HOMOLOGY OF NON ORIENTABLE REAL ALGEBRAIC VARIETIES

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Abstract. Let \( R \) be any commutative ring with unity and \( X \) a nonsingular compact real algebraic variety with a nonsingular projective complexification \( i : X \to X_C \). For a topological component \( X_0 \) of \( X \) we define \( KH_*(X_0, R) \) as the kernel of the induced homomorphism \( i_* : H_*(X_0, R) \to H_*(X_C, R) \) and \( ImH^*(X_0, R) \) as the image of the homomorphism \( i^* : H^*(X_C, R) \to H^*(X_0, R) \). In [6] the author showed that both \( KH_*(X_0, R) \) and \( ImH^*(X_0, R) \) are independent of the complexification \( X \subseteq X_C \) and thus (entire rational) isomorphism invariants of \( X \) provided that \( X_0 \) is \( R \)-orientable. In this note the same result is proved for non \( R \)-orientable \( X_0 \) under the assumption that \( 2 \in R \) is a unit. We have also some partial results for \( R = \mathbb{Z} \).

1. Introduction and the results

Let \( R \) be any commutative ring with unity. Let \( X \) be a nonsingular compact real algebraic variety and \( i : X \to X_C \) be the inclusion map into some nonsingular projective complexification. Define \( KH_*(X, R) \) as the kernel of the induced map

\[ i_* : H_*(X, R) \to H_*(X_C, R) \]

on homology and \( ImH^*(X, R) \) as the image of the induced map

\[ i^* : H^*(X_C, R) \to H^*(X, R). \]

In [6] it is shown that if \( X \) is \( R \)-orientable then both \( KH_*(X, R) \) and \( ImH^*(X, R) \) are independent of the complexification \( i : X \to X_C \) and thus are (entire rational) isomorphism invariants of \( X \) (see also [3]). Indeed the proof of this result enables us to define \( KH_*(X_0, R) \) and \( ImH^*(X_0, R) \) for any \( R \)-orientable (metric) topological component \( X_0 \) of the underlying smooth manifold \( X \). In other words, \( KH_*(X_0, R) \) and \( ImH^*(X_0, R) \) are independent of the complexification as long as \( X_0 \) is \( R \)-orientable.

Below is the main result of this note, which extends this result to non \( R \)-orientable varieties.
Theorem 1.1. Let $X_0$ be a topological component of any compact nonsingular real algebraic variety $X$ and $R$ is a commutative ring with unity. Then, both $KH_* (X_0, R)$ and $ImH^* (X_0, R)$ are independent of the choice of the smooth projective complexification $i : X \rightarrow X_\mathbb{C}$ provided that either $X_0$ is $R$-orientable or $R$ contains 2 as a unit.

Even though this theorem excludes the case $R = \mathbb{Z}$ for nonorientable topological component $X_0$, we have some results in this case also. For any positive integer $n$ consider the exact sequence of Abelian groups

$$0 \rightarrow \mathbb{Z} \overset{x^n}{\rightarrow} \mathbb{Z} \rightarrow \mathbb{Z}_n \rightarrow 0$$

and the corresponding Bockstein exact sequences for cohomology and homology (cf. see [4])

$$\cdots \rightarrow H^{i-1}(X_0, \mathbb{Z}_n) \overset{\beta}{\rightarrow} H^{i}(X_0, \mathbb{Z}) \overset{x^n}{\rightarrow} H^{i}(X_0, \mathbb{Z}) \rightarrow H^{i}(X_0, \mathbb{Z}_n) \overset{\beta}{\rightarrow} \cdots$$

and

$$\cdots \rightarrow H_{i+1}(X_0, \mathbb{Z}_n) \overset{\beta}{\rightarrow} H_{i}(X_0, \mathbb{Z}) \overset{x^n}{\rightarrow} H_{i}(X_0, \mathbb{Z}) \rightarrow H_{i}(X_0, \mathbb{Z}_n) \overset{\beta}{\rightarrow} \cdots .$$

Theorem 1.2. Let $X_0$ be a nonorientable topological component of a compact nonsingular real algebraic variety $X$. Then, $ImH^i (X_0, \mathbb{Z}) \otimes \mathbb{Q}$ and the image of the Bockstein homomorphism restricted to $ImH^{i-1} (X_0, \mathbb{Z}_n)$, $\beta(ImH^{i-1} (X_0, \mathbb{Z}_n))$, which is a subgroup of $n$-torsion elements in $H^i (X_0, \mathbb{Z})$, are independent of the complexification $i : X \rightarrow X_\mathbb{C}$, provided that $n = 2$ or is a positive odd integer.

Similarly for homology, $KH_i (X_0, \mathbb{Z}) \otimes \mathbb{Q}$ and the image of the Bockstein homomorphism restricted to $KH_{i+1} (X_0, \mathbb{Z}_n)$, $\beta(KH_{i+1} (X_0, \mathbb{Z}_n))$, which is a subgroup of $n$-torsion elements in $H_i (X_0, \mathbb{Z})$, are independent of the complexification $i : X \rightarrow X_\mathbb{C}$, provided that $n = 2$ or is a positive odd integer.

2. Proofs

All real algebraic varieties under consideration in this report are nonsingular. It is well known that real projective varieties are affine (Proposition 2.4.1 of [1] or Theorem 3.4.4 of [2]). Moreover, compact affine real algebraic varieties are projective (Corollary 2.5.14 of [1]) and therefore, we will not distinguish between real compact affine varieties and real projective varieties.

For real algebraic varieties $X \subseteq \mathbb{R}^r$ and $Y \subseteq \mathbb{R}^s$ a map $F : X \rightarrow Y$ is said to be entire rational if there exist $f_i, g_i \in \mathbb{R}[x_1, \ldots , x_r]$, $i = 1, \ldots , s$, such that each $g_i$ vanishes nowhere on $X$ and $F = (f_1/g_1, \ldots , f_s/g_s)$. We say $X$ and $Y$ are isomorphic if there are entire rational maps $F : X \rightarrow Y$ and $G : Y \rightarrow X$ such that $F \circ G = id_Y$ and $G \circ F = id_X$. Isomorphic algebraic varieties will be regarded the same. We refer the reader for the basic definitions and facts about real algebraic geometry to [1, 2].

We will only prove the statements of the above theorems involving cohomology, because proof of the statements about homology are completely analogous.
Let $X_0$ be a nonorientable topological component of a nonsingular real algebraic variety $X$. Since $X_0$ is nonorientable so is $X$. The smooth orientation double cover of $X$ is diffeomorphic to a nonsingular real algebraic variety $\tilde{X}$, possibly not unique, on which the corresponding $\mathbb{Z}_2$ deck transformation group acts algebraically and the quotient map $p : \tilde{X} \to X$ is entire rational. This can be seen as follows: The determinant line bundle of $X$ is (strongly) algebraic and is non trivial on $X_0$. Hence, the $f : X \to \mathbb{R}P^N$ be an entire rational map classifying this line bundle. Now, $\tilde{X} \to X$ can be taken to be the pull back of the algebraic double covering $S^N \to \mathbb{R}P^N$ (cf. see [7, 8]). Now, we have the following result:

**Lemma 2.1.** Assume that $X$ is a nonsingular compact real algebraic variety and $X_0$ a nonorientable topological component of $X$. Let $i : X \to X_C$ be any smooth projective complexification and $p : \tilde{X} \to X$ any real algebraic orientation double cover as above. Then if $R$ contains 2 as a unit then $(p^*)^{-1}(\text{Im}H^* (X, R)) = \text{Im}H^* (X_0, R)$. Moreover, $\text{Im}H^* (X_0, R)$ is independent of the complexification.

One can easily state and prove the above lemma for homology.

**Proof.** Let $i : X \to X_C$ be any fixed smooth projective complexification. Then the entire rational covering map $p : \tilde{X} \to X$ will extend to a rational map, and after blowing up some smooth centers away from the real locus, to some smooth projective complexification $p_C : \tilde{X}_C \to X_C$ (possibly a branched double covering projection) making the diagram below commutative:

\[
\begin{array}{ccc}
\tilde{X}_0 & \stackrel{j}{\longrightarrow} & \tilde{X}_C \\
p \downarrow & & \downarrow p_C \\
X_0 & \stackrel{i}{\longrightarrow} & X_C
\end{array}
\]

where $\tilde{X}_0$ is $p^{-1}(X_0)$, clearly a topological component of $\tilde{X}$. The $\mathbb{Z}_2$ action on $\tilde{X}$ extends to an algebraic action on $\tilde{X}_C$ so that $p_C : \tilde{X}_C \to X_C$ is a, possibly branched, double covering, and the vertical maps are equivariant. This diagram yields the following commutative diagram

\[
\begin{array}{ccc}
H^i(\tilde{X}_C, R) & \stackrel{j^*}{\longrightarrow} & H^i(\tilde{X}_0, R) \\
p_C^* \uparrow & & \uparrow p^* \\
H^i(X_C, R) & \stackrel{i^*}{\longrightarrow} & H^i(X_0, R).
\end{array}
\]

Since the smooth projective complexification $i : X \to X_C$ is arbitrary the commutativity of the above diagram implies that

\[
\text{Im}H^* (X_0, R) \subseteq (p^*)^{-1}(\text{Im}H^* (\tilde{X}_0, R)).
\]

To see the other inclusion, let $\tau$ and $\tau_C$ denote the involutions of the $\mathbb{Z}_2$-actions on $\tilde{X}$ and $\tilde{X}_C$, respectively. It is well known that the vertical maps in
the above diagrams are injective with images $H^i(\tilde{X}_0, R)^{\tau^*}$ and $H^i(\tilde{X}_C, R)^{\tau C}$, the subgroups of invariant classes (cf. see page 193 of [5]).

Note that to finish the proof of the first assertion we need to prove the following: Let $a \in H^i(X_0, R)$ be such that $p^*(a) \in \text{Im} H^i(\tilde{X}_0, R)$. Then $a \in \text{Im} H^i(X_0, R)$. To prove this let $b \in H^i(\tilde{X}_C, R)$ be such that $j^*(b) = p^*(a)$. By the above paragraph $2p^*(a) = j^*(b + \tau C(b)) = p^*_C(c)$, for some $c \in H^i(X_C, R)$. It follows from the commutativity of the above diagram and the injectivity of $p^*$ that $2a = i^*(c)$. Since 2 is a unit we have $a = i^*(\frac{c}{2}) \subseteq \text{Im}(i^*)$.

For the last assertion just note that the complexification $i : X \rightarrow X_C$ is arbitrary and independent from the choice of the double covering $p : \tilde{X} \rightarrow X$. $\square$

Note that Theorem 1.1 follows from the above Lemma 2.1.

Proof of Theorem 1.2. Consider the exact sequence of Abelian groups

$$0 \rightarrow Z \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/Z \rightarrow 0$$

and the corresponding Bockstein exact sequences for $X_0$ and $X_C$

$$\cdots \rightarrow H^{i-1}(X_0, \mathbb{Q}/Z) \rightarrow H^i(X_0, \mathbb{Z}) \rightarrow H^i(X_0, \mathbb{Q}) \rightarrow H^i(X_0, \mathbb{Q}/Z) \rightarrow \cdots$$

$$\uparrow i^* \quad \uparrow i^* \quad \uparrow i^* \quad \uparrow i^*$$

$$\cdots \rightarrow H^{i-1}(X_C, \mathbb{Q}/Z) \rightarrow H^i(X_C, \mathbb{Z}) \rightarrow H^i(X_C, \mathbb{Q}) \rightarrow H^i(X_C, \mathbb{Q}/Z) \rightarrow \cdots$$

Tensoring the above sequences with $\mathbb{Q}$ and using Theorem 1.1 we get that $\text{Im} H^i(X_0, \mathbb{Z}) \otimes \mathbb{Q}$ is independent of the complexification $i : X \rightarrow X_C$.

Now let us concentrate on torsion elements in $H^i(X_0, \mathbb{Z})$. Let $n$ be a positive odd integer. Then 2 is a unit in $\mathbb{Z}_n$. Considering a similar diagram as above corresponding to the short exact sequence of Abelian groups

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \rightarrow \mathbb{Z}_n \rightarrow 0$$

and the corresponding Bockstein sequences

$$\cdots \rightarrow H^{i-1}(X_0, \mathbb{Z}_n) \xrightarrow{\beta} H^i(X_0, \mathbb{Z}) \xrightarrow{\times n} H^i(X_0, \mathbb{Z}) \rightarrow H^i(X_0, \mathbb{Z}_n) \rightarrow \cdots$$

$$\uparrow i^* \quad \uparrow i^* \quad \uparrow i^* \quad \uparrow i^*$$

$$\cdots \rightarrow H^{i-1}(X_C, \mathbb{Z}_n) \xrightarrow{\beta} H^i(X_C, \mathbb{Z}) \xrightarrow{\times n} H^i(X_C, \mathbb{Z}) \rightarrow H^i(X_C, \mathbb{Z}_n) \rightarrow \cdots$$

we deduce that the image of the Bockstein homomorphism restricted to $\text{Im} H^{i-1}(X_0, \mathbb{Z}_n)$, $\beta(\text{Im} H^i(X_0, \mathbb{Z}_n))$, which is a subgroup of $n$-torsion elements in $H^i(X_0, \mathbb{Z})$, is independent of the complexification $i : X \rightarrow X_C$ by Theorem 1.1. On the other hand, since $\mathbb{Z}_2$ is a field $\text{Im} H^i(X_0, \mathbb{Z}_2)$ is also independent of the complexification ([6]). Now using the exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \rightarrow \mathbb{Z}_2 \rightarrow 0,$$
in a similar fashion, we see that the image of the Bockstein homomorphism restricted to $\beta(ImH^{i-1}(X_0,\mathbb{Z}_2))$, which is a subgroup of 2-torsion elements in $H^i(X_0,\mathbb{Z})$, is also independent of the complexification $i : X \to X_C$. □

REFERENCES


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