{j}$ is a unit vector, then the directional derivative of f at (a,b) in the direction of u is given by

$$D_u f(a,b) = u \cdot \nabla f(a,b)$$

Proof Apply Chain Rule:

$$D_u f(a,b) = \frac{d}{dt}\left(f(a+tu, b+tv)\right)\Big|_{t=0}$$

$$= f_1 \cdot u + f_2 \cdot v$$

$$= (f_1, f_2) \cdot (u,v)$$

$$= \nabla f(a,b) \cdot (u,v)$$

Remark: If $\vec{u} = (u,v)$ is not a unit vector then we define the directional derivative along $\vec{u}$ as follows:

$$D_{\frac{\vec{u}}{|\vec{u}|}} f(a,b).$$

Example: Find the rate of change of $f(x,y) = y^4 + 2xy^3 + x^2y^2$ at $(0,1)$ measured in each of the following directions:

a) $\hat{i}+2\hat{j}$ b) $-2\hat{i}+\hat{j}$ c) $3\hat{i}$ d) $\hat{i}+\hat{j}$.

Solution: Since $f(x,y)$ is a polynomial it is "differentiable" we may use the above theorem to compute the directional derivatives:

$$\nabla f = (f_x, f_y) = (2y^3 + 2xy^2,\ 4y^3 + 6xy^2 + 2x^2y)$$

$(a,b) = (0,1)$, $\nabla f(a,b) = \nabla f(0,1) = (2,4)$.

a) $u = \hat{i} + 2\hat{j}$, $\dfrac{u}{|u|} = \dfrac{\hat{i}+2\hat{j}}{\sqrt{1+4}} = \dfrac{1}{\sqrt{5}}\hat{i} + \dfrac{2}{\sqrt{5}}\hat{j}$

$$D_{u/|u|} f(0,1) = \nabla f(0,1) \cdot \frac{u}{|u|}$$
$$= (2,4) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$
$$= \frac{10}{\sqrt{5}} = 2\sqrt{5}.$$

b) $u = -2\hat{i} + \hat{j}$, $D_{u/|u|} f(0,1) = (2,4) \cdot \left(\dfrac{-2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}\right)$
$$= 0.$$

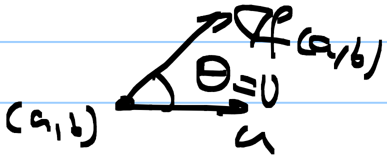
c) $u = 3\hat{i}$, $|u| = 3$, $u/|u| = \hat{i} = (1,0)$

$$D_{u/|u|} f(0,1) = (2,4) \cdot (1,0) = 2.$$

d) $u = \hat{i} + \hat{j}$, $|u| = \sqrt{2}$

$$D_{u/|u|} f(0,1) = (2,4) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{6}{\sqrt{2}} = 3\sqrt{2}.$$

<u>Geometric Observation:</u>



$$D_u f(a,b) = \nabla f(a,b) \cdot u$$
$$= |\nabla f(a,b)| \cdot |u| \cdot \cos\theta$$
$$= |\nabla f(a,b)| \cos\theta$$

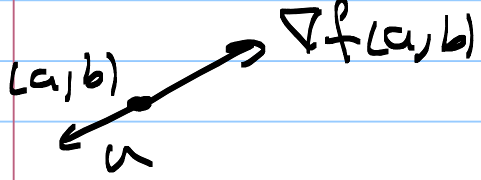
1) $D_u f(a,b)$ takes its maximal value in the direction $u = \nabla f(a,b)/|\nabla f(a,b)|$. In this case

$$D_u f(a,b) = |\nabla f(a,b)| \cdot \underbrace{\cos 0}_{=1} = |\nabla f(a,b)|$$

$u \nearrow \nabla f(a,b)$

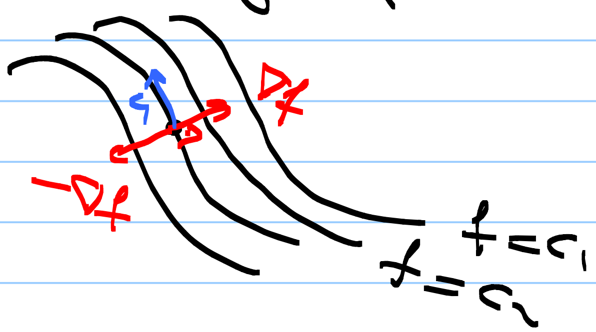
2) $D_u f(a,b)$ takes its minimal value in the direction $u = -\nabla f(a,b)/|\nabla f(a,b)|$. In this case

$$D_u f(a,b) = |\nabla f(a,b)| \cos \pi = -|\nabla f(a,b)|$$



3) If $u \perp \nabla f(a,b)$ then $D_u f(a,b) = 0$ so that the values of f does not change in the direction of u.

Example: If $z = f(x,y)$ is the height of a point on the ground from the sea level then the height increases most rapidly in the direction of ∇f. It decreases most rapidly in the direction of $-\nabla f$. Finally, it does not change if u is $\perp \nabla f$.



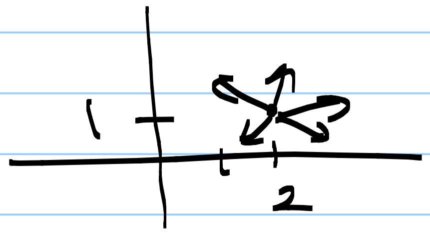
Video 39

Example: The temperature at position (x,y) in a region of the xy-plane is T°C, where

$$T(x,y) = x^2 e^{-y}.$$

In what direction at the $(2,1)$ does the temperature increase most rapidly? What is the rate of increase of T in that direction?

Solution $(a,b) = (2,1)$ $T(x,y) = x^2 e^{-y}$, $(x,y) \in \mathbb{R}^2$



$D_u T(a,b) = ?$

$$\nabla T(a,b) = (T_x, T_y)(a,b) = (2x e^{-y}, -x^2 e^{-y})(2,1)$$
$$= (4e^{-1}, -4e^{-1}). \qquad |\nabla T(a,b)| = 4\sqrt{2}e^{-1}$$

$$u = \frac{\nabla T(a,b)}{|\nabla T(a,b)|} = \frac{1}{\sqrt{2}}(1,-1).$$

$$D_u T(a,b) = \nabla T(a,b) \cdot u$$
$$= (4e^{-1}, -4e^{-1}) \cdot \frac{1}{\sqrt{2}}(1,-1) = \frac{8e^{-1}}{\sqrt{2}}$$

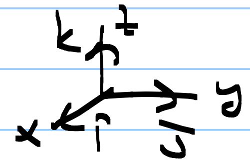
Hence, the temperature increases most rapidly at the point $(1,2)$ in the direction $u = 1/\sqrt{2}(1,-1)$ and the rate of increase in that direction is $8e^{-1}/\sqrt{2}$.

The Gradient of $f(x_1, x_2, \dots, x_n)$ is defined as

$$\nabla f(x_1,\dots,x_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right).$$

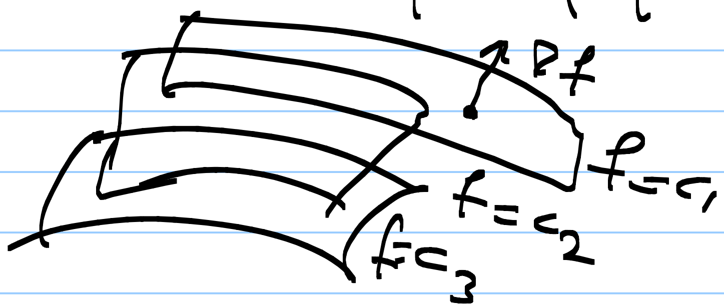
If $n=3$, then we can write $\nabla f(x,y,z)$ as

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

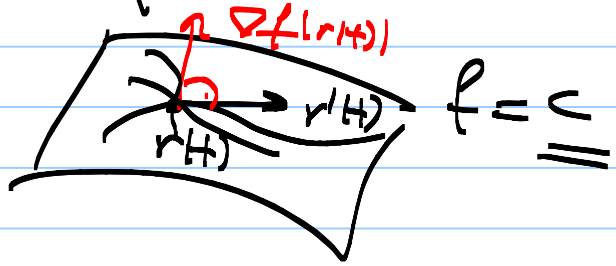


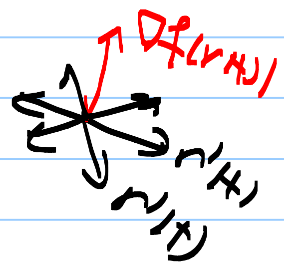
$$= \frac{\partial f}{\partial x}\,\vec{i} + \frac{\partial f}{\partial y}\,\vec{j} + \frac{\partial f}{\partial z}\,k.$$

<u>Fact</u>: If $f = f(x,y,z)$ is a differentiable function then $\nabla f(a,b,c)$ is perpendicular to the level surface of f at (a,b,c).



<u>Proof</u>. Let $r(t) = (x(t), y(t), z(t))$ lies in a level surface $f = c$ then





So, $f(x(t), y(t), z(t)) = c$ for all t.

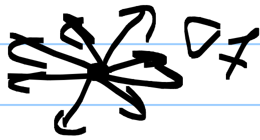
$$\frac{d}{dt}\big(f(x(t), y(t), z(t))\big) = 0$$

$$f_x \cdot x'(t) + f_y\, y'(t) + f_z \cdot z'(t) = 0$$

$$\Rightarrow (f_x, f_y, f_z)(r(t)) \cdot \overbrace{(x'(t), y'(t), z'(t))}^{r'(t)} = 0 \quad \forall t$$

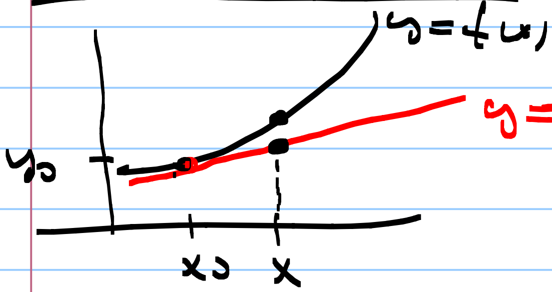
So, $\nabla f(r(t))$ is $\perp$ to the level surface $f(x,y,z) = c$.

$f(x,y,t)$



§ 12.6. Linear Approximations, Differentiability and Differentials

One Variable Case:



$y = L(x) = f(x_0) + f'(x_0)(x - x_0)$

Equation of the tangent line

$L(x) = f(x_0) + f'(x_0)(x - x_0)$

$y_0 = f(x_0) = L(x_0)$

Recall that $L(x)$ is called the linearization of $f(x)$ at x_0.

In two variable case the linearization at (x_0, y_0) is defined as

$z = f(x,y)$

$$z = L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

Note that $z = L(x,y)$ is the equation of the tangent plane to the graph of $z = f(x,y)$ at (x_0, y_0).

Example: Find an approximate value for

$f(x,y)=\sqrt{2x^2+e^{2y}}$ at $(2.2, -0.2)$

Solution $(x_0,y_0)=(2,0)$, $(x,y)=(2.2,-0.2)$

Linearization of f at $(x_0,y_0)=(2,0)$

$L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

$f(x,y)=\sqrt{2x^2+e^{2y}}$ $\quad f(x_0,y_0)=f(2,0)=\sqrt{9}=3$

$f_x(x,y)=\dfrac{4x}{2\sqrt{2x^2+e^{2y}}}$, $\quad f_x(x_0,y_0)=f_x(2,0)=\dfrac{8}{2.3}=4/3$

$f_y(x,y)=\dfrac{2e^{2y}}{2\sqrt{}}$, $\quad f_y(x_0,y_0)=f_y(2,0)=\dfrac{1}{3}$

$L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

$=3+\frac{4}{3}(x-2)+\frac{1}{3}(y-0)$

$f(2.2,-0.2)\approx L(2.2,-0.2)=3+\frac{4}{3}(2.2-2)$
$+\frac{1}{3}(-0.2-0)$

$\approx 3+\dfrac{0.8}{3}-\dfrac{0.2}{3}=3+0.2=3.2$

Definition: We say that the function $f(x,y)$ is differentiable at a point (a,b) if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k)-f(a,b)-hf_1(a,b)-kf_2(a,b)}{\sqrt{h^2+k^2}}=0.$$

Remark: For (h,k) define

$$a(h,k)=\left(f(a+h,b+k)-f(a,b)-hf_1(a,b)-kf_2(a,b)\right)/\sqrt{h^2+k^2}.$$

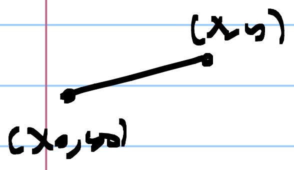
Then $f(a+h,b+k)=f(a,b)+hf_1(a,b)+kf_2(a,b)+\sqrt{h^2+k^2}\,a(h,k)$.

$(x_0,y_0)=(a,b)$, $(x,y)=(a+h,b+k)$, then

$$f(x,y)=f(x_0,y_0)+f_1(x_0,y_0)(x-x_0)+f_2(x_0,y_0)(y-y_0)+\sqrt{(x-x_0)^2+(y-y_0)^2}\,a(x-x_0,y-y_0)$$

so that $\lim_{(h,k)\to(0,0)} a(h,k)=0$.

$$f(x,y)=L(x,y)+\underbrace{\sqrt{(x-x_0)^2+(y-y_0)^2}\cdot a(x-x_0,y-y_0)}_{\text{"Error term"}}.$$



Theorem: If $f_1(x,y)$ and $f_2(x,y)$ are continuous in a neighborhood of the point (a,b), and if the absolute values of h and k are sufficiently small, then there exist numbers θ_1 and θ_2, each between 0 and 1, such that

$$f(a+h,b+k)-f(a,b)=hf_1(a+\theta_1h,b+\theta_2k)+kf_2(a+\theta_1h,b+\theta_2k).$$

Proof: Consider the difference

$$f(a+h, b+k) - f(a,b) = f(a+h, b+k) - f(a, b+k) + f(a, b+k) - f(a,b)$$

Let $g(\theta)$ be the function defined by
$$g(\theta) = f(a+\theta h, b+k).$$ Note that $g(\theta)$ is differentiable since f_1 exists.

So $g'(\theta) = h f_1(a+\theta h, b+k)$. Apply M.V.T. to $g(\theta)$ on $[0,1]$. Then
$$g(1) - g(0) = g'(\theta_1)(1-0) \text{ for some } \theta_1 \in (0,1).$$

$$\Rightarrow f(a+h, b+k) - f(a, b+k) = h f_1(a+\theta_1 h, b+k)$$

Similarly, let $J(\theta) \doteq f(a, b+\theta k)$. Then J is differentiable with derivative
$$J'(\theta) = f_2(a, b+\theta k) \cdot k.$$

Let's apply M.V.T. to $J(\theta)$ on $[0,1]$. Then

$$J(1) - J(0) = J'(\theta_2)(1-0) \text{ for some } \theta_2 \in (0,1).$$

$$f(a, b+k) - f(a,b) = k f_2(a, b+\theta_2 k)$$

Hence, $$f(a+h, b+k) - f(a,b) = h f_1(a+\theta_1 h, b+k) + k f_2(a, b+\theta_2 k)$$

for some $\theta_1, \theta_2 \in (0,1)$. ∎

Theorem: If f_1 and f_2 are continuous in a neighborhood of the point (a,b) then f is differentiable at (a,b).

Proof: For $(h,k)\neq(0,0)$ we have

$\left|\frac{h}{\sqrt{h^2+k^2}}\right|\le 1$ and $\left|\frac{k}{\sqrt{h^2+k^2}}\right|\le 1$. So

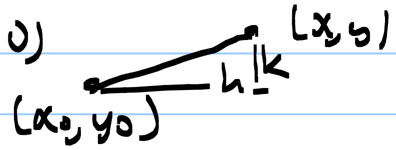
$$\left|\frac{f(a+h,b+k)-f(a,b)-hf_1(a,b)-kf_2(a,b)}{\sqrt{h^2+k^2}}\right|$$

$$=\frac{|hf_1(a+\theta_1 h,b+k)+kf_2(a,b+\theta_2 k)-hf_1(a,b)-kf_2(a,b)|}{\sqrt{h^2+k^2}}$$

$$\le \frac{|h|}{\sqrt{h^2+k^2}}|f_1(a+\theta_1 h,b+k)-f_1(a,b)|+\frac{|k|}{\sqrt{h^2+k^2}}|f_2(a,b+\theta_2 k)-f_2(a,b)|$$

$$\le \underbrace{|f_1(a+\theta_1 h,b+k)-f_1(a,b)|+|f_2(a,b+\theta_2 k)-f_2(a,b)|}_{R.H.S.}$$

Taking limit as $(h,k)\to(0,0)$



Since f_1 and f_2 are continuous we obtain

$\lim_{(h,k)\to(0,0)} R.H.S=0$. By the squeezing lemma

then we have

$$\lim_{(h,k)\to(0,0)}\left|\frac{f(a+h,b+k)-f(a,b)-hf_1(a,b)-kf_2(a,b)}{\sqrt{h^2+k^2}}\right|=0$$

Hence, f is differentiable at (a,b). $\blacksquare$

Video 41

Example Any polynomial function $f(x,y)$ is continuous. Hence, any polynomial (indeed, rational) function is differentiable.

Differentials: For a function of single variable, say $f(x)$, the differential of f is defined to be the object
$$df \doteq f'(x)\, dx.$$

For a function of several variables say
$f = f(x_1, x_2, \ldots, x_n)$, its differential is defined to be the object

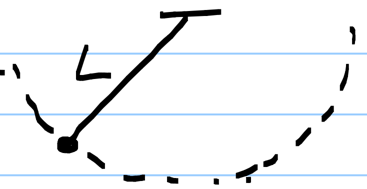
$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

Remark: The linearization of a two variable function $f(x,y)$ is as follows:

$$L(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\underline{(x-x_0)} + \frac{\partial f}{\partial y}(x_0,y_0)\underline{(y-y_0)}$$
$$\wr\wr$$
$$f(x,y) - f(x_0,y_0) \simeq \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$
$$\|$$
$$\Delta f(x,y) \simeq \frac{\partial f}{\partial x}(x_0,y_0)\,\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\,\Delta y$$
$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Example: Estimate the percentage change in the period $T = 2\pi\sqrt{L/g}$ of a simple pendulum if the length, L, of the pendulum increases by 2% and the acceleration of gravity, g, decreases 0.6 %.

Solution:



$$T = 2\pi\sqrt{L/g}$$

$$\frac{\partial T}{\partial L} = 2\pi \frac{1}{2}\sqrt{g/L} = \pi\sqrt{g/L}\,\frac{1}{g}$$

$$\frac{\partial T}{\partial g} = 2\pi\frac{1}{2}\sqrt{g/L}\cdot\frac{-L}{g^2} = -\pi\sqrt{\frac{L}{g^3}}$$

$$T = T(L, g)$$

$$dT = \frac{\partial T}{\partial L}\,dL + \frac{\partial T}{\partial g}\,dg$$

$$dT = \frac{\pi}{g}\sqrt{g/L}\,dL - \pi\sqrt{L/g^3}\,dg \qquad dL = \frac{2L}{100}$$

$$= \pi\sqrt{g/L}\,\frac{2L}{g100} + \pi\sqrt{L/g^3}\,\frac{0.6g}{100} \qquad dg = \frac{-0.6g}{100}$$

$$= \pi\sqrt{L/g}\,\frac{2}{100} + \pi\sqrt{L/g}\,\frac{0.6}{100}$$

$$= \pi\sqrt{L/g}\left(\frac{2}{100} + \frac{0.6}{100}\right)$$

$$= 2\pi\sqrt{L/g}\,\frac{2.6}{100}\cdot\frac{1}{2}$$

$$= T\,\frac{1.3}{100} \implies \text{So } \frac{dT}{T} = \frac{1.3}{100}$$

Hence T increases 1.3 %.

Recitation:

1) If $f(x,y)$ is Harmonic, show $f\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$ is also Harmonic.

Solution: $f_{xx} + f_{yy} = 0$.

$$g(x,y) \doteq f\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right).$$

Show that $g_{xx} + g_{yy} = 0$.

$$g_x = f_1\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + f_2\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) \cdot \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right)$$

$$= \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} f_1 - y \cdot \frac{-2x}{(x^2+y^2)^2} \cdot f_2$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} f_1\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) + \frac{2xy}{(x^2+y^2)^2} f_2\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right).$$

$$g_{xx} = \frac{-2x(x^2+y^2)^2 - (y^2-x^2)\cdot 2\cdot 2x\,(x^2+y^2)}{(x^2+y^2)^4} f_1$$

$$+ \frac{y^2-x^2}{(x^2+y^2)^2}\left[\frac{y^2-x^2}{(x^2+y^2)^2} f_{11} + \frac{2xy}{(x^2+y^2)^2} f_{12}\right]$$

$$+ \frac{2y(x^2+y^2)^2 - 2xy\cdot 2\cdot 2x\,(x^2+y^2)}{(x^2+y^2)^4} f_2$$

$$+ \frac{2xy}{(x^2+y^2)^2}\left[\frac{y^2-x^2}{(x^2+y^2)^2} f_{12} + \frac{2xy}{(x^2+y^2)^2} f_{22}\right]$$

Remark: There is a shortcut which uses "Complex Calculus."

2) Let $F(x,y) = \begin{cases} \dfrac{2xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

a) Show that $F(x,y) = -F(y,x)$.

b) Show that $F_1(x,y) = -F_2(y,x)$ and

$F_{12}(x,y) = -F_{21}(y,x)$ for $(x,y) \neq (0,0)$.

c) Show that $F_1(0,y) = -2y$, for all y, and hence, that $F_{12}(0,0) = -2$.

d) Deduce that $F_2(x,0) = 2x$ and hence, $F_{21}(0,0) = 2$.

Solution $F(x,y) = \begin{cases} \dfrac{2xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

a) $F(y,x) = \begin{cases} \dfrac{2yx(y^2-x^2)}{x^2+y^2} & \text{if } (y,x) \neq (0,0) \\ 0 & \text{if } (y,x) = (0,0) \end{cases}$

$= -F(x,y)$

b) Since $F(x,y) = -F(y,x)$ we get

$$\frac{\partial}{\partial x}\left(F(x,y)\right) = -\frac{\partial}{\partial x}\left(F(y,x)\right)$$

$$F_1 \underbrace{\frac{\partial x}{\partial x}}_{1} + F_2 \underbrace{\frac{\partial y}{\partial x}}_{0} = -\left(F_1 \underbrace{\frac{\partial y}{\partial x}}_{0} + F_2 \underbrace{\frac{\partial x}{\partial x}}_{1}\right)$$

$\Rightarrow F_1(x,y) = -F_2(y,x)$

Video 42

Now, $F_{12}(x,y) = \frac{\partial}{\partial x}\left(F_2(x,y)\right)$

$$= \frac{\partial}{\partial x}\left(-F_1(y,x)\right)$$

$$= -\left(F_{11}\underbrace{\frac{\partial y}{\partial x}}_{0} + F_{21}\frac{\partial x}{\partial x}\right)$$

$$= -F_{21}(y,x)$$

c) $F(x,y) = \begin{cases} \frac{2xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

$$F_1(0,y) = \lim_{h\to 0} \frac{F(0+h,y) - F(0,y)}{h} \qquad \left(\begin{matrix} y\neq 0 \\ y=0 \end{matrix}\right)$$

$$= \lim_{h\to 0} \frac{\frac{2\cancel{h}y(h^2-y^2)}{h^2+y^2} - 0}{\cancel{h}}$$

$$= \lim_{h\to 0} \frac{2y(h^2-y^2)}{h^2+y^2} = \frac{-2y^3}{y^2} = -2y \qquad (y\neq 0)$$

Show that also $F_{12}(0,0) = 2$.

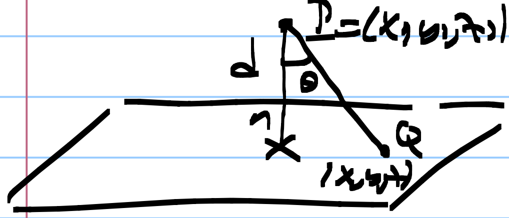
$$F_{12}(0,0) = -F_{21}(0,0) = -\frac{\partial}{\partial y}\left(F_1(0,y)\right)\Big|_{y=0}$$

$$= -\frac{\partial}{\partial y}(-2y)$$

$$= 2.$$

d) $F_{12}(0,0) = 2$ and $F_{21}(0,0) = -F_{12}(0,0) = -2$ and hence $F_{12}(0,0) \neq F_{21}(0,0)$.

3) Find the distance of the origin to the plane $x+2y+3z-4=0$



$Ax+By+Cz+D=0$

$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}} \qquad (x_1,y_1,z_1)=(0\,0\,0)$$

$$=\frac{|D|}{\sqrt{A^2+B^2+C^2}}=\frac{|-4|}{\sqrt{1+4+9}}=\frac{4}{\sqrt{14}}$$

4) $$f(x,y)=\begin{cases}\dfrac{x^3\sin(\pi/2+y)}{x^2+y^2}, & (x,y)\neq(0\,0)\\ 0, & (x,y)=(0\,0).\end{cases}$$

Is f continuous at $(0,0)$?

$$|f(x,y)|=\left|\frac{x^3\sin(\pi/2+y)}{x^2+y^2}\right|$$

$$=\underbrace{\left|\frac{x^2}{x^2+y^2}\right|}_{\leq 1}|x|\,\underbrace{\left|\sin\left(\frac{\pi}{2}+y\right)\right|}_{\leq 1}$$

$$\leq 1.1.|x|$$

So if $(x,y)\to(0,0)$ then $|x|\to 0$ and hence

$$|f(x,y)|\to 0 : \lim_{(x,y)\to(0,0)} f(x,y)=0$$

Since $f(0,0)=0=\lim_{(x,y)\to(0,0)} f(x,y)$ we see that

f is continuous at $(0,0)$.

5) Consider the planes $x+y+z=0$ and $2x-y-3z=2$. Find another plane containing the intersection of these planes.

If (x_0, y_0, z_0) is a point in the intersection of these planes then

$$x_0+y_0+z_0=0 \quad \text{and} \quad 2x_0-y_0-3z_0-2=0.$$

Then for any $\lambda, \mu \in \mathbb{R}$ we have

$$\lambda(x_0+y_0+z_0)+\mu(2x_0-y_0-3z_0-2)=0$$

Hence, (x_0, y_0, z_0) is on the plane

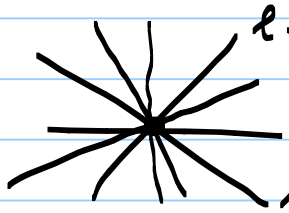
$$\lambda(x+y+z)+\mu(2x-y-3z-2)=0$$

a pencil of planes

In particular, let $\lambda=2$ and $\mu=3$. Then the plane
$2(x+y+z)+3(2x-y-3z-2)=0$ contains the line of intersection of the original planes.

$$8x-y-8z-6=0.$$

Pencil of lines:



$\lambda(ax+by+c)+\mu(dx+ey+f)=0$ is the pencil of lines determined by the point of intersection of l_1 and l_2.

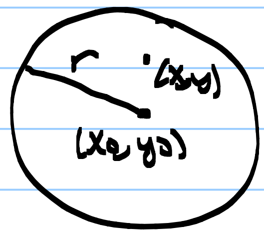
CHAPTER 13. Applications of Partial Derivatives:

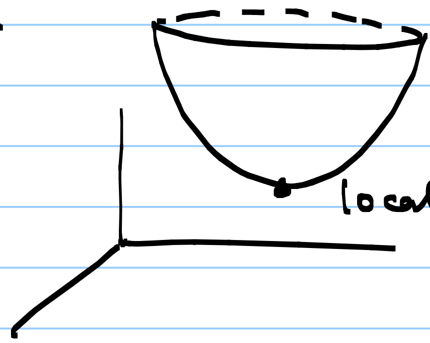
§13.1. Extreme Values:

Given a function $f=f(x,y)$ (or $f=f(x_1,\dots,x_n)$) we say that f has a local maximum (or minimum) at a point (x_0,y_0) if there is a ball $B=\{(x,y) \mid (x-x_0)^2+(y-y_0)^2<r^2\}$ with center (x_0,y_0) and some positive radius $r>0$ so that

$$f(x_0,y_0) \geq f(x,y) \quad (\text{or, } f(x_0,y_0) \leq f(x,y), \text{ respectively})$$

for all $(x,y) \in B$.





We say that f has absolute (or global) maximum (or minimum) at a point (x_0,y_0) if

$$f(x_0,y_0) \geq f(x,y) \quad (\text{or } f(x_0,y_0) \leq f(x,y), \text{ respectively})$$

for all (x,y) in the domain of f.

Such local or absolute maximum and minimum values of f are called extreme values of f.

<u>Remark:</u> It is clear from the definitions that any absolute maximum or minimum is a local maximum or minimum.

As in the case single variable functions, a several variable function has an extreme value at one of the following types of points:

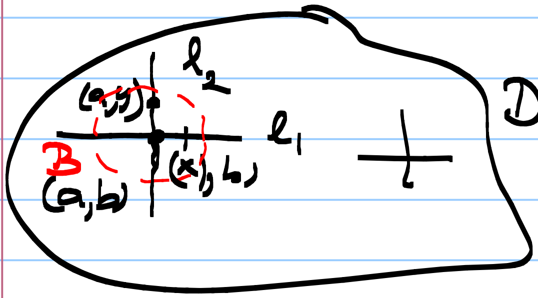
a) Critical points (single variable $f'(x_0)=0$)
b) Singular points ($f'(x_0)$ does not exist)
c) Boundary points (End points of intervals)

Theorem: A function $f(x,y)$ can have a local or absolute extreme value at point (a,b) in its domain only if (a,b) is one of the following:

a) a critical point, that is a point satisfying $\nabla f(a,b)=0$.
b) a singular point of f, that is a point where $\nabla f(a,b)$ does not exist.
c) a boundary point of the domain of f.

Proof: It is enough to show that if (a,b) is not a singular point or a boundary point of the domain of f then (a,b) must a critical point.

So assume that (a,b) is in the domain of f but not on the boundary of the domain and $\nabla f(a,b) = (f_x(a,b), f_y(a,b))$ exists.



D = domain of f

without loss of generality assume that f has a local maximum at (a,b).

Let $g(x) = f(x,b)$ which is a single variable function and $g(a) = f(a,b) \geq f(x,b) = g(x)$, for all $a \in l_1 \cap B$. Hence, g has a local maximum at $x=a$. Clearly, a is not an end point of l_1

and $g'(a) = f_1(a,b)$ exists. Hence, $x=a$ is must be a critical point for $g(x)$. In particular,

$f_1(a,b) = g'(a) = 0.$

Similarly, by restricting f to the line l_2 we see that $f_2(a,b) = 0$.

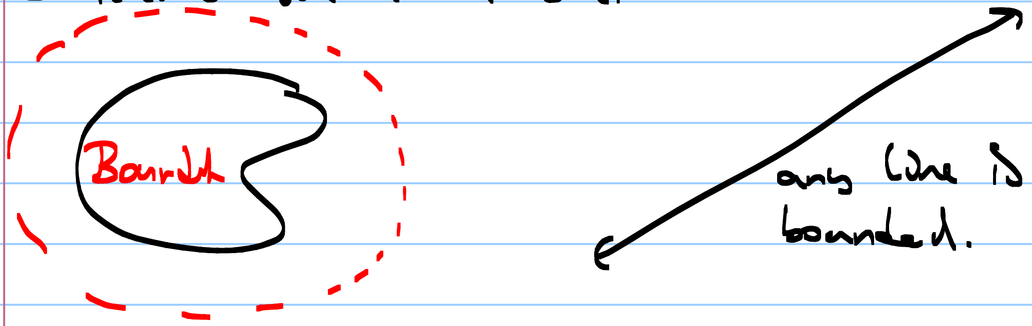
Thus, $\nabla f(a,b) = (f_1(a,b), f_2(a,b)) = (0,0)$, i.e., (a,b) is a critical point for f. This finishes the proof. ∎

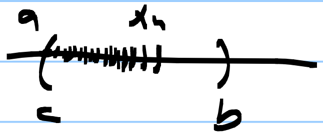
Theorem (Sufficient Conditions for Extreme Values)

If f is a continuous function of n variables whose domain is a closed and bounded set in $\mathbb{R}^n$, then the range of f is a bounded set of real numbers, and there are points in its domain where f takes absolute maximum and minimum values.

Remark: A closed a bounded interval is of type $[a,b]$.

A closed and bounded set in $\mathbb{R}^2$, $\mathbb{R}^3$ or in $\mathbb{R}^n$ is a region so that it lies inside a ball and it contains the limit points of all sequences contained in that set.



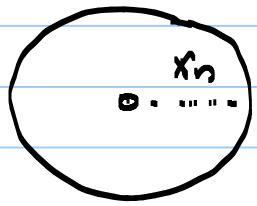


$x_n = a + \frac{1}{n} \in (a,b)$

$\lim x_n = \lim a + \frac{1}{n} = a \notin (a,b)$

$\Rightarrow (a,b)$ is not closed.

Examples $D = \{(x,y) \in \mathbb{R}^2 \mid 0 < x^2+y^2 \leq 1\}$.



not closed

$x_n = (\frac{1}{n}, 0)$ $n = 1,2,\dots$

$x_n \in D$ $\forall n$ but

$\lim x_n = \lim (1/n, 0) = (0,0) \notin D$.

Ex $\mathbb{R}^2$ is closed but not bounded.

Remark: The proof of the above theorem is out of scope of this course and thus it is omitted. It is proved in so called Mathematical Analysis.

Examples 1) The function $f(x,y) = x^2+y^2$ has a unique critical point.

$\nabla f = (f_x, f_y) = (2x, 2y) = (0,0) \Rightarrow x=0, y=0$

So $(0,0)$ is the only critical point.

$f(0,0) = 0 \leq x^2+y^2 = f(x,y)$, for all (x,y).

Hence $(0,0)$ is indeed the absolute minimum of f.

2) Let $h(x,y) = y^2 - x^2$ on $\mathbb{R}^2$. $\nabla h = (h_x, h_y) = (-2x, 2y)$ and hence $\nabla h = (0,0) \Rightarrow (-2x, 2y) = (0,0)$ so that $(x,y) = (0,0)$. Thus again h has a unique critical point. $h(0,0) = 0^2 - 0^2 = 0$ but 0 is neither a

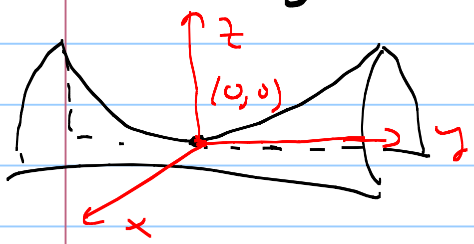
local maximum nor a local minimum.

$h(x,y)=y^2-x^2$ takes all real values since $h(0,y)=y^2$ and $h(x,0)=-x^2$.

Hence, this example shows that a function may not have a local extreme value at a critical point.

A critical point of a function say (a,b) which is inside the domain of the function is called a saddle point of f has no local extrema at that point.

In particular, $(0,0)$ is a saddle point for $h(x,y)=y^2-x^2$ on $\mathbb{R}^2$.



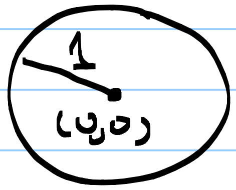
3) $g(x,y)=\sqrt{x^2+y^2}$ on $\mathbb{R}^2$.

$\nabla g=(g_x, g_y)$ $\quad g_x=\dfrac{x}{\sqrt{x^2+y^2}}$ and $g_y=\dfrac{y}{\sqrt{x^2+y^2}}$

Note that $\nabla g \neq (0,0)$ but the point $(0,0)$ is a singular point.

$g(0,0)=0\leq\sqrt{x^2+y^2}=g(x,y)$ so that $0=g(0,0)$ is the absolute minimum of $g(x,y)$. Hence, $g(x,y)$ has its extreme value at a singular point.

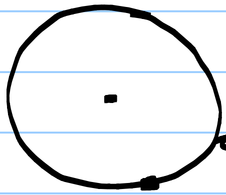
4) Let $f:D\to\mathbb{R}$, $f(x,y)=1-x$, where D is the disc $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 1\}$.



$\xrightarrow{f} \mathbb{R}$

$f(x,y) = 1-x$. So $f_x = -1$, $f_y = 0$, $\nabla f(x,y) = (-1,0)$ exists at every point $(x,y) \in \mathbb{R}^2$ and $\nabla f \neq (0,0)$.
Hence, f has <u>no</u> critical or singular points.

Since D is closed and bounded, f must have both absolute maximum and minimum on D.
Hence, f has its extreme values on the boundary of its domain D. (f is clearly cont!)



$D = \{(x,y) \mid x^2+y^2 \leq 1\}$.

$\leftarrow$ Boundary of D, $\partial D = \{(x,y) \mid x^2+y^2 = 1\}$.

$(x,y) = (\cos\theta, \sin\theta) \quad 0 \leq \theta \leq 2\pi$

$f|_{\partial D}(x,y) = f(\cos\theta, \sin\theta) = 1 - \cos\theta, \quad 0 \leq \theta \leq 2\pi$.

Note that $f|_{\partial D}$ is a single variable function defined on the closed and bounded interval $[0, 2\pi]$.

Let $h(\theta) = f(\cos\theta, \sin\theta) = 1 - \cos\theta$, $\theta \in [0, 2\pi]$.

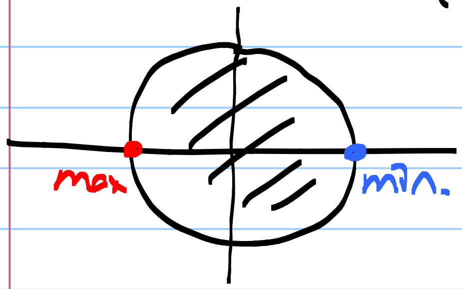
1) Critical points of $h(\theta)$: $h'(\theta) = \sin\theta = 0$
$\Rightarrow \theta = 0, \pi, 2\pi$.

2) Singular points of $h(\theta)$: $h'(\theta) = \sin\theta$ exists at all points so that $h(\theta)$ has <u>no</u> singular point.

3) End points: 0, 2π.

Check: $h(0) = 1 - \cos 0 = 1 - 1 = 0$
$h(\pi) = 1 - \cos\pi = 1-(-1) = 2$ ← Abs. max.
$h(2\pi) = 1 - \cos 2\pi = 1 - 1 = 0.$ abs. min

$\theta = 0 \Rightarrow (x,y) = (\cos 0, \sin 0) = (1,0)$
$\theta = 2\pi \Rightarrow (x,y) = (\cos 2\pi, \sin 2\pi) = (1,0)$
$\theta = \pi \Rightarrow (x,y) = (\cos\pi, \sin\pi) = (-1,0)$



Classifying the critical points:

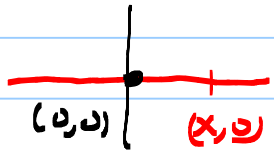
Consider the following function $f(x,y) = 2x^3 - 6xy + 3y^2$ on $\mathbb{R}^2$.
Clearly, f has no singular points. Let's find the critical points:

$$f_x = 6x^2 - 6y = 0 \Rightarrow y = x^2$$
$$f_y = -6x + 6y = 0 \Rightarrow x = y$$
$$\Rightarrow x^2 = x$$

$\Rightarrow x(x-1) = 0 \Rightarrow x = 0$ or $x = 1$.
$\Downarrow$ $y = 0$, $\Downarrow$ $y = 1$

Hence, $(0,0)$ and $(1,1)$ are the only critical values of $f(x,y)$.
Let's check if f has a local extrema at these critical points.

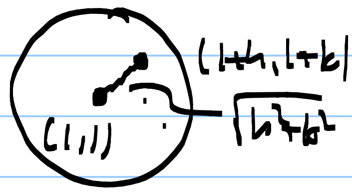
$(0,0)$: $f(x,y) = 2x^3 - 6xy + 3y^2$ when restricted to the x-axis becomes $f|_{x\text{-axis}} = 2x^3$ and has no local extrema at $x=0$.



Hence f has no local extrema at $(0,0)$.

$\underline{\underline{(1,1)}}$: $f(x,y) = 2x^3 - 6xy + 3y^2$

$f(1,1) = 2 - 6 + 3 = -1$.



To see whether f has a local max/min at $(1,1)$ let's compare the value $f(1+h,1+k)$ with $f(1,1)$ for small values of h and k.

$$\begin{aligned} f(1+h,1+k) &= 2(1+h)^3 - 6(1+h)(1+k) + 3(1+k)^2 \\ &= 2(1+\cancel{3h}+3h^2+h^3) - 6 - \cancel{6h} - \cancel{6k} - 6hk \\ &\qquad + 3 + \cancel{6k} + 3k^2. \\ &= 2h^3 + 6h^2 + 3k^2 - 6hk - 1 \end{aligned}$$

$$\underbrace{f(1+h,1+k) - f(1,1)}_{\Delta f} = 2h^3 + 6h^2 + 3k^2 - 6hk - 1 - (-1)$$

$$\begin{aligned} \Delta f &= 2h^3 + 6h^2 + 3k^2 - 6hk \\ &= (3h^2 + 3k^2 - 6hk) + (2h^3 + 3h^2) \\ &= \underbrace{3(h-k)^2}_{\geq 0} + \underbrace{h^2}_{\geq 0}\underbrace{(3+2h)}_{> 0} \text{ if } h \text{ is small!} \end{aligned}$$

≥ 0 is (h,k) is small.

Hence, $f(1+h,1+k) \geq f(1,1)$ if (h,k) is small.

So f has a local minimum at $(1,1)$.

Video 45

Theorem (A second Derivative Test for Critical Points)

Suppose that $a=(a_1,a_2)$ is a critical point of $f(x,y)$ and it is in the interior of the domain of f. Also assume that all second partial derivatives of f are continuous throughout a neighborhood of $a=(a_1,a_2)$. Consider the Hessian matrix

$$H(x,y)=\begin{pmatrix} f_{11}(x,y) & f_{12}(x,y) \\ f_{21}(x,y) & f_{22}(x,y) \end{pmatrix},$$ which is a continuous function

Then we have

a) If $H(a_1,a_2)$ is positive definite then f has a local minimum at (a_1,a_2).

b) If $H(a_1,a_2)$ is negative definite then f has a local maximum at (a_1,a_2).

c) If $H(a_1,a_2)$ is indefinite then f has a saddle point at (a_1,a_2).

d) If $H(a_1,a_2)$ is neither positive nor negative definite nor indefinite, then the test fails and thus gives no information.

$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det A = ad-bc$

Some Linear Algebra:

$$H=\begin{bmatrix} a & b \\ b & d \end{bmatrix}$$ Symmetric 2×2-matrix.

1) H is called positive definite if $a>0$ and $\det H = ad-b^2>0$.

2) H is called negative definite if $a<0$ and
$\det H = ad - b^2 > 0$.

3) H is called indefinite if $\det H < 0$.

Examples 1) Find and classify the critical points of the function

$$f(x,y) = xy\, e^{-(x^2+y^2)/2}$$

Solution: Let's find first the critical points:

$$f_x = y\, e^{-(x^2+y^2)/2} + xy \cdot (-2x)\frac{1}{2}\, e^{-(x^2+y^2)/2}$$

$$f_x = (y - x^2 y)\, e^{-(x^2+y^2)/2} = y(1-x^2)\, e^{-(x^2+y^2)/2}$$

$$f_y = (x - y^2 x)\, e^{-(x^2+y^2)/2} = x(1-y^2)\, e^{-(x^2+y^2)/2}$$

$$f_x = 0 \Rightarrow y(1-x^2)\underbrace{e^{\cdots}}_{\neq 0} = 0 \Rightarrow y(1-x^2) = 0.$$
$$\Rightarrow y = 0 \text{ or } 1-x^2 = 0 \Rightarrow x = \pm 1.$$

$$f_y = 0 \Rightarrow x(1-y^2)e^{\cdots} = 0 \Rightarrow x(1-y^2) = 0$$
$$\Rightarrow x = 0 \text{ or } y = \pm 1.$$

$$\big((\pm 1, y) \text{ or } (x, 0)\big) \text{ and } \big((x, \pm 1) \text{ or } (0, y)\big)$$

$(0,0)$

$(\pm 1, \pm 1)$

no points

no points

So we have 5 critical points: $(\pm 1, \pm 1), (0,0)$

$(1,1)$, $(-1,1)$, $(-1,1)$, $(-1,-1)$ and $(0,0)$.

To classify the critical points we use the 2nd derivative test:

$$H(x,y) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

$f_x = y(1-x^2)e^{-(x^2+y^2)/2}$, $f_y = x(1-y^2)e^{-(x^2+y^2)/2}$

$f_{xx} = xy(x^2-3)e^{-(x^2+y^2)/2}$, $f_{xy} = f_{yx} = (1-x^2)(1-y^2)e^{-(x^2+y^2)/2}$

$f_{yy} = xy(y^2-3)e^{-(x^2+y^2)/2}$.

Let's check the critical points:

$(1,1)$: $f_{xx}(1,1) = -2e^{-1}$, $f_{yy}(1,1) = 2e^{-1}$, $f_{xy}(1,1) = 0$

$$H(1,1) = \begin{pmatrix} f_{xx}(1,1) & f_{xy}(1,1) \\ f_{yx}(1,1) & f_{yy}(1,1) \end{pmatrix} = \begin{pmatrix} -2e^{-1} & 0 \\ 0 & -2e^{-1} \end{pmatrix}$$

$\det H(1,1) = (-2e^{-1})\cdot(-2e^{-1}) - 0\cdot 0 = 4e^{-2} > 0$
and $-2e^{-1} < 0$.
Hence $H(1,1)$ is negative definite.
Therefore, f has a local maximum at $(1,1)$.

$(1,-1)$ or $(-1,1)$ $\quad f_{xx} = f_{yy} = 2e^{-1}$ and
$f_{xy} = f_{yx} = 0$ at $(1,-1)$
and $(-1,1)$.

So $H(1,-1) = H(-1,1) = \begin{pmatrix} 2e^{-1} & 0 \\ 0 & 2e^{-1} \end{pmatrix}$

$\det H(1,-1) = \det H(-1,1) = 4e^{-2} > 0$ and $2e^{-1} > 0$ and thus $H(1,-1) = H(-1,1)$ is positive definite and thus f has local minimum at $(1,-1)$ and $(-1,1)$.

$(-1,-1)$: $f_{xx}(-1,-1) = -2e^{-1} = f_{yy}(-1,-1)$ and

$f_{xy}(-1,-1) = 0.$

$$H(-1,-1) = \begin{pmatrix} -2e^{-1} & 0 \\ 0 & -2e^{-1} \end{pmatrix}$$

$\det H(-1,-1) = 4e^{-2} > 0$ and $-2e^{-1} < 0.$

Hence, H is negative definite at $(-1,-1)$ and hence, f has a local maximum at $(-1,-1)$.

$(0,0)$: $f_{xx}(0,0) = 0 = f_{yy}(0,0)$ and $f_{xy}(0,0) = 1 = f_{yx}(0,0)$

$$H(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

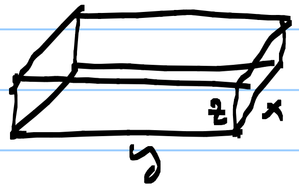
$\det H = 0 \cdot 0 - 1 \cdot 1 = -1 < 0$

Hence, H is indefinite and thus $(0,0)$ is a saddle point for $f(x,y)$.

Video 46

Example: Find the shape of a rectangular box with no top having given volume V and the least possible total surface area of its five faces.

Solution:



$V = xyz$ — fixed volume

S = sum of areas of the 5-faces
$= xy + 2yz + 2xz$

Find x, y, z so that S is minimal and $\overset{0}{\hat{V}} = xyz$.

$$z = \frac{V}{xy} \Rightarrow S = S(x,y) = xy + 2y\frac{V}{xy} + 2x \cdot \frac{V}{xy}$$

$$S = S(x,y) = xy + 2V\left(\frac{1}{x} + \frac{1}{y}\right), \quad x, y > 0$$

$D = \{(x,y) \in \mathbb{R}^2 \mid x, y > 0\}$ has no boundary and thus

if S has any extreme value then that should be attained at either at a critical point or at a singular point.

$$\nabla S = \left(\frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}\right) \qquad \frac{\partial S}{\partial x} = y - 2V\frac{1}{x^2}$$

$$\frac{\partial S}{\partial y} = x - 2V\frac{1}{y^2}$$

Both exists on D and thus there is no singular point.

Critical points: $\frac{\partial S}{\partial x} = 0$ and $\frac{\partial S}{\partial y} = 0$

$$y - 2V\frac{1}{x^2} = 0 \quad \text{and} \quad x - \frac{2V}{y^2} = 0$$

$\Rightarrow x^2y = 2V$ and $xy^2 = 2V$

$\Rightarrow x^2y = xy^2 \Rightarrow x = y. \Rightarrow 2V = x^2y = x^3$

$\Rightarrow x = (2V)^{1/3} = y$. Hence, $z = \frac{V}{xy} = \frac{V}{(2V)^{2/3}}$

$\Rightarrow z = \frac{V^{1/3}}{2^{2/3}}$.

There is only one critical point

$\boxed{(x, y, z) = \left((2V)^{1/3}, (2V)^{1/3}, 2^{-2/3} V^{1/3}\right).}$ Dimensions of the desired box

Let's use the 2nd derivative test to decide whether we have a local max. or min. at this critical point.

$$S = S(x,y) = xy + 2V\left(\frac{1}{x} + \frac{1}{y}\right).$$

$$S_x = y - \frac{2V}{x^2}, \quad S_y = x - \frac{2V}{y^2}$$

$$S_{xx} = \frac{4V}{x^3}, \quad S_{yx} = 1, \quad S_{yy} = \frac{4V}{x^3}.$$

$$H(x,y) = \begin{pmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{pmatrix} = \begin{pmatrix} \frac{4V}{x^3} & 1 \\ 1 & \frac{4V}{y^3} \end{pmatrix}$$

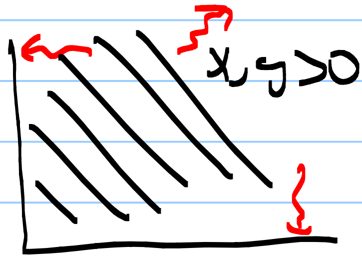
$\det H(x,y) = \frac{16V^2}{x^3y^3} - 1.$ $\qquad x = y = (2V)^{1/3}$

$$H\left((2V)^{1/3}, (2V)^{1/3}\right) = \frac{16V^2}{2V \cdot 2V} - 1 = 4 - 1 = 3 > 0.$$

$$S_{xx}\left((2V)^{1/3},(2V)^{1/3}\right) = \frac{4V}{2V} = 2 > 0.$$

Hence, H is positive definite and thus we have a local minimum at the critical point.

$$S\left((2V)^{1/3},(2V)^{1/3}\right) = (2V)^{2/3} + 2V\left((2V)^{-1/3}+(2V)^{-1/3}\right)$$
$$= (2V)^{2/3}+(2V)^{2/3}+(2V)^{2/3}$$
$$= 3\,(2V)^{2/3}.$$



$$S(x,y) = xy + 2V\left(\frac{1}{x}+\frac{1}{y}\right)$$

If x or $y \to 0$ then $S \to +\infty$.

If x or $y \to \infty$ then again $S \to \infty$.

This implies that the local minimum is the absolute minimum.

<u>Idea of the proof of the 2nd derivative test:</u>

First consider a one variable function $f:\mathbb{R}\to\mathbb{R}$ having a critical point at x_0. Assume f has continuous 2^{nd} derivative. By Taylor's theorem we have at any point $x\in\mathbb{R}$ we have

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(\xi)(x-x_0)^2, \text{ for}$$

some $\xi \in (x, x_0)$.

Since x_0 is a critical point we have $f'(x_0)=0$ so that

$$\underline{f(x) - f(x_0)} = \underbrace{\tfrac{1}{2}}_{} \underbrace{f''(\xi)}_{\geq 0} \underbrace{(x-x_0)^2}_{\geq 0}$$

If $f''(x_0) > 0$ then $f''(x) > 0$ for x close enough to x_0. In particular if $|x-x_0|$ is small enough then $f''(\xi) > 0$.

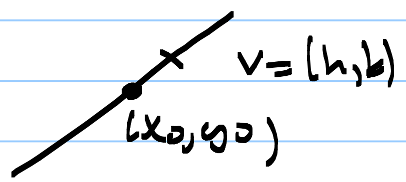
So if $|x-x_0|$ is small enough then $f(x)-f(x_0) \geq 0$.

$\Rightarrow f(x) \geq f(x_0)$ for all x closed enough to x_0.

In other words f has a local minimum at x_0.

Now let's apply this idea to several variable functions:

Assume that $f = f(x,y)$ has a critical point (x_0, y_0). Then consider any line through (x_0, y_0):



$$(x,y) = (x_0, y_0) + tv = (x_0 + th, y_0 + tk)$$

$$g(t) = f(x,y) = f(x_0+th, y_0+tk), \quad t \in \mathbb{R}.$$

Then $g(t) = g(0) + g'(0)(t-0) + g''(\xi)(t-0)^2$

for some $\xi \in (0,t)$.

$g(t) = f(x_0+th, y_0+tk)$, $g'(t) = f_1 \cdot h + f_2 \cdot k$

$g'(t) = h f_1(x_0+th, y_0+tk) + k f_2(x_0+th, y_0+tk)$

$$g'(0) = h f_1(x_0,y_0) + k\cdot f_2(x_0,y_0)$$

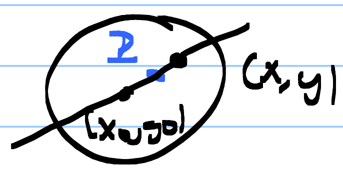
$$g''(t) = h(f_{11}\cdot h + f_{21}\cdot k) + k(f_{12}\cdot h + f_{22}\cdot k)$$

$$= h^2 f_{11} + hk(\underbrace{f_{12} + f_{21}}_{2f_{12}}) + k^2 f_{22}$$

$$g(t) = g(0) + g'(0)t + g''(\xi)\, t^2$$

$$f(\underbrace{x_0+th}_{x}, \underbrace{y_0+tk}_{y}) = f(x_0,y_0) + (h\underbrace{f_1(x_0,y_0)}_{0} + k\underbrace{f_2(x_0,y_0)}_{0})t$$
$$+ (h^2 f_{11}(x_0+\xi h, y_0+\xi k) + 2hk f_{12}(x_0+\xi h, y_0+\xi k)$$
$$+ k^2 f_{22}(x_0+\xi h, y_0+\xi k))$$

Since (x_0,y_0) is a critical point for $f(x,y)$
$f_1(x_0,y_0) = 0 = f_2(x_0,y_0)$.



$$f(x,y) - f(x_0,y_0) = (h^2 f_{11} + 2hk f_{12} + k^2 f_{22})(p)$$

evaluated at some point on the line segment joining (x_0,y_0) to (x,y).

$$f(x,y) - f(x_0,y_0) = [h \;\; k] \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix} (p) \begin{bmatrix} h \\ k \end{bmatrix}$$

$$H(p) = \begin{bmatrix} f_{11}(p) & f_{12}(p) \\ f_{21}(p) & f_{22}(p) \end{bmatrix}$$

$H(p)$ is called positive (negative) definite if

$$[h \;\; k]\, H(p) \begin{bmatrix} h \\ k \end{bmatrix} > 0 \;(<) \text{ for all } (h,k) \neq (0,0)$$

Hence, if $H(p)$ is positive definite then

$f(x,y) - f(x_0,y_0) \geq 0$ for all (x,y) close enough to (x_0,y_0).

Hence, f has a local minimum at (x_0,y_0).

If $H(p)$ is negative definite then

$f(x,y) - f(x_0,y_0) \leq 0$ so that f has a local maximum at (x_0,y_0).

$H(p)$ is called indefinite if $[h\ k]\, H(p) \begin{bmatrix} h \\ k \end{bmatrix}$ is > 0 for some

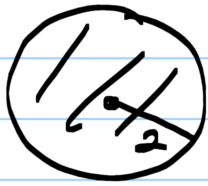
(h,k) and < 0 for some other (h,k).
In this case, (x_0,y_0) is a saddle point for $f(x,y)$.

Video 47

§13.2. Extreme Values of Functions Defined on Restricted Domains

Example: Find the maximum and minimum values of $f(x,y) = 2xy$ on the closed disk $x^2+y^2 \le 4$.

Solution:

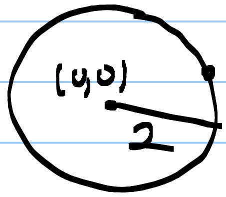


Since f is continuous on the closed and bounded disk $x^2+y^2 \le 4$, f has a maximum and minimum value on the disk.

$\nabla f = (f_x, f_y) = (2y, 2x)$ which exists at any point. Hence, there are no singular points.

Critical points. $\nabla f = (0,0) \Rightarrow 2y = 0,\ 2x = 0$
$\Rightarrow (x,y) = (0,0)$, the only critical point.

Boundary Points:



$(x,y) = (2\cos\theta, 2\sin\theta)$

$$f(x,y) = f(2\cos\theta, 2\sin\theta) = 2\cdot(2\cos\theta)(2\sin\theta)$$
$$= 8\cos\theta\sin\theta$$
$$g(\theta) \doteq 4\sin 2\theta \quad , \quad 0 \le \theta \le 2\pi.$$

Hence, finding extreme values of f on the boundary circle is the same as find extreme values of $g(\theta) = 4\sin 2\theta$ on the closed interval $[0, 2\pi]$.

$g'(\theta) = 8\cos 2\theta$ exists at all points.
Hence, there are no singular points.

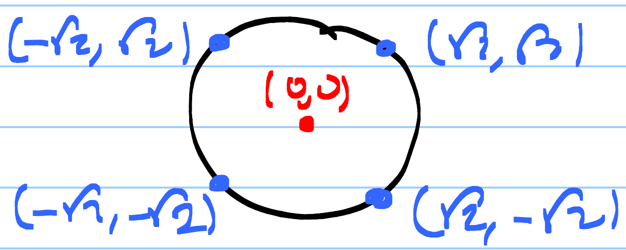
$g'(\theta) = 0 \Rightarrow 8\cos 2\theta = 0$ so we have

$2\theta = \pi/2,\ 2\theta = 3\pi/2,\ 2\theta = 5\pi/2,\ 2\theta = 7\pi/2$

because since $\theta \in [0, 2\pi]$, $2\theta \in [0, 4\pi]$.

$\Rightarrow \theta_1 = \pi/4,\ \theta_2 = 3\pi/4,\ \theta_3 = 5\pi/4,\ \theta_4 = 7\pi/4$.

$\theta_1 = \pi/4 \Rightarrow (x,y) = (2\cos \pi/4, 2\sin \pi/4) = (\sqrt{2}, \sqrt{2})$
$\theta = 3\pi/4 \Rightarrow (x,y) = (2\cos 3\pi/4, 2\sin 3\pi/4) = (-\sqrt{2}, \sqrt{2})$
$\theta = 5\pi/4 \Rightarrow (x,y) = (2\cos 5\pi/4, 2\sin 5\pi/4) = (-\sqrt{2}, -\sqrt{2})$
$\theta = 7\pi/4 \Rightarrow (x,y) = (2\cos 7\pi/4, 2\sin 7\pi/4) = (\sqrt{2}, -\sqrt{2})$.



One critical point for f and 4 boundary points.

$f(x,y) = 2xy$

$(x,y) = (0,0) \Rightarrow f(0,0) = 0$
$(x,y) = (\pm\sqrt{2}, \pm\sqrt{2})$, $f(\pm\sqrt{2}, \pm\sqrt{2}) = \pm 2\cdot\sqrt{2}\cdot\sqrt{2} = \pm 4$

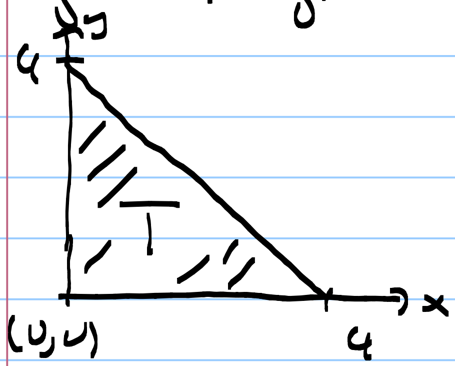
Hence, the maximum value of f is 4 and it is attained at the points $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$

The minimum value of f is -4 and is attained at the points $(-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$.

[Note that $f(0,0) = 0$ but $f(x,y) > 0$ and $f(x,y) < 0$ at points near $(0,0)$ so that $(0,0)$ is a saddle point for f.]

($f = 2xy$ $\quad f_x = 2y$ $\quad f_y = 2x$, $f_{xx} = 0$
$f_{yy} = 0$, $f_{xy} = 2$ $\quad \det\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = -4 < 0$ indefinite)

Example: Find the extreme values of the function $f(x,y) = x^2 y e^{-(x+y)}$ on the triangular region T given by $x \geq 0, y \geq 0, x+y \leq 4$.



Solution:

$$f(x,y) = x^2 y e^{-(x+y)}.$$

$$f_x = 2xy e^{-(x+y)} - x^2 y e^{-(x+y)} = (2xy - x^2 y) e^{-(x+y)}$$

$$f_y = x^2 e^{-(x+y)} - x^2 y e^{-(x+y)} = (x^2 - x^2 y) e^{-(x+y)}$$

∇f exists at every point so that f has no singular points.

Critical points: $f_x = 0$, $f_y = 0$

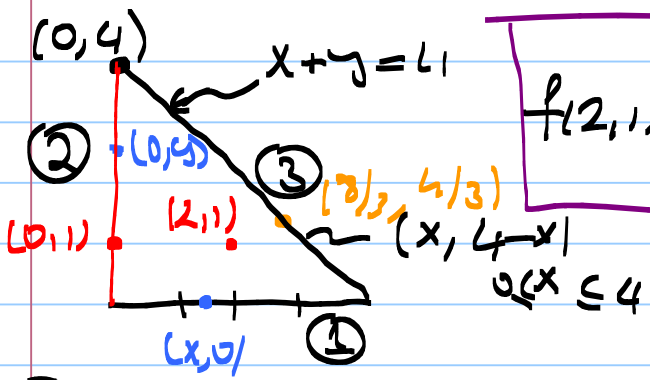
Since $e^{-(x+y)} > 0$ we see that $2xy - x^2 y = 0$ and $x^2 - x^2 y = 0$.

$\Rightarrow x^2(1-y) = 0. \Rightarrow x = 0$ or $y = 1$.

$x = 0 \Rightarrow 2xy - x^2 y = 0$, Hence any point of the form $(0,y)$ is a critical point.

$y = 1$. Then $2xy - x^2 y = 0$ becomes $2x - x^2 = 0$.
$\Rightarrow x(2-x) = 0 \Rightarrow x = 0$ or $x = 2$.

Hence, we get two critical points $(0,1)$ and $(2,1)$.



$$\boxed{\begin{aligned} f(2,1) &= x^2 y\, e^{-(x+y)} \\ &= 4e^{-3} \end{aligned}}$$

<u>Boundaries</u>: ① $(x,y) = (x,0) \quad 0 \le x \le 4$

$f(x,y) = (x,0) = \boxed{0}$

② $(x,y) = (0,y) \quad f(x,y) = f(0,y) = \boxed{0.}$

③ $(x,y) = (x, 4-x), \ 0 \le x \le 4$

$f(x,y) = x^2 y\, e^{-(x+y)} = x^2 (4-x)\, e^{-4}, \quad 0 \le x \le 4.$

$g(x) = e^{-4}(-x^3 + 4x^2)$ on $[0,4]$.

$g'(x) = e^{-4}(-3x^2 + 8x)$ exists everywhere.
so no singular points.

$g'(x) = 0 \Rightarrow -3x^2 + 8x = 0 \Rightarrow x(-3x+8) = 0$

$x = 0$ or $-3x + 8 = 0 \Rightarrow x = 8/3$

$\Downarrow \qquad\qquad \Downarrow$

$y = 4 - x = 4 \qquad\qquad y = 4 - x = 4 - \frac{8}{3} = 4/3$

$\Rightarrow (0,4)$ and $(8/3, 4/3)$

$f(8/3, 4/3) = x^2 y\, e^{-(x+y)} = \boxed{e^{-4}\, \frac{256}{27}}$

$\boxed{f(2,1) = 4e^{-3}}$ ← maximum value.

Video 48

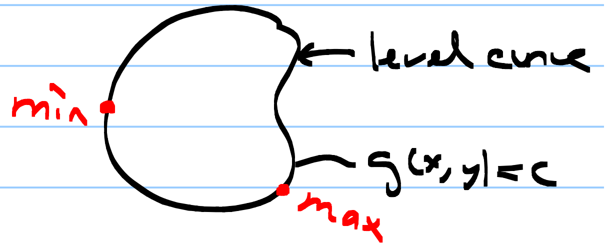
So the minimum value is zero and it is obtained on the right side of the triangle.

§ 13.3. Lagrange Multipliers:

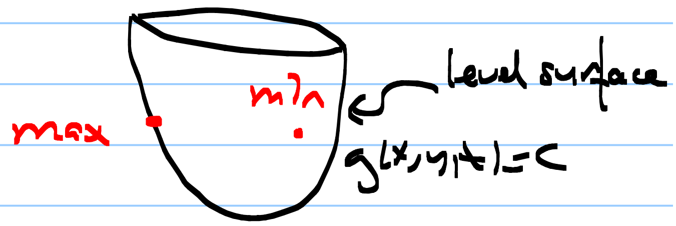
This is a method to find extreme values of a several variable function on a curve in $\mathbb{R}^2$ or a surface in $\mathbb{R}^3$.

Such as:

1) maximize $f(x,y)$ subject to $g(x,y)=c$
(minimize)



2) maximize $f(x,y,z)$ subject to $g(x,y,z)=c$.



3) minimize/maximize $f(x,y,z,w)$

subject to $g(x,y,z,w)=c_1$, $h(x,y,z,w)=c_2$.

Theorem: Suppose f and g have continuous first partial derivatives near the point $P_0=(x_0,y_0)$ on the curve $\mathcal{C}$ with equation $g(x,y)=0$.

Suppose also that, when restricted to the points on $\mathcal{C}$, the function $f(x,y)$ has a local maximum or a local minimum value at P_0. Finally, suppose that

i) P_0 is not an end point of $\mathcal{C}$

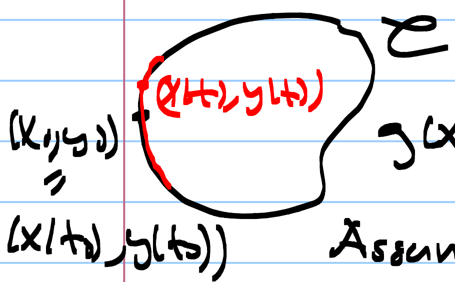
ii) $\nabla g(P_0) \neq (0,0)$.

Then there exists a number λ_0 such that (x_0, y_0, λ_0) is a critical point of the so called Lagrangian function

$$L(x,y,\lambda) = f(x,y) + \lambda g(x,y).$$

Hence, in order to find the extreme values of $f(x,y)$ on the curve $\mathcal{C}$ given by $g(x,y)=0$ one may find the critical points (x_0, y_0, λ_0) of the function $L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$.

Idea of the proof



$g(x,y) = 0$, $\nabla g = (g_x, g_y)$

Assume that f has an extreme value at (x_0, y_0).

Suppose that $(x(t), y(t))$ is a parametrization for the curve $\mathcal{C}$ near (x_0, y_0). Say $(x_0, y_0) = (x(t_0), y(t_0))$ for some t_0.

Then $g(x(t), y(t)) = 0$ for all t and the function $h(t) = f(x(t), y(t))$ has extreme value at t_0. Since the derivatives exist there is no singular point. Also we know that (by the assumption) P_0 is not an end point of $\mathcal{C}$. Hence, t_0 must be a critical point of $h(t)$.

Hence, $h'(t_0)=0$.

$$h'(t) = f_x(x(t),y(t))\,x'(t) + f_y(x(t),y(t))\,y'(t)$$

$$0 = h'(t_0) = f_x(x(t_0),y(t_0))\,x'(t_0) + f_y(x(t_0),y(t_0))\,y'(t_0)$$

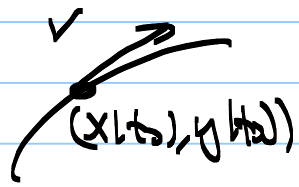
$$\boxed{0 = f_x(P)\cdot x'(t_0) + f_y(P)\,y'(t_0).}$$

Since $g(x(t),y(t))=0$ for all t, we see that by taking d/dt of both sides

$$0 = g_x(\underline{x(t),y(t)})\,x'(t) + g_y(x(t),y(t))\,y'(t),$$

for all t.

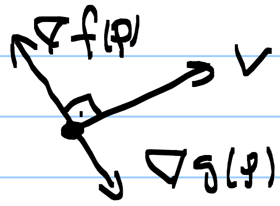
Plug $t=t_0$ $\Longrightarrow$ $\boxed{0 = g_x(P)\,x'(t_0) + g_y(P)\,y'(t_0).}$



$$f_x(P)\,x'(t_0) + f_y(P)\,y'(t_0) = 0$$

$$\nabla f(P)\cdot V = 0, \qquad V = (x'(t_0), y'(t_0))$$

Similarly, $\nabla g(P)\cdot V = 0$



Then $\nabla f(P) + \lambda_0 \nabla g(P) = 0$ for some λ_0.

Video 49

Now (x_0, y_0, λ_0) is a critical point of $L(x,y,\lambda)$.

$$L(x,y,\lambda) = f(x,y) + \lambda g(x,y).$$

$$L_x = f_x + \lambda g_x, \quad L_y = f_y + \lambda g_y$$

$$L_\lambda = 0 + g(x,y).$$

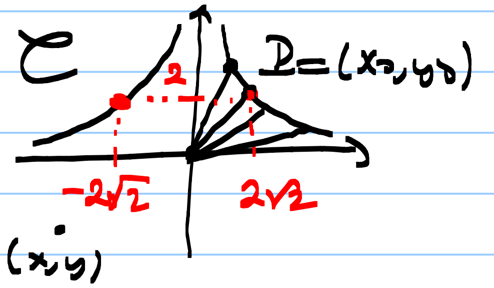
$$L_x(x_0, y_0, \lambda_0) = f_x(x_0, y_0) + \lambda_0 g_x(x_0, y_0) = 0$$

$$L_y(x_0, y_0, \lambda_0) = f_y(x_0, y_0) + \lambda_0 g_y(x_0, y_0) = 0$$

$$L_\lambda = g(x_0, y_0) = 0.$$

Example: Find the shortest distance from the origin the curve $x^2 y = 16$.

Solution: $x^2 y = 16 \Rightarrow y = \dfrac{16}{x^2}$



$f(x,y) = x^2 + y^2$ the distance^2 of the point (x,y) to the origin.

Our problem: Find the $(\text{minimum})^2$ of $f(x,y) = x^2 + y^2$

subject to the condition $g(x,y) = x^2 y - 16 = 0$

$f_x = 2x, \; f_y = 2y, \; g_x = 2xy, \; g_y = x^2$ are all continuous. The level curve C has no end points. $\nabla g = (g_x, g_y) = (2xy, x^2) \neq (0,0)$ when $x^2 y = 16$.

Hence, we may apply the theorem. If (x_0, y_0) is a point on C at which f has its maximum or minimum then there is some $\lambda_0 \in \mathbb{R}$ so that (x_0, y_0, λ_0) is a critical point for

$$h(x,y,\lambda) = f(x,y) + \lambda g(x,y) = x^2+y^2+\lambda(x^2y-16)$$

$$h_x = 2x + 2\lambda xy = 0 \Rightarrow 2x(1+\lambda y) = 0$$

$$h_y = 2y + \lambda x^2 = 0 \qquad \text{Hence } 2x = 0 \text{ or } 1+\lambda y = 0.$$

$$h_\lambda = x^2 y - 16 = 0$$

$2x = 0 \Rightarrow x = 0$, then $h_\lambda = 0 - 16 \neq 0$ a contradiction. So $x = 0$ gives no solution.

Thus, $1+\lambda y = 0 \Rightarrow \lambda y = -1 \Rightarrow y = -1/\lambda$

$$0 = h_y = 2y + \lambda x^2 = -\frac{2}{\lambda} + \lambda x^2 = 0 \Rightarrow \lambda x^2 = \frac{2}{\lambda}$$

$$x^2 y = 16 \Rightarrow x^2 \cdot \left(-1/\lambda\right) = 16 \qquad \Rightarrow x^2 = \frac{2}{\lambda^2}$$

$$\Downarrow$$

$$\frac{2}{\lambda^2}\left(-\frac{1}{\lambda}\right) = 16 \Rightarrow -\frac{1}{\lambda^3} = 8$$

$$\Rightarrow \lambda^3 = -\frac{1}{8} \Rightarrow \lambda = -\frac{1}{2}.$$

Then $y = -1/\lambda = 2$. And $x^2 = \frac{2}{\lambda^2} = 8$

$\Rightarrow x = \pm 2\sqrt{2}$.

Hence, we get two critical points of $h(x,y,\lambda)$.

$(x_0, y_0, \lambda_0) = (-2\sqrt{2}, 2, -1/2)$ and

$(x_1, y_1, \lambda_1) = (2\sqrt{2}, 2, -1/2)$.

$f(x_0, y_0) = f(-2\sqrt{2}, 2) = (-2\sqrt{2})^2 + (-2)^2 = 12$

$f(x_1, y_1) = f(2\sqrt{2}, 2) = (-2\sqrt{2}) + 2^2 = 12$.

Hence, the distance from these points to the origin is $\sqrt{12}$.

__Remark:__ The method and the theorem above work for functions of three or more variables also. So, for $f = f(x,y,z)$ and $g = g(x,y,z)$ the statement would be the same: If f and g have continuous partial derivatives and if f has a maximum/minimum at point (x_0, y_0, z_0) subject to the condition that $g(x,y,z) = c$ is such that (x_0, y_0, z_0) is __not__ an end point of the level surface $g(x,y,z) = c$ and $\nabla g(x_0, y_0, z_0) \neq 0$ then

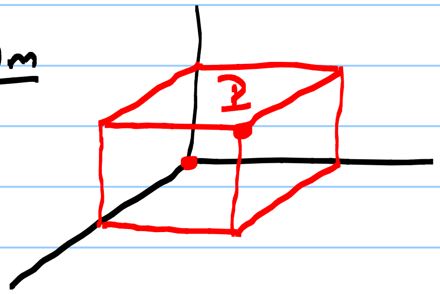
$(x_0, y_0, z_0, \lambda_0)$ would be a critical point for the Lagrange function

$L(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z)$.

__Example:__ Find the maximum volume of a rectangular box with faces parallel to the coordinate planes if one corner is at the origin and the diagonally opposite corner is on the first octant part of the surface $xy + 2yz + 3xz = 18$.

Video 50

Solution



$P = (x, y, z)$

$xy + 2yz + 3xz = 18.$

Maximize $V(x,y,z) = xyz$ subject to

$g(x,y,z) = xy + 2yz + 3xz - 18 = 0$

If (x_0, y_0, z_0) is a point on $g(x,y,z) = 0$ at which $V(x,y,z)$ has its maximum then $(x_0, y_0, z_0, \lambda_0)$ is a critical point for

$$L(x,y,z,\lambda) = V + \lambda g = xyz + \lambda(xy + 2yz + 3xz - 18).$$

$L_x = 0 \Rightarrow yz + \lambda(y + 3z) = 0 \Rightarrow yz = -\lambda(y+3z)$

$L_y = 0 \Rightarrow xz + \lambda(x + 2z) = 0 \Rightarrow xz = -\lambda(x+2z)$

$L_z = 0 \Rightarrow xy + \lambda(2y + 3x) = 0 \Rightarrow xy = -\lambda(2y+3x)$

$L_\lambda = 0 \Rightarrow xy + 2yz + 3xz - 18 = 0$

$\dfrac{y}{x} = \dfrac{y+3z}{x+2z}$

$xy + 2yz = xy + 3xz$

$2yz = 3xz$

$1 \cdot L_z + 3L_y + 2L_x = 0$

$xy + \lambda(2y + 3x) + 3xz + 3\lambda(x + 2z) + 2yz + 2\lambda(y + 3z) = 0$

$$\boxed{18 + \lambda(6x + 4y + 12z) = 0}$$

$\lambda \neq 0$

$2yz = 3xz \Rightarrow z(3x - 2y) = 0.$

So either $z = 0$ or $3x - 2y = 0.$

$\Downarrow$ $\qquad$ $\Downarrow$

$x = 0,\ y = 0$ $\qquad$ So $3x = 2y.$

$\Downarrow$ $\qquad$ $y = \dfrac{3x}{2}.$

contradiction.

$18 + 12\lambda$

$(x+z) = 0$

$\Rightarrow 18 + 12\lambda(x + z) = 0.$

Also $6x\lambda + xy = 0$ $\quad\quad$ $2y = 3x$

$\Rightarrow 6x\lambda + \frac{3x^2}{2} = 0$ $\quad\quad$ $y = \frac{3x}{2}$

$\Rightarrow x\left(6\lambda + \frac{3x}{2}\right) = 0 \Rightarrow x = 0 \Rightarrow y = 0 \Rightarrow z = 0$

$\Rightarrow$ contradiction.

$\Rightarrow 6\lambda + \frac{3x}{2} = 0 \Rightarrow x = -4\lambda.$

$\Rightarrow y = -6\lambda$

$18 + \lambda(6x + 4y + 12z) = 0$

$18 + \lambda(-24\lambda - 24\lambda + 12z) = 0$

$48\lambda^2 - 12\lambda z - 18 = 0$

$8\lambda^2 - 2\lambda z - 3 = 0 \Rightarrow z = \frac{8\lambda^2 - 3}{2\lambda}$

Plug them in $xy + 2yz + 3xz = 18$

$24\lambda^2 + 2(-6\cancel{\lambda})\left(\frac{8\lambda^2-3}{2\cancel{\lambda}}\right) + 3(-4\cancel{\lambda})\left(\frac{8\lambda^2-3}{2\cancel{\lambda}}\right) = 18$

$24\lambda^2 - 12(8\lambda^2 - 3) - 12(8\lambda^2 - 3) = 18$

$(24 - 192)\lambda^2 + 36 + 36 = 18$

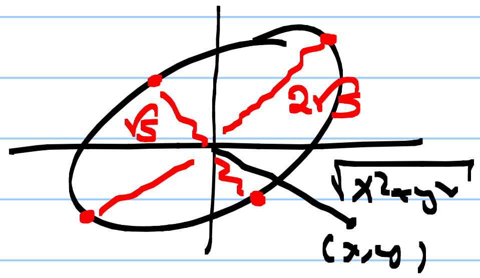
$\lambda^2 = \frac{54}{168} = \frac{9}{28} \Rightarrow \lambda_0 = \pm\frac{3}{\sqrt{28}}$

$x = -4\lambda, \; y = -6\lambda, \; z = \frac{8\lambda^2 - 3}{2\lambda}$ $\quad$ $\frac{72}{28} - 3$

$x_0 = \frac{12}{\sqrt{28}}, \; y_0 = \frac{18}{\sqrt{28}}, \; z_0 = \frac{-12}{-6/\sqrt{28}} = 2\sqrt{28}$

$V(x_0, y_0, z_0) = xyz = \frac{432}{\sqrt{28}}$

Example: Find the points on the curve
$17x^2+12xy+8y^2=100$ that are
closest to and farthest away from the origin.



Extremize $f(x,y)=x^2+y^2$
subject to
$g(x,y)=17x^2+12xy+8y^2-100=0.$

Solution: $L(x,y,\lambda)=f(x,y)+\lambda\, g(x,y)$
$= x^2+y^2+\lambda(17x^2+12xy+8y^2-100)$

Look for critical points of $L(x,y,\lambda)$.

$$L_x = 2x+\lambda(34x+12y)=0 \Rightarrow \lambda = \frac{-2x}{34x+12y}$$
$$L_y = 2y+\lambda(12x+16y)=0 \Rightarrow \lambda = \frac{-2y}{12x+16y}$$
$$L_\lambda = 17x^2+12xy+8y^2-100=0$$

$$\Rightarrow \frac{\cancel{-2}x}{34x+12} = \frac{\cancel{-2}y}{12x+16y} \Rightarrow 12x^2+16xy = 34xy+12y^2$$

$$\Rightarrow 2x^2-3xy-2y^2=0 \quad \Big|\, 4$$
$$+\ 17x^2+12xy+8y^2=100 \quad \Big|\, 1$$

$$25x^2+0\cdot xy+0\cdot y^2 = 100 \Rightarrow x^2=4 \Rightarrow x=\pm 2.$$

$x=2 \Rightarrow 2\cdot x^2-3xy-2y^2=0 \Rightarrow 8-6y-2y^2=0$

$$y^2+3y-4=0 \Rightarrow y=\frac{-3\pm\sqrt{9+16}}{2}$$

$8+6y-2y^2=0$

$$= \frac{-3\pm 5}{2} = -4,\ 1$$

VNeo 51

$x=2 \Rightarrow y=-4$ or $1 \Rightarrow (x,y) = \boxed{(2,-4)}$ and $\boxed{(2,1)}$.

$x=-2 \Rightarrow 8+6y-2y^2=0 \Rightarrow y^2-3y-4=0$

$$y = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3\pm 5}{2} = 4, -1$$

$\boxed{(-2,4)}$ and $\boxed{(-2,-1)}$.

Distance: $(2,-4) \Rightarrow \sqrt{4+16} = 2\sqrt{5} \leftarrow$ maximal
$(-2,4) \Rightarrow \sqrt{4+16} = 2\sqrt{5} \leftarrow$
$(-2,-1) \Rightarrow \sqrt{4+1} = \sqrt{5} \leftarrow$ minimal.
$(2,1) \Rightarrow \sqrt{4+1} = \sqrt{5} \leftarrow$

More than on Constraint:

Problem: Extremize $f(x,y,z)$ subject to $g(x,y,z)=c_1$ and $h(x,y,z)=c_2$.

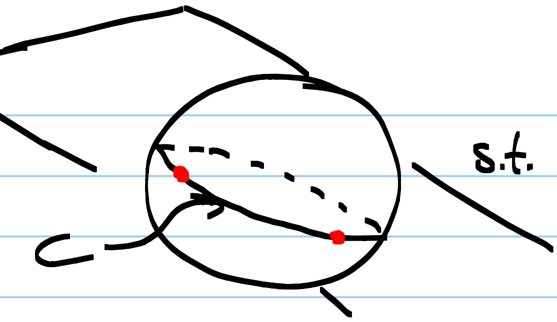
In this case the Lagrange function has two Lagrange multipliers:

$$L(x,y,z,\lambda,\mu) = f(x,y,z) + \lambda(g(x,y,z)-c_1) + \mu(h(x,y,z)-c_2)$$

Then look for critical points of $L(x,y,z,\lambda,\mu)$.

Example: Find the maximum and minimum values of $f(x,y,z) = xy+2z$ on the circle that is the intersection of the plane $x+y+z=0$ and the sphere $x^2+y^2+z^2=24$.

Solution



$f(x,y,z) = xy + 2z$

s.t. $g(x,y,z) = x+y+z = 0$

$h(x,y,z) = x^2+y^2+z^2-24=0$

Lagrange function

$$L(x,y,z,\lambda,\mu) = f + \lambda g + \mu h$$
$$= xy + 2z + \lambda(x+y+z) + \mu(x^2+y^2+z^2-24)$$

Critical points:

$$L_x = y + \lambda + 2\mu x = 0$$
$$L_y = x + \lambda + 2\mu y = 0$$
$$L_z = 2 + \lambda + 2\mu z = 0$$
$$L_\lambda = x+y+z = 0$$
$$L_\mu = x^2+y^2+z^2-24 = 0$$

$$y - x + 2\mu(x-y) = 0$$
$$\Rightarrow (x-y)(2\mu-1) = 0$$
$$\Rightarrow 1)\ x = y \quad \text{or}$$
$$2)\ \mu = 1/2.$$

Case 2: $\mu = 1/2 \Rightarrow x+y+\lambda = 0 \Rightarrow x+y = -\lambda$

$2 + \lambda + z = 0 \Rightarrow 2+z = -\lambda$

$\Rightarrow x+y = 2+z$

We have also $(x+y)+z = 0$ $\Rightarrow 2+z+z = 0$

$\Rightarrow 2+2z = 0 \Rightarrow z = -1.$

$x+y = 2+(-1) = 1. \Rightarrow x+y = 1$

From the last equation we get $x^2+y^2 = 23$

Let $y = 1-x. \Rightarrow x^2 + (1-x)^2 = 23$

$\Rightarrow 2x^2 - 2x - 22 = 0 \Rightarrow x^2 - x - 11 = 0$

$$x = \frac{1 \pm \sqrt{45}}{2} = \frac{1 \pm 3\sqrt{5}}{2}$$

$$y = 1 - x = \frac{1 \mp 3\sqrt{5}}{2}$$

$$\boxed{\left(\frac{1+3\sqrt{5}}{2}, \frac{1-3\sqrt{5}}{2}, -1\right)} \text{ or } \boxed{\left(\frac{1-3\sqrt{5}}{2}, \frac{1+3\sqrt{5}}{2}, -1\right)}.$$

Case 1 $\quad x=y \Rightarrow 2x+z=0$ and $2x^2+z^2=24$

$\Rightarrow z=-2x \Rightarrow 2x^2+4x^2=24$

$\Rightarrow x^2=4$

$\Rightarrow x=\pm 2.$

$y=x=\pm 2$ and $z=\pm 4.$

$\boxed{(2,2,-4)}$ and $\boxed{(-2,-2,4).}$

$f(x,y,z)=xy+2z$

$f(2,2,-4)=-4$, $f(-2,-2,4)=\boxed{12.}$ maximum

$f\left(\frac{1+3\sqrt{5}}{2},\frac{1-3\sqrt{5}}{2},-1\right)=\frac{1-45}{4}+2(-1)=\boxed{-13}$

$f\left(\frac{1-3\sqrt{5}}{2},\frac{1+3\sqrt{5}}{2},-1\right)=\frac{1-45}{4}+2(-1)=\boxed{-13.}$

minimum

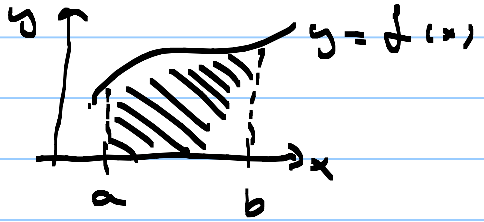
Video 52

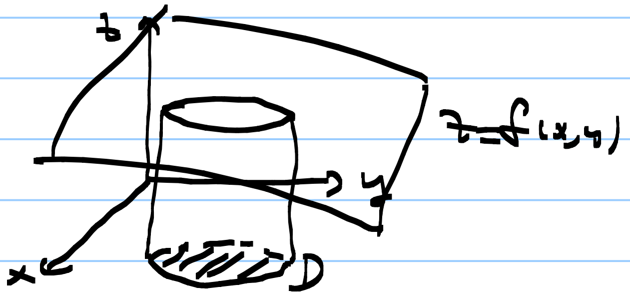
CHAPTER 14. Multiple Integration:

§14.1. Double Integration:

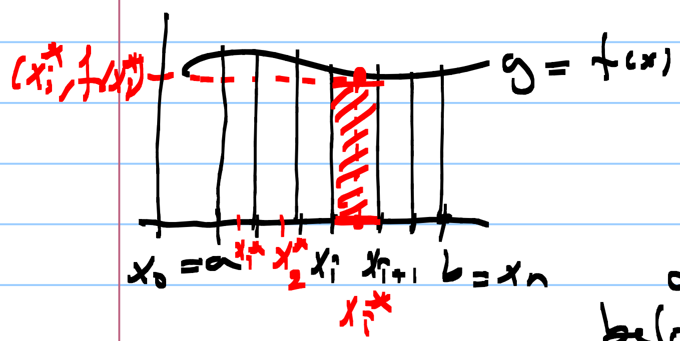
Motivation: For a one variable function $y=f(x)$ the integration

$\int_a^b f(x)\,dx$ is the area of the region between the graph of $y=f(x)$ and the x-axis, provided that $f(x)\geq 0$ on $[a,b]$ and $f(x)$ is a reasonable function.

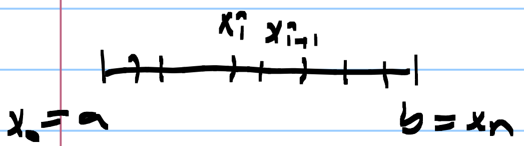




Now let $z=f(x,y)\geq 0$ be a function defined on a closed and bounded region D. Then we may try to compute the volume of the solid region above D between the graph of $z=f(x,y)$ and the xy-plane.

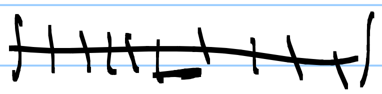


$\sum_{i=0}^{n-1} f(x_i^*)(x_{i+1}-x_i)$ is an approximation for the area below the graph of $y=f(x)$ above the interval $[a,b]$.



$P=\{x_0=a<x_1<\dots<x_n=b\}$

$\|P\|=\max\{|x_{i+1}-x_i|\}$ = the norm of the partition P.

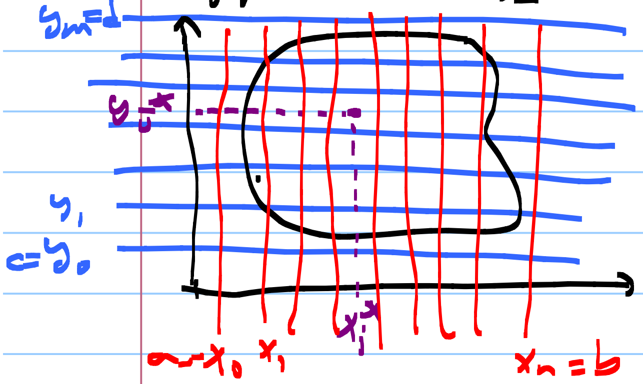


$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*) \doteq \int_a^b f(x)\,dx$$ provided that the limit exists as $\|P\| \to 0$. In this case, we say that f is integrable over $[a,b]$.

We know that if $f(x)$ is continuous on $[a,b]$ then f is integrable over $[a,b]$.

In two variables we'll mimic the same construction as follows:

Suppose $D \subseteq \mathbb{R}^2$ is a reasonable region.



$D \subseteq [a,b] \times [c,d]$

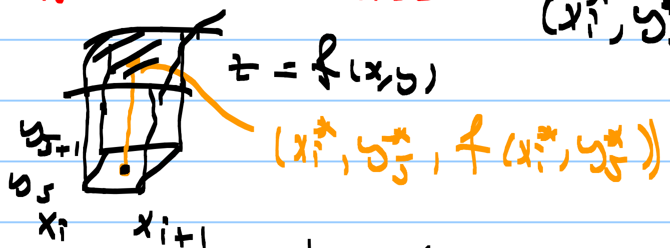
$a = x_0 < x_1 < x_2 - - - < x_n = b$

$c = y_0 < y_1 < y_2 < - - < y_m = d$

(x_i^*, y_j^*)

$x_i \leq x_i^* \leq x_{i+1}$

$y_j \leq y_j^* \leq y_{j+1}$

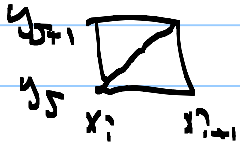


$$\text{vol}_{ij} = (x_{i+1} - x_i)(y_{j+1} - y_j) \cdot f(x_i^*, y_j^*)$$

$\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m} \text{vol}_{ij}$: an approximation for the volume of the solid region bounded from above by the graph of $z = f(x,y)$ and below by the xy-plane.

$$P = \{(x_i, y_j) \mid a = x_0 < x_1 - - < x_n = b,\ c = y_0 < y_1 - < y_m = d\}$$

$$\|P\| = \max_{i,j} \left\{ \sqrt{(x_{i+1}-x_i)^2 + (y_{j+1}-y_j)^2} \right\}$$



$$\lim_{\|P\|\to 0} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)(x_{i+1}-x_i)(y_{j+1}-y_j)$$

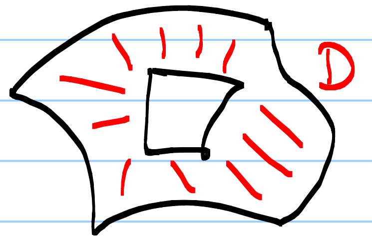
If this limit exists then we say that f is integrable over the region D and write

$$\iint_D f(x,y)\,\underbrace{dA}_{\text{area element}} = \lim_{\|P\|\to 0} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)(x_{i+1}-x_i)(y_{j+1}-y_j)$$

where x_i^*, y_j^* are arbitrarily chosen in the intervals $[x_i, x_{i+1}]$ and $[y_j, y_{j+1}]$, respectively.

The sum in the above definition is called a Riemann sum for the integral of $f(x,y)$ over the region D.

__Theorem:__ If $f(x,y)$ is a continuous function on a closed, bounded domain D whose boundary consists of finitely many curves of finite length, then f is integrable on D.

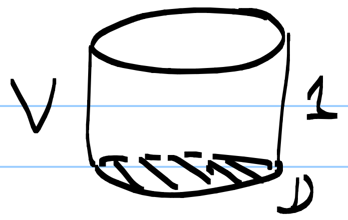


__Properties of the Double Integral:__

Suppose f and g are integrable functions over D and L and M are constants. Then

a) $\iint_D f(x,y)\,dA = 0$ if D has zero area.

b) $\iint_D 1 \cdot dA = $ Area of D



vol(V) = Area (D)

c) Integral representing Volume

If $f(x,y) \geq 0$ on D then $\iint_D f(x,y)\,dA = V \geq 0$,

where V is the volume of the solid lying vertically above D and below the surface $z = f(x,y)$

d) If $f(x,y) \leq 0$ on D then $\iint_D f(x,y)\,dA = -V \leq 0$

when V is the volume between the graph of $z = f(x,y)$ and the xy-plane.

e) Linear Dependence:

$$\iint_D (L f(x,y) + M g(x,y))\,dA = L\iint_D f(x,y)\,dA + M\iint_D g(x,y)\,dA$$

f) Inequalities are preserved:

If $f(x,y) \leq g(x,y)$ on D then $\iint_D f(x,y)\,dA \leq \iint_D g(x,y)\,dA$

g) Triangle Inequality

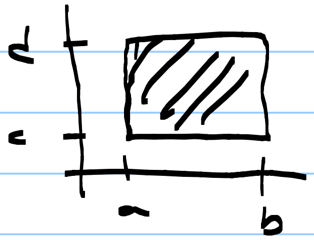
$$\left| \iint_D f(x,y)\,dA \right| \leq \iint_D |f(x,y)|\,dA$$

h) If $D = D_1 \cup \cdots \cup D_k$ is the union of non overlapping regions so that f is integrable over each D_i then

$$\iint_D f\,dA = \sum_{i=1}^{k} \iint_{D_i} f\,dA.$$

Video 53

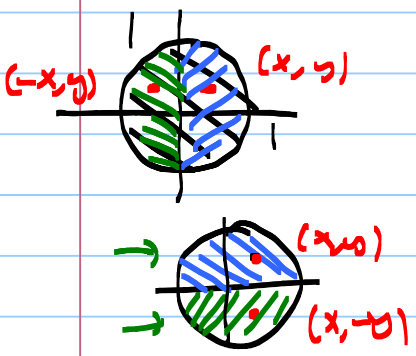
Example: 1) $\iint_R 3\, dA$, where $R: \begin{array}{l} a \le x \le b \\ c \le y \le d \end{array}$



$$\iint_R 3\, dA = 3 \iint_R 1\, dA = 3\, \text{Area}(R)$$

$$= 3\,(b-a)(d-c)$$

2) $$I = \iint_{x^2+y^2 \le 1} (\sin x + y^3 + 4)\, dA$$



$$= \iint_{x^2+y^2\le 1} \sin x\, dA + \iint_{x^2+y^2\le 1} y^3\, dA + \iint_{x^2+y^2\le 1} 4\, dA$$

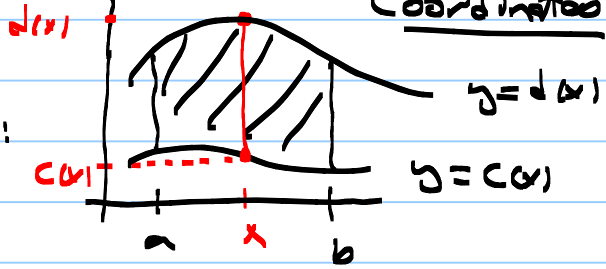
$$= 0 + 0 + 4 \iint_{x^2+y^2\le 1} 1\, dA$$

$$= 4\, \text{Area}(x^2+y^2 \le 1)$$

$$= 4\,\pi \cdot 1^2 = 4\pi.$$

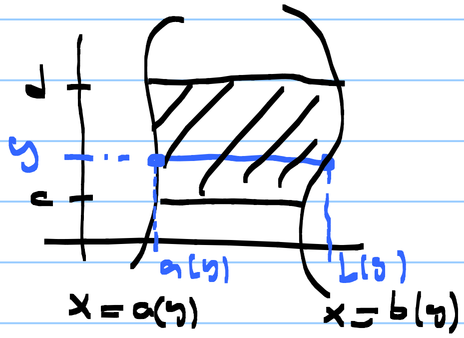
§14.2. Iteration of Double Integrals in Cartesian Coordinates:

y-simple region D:



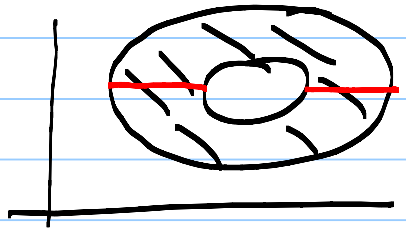
$$D = \{(x,y) \mid a \le x \le b,\ c(x) \le y \le d(x)\}$$

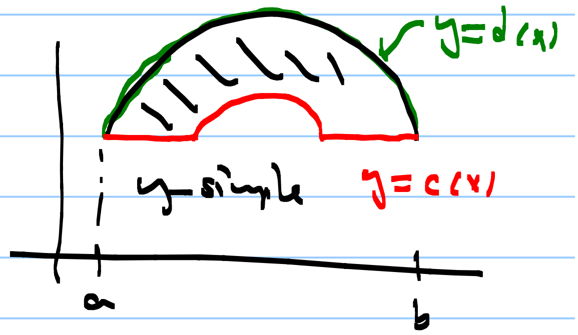
x-simple region D:



A region D which is the union of non-overlapping finitely many simple regions is called a regular region.

Ex





Similarly the lower part is also y-simple.

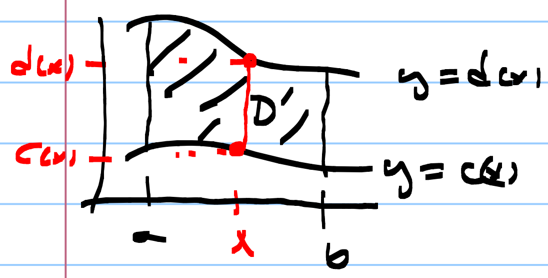
Theorem (Iteration of Double Integrals)

If $f(x,y)$ is continuous on the bounded y-simple region D given by $a \le x \le b$ and $c(x) \le y \le d(x)$, then

$$\iint_D f(x,y)\,dA = \int_a^b dx \int_{c(x)}^{d(x)} f(x,y)\,dy.$$

Similarly, if f is continuous on the x-simple region D given by $c \le y \le d$ and $a(y) \le x \le b(y)$, then

$$\iint_D f(x,y)\,dA = \int_c^d dy \int_{a(y)}^{b(y)} f(x,y)\,dx.$$



$$\iint_D f(x,y)\,dA = \int_a^b dx \int_{c(x)}^{d(x)} f(x,y)\,dy$$

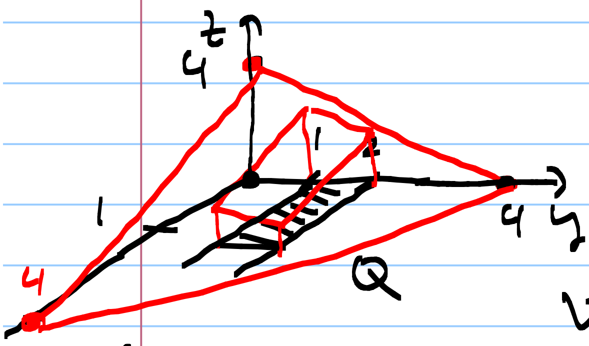
Notation: $\displaystyle \iint_D f(x,y)\,dA = \iint_D f(x,y)\,dx\,dy = \iint_D f(x,y)\,dy\,dx$

$\int_a^b dx \int_{c(x)}^{d(x)} f(x,y)\,dy$ is also denoted as $\int_a^b \left(\int_{c(x)}^{d(x)} f(x,y)\,dy \right) dx$

Similarly, $\int_c^d dy \int_{a(x)}^{b(x)} f(x,y)\,dx$ is denoted as $\int_c^d \left(\int_{a(x)}^{b(x)} f(x,y)\,dx \right) dy$

Example(s) 1) Find the volume of the solid lying above the square Q defined by $0 \le x \le 1,\ 1 \le y \le 2$ and below the plane $z = 4 - \underline{\underline{x}} - \underline{\underline{y}}$.

$x = 0,\ y = 4$
$z = 0$

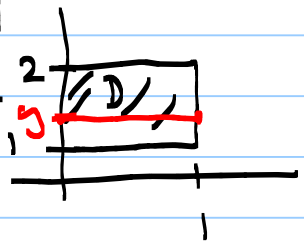


$z = f(x,y) = 4 - x - y$

height function

$$\text{Volume} = \iint_D f(x,y)\,dA$$

D: $0 \le x \le 1$
$1 \le y \le 2$



$$= \int_1^2 dy \int_0^1 (4 - x - y)\,dx$$

$$= \int_1^2 dy \left(\left(4x - \frac{x^2}{2} - yx\right)\Big|_0^1 \right)$$

$$= \int_1^2 dy \left[\left(4 - \frac{1}{2} - y\right) - (0 - 0 - 0) \right]$$

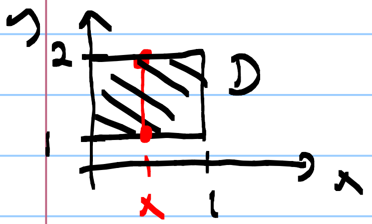
$$= \int_1^2 \left(\tfrac{7}{2} - y\right) dy$$

$$= \frac{7y}{2} - \frac{y^2}{2} \Big|_1^2$$

$$= (7 - 2) - \left(\tfrac{7}{2} - \tfrac{1}{2}\right) = 5 - 3 = 2.$$

Video 54

Since the region D also x-simple we can compute the integral as follows also:



$$\iint_D (4-x-y)\,dA = \int_0^1 dx \underbrace{\int_1^2 (4-x-y)\,dy}_{||}$$

$$= \int_0^1 dx \left(4y - xy - \frac{y^2}{2}\right)\Bigg|_{1=y}^{2=y}$$

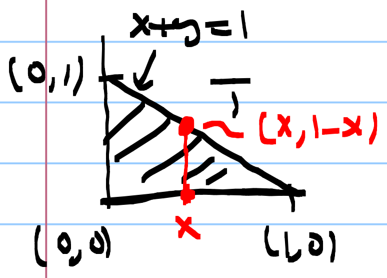
$$= \int_0^1 dx \left[(8-2x-2)-(4-x-1/2)\right]$$

$$= \int_0^1 \left(-x + \frac{5}{2}\right) dx$$

$$= -\frac{x^2}{2} + \frac{5x}{2}\Bigg|_0^1$$

$$= -1/2 + 5/2 - 0 = 2.$$

2) Evaluate $\iint_T xy\,dA$ over the triangle T with vertices $(0,0)$, $(1,0)$ and $(0,1)$.



$$\iint_T xy\,dA = \int_0^1 dx \int_0^{1-x} xy\,dy$$

$$= \int_0^1 dx \left(\frac{xy^2}{2}\Bigg|_0^{1-x}\right)$$

$$= \int_0^1 \left(\frac{x(1-x)^2}{2} - 0\right) dx$$

$$= \int_0^1 \frac{x(x^2-2x+1)}{2}\,dx$$

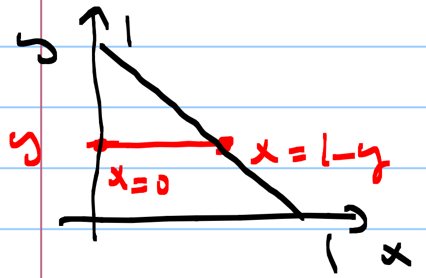
$$= \frac{1}{2}\int_0^1 (x^3 - 2x^2 + x)\, dx$$

$$= \frac{1}{2}\left(\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right)\Big|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\frac{3}{4} - \frac{2}{3}\right) = \frac{1}{24}$$

Also it can be computed when the triangle T is considered x-simple:



$$\iint_T xy\, dA = \int_0^1 dy \int_0^{1-y} xy\, dx$$

$$= \int_0^1 dy \left(\frac{x^2 y}{2}\Big|_0^{1-y}\right)$$

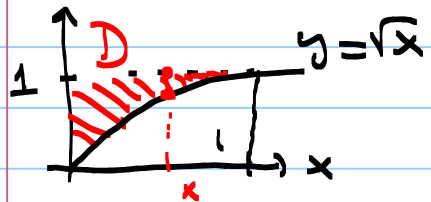
$$= \frac{1}{2}\int_0^1 dy \left((1-y)^2 y - 0\right)$$

$$= \frac{1}{2}\int_0^1 (y^3 - 2y^2 + y)\, dy$$

$$= \frac{1}{2}\left(\frac{y^4}{4} - \frac{2}{3}y^3 + \frac{y^2}{2}\right)\Big|_0^1$$

$$= 1/24.$$

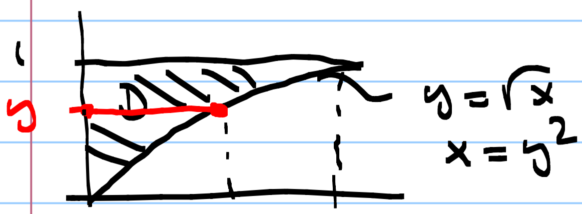
3) Evaluate the integral $I = \int_0^1 dx \int_{\sqrt{x}}^1 e^{y^3}\, dy$.



$$y = \sqrt{x} \Rightarrow x^2 = y$$

First let's write this iterated integral as a double integral over a region D.

$$I = \int_0^1 dx \int_{\sqrt{x}}^1 e^{y^3}\, dy = \iint_D e^{y^3}\, dA = \int_0^1 dy \int_0^{y^2} \underbrace{e^{y^3}}\, dx$$



$$= \int_0^1 dy\, e^{y^3} \int_0^{y^2} 1\, dx$$

$$= \int_0^1 dy\, e^{y^3} \left(x \Big|_0^{y^2} \right)$$

$$= \int_0^1 dy\, e^{y^3} (y^2 - 0)$$

$u = y^3$

$du = 3y^2\, dy$

$\frac{du}{3} = y^2\, dy$

$y = 0 \Rightarrow u = 0$

$y = 1 \Rightarrow u = 1$

$$= \int_0^1 y^2 e^{y^3}\, dy$$

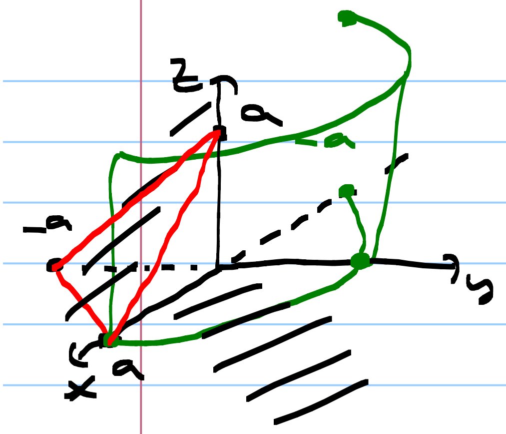
$$= \int_0^1 e^u \frac{du}{3}$$

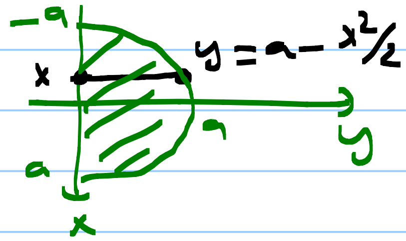
$$= \frac{e^u}{3}\Big|_0^1 = \frac{e-1}{3}.$$

Remark: This technique is called the change of order of integration.

4) Sketch and find the volume of the solid bounded by the planes $y=0$, $z=0$ and $z=a-x+y$ and the parabolic cylinder $y = a - \frac{x^2}{a}$, where a is a positive constant.

$y=0 \quad z=0 \quad z = \underset{\overset{\shortparallel}{u}}{a-x+y} \quad y = a - \frac{x^2}{a}$





$D: \; -a \leq x \leq a$
$\quad\;\; 0 \leq y \leq a - \frac{x^2}{a}$

$$\text{Volume} = \iint_D \overset{a-x+y}{\overset{\shortparallel}{z}}\, dA = \iint_D (a-x+y)\, dA$$

$$= \int_{-a}^{a} dx \int_0^{a - x^2/a} (a-x+y)\, dy$$

$$= \int_{-a}^{a} dx \left(ay - xy + \frac{y^2}{2} \right) \Bigg|_{y=0}^{y=a-\frac{x^2}{a}}$$

$$= \int_{-a}^{a} dx \left(a\left(a - \frac{x^2}{a}\right) - x\left(a - \frac{x^2}{a}\right) + \frac{1}{2}\left(a - \frac{x^2}{a}\right)^2 \right)$$

$$= \int_{-a}^{a} \left(a^2 - x^2 - xa + \frac{x^3}{a} + \frac{1}{2}a^2 - x^2 + \frac{1}{2}\frac{x^4}{a^2} \right) dx$$

$$= \int_{-a}^{a} \left(\frac{3}{2}a^2 - xa - 2x^2 + \frac{x^3}{a} + \frac{1}{2}\frac{x^4}{a^2} \right) dx$$

$$= \left(\frac{3x}{2}a^2 - \cancel{\frac{x^2 a}{2}} - \frac{2x^3}{3} + \cancel{\frac{x^4}{4a}} + \frac{x^5}{10a^2} \right) \Bigg|_{-a}^{a}$$

$$= 3a^3 - \frac{4a^3}{3} + \frac{a^3}{5}$$

$$= \frac{45 - 20 + 3}{15} a^3 = \frac{28}{15} a^3.$$

Video 55

§14.4. Double Integrals in Polar Coordinates:

Consider the following Integral

$$\int_0^{\sqrt{\pi}} x\sin x^2\,dx = \int_0^{\pi} \sin u\,du = -\cos u\Big|_0^{\pi} = 2.$$

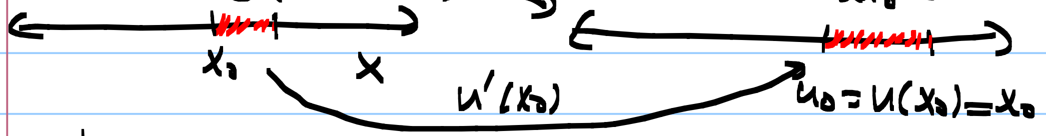
$u = x^2,\ du = 2x\,dx \qquad x=0 \Rightarrow u = x^2 = 0$

$\frac{1}{2}du = x\,dx \qquad x=\sqrt{\pi} \Rightarrow u = x^2 = \pi$

Understanding the Variable Change:

$u: \mathbb{R} \to \mathbb{R},\quad u(x) = x^2$

$\Delta u = 2x_0\,\Delta x = u'(x_0)\,\Delta x$

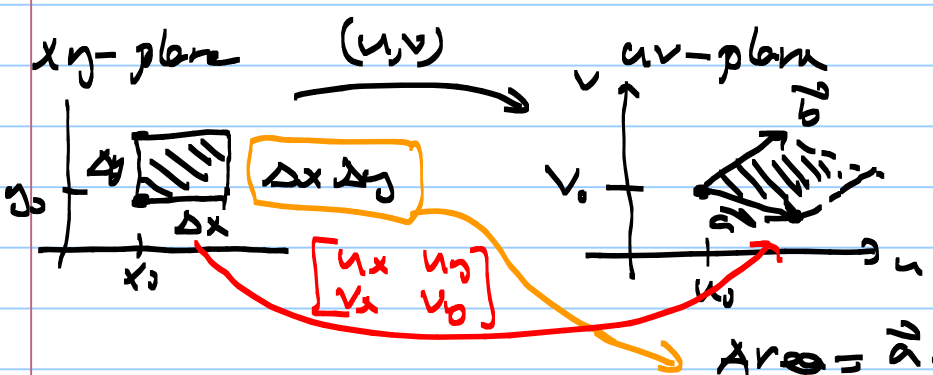


$u'(x_0) = 2x_0$

$u(x_0+\Delta x) = x_0^2 + 2x_0\,\Delta x + (\Delta x)^2 \simeq u(x_0) + 2x_0\,\Delta x$

$\Delta u = 2x_0\,\Delta x$ is written as $du = 2x_0\,dx.$

What about two variables?



$$\iint_D f(x,y)\,dA$$

$$\text{Area} = \vec{a}\times\vec{b}$$

$u = u(x,y) \qquad u_0 = u(x_0,y_0)$

$v = v(x,y) \qquad v_0 = v(x_0,y_0)$

$$= \big(u_x(x_0,y_0)\,v_y(x_0,y_0) - u_y(x_0,y_0)\,v_x(x_0,y_0)\big)\,\Delta x\,\Delta y$$

$$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}(x_0,y_0)\,\Delta x\,\Delta y$$

$$u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0) \simeq u_x(x_0,y_0)\,\Delta x + u_y(x_0,y_0)\,\Delta y$$
$$v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0) \simeq v_x(x_0,y_0)\,\Delta x + v_y(x_0,y_0)\,\Delta y$$

$$\Delta x \longmapsto (\Delta u, \Delta v) = (u_x(x_0,y_0), v_x(x_0,y_0))\,\Delta x = \vec{a}$$
$$\Delta y \longmapsto (\Delta u, \Delta v) = (u_y(x_0,y_0), v_y(x_0,y_0))\,\Delta y = \vec{b}$$

$$\hat{i}\, u_x + \hat{j}\, v_x = \vec{a}$$
$$\hat{i}\, u_y + \hat{j}\, v_y = \vec{b}$$

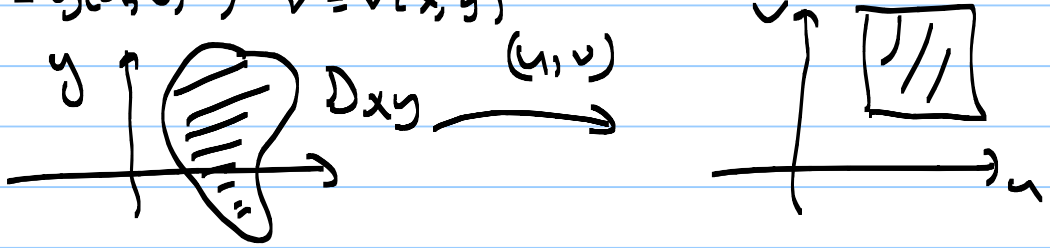
Conclusion: If we use a change of variable from the xy-plane to uv-plane which transform a region D_{xy} to D_{uv} then an integral of the form

$\displaystyle\iint_{D_{xy}} f(x,y)\,dx\,dy$ is transformed into the

integral $\displaystyle\iint_{D_{uv}} f(x(u,v), y(u,v)) \left|\begin{matrix} u_x & u_y \\ v_x & v_y \end{matrix}\right|^{-1} du\,dv$

$$= \left|\begin{matrix} x_u & x_v \\ y_u & y_v \end{matrix}\right|$$

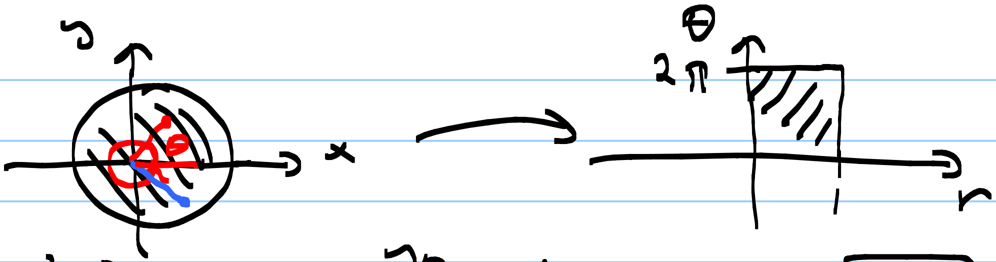
$$\begin{cases} x = x(u,v), & u = u(x,y) \\ y = y(u,v), & v = v(x,y) \end{cases}$$



Apply this idea to Polar Coordinates:

Let's try to compute the integral below

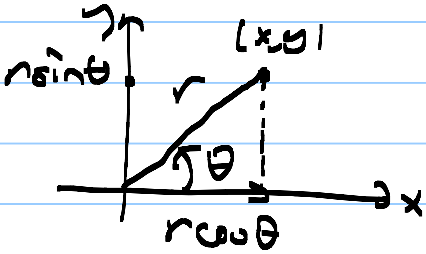
$$\iint_{x^2+y^2\le 1} (1-x^2-y^2)\,dA = \iint_{x^2+y^2\le 1} (1-x^2-y^2)\,dx\,dy$$



$x^2+y^2 \le 1$

$x = r\cos\theta$

$y = r\sin\theta$



$r=\sqrt{x^2+y^2}$

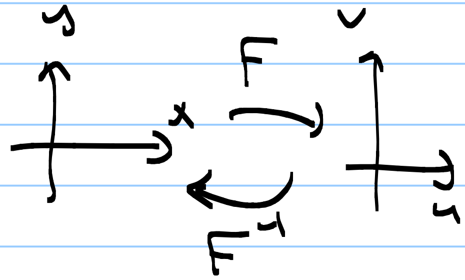
$\theta=\tan^{-1}\frac{y}{x}$

$0 \le \theta \le 2\pi$

$0 \le r \le 1$

$u=u(x,y)$

$v=v(x,y)$ $\quad dF=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$



$x=x(u,v)$

$y=y(u,v)$ $\quad dF^{-1}=\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$F(x,y)=(u,v)$

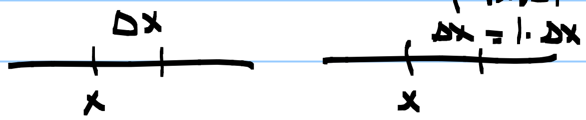
$F^{-1}(u,v)=(x,y)$

$(F^{-1}\circ F)(x,y)=(x,y) \Rightarrow F^{-1}\circ F = I(x,y)=(x,y)$

$d(F^{-1}\circ F) = d(I(x,y)) = I$ $\qquad f:\mathbb{R}\to\mathbb{R},\ f(x)=x$

$f'(x)=1$

$\underbrace{dF^{-1}(F)}\cdot\underbrace{dF} = I$



$dF^{-1}(F) = (dF)^{-1}$

$f^{-1}(f(x))=x \Rightarrow (f^{-1})'(f(x))\cdot f'(x) = 1$

$\Rightarrow (f^{-1})'(f(x)) = \frac{1}{f'(x)} = (f'(x))^{-1}$

$[f'(x)]^{-1} = [1/f'(x)]$

So, $\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = Id = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det[\ \ ]\cdot\det[\ \ ] = \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1.$

$$(x_u y_v - x_v y_u)\cdot(u_x u_y - v_x v_y) = 1.$$

In our case, we replace x, y with r, θ.

$x = r\cos\theta$
$y = r\sin\theta$

$$dx\,dy = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} dr\,d\theta$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr\,d\theta$$

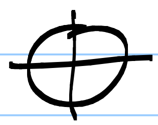
$$= (r\cos^2\theta + r\sin^2\theta)\,dr\,d\theta$$

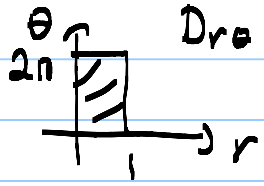
$$= r\,dr\,d\theta$$

$$\iint_{D_{xy}} f(x,y)\,dx\,dy = \iint_{D_{r\theta}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Hence,

$$\iint_{x^2+y^2\le 1} (1-x^2-y^2)\,dx\,dy = \iint_{D_{r\theta}} (1 - r^2\cos^2\theta - r^2\sin^2\theta)\,r\,dr\,d\theta$$





$$= \iint_{D_{r\theta}} (1-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r - r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1\right) d\theta$$

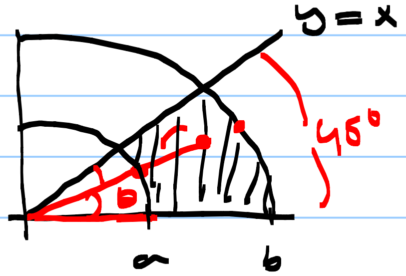
$$= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{4}\right) d\theta$$

Video 56

$$= \int_0^{2\pi} \frac{1}{4}\, d\theta = \frac{\theta}{4}\Big|_0^{2\pi} = \frac{\pi}{2}.$$

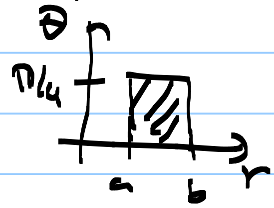
Example: Compute the Integral

$I = \iint_R \frac{y^2}{x^2}\, dA$, where $R: 0 < a^2 \le x^2+y^2 \le b^2$, below the $y=x$ line and above the x-axis.



Use polar coordinate,

$R_{r,\theta}$: $a \le r \le b$, $0 \le \theta \le \pi/4$



$$\iint_{R_{xy}} \frac{y^2}{x^2}\, dA = \iint_{R_{xy}} \frac{y^2}{x^2}\, dx\,dy = \iint_{R_{r\theta}} \frac{r^2 \sin^2\theta}{r^2\cos^2\theta}\, r\,dr\,d\theta$$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ dx\,dy &= \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} dr\,d\theta \\ &= r\,dr\,d\theta \end{aligned}}$$

$$= \int_0^{\pi/4}\int_a^b r\tan^2\theta\, dr\,d\theta$$

$$= \int_0^{\pi/4} \tan^2\theta \left(\frac{r^2}{2}\Big|_a^b\right) d\theta$$

$$= \int_0^{\pi/4} \left(\frac{b^2-a^2}{2}\right)\tan^2\theta\, d\theta$$

$(\tan\theta)' = 1+\tan^2\theta$

$$= \frac{b^2-a^2}{2}\int_0^{\pi/4}\left[(\tan\theta)' - 1\right] d\theta$$

$$= \frac{b^2-a^2}{2}(\tan\theta - \theta)\Big|_0^{\pi/4}$$

$$= \frac{b^2-a^2}{2}\left[\left(1-\frac{\pi}{4}\right) - 0\right] = \left(1-\frac{\pi}{4}\right)\frac{b^2-a^2}{2}$$

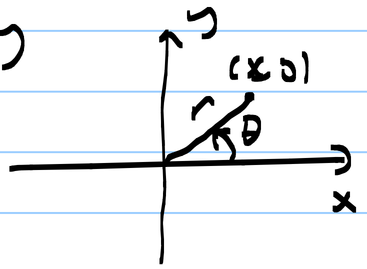
Ex: Show that $\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

Solution: Let $I = \int_{-\infty}^{+\infty} e^{-x^2}\,dx$.

Then
$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2}\,dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}\cdot e^{-y^2}\,dx\,dy$$

$$= \iint_{\mathbb{R}^2_{xy}} e^{-(x^2+y^2)}\,dx\,dy$$



$x = r\cos\theta$
$y = r\sin\theta$

$0 \leq r < \infty$
$0 \leq \theta < 2\pi$

$$= \iint_{\mathbb{R}^2_{r,\theta}} e^{-r^2}(r\,dr\,d\theta)$$

$$= \int_0^{2\pi}\int_0^{\infty} r e^{-r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{e^{-r^2}}{-2}\Big|_0^{\infty}\right)d\theta$$

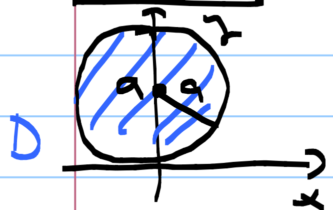
$$= \int_0^{2\pi}\left(\lim_{r\to\infty} e^{-r^2} - \frac{e^0}{-2}\right)d\theta$$

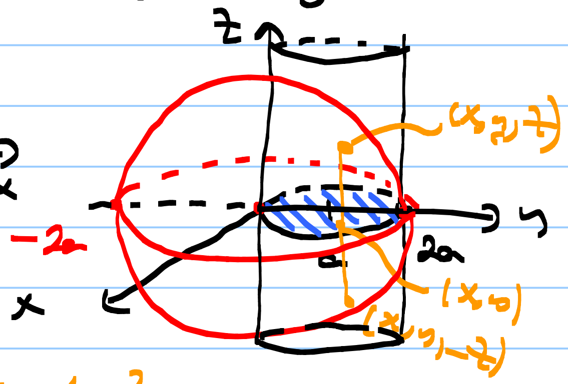
$$= \int_0^{2\pi}\left(0+\frac{1}{2}\right)d\theta = \frac{1}{2}\left(\theta\Big|_0^{2\pi}\right) = \frac{2\pi}{2} = \pi.$$

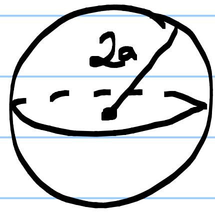
Hence, $I^2 = \pi$ and therefore
$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = I = \sqrt{\pi}.$$

Example: Find the volume of the solid region lying inside both the sphere $x^2+y^2+z^2=4a^2$ and the cylinder $x^2+y^2=2ay$, when $a>0$.

Solution: $x^2+y^2=2ay \Rightarrow x^2+(y-a)^2=a^2$



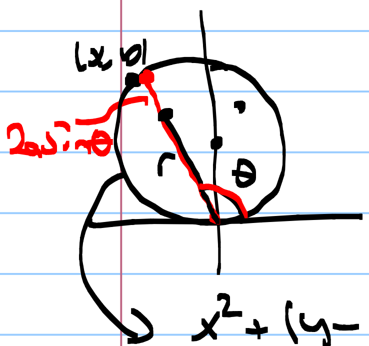




$x^2+y^2+z^2=4a^2$

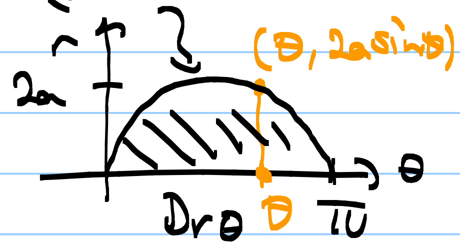
$z=\left[4a^2-(x^2+y^2)\right]^{1/2}$; $\iint \underline{\underline{f(x,y)}}\,dx\,dy$

$$\text{Volume} = 2\iint_{D_{xy}} \left[4a^2-(x^2+y^2)\right]^{1/2} dA$$



$x = r\cos\theta$ $\quad$ $0 \le \theta \le \pi$

$y = r\sin\theta$ $\quad$ $0 \le r \le 2a\sin\theta$: $D_{r\theta}$



$x^2+(y-a)^2=a^2$

$(r\cos\theta)^2+(r\sin\theta-a)^2=a^2$

$\underline{r^2\cos^2\theta+r^2\sin^2\theta}-2ar\sin\theta+a^2=a^2$

$r^2-2ar\sin\theta=0 \Rightarrow \overset{\neq 0}{r}(r-2a\sin\theta)=0$

$\Rightarrow r=2a\sin\theta$

$$\text{Volume} = 2\iint_{D_{xy}} \sqrt{4a^2-(x^2+y^2)}\,dx\,dy$$

$$= 2\iint_{D_{r\theta}} (4a^2-r^2)^{1/2}\,r\,dr\,d\theta$$

Video 57

$$\text{Volume} = 2\int_0^{\pi}\int_0^{2a\sin\theta} r\sqrt{4a^2-r^2}\,dr\,d\theta$$

Let $u = 4a^2 - r^2$, $du = -2r\,dr$

$$\text{Volume} = 2\int_0^{\pi}\int_0^{2a\sin\theta} (-1/2)\sqrt{u}\,du\,d\theta$$

$$= 2\int_0^{\pi} \frac{u^{3/2}}{-3}\Big|_?^? \,d\theta$$

$$= 2\int_0^{\pi} \frac{(4a^2-r^2)^{3/2}}{-3}\Big|_0^{2a\sin\theta} d\theta$$

$$= \frac{2}{-3}\int_0^{\pi} \Big((\underbrace{4a^2 - 4a^2\sin^2\theta}_{4a^2\cos^2\theta \geq 0})^{3/2} - (4a^2)^{3/2} \Big)\,d\theta$$

$\sqrt{4a^2\cos^2\theta} = |2a\cos\theta|$

$$= \frac{2\cdot 2}{-3}\int_0^{\pi/2} 8a^3\cos^3\theta - 8a^3)\,d\theta$$

$$= \frac{32a^3}{-3}\int_0^{\pi/2} (\cos^3\theta - 1)\,d\theta$$

$\cos^3\theta = (1-\sin^2\theta)\cos\theta$

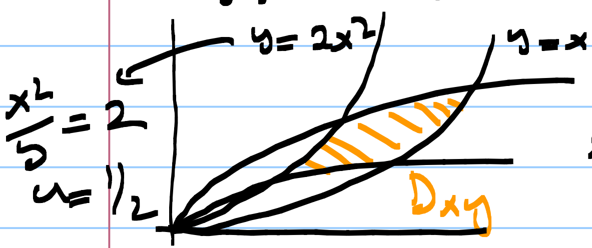
$$= \frac{32a^3}{-3}\int_0^{\pi/2} (\cos\theta - \cos\theta\sin^2\theta - 1)\,d\theta$$

$$= \frac{32a^3}{-3}\left(\sin\theta - \frac{\sin^3\theta}{3} - \theta\right)\Big|_0^{\pi/2}$$

$$= \frac{32a^3}{-3}\left(1 - \frac{1}{3} - \pi/2\right)$$

$$= \frac{16a^3\pi}{3} - \frac{64}{9}a^3$$

Example Find the area of the finite plane region bounded by the four parabolas $y=x^2$, $y=2x^2$, $x=y^2$, $x=3y^2$.

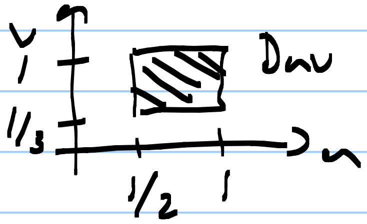


$\to \frac{x^2}{y}=1 \Rightarrow u=1$

$x=y^2 \leftrightarrow \frac{y^2}{x}=1 \leftrightarrow v=1$

$x=3y^2 \leftrightarrow \frac{y^2}{x}=1/3 \leftrightarrow v=1/3$

Solution: Let $u=u(x,y)=\frac{x^2}{y}$ and $v=v(x,y)=\frac{y^2}{x}$



$$\text{Area}=\iint_{D_{xy}} \underline{\underline{1}}\cdot dA=\iint_{D_{xy}} 1\cdot dx\,dy$$

$$dx\,dy=\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du\,dv$$

$$=\frac{\partial(x,y)}{\partial(u,v)} \leftarrow \text{Jacobian}$$

$$=\iint_{D_{uv}} 1\cdot 1/3 \; du\,dv$$

$$u=\frac{x^2}{y} \qquad v=\frac{y^2}{x}$$

$$\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} \qquad \frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}=\begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix}$$

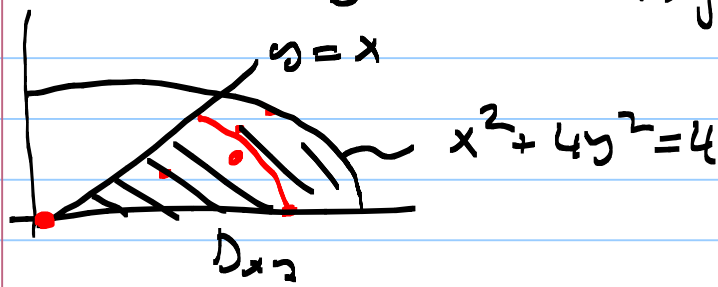
$$=\frac{1}{4-1}=1/3$$

$$\text{Area}=\frac{1}{3}\int_{1/3}^{1}\int_{1/2}^{1} du\,dv=\frac{1}{3}\int_{1/3}^{1}\left(u\Big|_{1/2}^{1}\right)dv$$

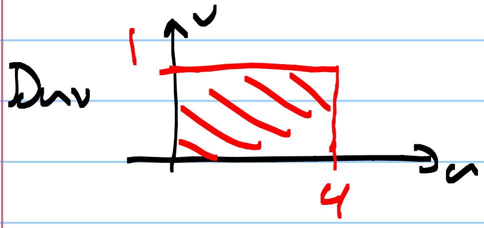
$$=\frac{1}{3}\left(1-1/2\right)\int_{1/3}^{1} dv$$

$$=1/3\cdot 1/2\cdot\left(v\Big|_{1/3}^{1}\right)=\overbrace{1/3\cdot 1/2\cdot 2/3}^{1/9\ =}$$

Example: $I = \iint_D \frac{y}{x}\, dx\, dy$ where D is the region below:



Solution: $u \doteq x^2+4y^2$, $v \doteq y/x$



$$\iint_{D_{xy}} y/x \, dx\, dy = \iint_{D_{uv}} v \frac{\partial(x,y)}{\partial(u,v)}\, du\, dv$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & 8y \\ -y/x^2 & 1/x \end{vmatrix} = 2 + \frac{8y^2}{x^2} = 2+8v^2$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{2+8v^2}$$

$$\iint_{D_{xy}} y/x \, dx\, dy = \iint_{D_{uv}} \frac{v}{2+8v^2}\, du\, dv$$

$$= \int_0^4 \left(\int_0^1 \frac{v}{2+8v^2}\, dv \right) du$$

$$= \int_0^4 \left(\frac{1}{16} \ln(2+8v^2) \Big|_0^1 \right) du$$

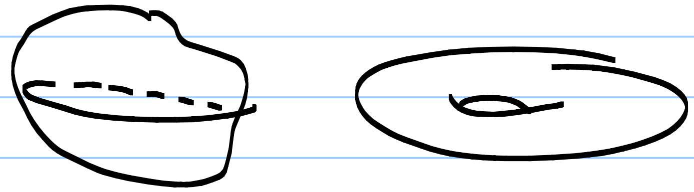
$$= \int_0^4 \frac{1}{16} \left(\overbrace{\ln 10 - \ln 2}^{\ln 5} \right) \underline{\underline{du}}$$

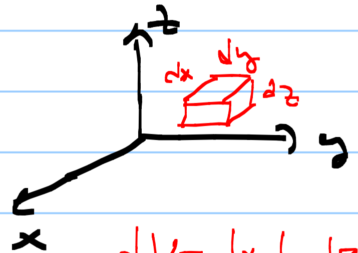
$$= \frac{\ln 5}{16} \left(u \Big|_0^4 \right) = \frac{4 \ln 5}{16} = \frac{\ln 5}{4}$$

Video 5P

§14.5. Triple Integrals: Both the theory and practice of triple integrals is very similar to those of double integrals.

Let $D \subseteq \mathbb{R}^3$ be a domain in $\mathbb{R}^3$.

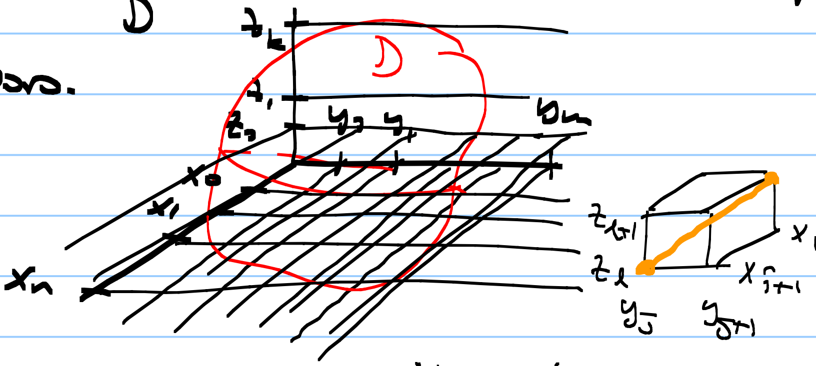




If D is closed and bounded with "reasonable" boundary.

$dV = dx\,dy\,dz$
volume element

Volume of $D = \iiint_D dV$. This is defined again using partitions.



$\Delta V_{ijl} = (x_{i+1}-x_i)(y_{j+1}-y_j)(z_{l+1}-z_l)$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{k} \Delta V_{ijl}$$

You take V_{ijl} into the summation if $V_{ijl} \cap D \neq \emptyset$.

Then the sum is an approximation for the volume of D. We define norm of the partition

$P = \{a = x_0 < x_1 < \cdots < x_n = b,\ c = y_0 < y_1 < \cdots < y_m = d,$
$e = z_0 < z_1 < \cdots < z_k = f\}$

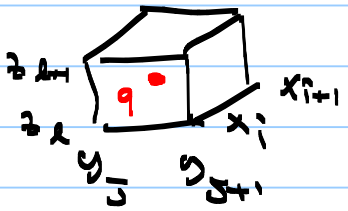
$$\|P\| = \max_{i,j,l} \left\{ \sqrt{(x_{i+1}-x_i)^2 + (y_{j+1}-y_j)^2 + (z_{l+1}-z_l)^2} \right\}$$

$$\sum_{x_i, y_j, z_l \in P} \Delta V_i = \sum_{i,j,l} (x_{i+1}-x_i)(y_{j+1}-y_j)(z_{l+1}-z_l)$$

If $\sum \Delta V_i$ has limit as $\|P\| \to 0$ then we say that the triple integral $\iiint_D dV$ exists and equal to the limit.

If $f(x,y,z)$ is a function on D then we may form Riemann sums as follows:

$D = \{x_i, y_j, z_l\}$ $\quad x_i^* \in (x_i, x_{i+1})$, $y_j^* \in (y_j, y_{j+1})$
$z_l^* \in (z_l, z_{l+1})$



$q_{ijl} = (x_i^*, y_j^*, z_l^*)$

$$R(f,P) = \sum_{i,j,l} f(x_i^*, y_j^*, z_l^*)\,\Delta V_{ijl}$$

If the Riemann sum $R(f,P)$ converges to a real number as $\|P\| \to 0$ then we say that f is Riemann integrable over D and we write

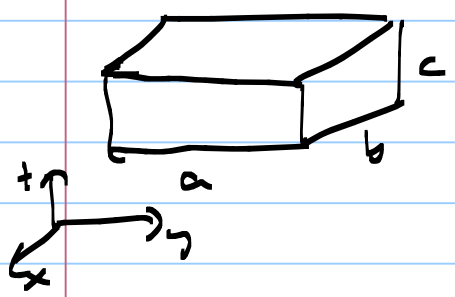
$$\iiint_D f\,dV = \lim_{\|P\| \to 0} R(f,P).$$

Remark: If $f(x,y,z) \geq 0$ on D then we may consider f as the density of the object at the point (x,y,z) and thus the integral above becomes the mass of the solid object D.

As in the case of double integrals we compute triple integrals by writing them as iterated integrals.

Example Let B be the rectangular box $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. Evaluate

$$I = \iiint_B (xy^2 + z^3)\, dV = \int_0^c \int_0^b \underbrace{\int_0^a (xy^2+z^3)\, dx}\, dy\, dz$$



$$= \int_0^c \int_0^b \left(\frac{x^2}{2}y^2 + xz^3\right)\Big|_0^a \, dy\, dz$$

$$= \int_0^c \int_0^b \left(\frac{a^2y^2}{2} + az^3 - 0\right) dy\, dz$$

$$= \int_0^c \left(\frac{a^2y^3}{6} + az^3y\right)\Big|_0^b \, dz$$

$$= \int_0^c \left(\frac{a^2b^3}{6} + az^3b - 0\right) dz$$

$$= \left(\frac{a^2b^3}{6}z + \frac{ab\,z^4}{4}\right)\Big|_0^c$$

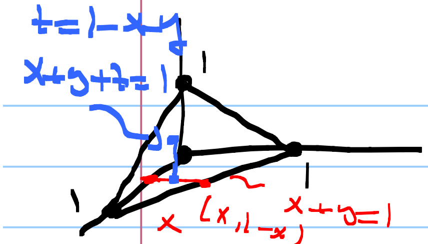
$$= \frac{a^2b^3c}{6} + \frac{abc^4}{4}$$

__Another notation__ for __iterated integrals__:

$$I = \int_0^c dz \int_0^b dy \int_0^a (xy^2+z^3)\, dx$$

__Example:__ Compute $2 = \iiint_T y\, dV$, where T is the tetrahedron in $\mathbb{R}^3$ with corners $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Video 59



$$I=\iiint_T y\,dV=\int_0^1\int_0^{1-x}\int_0^{1-x-y} y\,dz\,dy\,dx$$

$$=\int_0^1\int_0^{1-x}\left(yz\Big|_0^{1-x-y}\right)dy\,dx$$

$$=\int_0^1\int_0^{1-x}\left[y(1-x-y)-y\cdot 0\right]dy\,dx$$

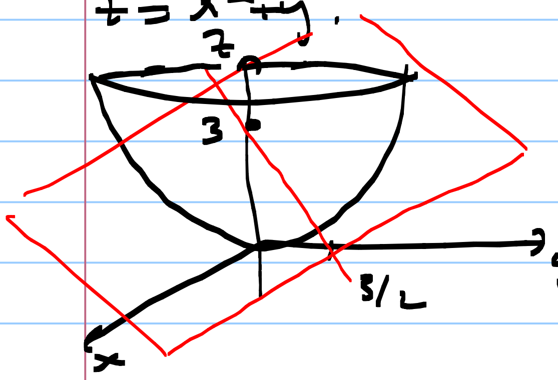
$$I=\int_0^1\int_0^{1-x}(y-yx-y^2)\,dy\,dx$$

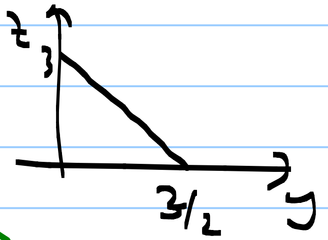
$$=\int_0^1\left(\frac{y^2}{2}-\frac{y^2x}{2}-\frac{y^3}{3}\right)\Big|_0^{1-x}dx$$

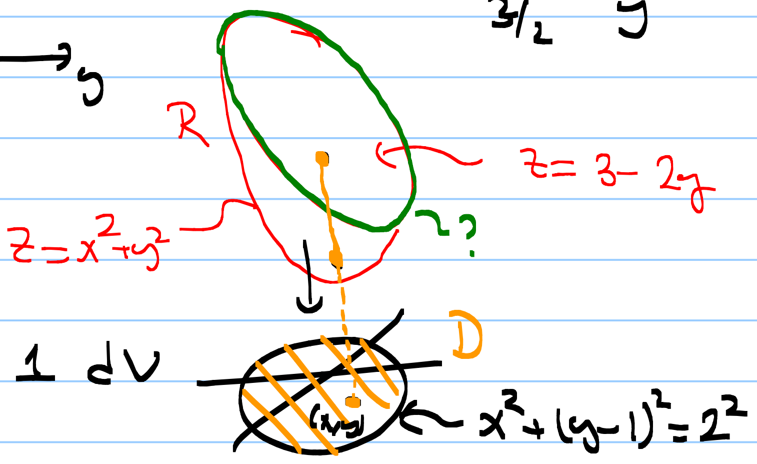
$$=\int_0^1\left[\frac{(1-x)^2}{2}-\frac{x(1-x)^2}{2}-\frac{(1-x)^3}{3}\right]dx$$

$$=\int_0^1\frac{(1-x)^3}{6}\,dx=-\frac{(1-x)^4}{24}\Big|_0^1=0-\left(-\frac{1}{24}\right)=\frac{1}{24}$$

Ex: Find the volume of the region R lying below the plane $z=3-2y$ and above the paraboloid $z=x^2+y^2$.



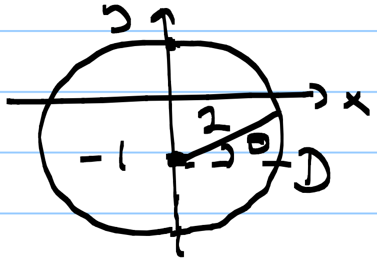




$$\text{Volume}=\iiint_R 1\,dV$$

$z = x^2+y^2$, $z = 3-2y \Rightarrow x^2+y^2 = 3-2y$

$\Rightarrow x^2+y^2+2y+1=4 \Rightarrow x^2+(y+1)^2 = 2^2$



$x = r\cos\theta$, $y+1 = r\sin\theta$

$0 \le r \le 2 \qquad 0 \le \theta \le 2\pi$

$$\text{Volume} = \iiint_R dV = \iint_D \int_{x^2+y^2}^{3-2y} 1 \; \underbrace{dz\,\overbrace{dA}^{dx\,dy}}_{dV}$$

$$= \iint_D \left(z \Big|_{x^2+y^2}^{3-2y} \right) dA$$

$dA = dx\,dy$

$x = r\cos\theta$

$\boxed{dx = \cos\theta\,dr - r\sin\theta\,d\theta}$

$y+1 = r\sin\theta$

$\boxed{dy + 0 = \sin\theta\,dr + r\cos\theta\,d\theta}$

$$= \iint_D (3-2y-x^2-y^2)\,dA$$

$$= \int_0^{2\pi}\int_0^2 (3-2y-x^2-y^2)\,r\,dr\,d\theta$$

$dx\,dy = r\,dr\,d\theta$

$3-2y-x^2-y^2 = 4-(x^2+(y+1)^2)$

$= 4-r^2$

So $$\text{Volume} = \int_0^{2\pi}\int_0^2 (4-r^2)\,r\,dr\,d\theta$$

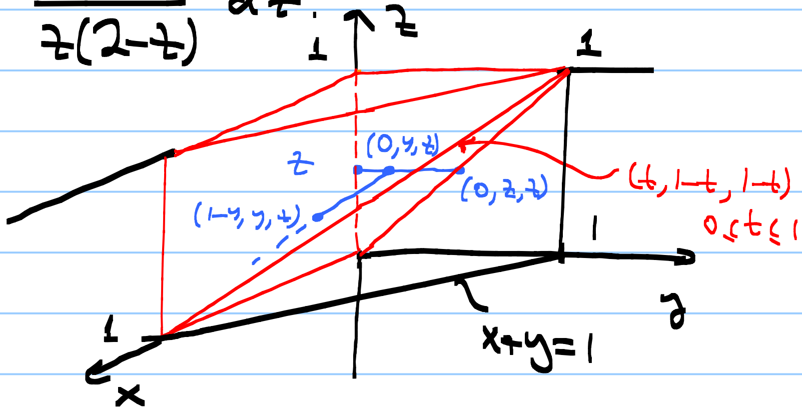
$$= \int_0^{2\pi}\int_0^2 (4r-r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(2r^2 - \frac{r^4}{4} \Big|_0^2 \right) d\theta$$

$$= \int_0^{2\pi} (8-4)\,d\theta = 4\int_0^{2\pi} d\theta = 4\cdot 2\pi = 8\pi.$$

Example: Compute the Iterated Integral

$$I=\int_0^1 dx\int_0^{1-x} dy\int_y^1 \frac{\sin(\pi z)}{z(2-z)}\,dz$$



Solution:

$x+y=1$

$y=z$

$x=t,\ y=1-t=z$

$x+z=1$

$$I=\int_0^1 dz\int_0^z dy\int_0^{1-y} dx\ \frac{\sin\pi z}{z(2-z)}$$

$$=\int_0^1 dz\int_0^z dy\ (1-y)\frac{\sin\pi z}{z(2-z)}$$

$$=\int_0^1 dz\ \frac{\sin\pi z}{z(2-z)}\left(-\frac{(1-y)^2}{2}\Big|_0^z\right)$$

$$=\int_0^1 \frac{\sin\pi z}{z(2-z)}\left(-\frac{(z-1)^2-1}{2}\right)dz$$

$$=\int_0^1 \frac{\sin\pi z}{z(2-z)}\left(-\frac{z^2-2z}{2}\right)dz=\frac{1}{2}\int_0^1 \sin\pi z\,dz$$

$$=-\frac{1}{2}\,\frac{\cos\pi z}{\pi}\Big|_0^1$$

$$=-\frac{1}{2\pi}(-1-1)$$

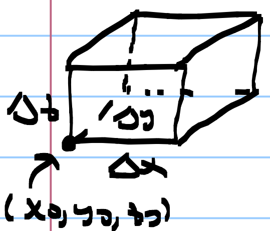
$$=\frac{1}{\pi}.$$

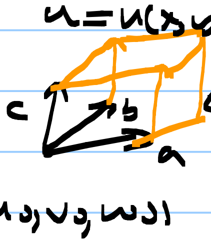
Video 60

§14.6. Change of Coordinates in Triple Integrals:

$x = x(u,v,w)$
$y = y(u,v,w)$ $\qquad dV = dx\,dy\,dz$
$z = z(u,v,w)$

$u = u(x,y,z),\ v = v(x,y,z),\ w = w(x,y,z)$





$\text{vol} = \det\begin{bmatrix} \vec a \\ \vec b \\ \vec c \end{bmatrix}$

$$u(x+\Delta x, y+\Delta y, z+\Delta z) \cong u(x_0,y_0,z_0) + u_x(x_0,y_0,z_0)\Delta x + u_y(x_0,y_0,z_0)\Delta y + u_z(x_0,y_0,z_0)\Delta z$$

$$v(x+\Delta x, y+\Delta y, z+\Delta z) \cong v(x_0,y_0,z_0) + v_x(x_0,y_0,z_0)\Delta x + v_y(x_0,y_0,z_0)\Delta y + v_z(x_0,y_0,z_0)\Delta z$$

$$w(x+\Delta x, y+\Delta y, z+\Delta z) \cong w(x_0,y_0,z_0) + w_x(x_0,y_0,z_0)\Delta x + w_y(x_0,y_0,z_0)\Delta y + w_z(x_0,y_0,z_0)\Delta z$$

$$\Delta x \longmapsto \vec a = (u_x(x_0,y_0,z_0)\Delta x,\ v_x(x_0,y_0,z_0)\Delta x,\ w_x(x_0,y_0,z_0)\Delta x)$$
$$\Delta y \longmapsto \vec b = (u_y(\quad)\Delta y,\ v_y(\quad)\Delta y,\ w_y(\quad)\Delta y)$$
$$\Delta z \longmapsto \vec c = (u_z(\quad)\Delta z,\ v_z(\quad)\Delta z,\ w_z(\quad)\Delta z)$$

So $dV = dx\,dy\,dz$ is changed to

$$\vec a \cdot (\vec b \times \vec c) = \det\begin{bmatrix} u_x\,\Delta x & v_x\,\Delta x & w_x\,\Delta x \\ u_y\,\Delta y & v_y\,\Delta y & w_y\,\Delta y \\ u_z\,\Delta z & v_z\,\Delta z & w_z\,\Delta z \end{bmatrix}(x_0,y_0,z_0)$$

$$= \Delta x\,\Delta y\,\Delta z\ \det\begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix}$$

$$\det \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix} = \frac{\partial(u,v,w)}{\partial(x,y,z)}$$ is called the

Jacobian determinant of u, v, w with respect to x, y, z.

Hence, $$\iiint_{R_{xyz}} f(x,y,z)\, dV = \iiint_{R_{xyz}} f(x,y,z)\, dx\, dy\, dz$$

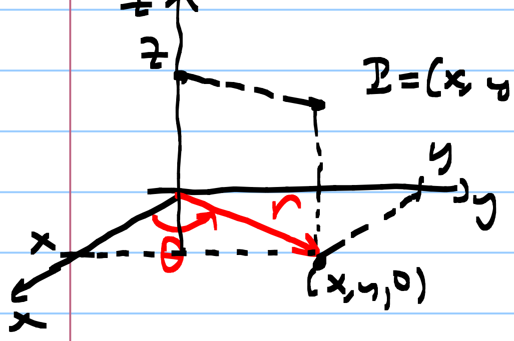
$$= \iiint_{R_{uvw}} g(u,v,w) \underbrace{\left| \frac{\partial(u,v,w)}{\partial(x,y,z)} \right|^{-1}}_{= \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right|} du\, dv\, dw$$

when $g(u,v,w) = f(x(u,v,w), y(u,v,w), z(u,v,w))$.

<u>Remark</u>: $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$ and
then $u = u(x,y,z)$, $v = v(x,y,z)$, $w = w(x,y,z)$.

$$\det \frac{\partial(x,y,z)}{\partial(u,v,w)} = \left(\det \frac{\partial(u,v,w)}{\partial(x,y,z)} \right)^{-1}$$

<u>Cylindrical Coordinates</u>:



$(x,y,z) \longmapsto (r, \theta, z)$

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1} y/x$.

$$dx\,dy\,dz = \det\left(\frac{\partial(x,y,z)}{\partial(r,\theta,z)} \right) dr\,d\theta\,dz$$

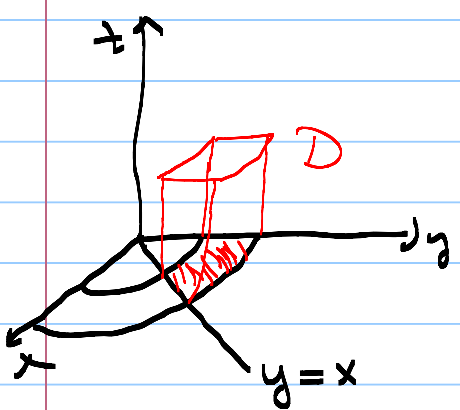
$$x = r\cos\theta \qquad \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{bmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{bmatrix}$$
$$y = r\sin\theta$$
$$z = z$$

$$= \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$dx\,dy\,dz = \det\left[\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right] dr\,d\theta\,dz$$
$$= (r\cos^2\theta + r\sin^2\theta)\,dr\,d\theta\,dz$$

So, $$\iiint_{R_{xyz}} f(x,y,z)\,dx\,dy\,dz = \iiint_{R_{r\theta z}} f(r\cos\theta, r\sin\theta, z)\, r\cdot dr\,d\theta\,dz.$$

Example $\iiint_D (x^2+y^2)\,dV$, where D is the region in the first octant bounded by the cylinders $x^2+y^2=1$ and $x^2+y^2+z^2=4$ and by planes $z=0$, $z=1$, $x=0$ and $x=y$.



$$1 \le r \le 2$$
$$\pi/4 \le \theta \le \pi/2$$
$$0 \le z \le 1$$

$$\iiint_{D_{xyz}} (x^2+y^2)\,dV = \iiint_{D_{xyz}} (x^2+y^2)\,dx\,dy\,dz$$

$$= \iiint_{D_{r\theta z}} r^2 \cdot (r\,dr\,d\theta\,dz)$$

$$= \int_0^1 \int_1^2 \int_0^{2\pi} r^3 \, dr\,d\theta\,dz$$

$$= \int_0^1 \int_1^2 r^3 \left(\theta \Big|_0^{2\pi}\right) dr\,dz$$

$$= 2\pi \int_0^1 \int_1^2 r^3 \, dr\,dz$$

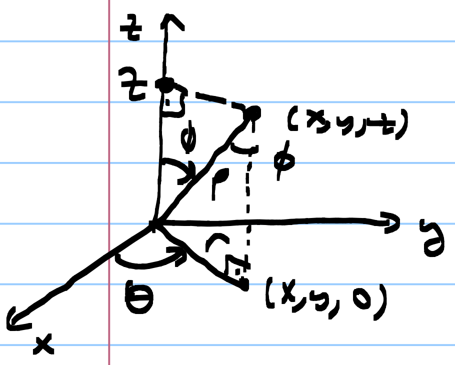
$$= 2\pi \int_0^1 \left(r^4/4 \Big|_1^2 \right) dz$$

$$= 2\pi \int_0^1 4 \, dz$$

$$= 8\pi \int_0^1 dz = 8\pi.$$

Video 61

Spherical Coordinates:



Spherical coordinates is the triple (ρ, ϕ, θ).

$$\rho = \sqrt{x^2+y^2+z^2}, \quad \cos\phi = \frac{z}{\rho}$$

$$\phi = \cos^{-1}\frac{z}{\rho}, \quad \theta = \tan^{-1}\frac{y}{x}.$$

$$\begin{cases} z = \rho\cos\phi \\ x = r\cos\theta = \rho\sin\phi\cos\theta \\ y = r\sin\theta = \rho\sin\phi\sin\theta \end{cases}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \det\begin{bmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{bmatrix}$$

$$= \det\begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}$$

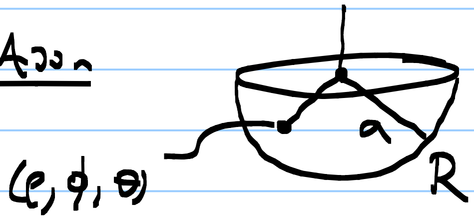
$$= \cos\phi\left(\rho^2\cos\phi\cos^2\theta\sin\phi + \rho^2\cos\phi\sin\phi\sin^2\theta\right)$$
$$+ \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)$$
$$= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi$$
$$= \rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right)$$
$$= \rho^2\sin\phi.$$

Then, $dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| d\rho\,d\phi\,d\theta$

$$= \rho^2\sin\phi\, d\rho\,d\phi\,d\theta.$$

Example: A solid half-ball H of radius a has density depending on the distance ρ from the centre of the base disc. The density is given by $k(2a-\rho)$, where k is a constant. Find the mass of the half-ball.

Solution



$$R: \quad 0 \leq \rho \leq a, \quad \pi/2 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi$$

$f(\rho, \phi, \theta) = k(2a-\rho)$ the density at the point (ρ, ϕ, θ).

$$\text{mass} = \iiint_R f(\rho, \phi, \theta)\, dV$$

$$= \int_0^a \int_0^\pi \int_0^{2\pi} \underbrace{k(2a-\rho)\rho^2 \sin\phi}\, \underline{d\theta}\, d\phi\, d\rho$$

$$= \int_0^a \int_0^\pi k(2a-\rho)\rho^2 \sin\phi \left(\theta \Big|_0^{2\pi}\right) d\phi\, d\rho$$

$$= 2\pi k \int_0^a \int_0^\pi (2a-\rho)\rho^2 \sin\phi\, d\phi\, d\rho$$

$$= 2\pi k \int_0^a (2a-\rho)\rho^2 \left(-\cos\phi \Big|_0^\pi\right) d\rho$$

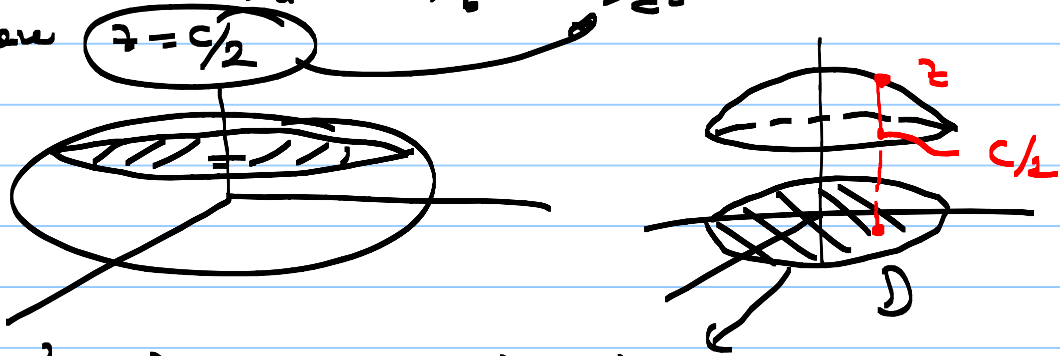
$$= 2\pi k \int_0^a (2a-\rho)\rho^2 (1+1)\, d\rho$$

$$= 4\pi k \int_0^a (2a\rho^2 - \rho^3)\, d\rho$$

$$= 4\pi k \left(\frac{2a\rho^3}{3} - \frac{\rho^4}{4} \Big|_0^a\right)$$

$$= 4\pi k \frac{5a^4}{12} = \frac{5\pi k a^4}{3}.$$

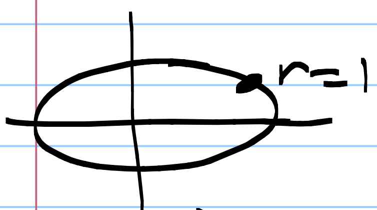
Example: Find the volume of the region inside the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ and above the plane $z = c/2$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{1}{4} = 1 \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{3}{4}$$

$$\frac{z}{c} = \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$$\text{Volume} = \iint_D \left(z - \frac{c}{2}\right) dA$$



$$x^2/a^2 + y^2/b^2 = 3/4 \Rightarrow \frac{x^2}{\left(a\frac{\sqrt{3}}{2}\right)^2} + \frac{y^2}{\left(b\sqrt{3}/2\right)^2} = 1$$

$$\frac{x}{a\sqrt{3}/2} = r\cos\theta, \quad \frac{y}{b\sqrt{3}/2} = r\sin\theta \qquad \begin{array}{l} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{array}$$

$$x = a\frac{\sqrt{3}}{2} r\cos\theta, \quad y = b\frac{\sqrt{3}}{2} r\sin\theta$$

$$dx\,dy = \frac{ab3}{4}\, r\,dr\,d\theta$$

$$\text{Volume} = \int_0^{2\pi}\int_0^1 \left(z - \frac{c}{2}\right)\frac{ab3}{4}\, r\,dr\,d\theta$$

$$\frac{z}{c} = \sqrt{1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2} = \sqrt{1 - \frac{3}{4}r^2\cos^2\theta - \frac{3}{4}r^2\sin^2\theta} = \sqrt{1 - \frac{3}{4}r^2}$$

$$\text{Volume} = \int_0^{2\pi}\int_0^1 \left(\sqrt{1-\frac{3r^2}{4}} - \frac{c}{2}\right)\frac{ab3}{4}\, r\,dr\,d\theta$$

$$= 2\pi \int_0^1 \sqrt{1-\frac{3r^2}{4}}\ \frac{ab3}{4}\, r\,dr - 2\pi\frac{3abc}{4}\int_0^1 r\,dr$$

$$= \frac{2\pi\, 3ab}{4}\int_0^1 \sqrt{1-\frac{3r^2}{4}}\, r\,dr - 2\pi\frac{3abc}{4}\ \frac{r^2}{2}\Big|_0^1$$

$$= \frac{\cancel{3}ab\pi}{2}\left(1-\frac{3r^2}{4}\right)^{3/2}\cdot\frac{1}{\cancel{3/2}\,(-6/4)}\Bigg|_0^1 - \frac{3abc\pi}{4}$$

$$= -\frac{4}{6}ab\pi\left[\frac{1}{8}-1\right] - \frac{3abc\pi}{4}$$

$$= \frac{7}{12}ab\pi - \frac{3abc\pi}{4}.$$

Video #2

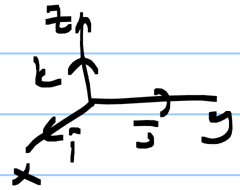
§11.1. Vector functions of One Variable:

$r = r(t) = (x(t), y(t), z(t))$ $x:\mathbb{R}\to\mathbb{R}$, $y:\mathbb{R}\to\mathbb{R}$, $z:\mathbb{R}\to\mathbb{R}$.

$$r:\mathbb{R}\to\mathbb{R}^3$$
$$t\longmapsto r(t)$$

Example: $r(t)=(t, t^2, t^3)$, $r(t)=(3\cos 2t, 4\cos 2t, 5\sin 2t)$

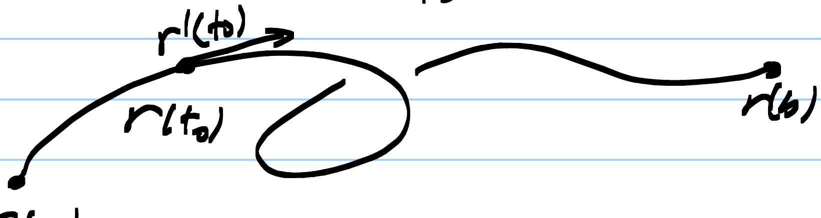
A vector valued function $r=r(t)=x(t)\,i+y(t)\,j+z(t)\,k$ will be continuous if its components x, y and z are continuous functions and similarly, it will be differentiable if its components are differentiable.



$r(t) = (x(t), y(t), z(t))$

If $r(t)$ is differentiable at a point $t_0\in\mathbb{R}$, then we write

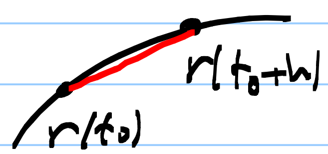
$$r'(t_0) = (x'(t_0), y'(t_0), z'(t_0)).$$



$r:[a,b]\longrightarrow\mathbb{R}^3$ $r'(t)$ can be considered as the velocity vector of a particle whose position is given by $r(t)$.

$$r'(t_0)=\lim_{h\to 0}\frac{r(t_0+h)-r(t_0)}{h}$$

$=$ velocity at t_0.



$||r'(t_0)|| = \sqrt{(x_1'(t_0))^2 + (y'(t_0))^2 + (z'(t_0))^2}$ is the speed at time t_0.

Differentiation Rules:

Theorem: Let $u(t)$ and $v(t)$ be differentiable vector-valued functions and let $\lambda(t)$ be a differentiable scalar valued function. Then $u(t)+v(t)$, $\lambda(t)u(t)$, $u(t)\cdot v(t)$, $u(t)\times v(t)$ and $u(\lambda(t))$ are differentiable and

a) $\dfrac{d}{dt}(u(t)+v(t)) = u'(t)+v'(t)$

b) $\dfrac{d}{dt}(\lambda(t)\, u(t)) = \lambda'(t)\, u(t) + \lambda(t)\, u'(t)$

c) $\dfrac{d}{dt}(u(t)\cdot v(t)) = u'(t)\cdot v(t) + u(t)\cdot v'(t)$

d) $\dfrac{d}{dt}(u(t)\times v(t)) = u'(t)\times v(t) + u(t)\times v'(t)$.

e) $\dfrac{d}{dt}(u(\lambda(t)) = u'(\lambda(t))\ \lambda'(t) = \lambda'(t)\ u'(\lambda(t))$.

Proof: c) $u(t) = (x_1(t), y_1(t), z_1(t))$ and
$v(t) = (x_2(t), y_2(t), z_2(t))$.

$u(t)\cdot v(t) = x_1(t)\,x_2(t) + y_1(t)\cdot y_2(t) + z_1(t)\,z_2(t)$

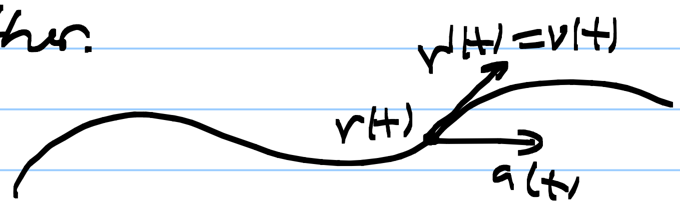
$$\frac{d}{dt}(u(t)\cdot v(t)) = (\qquad\qquad\qquad\qquad)'$$

$$= (x_1' x_2 + x_1 x_2') + (y_1' y_2 + y_1 y_2') + (z_1' z_2 + z_1 z_2')$$

$$= (x_1' x_2 + y_1' y_2 + z_1' z_2) + (x_1 x_2' + y_1 y_2' + z_1 z_2')$$
$$= (x_1', y_1', z_1')\cdot(x_2, y_2, z_2) + (x_1, y_1, z_1)\cdot(x_2', y_2', z_2')$$

$$= u'(t)\cdot v(t) + u(t)\cdot v'(t)$$

<u>Example:</u> Show that if an object has constant speed then the velocity vector and the acceleration vector are orthogonal to each other.



speed at time $t = \|v(t)\| = \|r'(t)\|$.

$r(t) = (x(t), y(t), z(t))$, $\quad r'(t) = (x'(t), y'(t), z'(t))$

$$\text{speed} = \|r'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

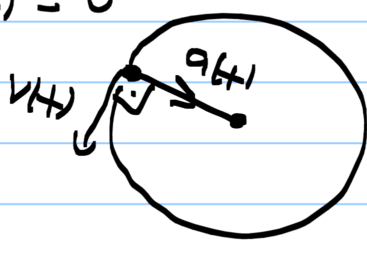
$$= \left(r'(t)\cdot r'(t)\right)^{1/2} = \text{constant}$$

$$r'(t)\cdot r'(t) = \text{constant}^2 = \text{Constant}$$

$$\frac{d}{dt}\left(r'(t)\cdot r'(t)\right) = \frac{d}{dt}(\text{Constant}) = 0.$$

$$r''(t)\cdot r'(t) + r'(t)\cdot r''(t) = 0$$

$$r'(t)\cdot r''(t) + r'(t)\cdot r''(t) = 0$$



$$\Rightarrow \underset{\substack{\parallel \\ v(t)}}{r'(t)} \cdot r''(t) = 0$$

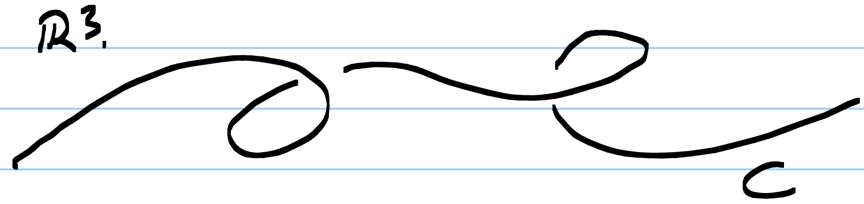
$$v(t)\cdot a(t) = 0$$

↖ acceleration vector

$\Rightarrow v(t) \perp a(t)$ for all t.

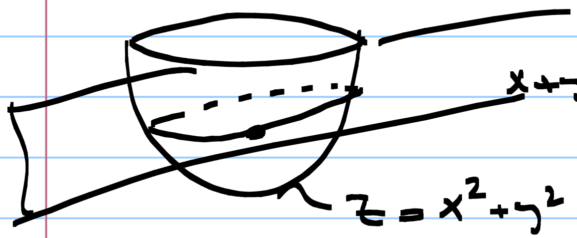
§11.3. Curves and Parametrization:

A vector valued function $r = r(t)$ traces a curve in $\mathbb{R}^3$.



In this case, we say that $r = r(t)$ is a parametrization of the curve C.

Ex: Find a parametrization for the intersection curve of the surfaces given by $x + y = 1$ and $z = x^2 + y^2$.



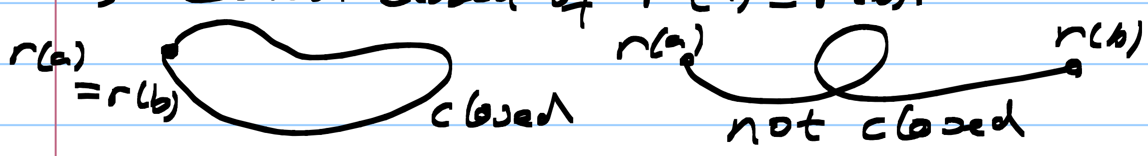
$r = r(t) = (x(t), y(t), z(t))$
$t \in \mathbb{R}$

$\begin{cases} z = x^2 + y^2 \\ x + y = 1 \end{cases}$ Let $x = t \Rightarrow y = 1 - t$, $z = x^2 + y^2$
$= t^2 + (1-t)^2$
$= 2t^2 - 2t + 1$

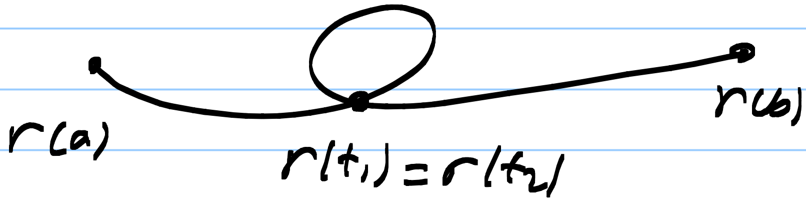
$r = r(t) = (t, 1-t, 2t^2 - 2t + 1)$

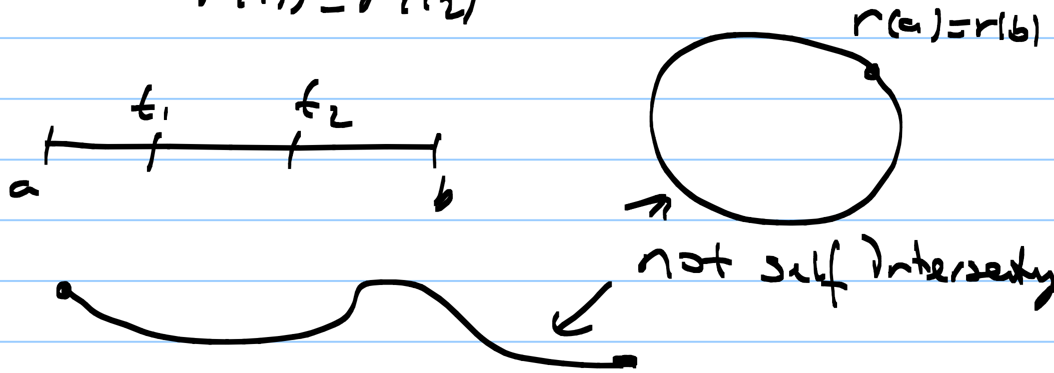
It is clear that there are infinitely many parametrizations for a given curve.

A curve $r = r(t)$ defined on $[a,b]$ ($r: [a,b] \to \mathbb{R}^3$) is called closed if $r(a) = r(b)$.

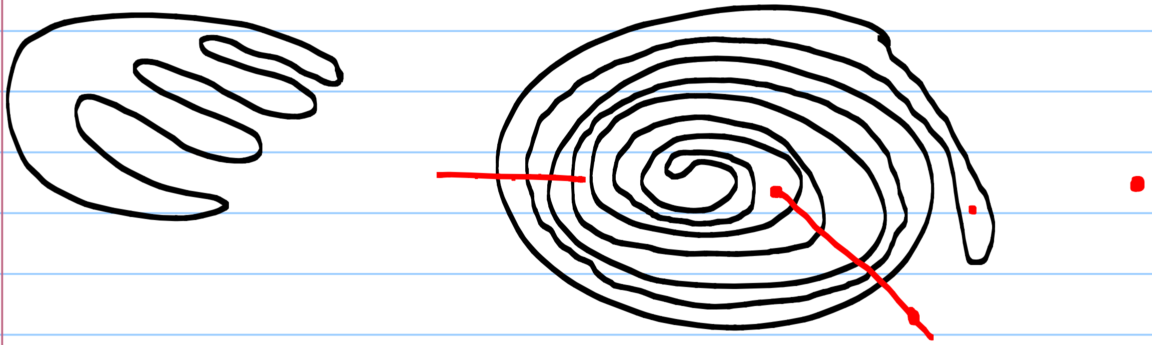


If $r(t_1) = r(t_2)$ for two points $t_1, t_2 \in (a,b)$ ($r : [a,b] \to \mathbb{R}$) then we say that $r(t)$ is self intersecting curve.





A not self-intersecting closed curve is called a simple closed curve.



If C is a simple closed curve in $\mathbb{R}^2$ then

$\mathbb{R}^2 \setminus C$ has two components one of which is an open disc and the other one is an open disc one point removed. This fact is known as "Jordan Closed Curve Theorem".

$$S^2 = \mathbb{R}^2 \cup \{\infty\}$$

Arc length:

If $r = r(t)$ is a "differentiable" curve then the arclength of the curve parametrized by $r = r(t)$ for $t \in [a,b]$ is defined to be the integral

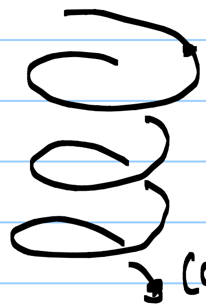
$$\int_a^b \|r'(t)\|\, dt = \text{Integral of the speed from } t=a \text{ to } t=b.$$

Remark: There are very weird "continuous" curves in the plane and therefore their arclength are <u>not</u> defined.

Example: Find the arclength of $r(t) = (a\cos t, a\sin t, bt)$ from the point $(a,0,0)$ to $(a,0,2\pi b)$.

Solution $r(t) = (x(t), y(t), z(t))$ $\quad a, b > 0.$

$x(t) = a\cos t$, $y(t) = a\sin t$, $z(t) = bt$



$(a,0,2\pi b) = r(2\pi)$

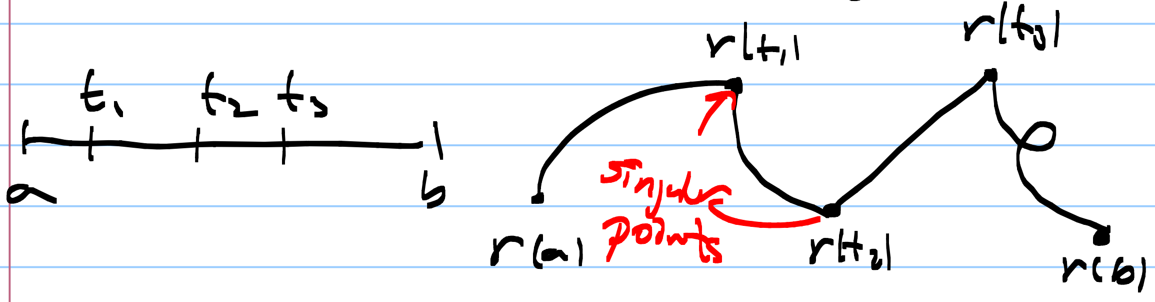
$(a,0,0) = r(0)$

$r'(t) = (-a\sin t, a\cos t, b)$

$$\|r'(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}$$
$$= \sqrt{a^2+b^2} \leftarrow \text{constant speed}$$

$$\text{Arc length} = \int_0^{2\pi} \|r'(t)\|\, dt$$
$$= \int_0^{2\pi} \sqrt{a^2+b^2}\, dt = \sqrt{a^2+b^2}\left(t\Big|_0^{2\pi}\right)$$
$$= 2\pi\sqrt{a^2+b^2}.$$

Definition: A continuous curve $r = r(t)$ defined on $[a, b]$ is called piecewise smooth if it is differentiable at all points except finitely many.

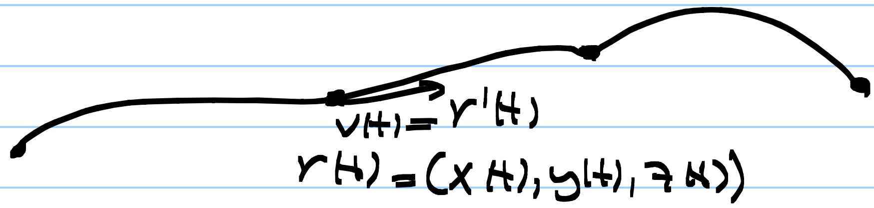


$$r'(t) = \lim_{h \to 0} \frac{r(t_0 + h) - r(t_0)}{h}$$ does not exists at $t = t_1, t_2, t_3$.

Video 64

§15.3. Line Integrals:

Let C be a bounded continuous parametric curve in $\mathbb{R}^3$. Also assume that C is piecewise smooth. Let $r=r(t)$ be a piecewise smooth parametrization for C. Then $v(t)=dr/dt$ is the velocity vector of $r(t)$.

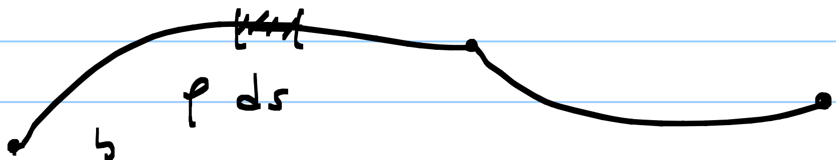


$v(t)=r'(t)$ the velocity vector (whenever it exists)

Arclength of C is given by: If $r:[a,b]\to\mathbb{R}^3$ then

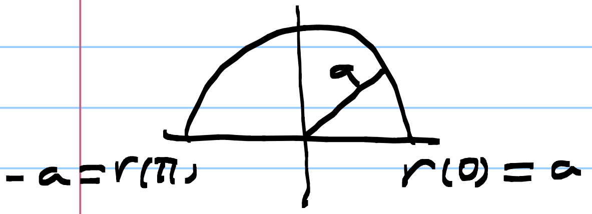
$$\text{Arclength} = \int_a^b \underbrace{\|r'(t)\|}_{\text{speed}}\,dt \quad , \|r'(t)\| = \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}$$

If $ds=\|r'(t)\|\,dt$ then we may compute the mass of the curve (a wire) if its density is given, say $\rho(\underline{x,y,z})$ gr/cm.



$\int_a^b \rho(r(t))\,ds$ is the mass of the wire.

Examples: 1) Calculate $\int_C y\,ds$, where C is parametrized as $r=r(t)=a\cos t\,\vec{i}+a\sin t\,\vec{j}$, $0\le t\le\pi$.



$r(t) = (a\cos t, a\sin t) = (x, y)$
$r'(t) = (-a\sin t, a\cos t)$
$ds = \|r'(t)\|\,dt$
$= \sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt$
$= a\,dt$

$$\int_C y\,ds = \int_0^{\pi} (a\sin t)(a\,dt) = a^2\int_0^{\pi} \sin t\,dt$$

$$= a^2\left(-\cos t\Big|_0^{\pi}\right)$$
$$= 2a^2.$$

2) Find the centroid of the circular helix C given by $r = r(t) = a\cos t\, i + a\sin t\, j + bt\, k$, $0 \le t \le 2\pi$.

Centroid = center of mass
Assume that it is homogeneous, i.e., the density is uniform.

Centroid $= (M_x, M_y, M_z)/\text{Arclength}$, where

$$M_x = \int_C x\,ds, \quad M_y = \int_C y\,ds \quad \text{and} \quad M_z = \int_C z\,ds.$$

$r(t) = (a\cos t, a\sin t, bt) = (x, y, z)$, $t \in [0, 2\pi]$
$r'(t) = (-a\sin t, a\cos t, b)$

$ds = \|r'(t)\|\,dt = \sqrt{a^2+b^2}\,dt$

$$M_x = \int_C x\,ds = \int_0^{2\pi} a\cos t\,\sqrt{a^2+b^2}\,dt = a\sqrt{a^2+b^2}\,(\sin t)\Big|_0^{2\pi}$$
$$= 0.$$

$$M_y = \int_C y\,ds = \int_0^{2\pi} a\sin t\,\sqrt{a^2+b^2}\,dt = a\sqrt{a^2+b^2}\left(-\cos t\Big|_0^{2\pi}\right) = 0$$

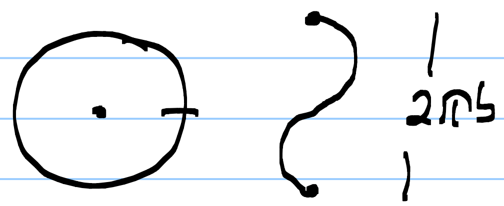
$$M_z = \int_C z\,ds = \int_0^{2\pi} bt\,\sqrt{a^2+b^2}\,dt = b\sqrt{a^2+b^2}\,\frac{t^2}{2}\Big|_0^{2\pi}$$

$$= 2b\sqrt{a^2+b^2}\,\pi^2$$

$$\text{Arclength} = \int_0^{2\pi} ds = \int_0^{2\pi} \sqrt{a^2+b^2}\,dt = \sqrt{a^2+b^2}\,2\pi$$

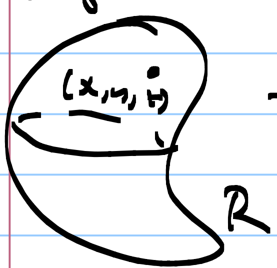
$$\text{Centroid} = (M_x, M_y, M_z) \,/\, \text{Arclength}$$
$$= (0, 0, 2b\sqrt{a^2+b^2}\,\pi^2) \,/\, (2\pi\sqrt{a^2+b^2})$$

$$= (0, 0, b\pi).$$



§15.1. Vector Fields and Scalar Fields:

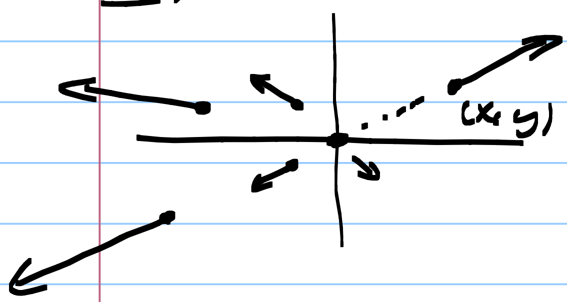
A function $f: R \to \mathbb{R}$, where R is a region in $\mathbb{R}^3$ is called a scalar field.



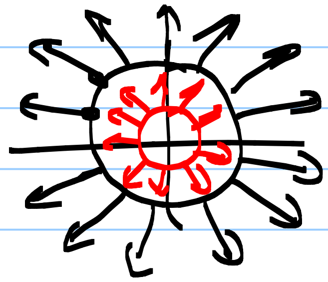
$f(x,y,z) \in \mathbb{R}$. f can be the temperature, mass or pressure at the point (x,y,z).

A function $F: R \to \mathbb{R}^2/\mathbb{R}^3$ depending on R is a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ is called a vector field on R.

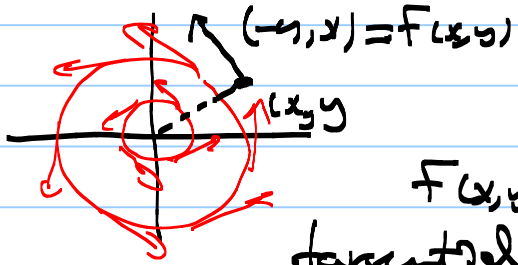
Ex 1) $R = \mathbb{R}^2$, $F(x,y) = x\hat{i} + y\hat{j} = (x,y)$



Radial vector field



2) $R = \mathbb{R}^2$, $F(x,y) = (-y, x) = -y\hat{i} + x\hat{j}$

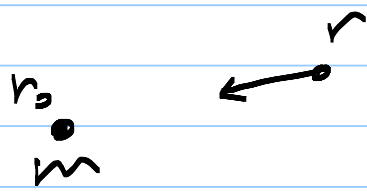


$(-y, x)\cdot(x,y) = 0$

$F(x,y) = (-y,x)$ is called a tangential vector field.

3) Suppose we have a mass m at the point r_0. Then the gravitational vector field induced by m is given by

$$F(x,y,z) = \frac{-km}{|r - r_0|^3}(r - r_0) \qquad r = (x,y,z), \quad r_0 = (x_0, y_0, z_0).$$



$$F(x,y,z) = \frac{-km\,(x - x_0, y - y_0, z - z_0)}{\left[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right]^{3/2}}$$

4) Gradient vector fields: If $\varphi : R \to \mathbb{R}$ is a function, where $R \subseteq \mathbb{R}^3/\mathbb{R}^2$ is a region then

$F = \mathrm{Grad}(\varphi) = (\varphi_x, \varphi_y, \varphi_z)$ is called the gradient vector field of the function φ.

Video 65

§ 16.1. Gradient, Divergence and Curl:

If $f(x,y,z)$ is a function whose first derivatives exist then the gradient vector field $\mathrm{grad}(f)$ is defined as

$$\mathrm{grad}(f) = (f_x, f_y, f_z).$$

Hence, if f is a scalar field then its gradient is a vector field.

Divergence and Curl: Take any vector field $F: R \to \mathbb{R}^3$, $R \subseteq \mathbb{R}^3$ (or $\mathbb{R}^2$). Say

$$F(x,y,z) = (F_1, F_2, F_3) = F_1 i + F_2 j + F_3 k$$

The div F is defined to be the scalar field given by

$$\mathrm{div}\, F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$
$$= \underbrace{\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)}_{\nabla} \cdot (F_1, F_2, F_3).$$

Hence, div F is a scalar field.

On the other hand, the curl F is the vector field defined by

$$\mathrm{curl}\, F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}, \frac{\partial F_2}{\partial x} - \frac{\partial F_3}{\partial y}\right)$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)k.$$

<u>Example:</u> Consider the vector field $F = xy\,i + (y^2 - z^2)\,j + yz\,k$. Compute div F and curl F.

<u>Solution:</u> $\operatorname{div} F = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \cdot (xy, y^2 - z^2, yz)$

$$= \frac{\partial(xy)}{\partial x} + \frac{\partial(y^2 - z^2)}{\partial y} + \frac{\partial(yz)}{\partial z}$$

$$= y + 2y + y = 4y.$$

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & y^2 - z^2 & yz \end{vmatrix}$$

$$= i\left(z - (-2z)\right) - j\,(0 - 0) + k(0 - x)$$

$$= 3z\,i - x\,k.$$

<u>Example:</u> If $\varphi = \varphi(x, y, z)$ is a twice differentiable function then div(grad φ) is called the Laplacian of φ, denoted as $\Delta\varphi$.

$$\Delta\varphi = \operatorname{div}(\operatorname{grad}\varphi) = \operatorname{div}(\varphi_x, \varphi_y, \varphi_z)$$

$$= (\partial/\partial x, \partial/\partial y, \partial/\partial z) \cdot (\varphi_x, \varphi_y, \varphi_z)$$

$$= \frac{\partial\varphi_x}{\partial x} + \frac{\partial\varphi_y}{\partial y} + \frac{\partial\varphi_z}{\partial z}$$

$$= \varphi_{xx} + \varphi_{yy} + \varphi_{zz}.$$

The equation $\Delta\varphi = 0$ is called the Laplace equation.

§ 15.2. Conservative Fields:

Definition: If a vector field $F(x,y,z) = \nabla\phi(x,y,z)$

$$F(x,y,z) = \nabla\phi(x,y,z) = \text{grad}(\phi) = (\phi_x, \phi_y, \phi_z),$$

then the vector field F is called conservative and ϕ is called a (scalar) potential for the vector field F.

Note that this definition makes sense in all dimensions.

Example: (Gravitational vector field of a point mass is conservative).

$$F(r) = \frac{-km(r-r_0)}{|r-r_0|^3}.$$

Let $\phi(x,y,z) = \dfrac{km}{|r-r_0|} = \dfrac{km}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}$

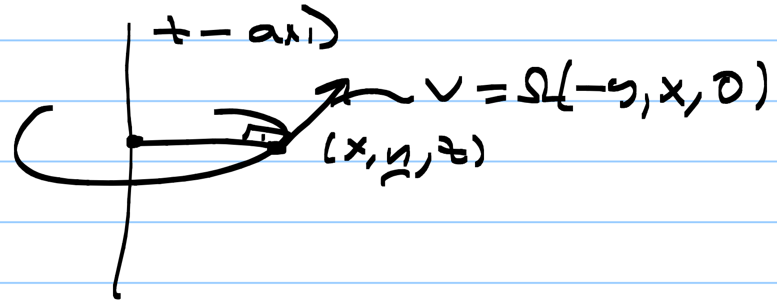
The potential $\phi: \mathbb{R}^3 \setminus \{(x_0,y_0,z_0)\} \to \mathbb{R}$ and

$$\phi_x = \frac{-(x-x_0)km}{\left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{3/2}} = -km\frac{x-x_0}{\|r-r_0\|^3}$$

Similarly, $\phi_y = -km\dfrac{y-y_0}{\|r-r_0\|^3}$ and $\phi_z = \dfrac{-km(z-z_0)}{\|r-r_0\|^3}$.

So, $F(x,y,z) = \text{grad}\,\phi = \nabla\phi$.

Example: The vector field $V = -\Omega y\,\hat{i} + \Omega x\,\hat{j}$ of rigid body rotation about the z-axis is <u>not</u> conservative.



To show that V is not conservative, assume on the contrary that it is conservative. Then try to reach at a contradiction.

So assume that $V = \nabla\phi$ for some function ϕ.

Then $V = (-y\Omega, x\Omega, 0) = (\phi_x, \phi_y, \phi_z)$.

$\Rightarrow \underline{\phi_x = -y\Omega}, \quad \underline{\phi_y = x\Omega}, \quad \underline{\phi_z = 0}.$

$\Rightarrow \phi_{yx} = -\Omega, \quad \phi_{xy} = \Omega,$

$\Rightarrow \phi_{xy} = \phi_{yx} \Rightarrow -\Omega = \Omega \Rightarrow \Omega = 0.$

So V is conservative only if $\Omega = 0$. Hence if $\Omega \neq 0$ then V cannot be conservative.

Proposition: If $F(x,y) = f_1(x,y)\hat{i} + f_2(x,y)\hat{j}$ is conservative then $\dfrac{\partial f_1}{\partial y} = \dfrac{\partial f_2}{\partial x}$.

Proof $F(x,y) = (F_1, F_2) = \nabla\phi = (\phi_x, \phi_y)$, for some ϕ. Then $F_1(x,y) = \phi_x(x,y)$ and

Video 66

$F_2(x,y) = \phi_y(x,y)$. then,

$$\phi_{xy} = \frac{\partial \phi_y}{\partial x} = \frac{\partial F_2}{\partial x} \quad \text{and} \quad \phi_{yx} = \frac{\partial \phi_x}{\partial y} = \frac{\partial F_1}{\partial y}.$$

If $F = (F_1, F_2)$ is differentiable then

$$\phi_{xy} = \phi_{yx} \Rightarrow \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

■

Proposition If $F = (F_1, F_2, F_3)$ is conservative then $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$, $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$.

Proof Let $F = (F_1, F_2, F_3) = \nabla \phi = (\phi_x, \phi_y, \phi_z)$.

Then $\frac{\partial F_1}{\partial y} = \frac{\partial \phi_x}{\partial y} = \phi_{yx} = \phi_{xy} = \frac{\partial \phi_y}{\partial x} = \frac{\partial F_2}{\partial x}$

Similarly, $\frac{\partial F_1}{\partial z} = \frac{\partial \phi_x}{\partial z} = \phi_{zx} = \phi_{xz} = \frac{\partial \phi_z}{\partial x} = \frac{\partial F_3}{\partial x}$

and $\frac{\partial F_2}{\partial z} = \frac{\partial \phi_y}{\partial z} = \phi_{zy} = \phi_{yz} = \frac{\partial \phi_z}{\partial y} = \frac{\partial F_3}{\partial y}$.

■

Example: Decide whether the vector field

$$F = (xy - \sin z)\,i + \left(\tfrac{1}{2}x^2 - \frac{e^y}{z}\right)j + \left(\frac{e^y}{z^2} - x\cos z\right)k$$

is conservative in $D = \{(x,y,z) \in \mathbb{R}^3 \mid z \neq 0\}$, and find a potential for F, if it exists.

Solution: So we look for a function $\phi = \phi(x,y,z)$ so that $F = \nabla\phi = (\phi_x, \phi_y, \phi_z)$.

$$\phi_x = F_1 = xy - \sin z$$
$$\phi_y = F_2 = \frac{1}{2}x^2 - \frac{e^y}{z}$$
$$\phi_z = F_3 = \frac{e^y}{z^2} - x\cos z.$$

If $\phi_x = xy - \sin z$, then $\phi = \frac{x^2y}{2} - x\sin z + h(y,z)$,

where $h(y,z)$ to be determined.

$$\phi_y = \frac{1}{2}x^2 - \frac{e^y}{z} \Rightarrow \frac{x^2}{2} - 0 + \frac{\partial h}{\partial y} = \frac{1}{2}x^2 - \frac{e^y}{z}.$$

$\Rightarrow \frac{\partial h}{\partial y} = -\frac{e^y}{z}$. So, $h(y,z) = -\frac{e^y}{z} + g(z)$,

for some $g(z)$ to be determined.

To determine $g(z)$ we use the last remaining equation, namely,

$\phi_z = \frac{e^y}{z^2} - x\cos z$, where

$$\phi(x,y,z) = \frac{x^2y}{2} - x\sin z - \frac{e^y}{z} + g(z).$$

$$\phi_z = 0 - x\cos z + \frac{e^y}{z^2} + g'(z) = \frac{e^y}{z^2} - x\cos z$$

$\Rightarrow g'(z) = 0$. So, $g(z) = C$, a constant function.

So, $\phi(x,y,z) = \frac{x^2y}{2} - x\sin z - \frac{e^y}{z} + C$, $C \in \mathbb{R}$.

We've one potential function for each $c \in \mathbb{R}$.

<u>Example:</u> For $(x,y) \neq (0,0)$ consider the vector field

$$F: \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}^2 \text{ given by}$$

$$F(x,y) = \left(\frac{-y}{x^2+y^2}\right)\hat{i} + \left(\frac{x}{x^2+y^2}\right)\hat{j}.$$

Find a potential for F if there is any.

<u>Solution</u> So we need to find $\varphi: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ so that

$$\varphi_x = F_1 = \frac{-y}{x^2+y^2} \quad \text{and} \quad \varphi_y = F_2 = \frac{x}{x^2+y^2}.$$

$$\varphi_y = \frac{x}{x^2+y^2} = \frac{1/x}{1+(y/x)^2}$$

$\varphi(x,y) = \tan^{-1}(y/x) + h(x)$, for some $h(x)$ to be determined.

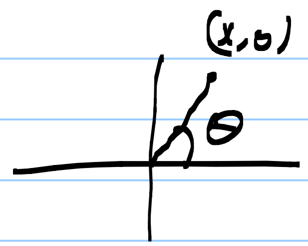
Then $\varphi_x = F_1 = \dfrac{-y}{x^2+y^2} \Rightarrow \dfrac{-y/x^2}{\underline{\underline{1+(y/x)^2}}} + h'(x) = \dfrac{-y}{x^2+y^2}$

$$\Rightarrow \frac{-y}{x^2+y^2} + h'(x) = -y/x^2+y^2 \Rightarrow h'(x) = 0.$$

So, $h(x) = C$ is a constant function.

Hence, $\varphi = \varphi(x,y) = \tan^{-1}(y/x) + C$

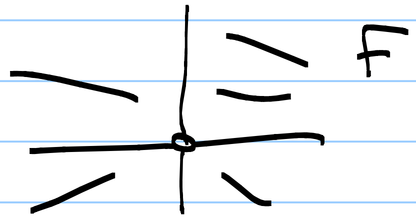
Thus, $F = \nabla\varphi = \nabla\left(\tan^{-1} y/x\right)$, which is defined on $\mathbb{R}^2 \setminus \{y\text{-axis}\}$.

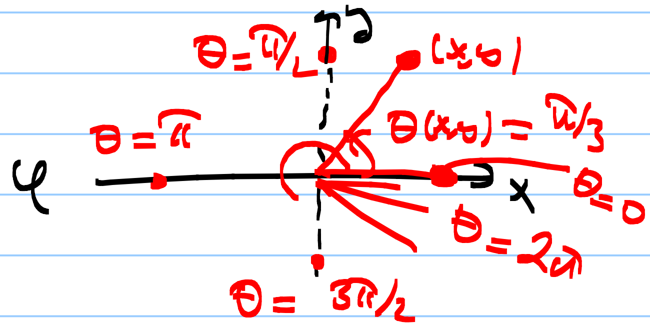


Indeed, $\varphi = \theta = \theta(x,y) = \tan^{-1} y/x$

Since $F : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ and

$\varphi = \theta(x,y) = \tan^{-1} y/x$ is defined only on $\mathbb{R}^2 \setminus \{y\text{-axis}\}$ F is not conservative on $\mathbb{R}^2 \setminus \{(0,0)\}$.





$\varphi = \theta = \tan^{-1} y/x$ cannot be even continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$.

<u>Remark</u>: Note that the above vector field

$F = (F_1, F_2) : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ satisfies

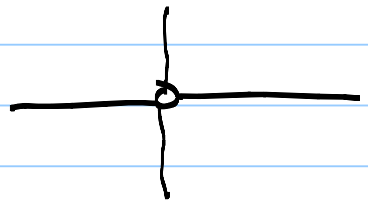
$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, the necessary criterion for being conservative.

$$F_1 = \frac{-y}{x^2+y^2}, \quad F_2 = \frac{x}{x^2+y^2}$$

$$\frac{\partial F_1}{\partial y} = \frac{-1\cdot(x^2+y^2) + y\,(2y)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial F_2}{\partial x}.$$

However, this is not sufficient for F to be conservative. In other words, the necessary condition that $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$ is <u>not</u> sufficient.

<u>Remark</u>: In the next lecture we'll see that this happens because of the "topology" of the domain the vector field is defined.



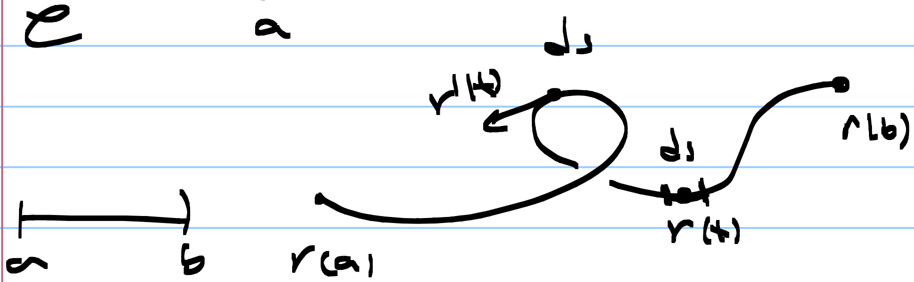
$\mathbb{R}^2 \setminus \{(0,0)\}$	$\mathbb{R}^2$
$\Downarrow$	$\Downarrow$
existence of $\theta(x,y) = \tan^{-1} y/x$	no such θ exists.

We'll see also that on $\mathbb{R}^2$ any vector field satisfying $\partial F_1/\partial y = \partial F_2/\partial x$ is indeed conservative.

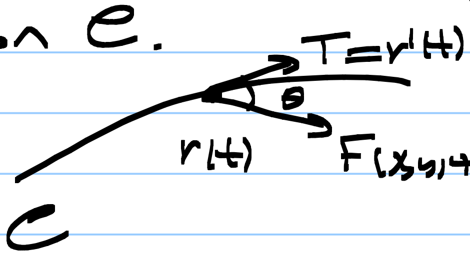
§15.4. Line Integrals of Vector Fields:

Last time we studied line integrals of scalar functions: Suppose we have a parametrized curve $\mathcal{C}$, $r = r(t)$, $t \in [a,b]$ and a function $f = f(x,y,t)$ defined on $\mathcal{C}$. Then

$$\int_{\mathcal{C}} f \, ds = \int_a^b f(r(t)) \, \|r'(t)\| \, dt$$



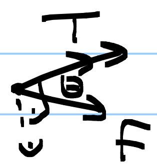
Today we have again a piecewise smooth parametrized curve $\mathcal{C}$, $r = r(t)$, $t \in [a,b]$, and a vector valued function $F = F(x,y,t)$ defined on $\mathcal{C}$.



$F(x,y,t) = (F_1(x,y,t), F_2(x,y,t), F_3(x,y,t))$

$dW = F \cdot r'(t) \, dt$ is the power the force $F(t)$ produces.

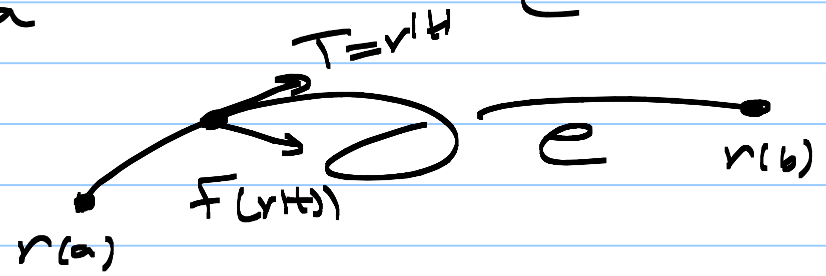
$$F \cdot r'(t) = \underline{\|F\| . \|r'(t)\| \cos\theta}$$



dW is just the work done during the period dt.

Hence the integral of $dW = F(r(t)) \cdot r'(t) \, dt$ over the interval $[a,b]$ is the total work done by the force F along the path $\mathcal{C}$.

$$\int_a^b F(r(t))\cdot r'(t)\,dt \doteq \int_{\mathcal{C}} F\cdot dr \qquad dr = r'(t)\,dt$$



Notation: $r(t) = (x(t), y(t), z(t))$ $\qquad \dfrac{dx}{dt} = x'(t)$

$\dfrac{dr}{dt} = (x'(t), y'(t), z'(t))$ $\qquad dx = x'(t)\,dt$

$$\Rightarrow dr = (x'(t), y'(t), z'(t))\,dt$$
$$= (x'(t)\,dt, y'(t)\,dt, z'(t)\,dt)$$
$$= (dx, dy, dz)$$

$$F\cdot dr = (F_1, F_2, F_3)\cdot(dx, dy, dz)$$
$$= F_1\,dx + F_2\,dy + F_3\,dz$$

$$\int_{\mathcal{C}} F\cdot dr = \int_{\mathcal{C}} F_1\,dx + F_2\,dy + F_3\,dz.$$

Example: let $F(x,y) = y^2\,\mathbf{i} + 2xy\,\mathbf{j} = (y^2, 2xy)$

$= (F_1, F_2)$, $F_1(x,y) = y^2$,

$F_2(x,y) = 2xy$.

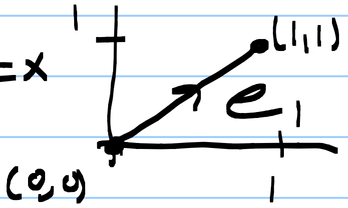
$\mathcal{C}$ is the curve in plane from $(0,0)$ to $(1,1)$ along

a) the straight line $y=x$
b) the curve $y=x^2$; and
c) the piecewise smooth path consisting of the

straight lines from (0,0) to (0,1) and from (0,1) to (1,1).

Solution: $F(x,y) = (F_1, F_2) = (y^2, 2xy)$.

a) $y = x$



$r(t) = (x(t), y(t))$

$x(t) = t,\ y(t) = t,\ t \in [0,1]$.

$r(t) = (t,t),\ r'(t) = (x'(t), y'(t))dt = (1,1)\,dt$

$$\int_{e_1} F \cdot dr = \int_0^1 F \cdot r'(t)\,dt$$

$$= \int_0^1 (y^2, 2xy) \cdot (1,1)\,dt$$

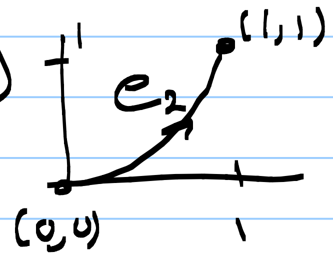
$$= \int_0^1 (y^2 + 2xy)\,dt$$

$$= \int_0^1 (t^2 + 2 \cdot t \cdot t)\,dt$$

$$= \int_0^1 3t^2$$

$$= t^3 \Big|_0^1 = 1.$$

b)



$y = x^2$, $r(t) = (x(t), y(t))$

$x(t) = t,\ y(t) = t^2,\ t \in [0,1]$

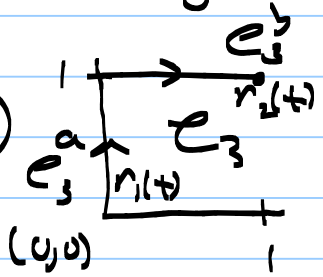
$r(t) = (t, t^2),\ dr = r'(t)\,dt$
$= (1, 2t)\,dt$

$$\int_{e_2} F \cdot dr = \int_0^1 (y^2, 2xy) \cdot (1, 2t)\,dt$$

$$\int_{C_2} F \cdot dr = \int_0^1 (t^4, 2t \cdot t^2) \cdot (1, 2t)\, dt$$

$$= \int_0^1 (t^4 + 4t^4)\, dt$$

$$= \int_0^1 5t^4\, dt = t^5 \Big|_0^1 = 1.$$

c) 

$C_3 = C_3^a \cup C_3^b$

C_3^a, $x = 0$, $y(t) = t$, $t \in [0,1]$.

$r_1(t) = (x(t), y(t))$
$= (0, t)$, $t \in [0,1]$

C_3^b: $x = t$, $y = 1$, $t \in [0,1]$.

$r_2(t) = (x(t), y(t)) = (t, 1)$, $t \in [0,1]$.

$$\int_{C_3} F \cdot dr = \int_{C_3^a} F \cdot dr + \int_{C_3^b} F \cdot dr$$

$$= \int_0^1 (\underline{y^2}, 2xy) \cdot (0,1)\, dt + \int_0^1 (y^2, 2xy) \cdot (1,0)\, dt$$

$$\boxed{\begin{aligned} r_1(t) &= (\overset{x}{0}, \overset{y}{t}) \\ dr_1 &= (0,1)\, dt \\ r_2(t) &= (\overset{x}{t}, \overset{y}{1}) \\ dr_2 &= (1,0)\, dt \end{aligned}}$$

$$= \int_0^1 (t^2, 0) \cdot (0,1)\, dt + \int_0^1 (1^2, 2t) \cdot (1,0)\, dt$$

$$= \int_0^1 (0 + 0)\, dt + \int_0^1 (1 + 0)\, dt$$

$$= 0 + t \Big|_0^1 = 1.$$

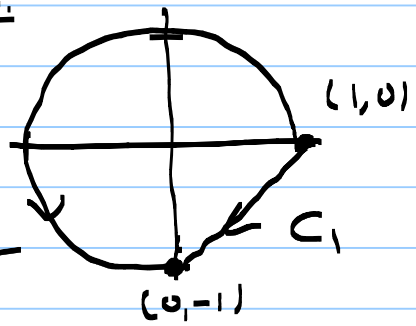
Video 6 P

Example 2: Let $F(x,y) = y\hat{i} - x\hat{j} = (y, -x)$

Find $\int_C F \cdot dr$ from $(1,0)$ to $(0,-1)$ along

a) the straight line segment joining these points
b) three-quarters of the circle of unit radius centered at the origin and traversed counterclockwise.

Solution:



$F(x,y) = (y, -x)$

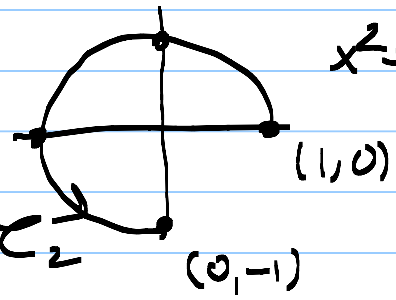
$$\frac{x}{1} + \frac{y}{-1} = 1 \Rightarrow x - y = 1$$
$$x = 1 + y$$

a) $C_1 : r(t) = (x(t), y(t)) = (\underbrace{1-t}_{x(t)}, \underbrace{-t}_{y(t)}),\ t \in [0,1]$

$$\int_{C_1} F \cdot dr = \int_0^1 (y, -x) \cdot (-1,-1)\, dt = \int_0^1 (t, -t-1) \cdot (-1,-1)\, dt$$

$$= \int_0^1 -t + (t+1)\, dt = \int_0^1 1\, dt = t \Big|_0^1 = 1.$$

b)



$x^2 + y^2 = 1$

$x(t) = \cos t,\ y(t) = \sin t$

$t \in [0, 3\pi/2]$

$r(t) = (\cos t, \sin t)$
$dr = (-\sin t, \cos t)\, dt$

$t = 0 \Rightarrow (x(t), y(t)) = (1, 0)$
$t = \pi/2 \Rightarrow (x(t), y(t)) = (0, 1)$
$t = 3\pi/2 \Rightarrow (x(t), y(t)) = (0, -1)$

$$\int_{C_2} F \cdot dr = \int_0^{3\pi/2} (y, -x) \cdot (-\sin t, \cos t)\, dt$$

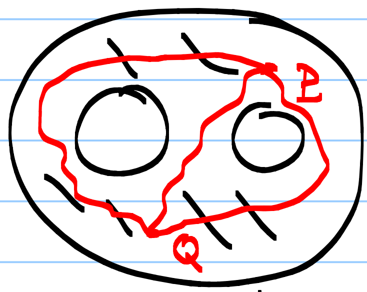
$$= \int_0^{3\pi/2} (\sin t, -\cos t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{3\pi/2} -dt = -t \Big|_0^{3\pi/2} = -3\pi/2.$$

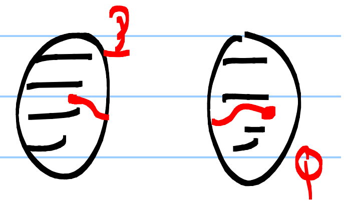
Connected and Simply Connected Domain:

Definition: A domain D is called connected if every pair of points P and Q in D can be joined by a piecewise smooth curve lying in D.

Connected

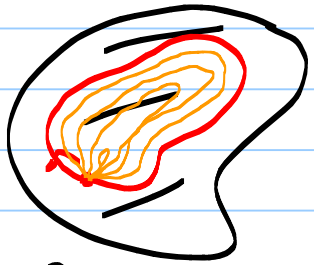


Connected



Not connected

Definition: A simply connected domain D is a connected domain in which every simple closed curve can be continuously shrunk to a point in D without any part ever passing out of D.

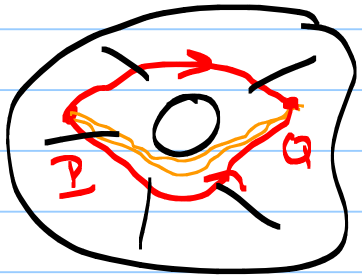


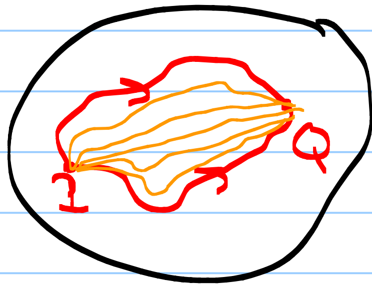
D

Simply Connected

connected but not simply connected

Remark: An equivalent formulation of simply connected domain is the following: A connected domain D is simply connected if any two path joining any two given points P and Q can be continuously deformed to each other.

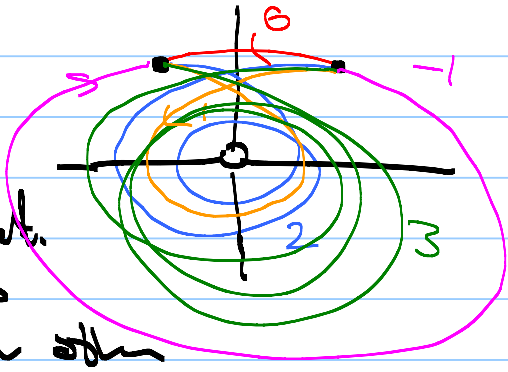




not simply connected (multiply connected)

simply connected

Let $D = \mathbb{R}^2 \setminus \{(0,0)\}$.



D is not simply connected. Indeed any two points can be joined to each other by $\mathbb{Z}$-many different "homotopy classes" of paths.

Theorem (Independence of Path)

Let D be an open, connected domain, and let F be a smooth vector field defined on D. Then the following three statements are equivalent in the sense that if any one of them is true then so are the other two:

a) F is conservative in D

b) $\oint_C F \cdot dr = 0$ for every piecewise smooth closed

Video 69

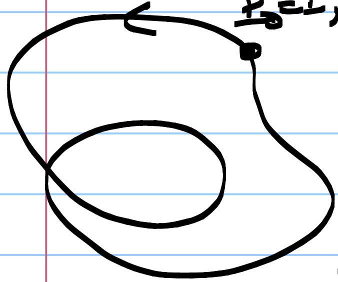
curve in D.

c) Given any two points P_0 and P_1 in D the integral

$\int_{\mathcal{C}} F\cdot dr$ has the same value for all piecewise smooth curves in D starting at P_0 and ending at P_1.

Proof: (a) $\Rightarrow$ (b): Assume that F is conservative.

must show: $\oint_{\mathcal{C}} F\cdot dr = 0$ for every closed curve $\mathcal{C}$ in D.



$r = r(t)$ a piecewise smooth parametrization for $\mathcal{C}$, $t \in [a,b]$.

Then $r(a) = P_0 = P_1 = r(b)$.

Since F is conservative we have $F = \nabla\phi$, for some function $\phi = \phi(x,y)$ so that

$F = (F_1, F_2) = (\phi_x, \phi_y) = \nabla\phi$.

Then $\oint_{\mathcal{C}} F\cdot dr = \int_a^b (\phi_x, \phi_y)\cdot(x'(t), y'(t))\,dt$, if

$r(t) = (x(t), y(t))$.

$$\oint_{\mathcal{C}} F\cdot dr = \int_a^b \Big(\phi_x(x(t),y(t))\,x'(t) + \phi_y(x(t),y(t))\,y'(t)\Big)dt$$

$$= \int_a^b \left(\frac{d}{dt}\phi(x(t),y(t))\right) dt$$

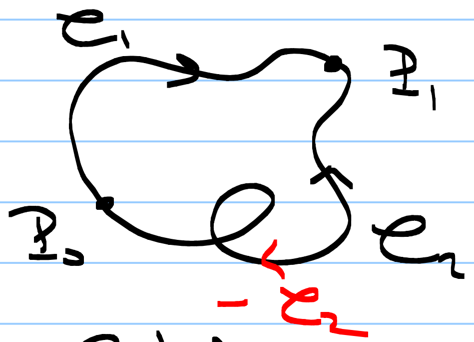
$$= \phi(x(t), y(t)) \Big|_{t=a}^{t=b} = \phi(x(b), y(b)) - \phi(x(a), y(a))$$

$$= \phi(P_1) - \phi(P_2) = \phi(P_2) - \phi(P_1) = 0.$$

(b) $\Rightarrow$ (c). So we assume that $\oint_{\mathcal{C}} F \cdot dr = 0$ for every closed (piecewise smooth) curve in D.

<u>must show</u> If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two paths starting at P_0 and ending at P_1 then

$$\int_{\mathcal{C}_1} F \cdot dr = \int_{\mathcal{C}_2} F \cdot dr$$



Note that $\mathcal{C} = \mathcal{C}_1 \cup (-\mathcal{C}_2)$ is a closed curve at P_0.

So since we assume (b) we have

$$\oint_{\mathcal{C}} F \cdot dr = 0.$$

However, $\oint_{\mathcal{C}} F \cdot dr = \int_{\mathcal{C}_1} F \cdot dr + \int_{-\mathcal{C}_2} F \cdot dr$

<u>Claim:</u> $\int_{-\mathcal{C}_2} F \cdot dr = -\int_{\mathcal{C}_2} F \cdot dr$

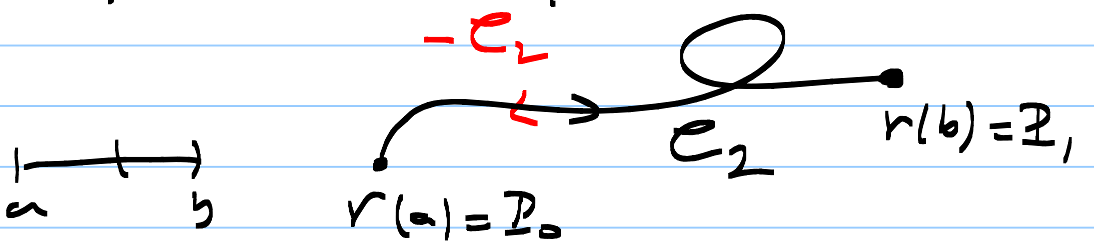
Note that "claim" implies that

$$0=\oint_{C} F\cdot dr=\int_{C_1} F\cdot dr+\int_{-C_2} F\cdot dr$$

$$=\int_{C_1} F\cdot dr-\int_{C_2} F\cdot dr$$

$$\Rightarrow \int_{C_1} F\cdot dr=\int_{C_2} F\cdot dr$$

<u>Proof</u> of <u>the claim</u>: Let $r=r(t)$, $t\in[a,b]$ be a parametrization for the curve C_2.



Then note that $\tilde{r}:[a,b]\longrightarrow D$ given by

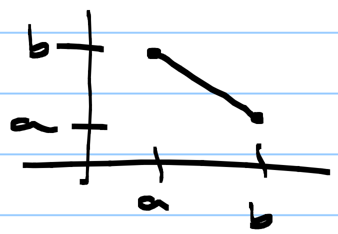
$\tilde{r}(t)=r(-t+a+b)$

$\tilde{r}(a)=r(b)=P_0$

$\tilde{r}(b)=r(a)=P_1$

$\tilde{r}(t)=r(-t+a+b)$

$=(x(-t+a+b),y(-t+a+b))$



$y(a)=b \quad y(b)=a$

$m=\dfrac{b-a}{a-b}=-1$

$(y-b)=-(x-a)$

$y=-x+(a+b)$

$$\int_{C_2} F\cdot dr = \int_a^b F(x(t),y(t))\cdot(x'(t),y'(t))\,dt$$

$$\int_{-C_2} F\cdot dr = \int_a^b F(\tilde{x}(t),\tilde{y}(t))\cdot(\tilde{x}'(t),\tilde{y}'(t))\,dt$$

$$\tilde{x}(t) = x(-t+a+b) \Rightarrow \tilde{x}'(t) = -x'(-t+a+b)$$
$$\tilde{y}(t) = y(-t+a+b) \Rightarrow \tilde{y}'(t) = -y'(-t+a+b)$$

So
$$\int_{-C_2} F\cdot dr = \int_{t=a}^{t=b} F(x(-t+a,b),y(-t+a+b))\cdot(-1)\,(x'(-t+a+b),y'(-t+a+b))\,dt$$

$$= -\int_{\xi=b}^{\xi=a} F(x(\xi),y(\xi))\cdot(x'(\xi),y'(\xi))\,(-d\xi)$$

Let $\xi = -t+a+b$

$d\xi = -dt$
$$= \int_b^a F(x(\xi),y(\xi))\cdot(x'(\xi),y'(\xi))\,d\xi$$

$$= \int_b^a F(x(t),y(t))\cdot(x'(t),y'(t))\,dt$$

$$= -\underbrace{\int_a^b F(x(t),y(t))\cdot(x'(t),y'(t))\,dt}$$

$$= -\int_{C_2} F\cdot dr.$$

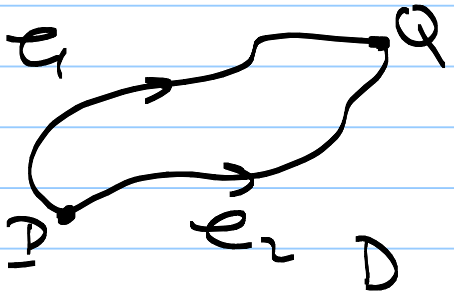
This finishes the proof of (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a): So we assume that the integral is path independent.

<u>must show:</u> F is conservative.

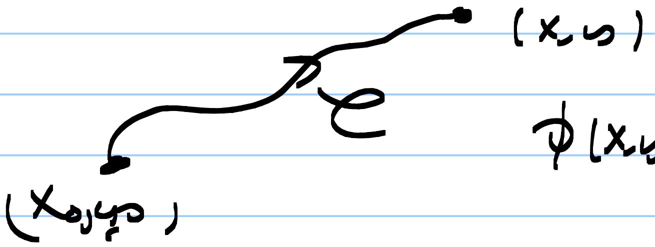
In other words, we must find a function $\phi = \phi(x,y)$ so that

$$F = (F_1, F_2) = (\phi_x, \phi_y).$$



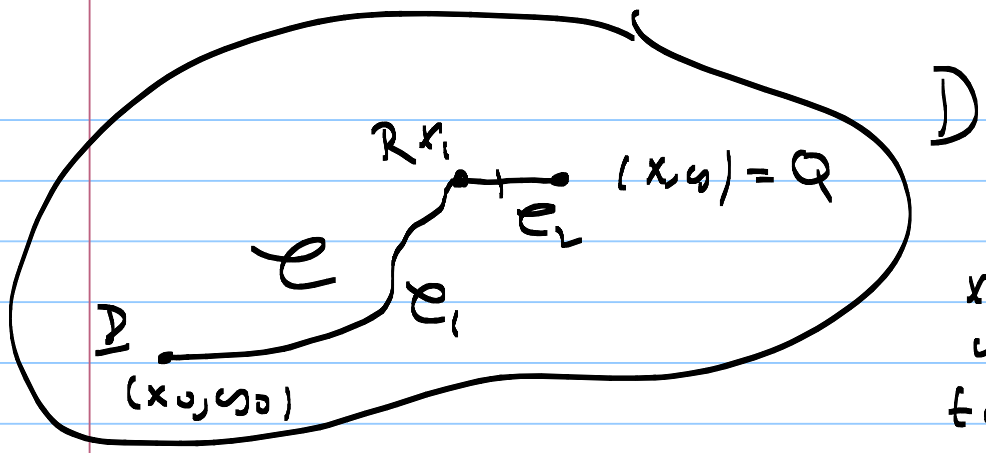
$$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$$

<u>Construction of $\phi = \phi(x,y)$:</u> Pick any point $P_0 = (x_0, y_0)$ in D. For any other point $Q = (x,y)$ let $\phi(x,y) \doteq \int_C F \cdot dr$, where C is any piecewise smooth curve starting at $P_0 = (x_0, y_0)$ and ending at $Q = (x,y)$.



$$\phi(x,y) = \int_C F \cdot dr$$

<u>must show:</u> $\phi_x = F_1$ and $\phi_y = F_2$.



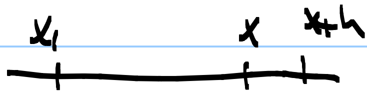
C_2:
$x(t) = x_1 + t$
$y(t) = y$
$t \in [0, x-x_1]$

$$\phi(x,y) = \int_C F\cdot dr = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr$$

$$\phi(x,y) = \int_{C_1} F\cdot dr + \int_0^{x_1-x_1} \underbrace{F}_{(F_1,F_2)}(x(t),y(t))\cdot(\overset{1}{x'(t)}, \overset{0}{y'(t)})\,dt$$

$$= \int_{C_1} F\cdot dr + \int_0^{x-x_1} F_1(x_1+t,y)\,dt$$

$$= \int_{C_1} F\cdot dr + \int_0^{x-x_1} F_1(x_1+t,y)\,dt$$



$$\phi_x(x,y) = \lim_{h\to 0} \frac{F(x+h,y) - F(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\int_0^{x+h-x_1} F_1(x_1+t,y)\,dt - \int_0^{x-x_1} F_1(x_1+t,y)\,dt}{t}$$

$$= \lim_{h\to 0} \frac{\int_{x-x_1}^{x-x_1+h} F_1(x_1+t,y)\,dt}{h}$$

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

$$\lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} \;\text{"}\; \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

$$= F_1(x_1 + \underline{\underline{(x-x_1)}}, y) = F_1(x,y).$$

Similarly, $\frac{\partial \phi}{\partial y}(x,y) = F_2(x,y)$, so that $\nabla\phi = (F_1, F_2) = F$, which finishes the proof ∎

Video 70

Example: For what values of the constants A and B is the vector field
$$F = Ax\sin(\pi y)\,i + (x^2\cos(\pi y) + Bye^{-z})\,j + y^2e^{-z}\,k$$
conservative? For this choice of A and B evaluate $\int_C F\cdot dr$, when C is

a) the curve $r = \cos t\, i + \sin 2t\, j + \sin^2 t\, k$, $0 \leq t \leq 2\pi$, and

b) the curve of intersection of the paraboloid $z = x^2 + 4y^2$ and the plane $z = 3x - 2y$, from $(0,0,0)$ to $(1, 1/2, 2)$.

Solution $F = Ax\sin(\pi y)\,i + (x^2\cos(\pi y) + Bye^{-z})\,j + y^2e^{-z}\,k$

A necessary condition for F to be conservative is the following: $F = (F_1, F_2, F_3)$.

$F = (F_1, F_2, F_3) = \nabla\phi = (\phi_x, \phi_y, \phi_z)$.

$$\frac{\partial F_1}{\partial y} = \phi_{yx} = \phi_{xy} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \phi_{zx} = \phi_{xz} = \frac{\partial F_3}{\partial x}$$
$$\frac{\partial F_2}{\partial z} = \phi_{zy} = \phi_{yz} = \frac{\partial F_3}{\partial y}$$

$F_1 = Ax\sin\pi y$, $F_2 = x^2\cos\pi y + Bye^{-z}$, $F_3 = y^2e^{-z}$

$$\frac{\partial F_1}{\partial y} = A\pi x\cos\pi y = 2x\cos\pi y = \frac{\partial F_2}{\partial x} \Rightarrow A\pi = 2 \Rightarrow A = 2/\pi.$$

$$\frac{\partial F_1}{\partial z} = 0 = 0 = \frac{\partial F_3}{\partial x} \quad \Bigg| \quad \frac{\partial F_2}{\partial z} = -Bye^{-z} = 2ye^{-z} = \frac{\partial F_3}{\partial y}$$
$$\Rightarrow -B = 2 \Rightarrow B = -2.$$

Here, $A = 2/\pi$ and $B = -2$.

$$F = \frac{2}{\pi} x \sin \pi y \, i + (x^2 \cos \pi y - 2y e^{-z}) j + y^2 e^{-z} k.$$

Look for potential function $\phi = \phi(x,y,z)$.

$$\phi_x = F_1 = \frac{2}{\pi} x \sin \pi y \Rightarrow \phi = \frac{x^2}{\pi} \sin \pi y + \underbrace{h(y,z)}_{?}$$

$$\phi_y = F_2 \Rightarrow x^2 \cos \pi y + h_y = x^2 \cos \pi y - 2y e^{-z}$$
$$\Rightarrow h_y = -2y e^{-z}$$
$$\Rightarrow h(y,z) = -y^2 e^{-z} + g(z), \quad g(z) = ?$$

$$\phi = \frac{x^2}{\pi} \sin \pi y - y^2 e^{-z} + g(z)$$

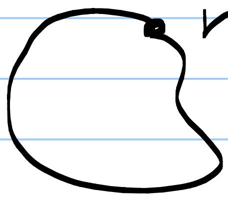
$$\phi_z = F_3 \Rightarrow y^2 e^{-z} + g'(z) = y^2 e^{-z} \Rightarrow g'(z) = 0$$
$$\Rightarrow g(z) = C \text{ const}$$

$$\phi = \phi(x,y,z) = \frac{x^2}{\pi} \sin \pi y - y^2 e^{-z} + \overset{0}{\overset{\shortparallel}{C}}$$

So $F = \nabla \phi$ and thus F is conservative.

a) $r = r(t) = \cos t \, i + \sin 2t \, j + \sin^2 t \, k, \quad 0 \le t \le 2\pi.$

$r(0) = i = r(2\pi)$ so that the curve is closed.

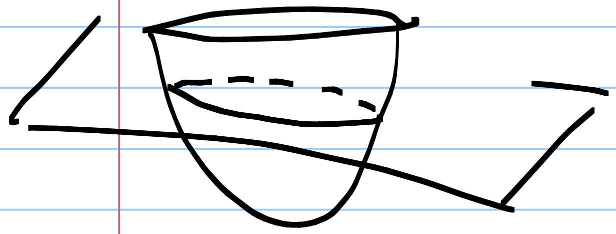


$r(0) = r(2\pi)$

$\int_C F \cdot dr = 0$ since F is conservative.

$$\int_C F \cdot dr = \int_C \nabla \phi \cdot dr = \phi(\underbrace{r(2\pi)}_{i}) - \phi(\underbrace{r(0)}_{i}) = 0.$$

b) $z = x^2+4y^2$, $z = 3x-2y$ $(0,0,0)$ to $(1,1/2,2)$.

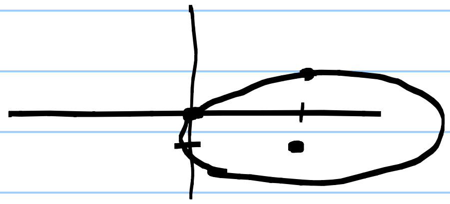


$$x^2+4y^2 = 3x-2y$$

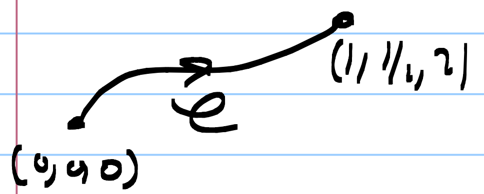
$$\Rightarrow x^2-3x+4\left(y^2+\frac{y}{2}\right)=0.$$

$$\Rightarrow \left(x-\frac{3}{2}\right)^2+4\left(y+\frac{1}{4}\right)^2 = \frac{9}{4}+\frac{1}{4}$$

$$\Rightarrow \left(x-\frac{3}{2}\right)^2+4\left(y+\frac{1}{4}\right)^2 = \frac{10}{4}$$



$$\int_C F\cdot dr = \int_C \nabla\phi\cdot dr$$



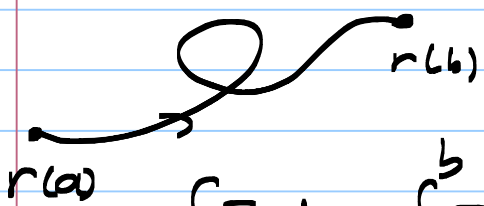
$$= \phi(1,1/2,2) - \phi(0,0,0)$$

$$= \frac{1}{\pi}\sin\frac{\pi}{2} - \frac{1}{4}e^{-2} - 0$$

$$\phi = \frac{x^2}{\pi}\sin\pi y - y^2 e^{-z}$$

$$= \frac{1}{\pi} - \frac{e^{-2}}{4}.$$

<u>**Remark**</u>, $F = \nabla\phi$, $C: r = r(t)$, $a \le t \le b$



$r = (x(t), y(t), z(t))$

$dr = (x', y', z')\,dt$

$$\int_C F\cdot dr = \int_a^b \nabla\phi\cdot dr = \int_a^b (\phi_x, \phi_y, \phi_z)\cdot(x', y', z')\,dt$$

$$= \int_a^b (\phi_x x' + \phi_y y' + \phi_z z')\,dt$$

$$= \int_a^b \frac{d}{dt}\left(\phi(x(t), y(t), z(t))\right)dt$$

$$= \phi(x(t), y(t), z(t))\Big|_{t=a}^{t=b} = \phi(r(b)) - \phi(r(a))$$

Example: Evaluate $I = \oint_C (e^x \sin y + 3y)\,dx + (e^x \cos y + 2x - 2y)\,dy$

counterclockwise around the ellipse $4x^2 + y^2 = 4$.

Solution $F = (e^x \sin y + 3y,\ e^x \cos y + 2x - 2y)$

Let's see if F is conservative?

$4x^2 + y^2 = 4 \Rightarrow x^2 + \left(\frac{y}{2}\right)^2 = 1$, $x = \cos t$, $\frac{y}{2} = \sin t$

$\Rightarrow x(t) = \cos t,\ y(t) = 2\sin t$

$F = (F_1, F_2) = (e^x \sin y + 3y,\ e^x \cos y + 2x - 2y) = (\phi_x, \phi_y)$

$\phi_x = e^x \sin y + 3y \Rightarrow \phi = e^x \sin y + 3xy + \underbrace{h(y)}_{?}$

$\phi_y = e^x \cos y + 2x - 2y \Rightarrow \phi_y = e^x \cos y + 3x + h'(y)$
$= e^x \cos y + 2x - 2y$

Hence, F is not conservative. However, we see that

$G = (e^x \sin y + 2y,\ e^x \cos y + 2x - 2y)$ is conservative.

where $G = \nabla \phi$, $\phi = e^x \sin y + 2xy - y^2$.

Moreover, $F = G + (y, 0)$.

So, $\oint_C F \cdot dr = \oint_C (G + (y, 0)) \cdot dr$

$= \oint_C \underbrace{G}_{\nabla\phi} \cdot dr + \oint_C (y, 0) \cdot dr$

$= 0 + \int_C (y, 0) \cdot dr,$

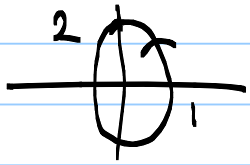
Video 71

because G is conservative and C is a closed loop.

Now, $\oint_C F \cdot dr = \oint_C (y, 0) \cdot dr$

C: $4x^2 + y^2 = 4 \Rightarrow x^2 + \left(\frac{y}{2}\right)^2 = 1$, $x = \cos t$, $y/2 = \sin t$

$r(t) = (\cos t, 2\sin t)$, $t \in [0, 2\pi]$.



$$\oint_C F \cdot dr = \oint_C (y, 0) \cdot dr$$

$$\boxed{\begin{aligned} x(t) &= \cos t \\ x'(t) &= -\sin t \\ y(t) &= 2\sin t \\ y'(t) &= 2\cos t \end{aligned}}$$

$$= \int_0^{2\pi} (2\sin t, 0) \cdot (x'(t), y'(t))\, dt$$

$$= \int_0^{2\pi} (2\sin t, 0) \cdot (-\sin t, 2\cos t)\, dt$$

$$\cos 2t = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t \quad \Big| \quad = \int_0^{2\pi} -2\sin^2 t \, dt$$

$$= \int_0^{2\pi} (\cos 2t - 1)\, dt$$

$$= -t \Big|_0^{2\pi} = -2\pi.$$

Remark:

Counterclockwise is regarded as the positive parametrization

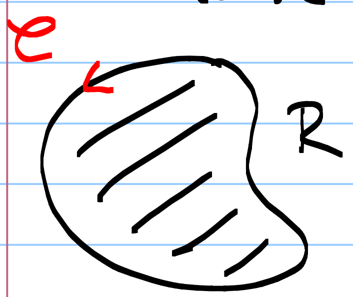
§16.3. Green's Theorem in the Plane:

Theorem: (Green's Theorem)

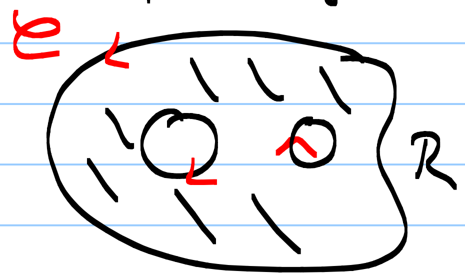
Let R be a regular, closed region in the xy-plane whose boundary $\mathcal{C}$, consists of one or more piecewise smooth, simple closed curves that are positively oriented with respect to R. If $F = (F_1, F_2)$ is a smooth vector field on R, then

$$\oint_{\mathcal{C}} F_1(x,y)\,dx + F_2(x,y)\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$

Remark: Here "positively oriented with respect to R" means the following:

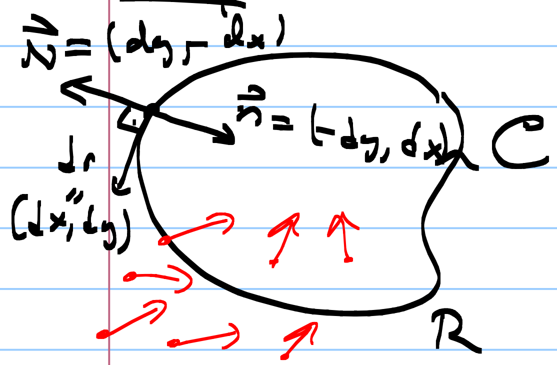


counterclockwise



$\mathcal{C}$ has 3-components

Proof: (Plumber's proof of Green's theorem)

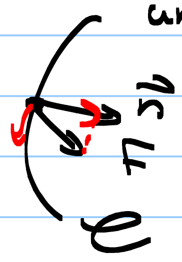


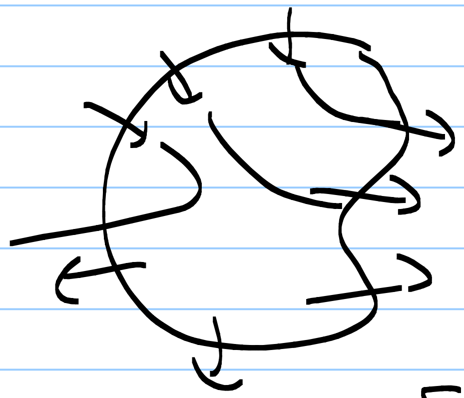
$$\begin{aligned} r = r(t) &= (x(t), y(t)) \\ dr &= (x'(t), y'(t))\,dt \\ &= (x'(t)\,dt,\ y'(t)\,dt) \\ &= (dx, dy) \end{aligned}$$

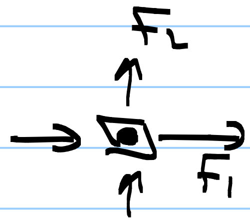
$F(x,y) = (F_1(x,y), F_2(x,y))$ Suppose that F is the velocity of the flow of some liquid on the xy-plane.

$\oint_C F\cdot\vec{n}$ = the total amount of liquid that enters the region per second.

C the amount of liquid that enters the region at a unit time.







$F=(F_1,F_2)$

The amount of liquid that accumulates at the point (x,y) is $-\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)$.

Thus we must have

$$\oint_C F\cdot n=\iint_A -\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)dA$$

$$\oint_C F\cdot n=\oint (F_1,F_2)\cdot(-dy,dx)$$

$$=\oint(-F_1)\,dy+F_2\,dx=\iint_A(-1)\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)dA$$

To make this look like Green's theorem write M for F_2 and N for $-F_1$.

Then

$$\oint_C M\,dx+N\,dy=\iint_R(-M_y+N_x)\,dA$$

Video 72

So $\displaystyle\oint_{\mathcal{C}} F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$

Example 1 (Area Bounded by a simple closed loop)

Let $F(x,y) = (F_1, F_2) = \left(-\frac{y}{2}, \frac{x}{2}\right)$. $\quad F_1 = -y/2$, $F_2 = x/2$

Let R be a region bounded by a simple closed loop $\mathcal{C}$.

$\mathcal{C}$ 

Let's apply Green's theorem to F and $\mathcal{C}$.

$$\oint_{\mathcal{C}} F \cdot dr = \oint_{\mathcal{C}} F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

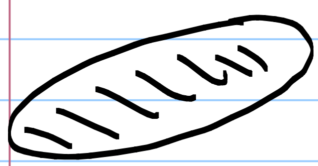
$$= \iint_R \left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right] dA$$

$$= \iint_R 1 \cdot dA$$

$$= \text{Area of } R.$$

Example: Compute the area of the region bounded by the curve

$r(t) = 3(\cos t + \sin t)\,\hat{i} + 2(\sin t - \cos t)\,\hat{j}, \quad 0 \le t \le 2\pi.$



$$\text{Area of } R = \oint_{\mathcal{C}} F \cdot dr \quad , \quad F = \left(-\frac{y}{2}, \frac{x}{2}\right)$$

$$r = r(t) = \underbrace{3(\cos t + \sin t)}_{x(t)}\, \hat{\imath} + \underbrace{2(\sin t - \cos t)}_{y(t)}\, \hat{\jmath}$$

$$\text{Area of } R = \oint_{\mathcal{C}} F \cdot dr = \int_0^{2\pi} \left(-\frac{y}{2}, \frac{x}{2}\right) \cdot (dx, dy)$$

$$= \frac{1}{2} \int_0^{2\pi} -y\, dx + x\, dy$$

$$\boxed{\begin{aligned} x &= 3(\cos t + \sin t) \\ dx &= 3(-\sin t + \cos t)\, dt \\ y &= 2(\sin t - \cos t) \\ dy &= 2(\cos t + \sin t)\, dt \end{aligned}}$$

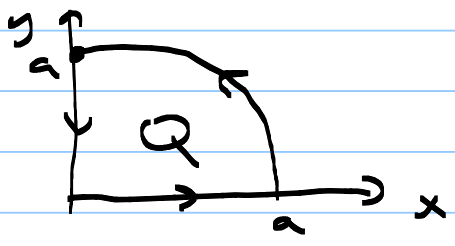
$$= \frac{1}{2} \int_0^{2\pi} \Big[2(-\sin t + \cos t) \cdot 3(-\sin t + \cos t) + 3(\cos t + \sin t) \cdot 2(\cos t + \sin t) \Big] dt$$

$$= \frac{1}{2} \int_0^{2\pi} 6 \left[1 - \cancel{2 \sin t \cos t} + 1 + \cancel{2 \cos t \sin t} \right] dt$$

$$= 6 \int_0^{2\pi} dt = 12\pi.$$

<u>Example 3.</u> Evaluate the integral

$I = \oint_{\mathcal{C}} (x - y^3)\, dx + (y^3 + x^3)\, dy$, where $\mathcal{C}$ is the positively oriented boundary of the quarter disk Q, $0 \le x^2 + y^2 \le a^2$, $x \ge 0$, $y \ge 0$.



Solution By Green's theorem

$$\oint_{C} F_1\,dx + F_2\,dy = \iint_Q \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

$$\oint_{C} \underbrace{(x-y^3)}_{F_1}\,dx + \underbrace{(y^3+x^3)}_{F_2}\,dy = \iint_Q (3x^2+3y^2)\,dA$$

$x = r\cos\theta,\ 0 \le r \le a$
$y = r\sin\theta,\ 0 \le \theta \le \pi/2$

$$= \int_0^{\pi/2}\int_0^a 3r^2 \cdot r\,dr\,d\theta$$

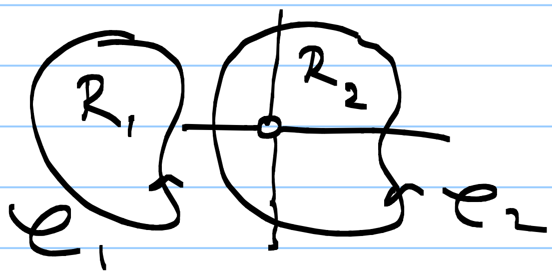
$$= \int_0^{\pi/2} \left(\frac{3}{4} r^4 \Big|_0^a\right) d\theta$$

$$= \frac{3a^4}{4}\,\theta\Big|_0^{\pi/2} = \frac{3a^4\pi}{8}.$$

Example: Let C be a positively oriented simple closed curve in the xy-plane bounding a region R and not passing through the origin. Then show that

$$\oint_{C} \frac{-y\,dx + x\,dy}{x^2+y^2} = \begin{cases} 2\pi & \text{if } R \text{ contains the origin} \\ 0 & \text{if } R \text{ does not contain the origin.} \end{cases}$$

Solution:



$\theta = \tan^{-1} y/x$

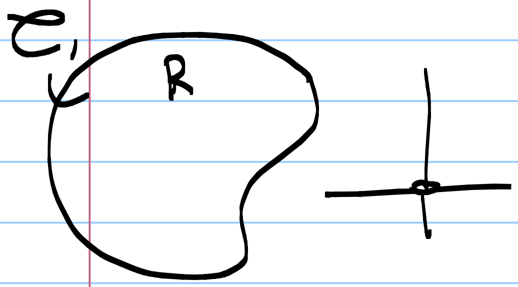
$d\theta = \dfrac{-y\,dx + x\,dy}{x^2+y^2}$

$$\oint_{C_1} \frac{-y\,dx + x\,dy}{x^2+y^2} = 0, \quad \oint_{C_2} \frac{-y\,dx + x\,dy}{x^2+y^2} = 2\pi.$$

$$F = (F_1, F_2) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) \qquad \frac{\partial F_2}{\partial x} = \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial F_1}{\partial y}$$

Note that F is defined on $\mathbb{R}^2 \setminus \{(0,0)\}$.

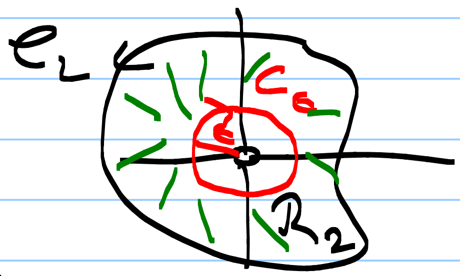


Hence we may apply Green's theorem to $\mathcal{C}_1$ and R_1.

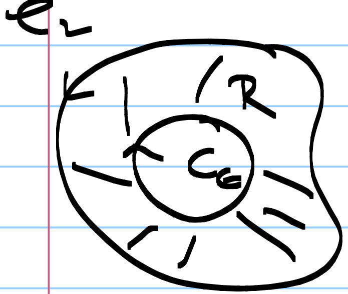
So
$$\oint_{\mathcal{C}_1} \frac{-y\,dx + x\,dy}{x^2+y^2} = \oint_{\mathcal{C}_1} F_1\,dx + F_2\,dy = \iint_R \overbrace{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)}^{=0} dA$$

$$= 0.$$

For the other integral note that $(0,0)$ is contained in R_2 and thus we cannot apply Green's theorem directly.



Let R be the region between $\mathcal{C}_2$ and C_ϵ, where C_ϵ is oriented positively (clockwise) w.r.t. the region R

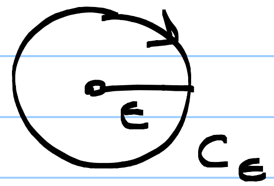


$$\oint_{\mathcal{C}_2 \cup C_\epsilon} \frac{-y\,dx + x\,dy}{x^2+y^2} \overset{\text{Green's Th}}{=} \iint_R \overbrace{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)}^{=0} dA$$

$$= 0$$

So,
$$\oint_{\mathcal{C}_2} \cdots + \oint_{C_\epsilon} \cdots = 0.$$

So, $\displaystyle\oint_{\mathcal{C}_2} \frac{-y\,dx + x\,dy}{x^2+y^2} = -\oint_{C_\epsilon} \frac{-y\,dx + x\,dy}{x^2+y^2}$



$\displaystyle = -\int_{C_\epsilon} \frac{-y\,dx + x\,dy}{x^2+y^2}$

$0 \le t \le 2\pi$

$x = \epsilon \cos t$

$y = -\epsilon \sin t$

$dx = -\epsilon \sin t\,dt$

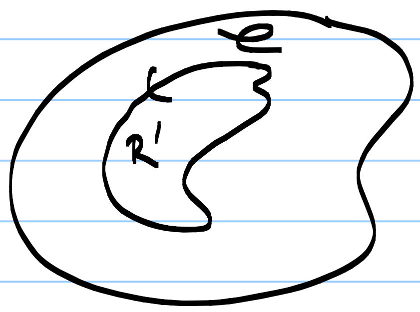
$dy = -\epsilon \cos t\,dt$

$$= -\int_0^{2\pi} \frac{\left[\epsilon \sin t\,(-\epsilon \sin t) + \epsilon \cos t\,(-\epsilon \cos t)\right]}{\epsilon^2(\cos^2 t + \sin^2 t)}\,dt$$

$$= -\int_0^{2\pi} \frac{-\epsilon^2}{\epsilon^2}\,dt$$

$$= \int_0^{2\pi} dt = 2\pi.$$

<u>Remark:</u> If $F = (F_1, F_2)$ is a smooth vector field with $\partial F_2/\partial x = \partial F_1/\partial y$ then F is conservative on any simply connected region.



If R is simply connected then for any simple closed curve $\mathcal{C}$ in R we can Green's Thm:

$$\oint_{\mathcal{C}} F\cdot dr = \oint_{\mathcal{C}} F_1\,dx + F_2\,dy = \iint_{R'} \underbrace{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)}_{=0} dA = 0$$

for any closed curve $\mathcal{C}$ in R.

Hence, F is conservative.