1) Write a constant coefficient linear homogeneous recurrence relation whose characteristic equation is 
\[ x^4 - x^3 + 2x^2 + 6x - 1 = 0. \]

\[ a_{n+4} = a_{n+3} - 2a_{n+2} - 6a_{n+1} + a_n \]

2) Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence whose elements satisfy a linear homogeneous recurrence relation of order four. You are given that the characteristic equation of this recurrence relation is \((x - 2)^2(x + 3)(x - 5) = 0\).

Express the general term of the sequence \( \{a_n\}_{n=1}^{\infty} \) explicitly. (Hint: Your answer shall involve some arbitrary constants.)

Since -3 and 5 are simple roots, and 2 is a root with multiplicity 2, the general term is of the form

\[ a_n = \alpha_1 2^k + \alpha_2 k 2^k + \alpha_3 (-3)^k + \alpha_4 5^k, \]

where \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R} \) are constants.

3) Given the constant coefficient recurrence relation 
\[ a_{n+5} = 2a_{n+4} + 3a_{n+2} + 6a_{n+1} + a_n, \]
find its characteristic equation. (Do not try to find the general term!)

The characteristic equation is
\[ x^5 - 2x^4 - 3x^2 - 6x - 1 = 0. \]

4) Reduce the relation \( c_{n+1} = 2c_n + n + 3 \) to a linear homogeneous recurrence relation of order 3.

\[
\begin{align*}
c_{n+2} &= 2c_{n+1} + n + 1 + 3, \\
c_{n+1} &= 2c_n + n + 3, \\
c_{n+2} - c_{n+1} &= 2c_{n+1} - 2c_n + 1, \\
c_{n+3} &= 3c_{n+2} - 2c_{n+1} + 1, \\
c_{n+3} - c_{n+2} &= 3c_{n+2} - 5c_{n+1} + 2c_n \\
\end{align*}
\]

so \( c_{n+2} = 3c_{n+1} - 2c_n + 1 \).

hence
\[ c_{n+3} = 4c_{n+2} - 5c_{n+1} + 2c_n. \]
5) Describe the solution set of the system

\[
\begin{align*}
2x + 2y + 3z + w &= 0 \\
4x + 4y + 6z + 5w &= 0
\end{align*}
\]

We apply row reduction to the corresponding matrix as follows

\[
\begin{bmatrix}
1 & 2 & 3 & 1 \\
2 & 2 & 1 & 3 \\
1 & 0 & 2 & 1 \\
4 & 4 & 6 & 5
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 3 & 1 \\
0 & -2 & -5 & 1 \\
0 & -2 & -1 & 0 \\
0 & -4 & -6 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 3 & 1 \\
0 & -2 & -5 & 1 \\
0 & 0 & 4 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 3 & 1 \\
0 & 0 & 4 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Let \( z = t \), then \( w = 4t \). From the second row, we get \( y = -1/2t \). From the first row, we get \( x = -6t \).

Thus, the solution set is

\[ S = \{ (-6t, -1/2t, t, 4t) : t \in \mathbb{R} \}. \]

6) Describe the solution set of the system

\[
\begin{align*}
x + 2y + 3z + 4w &= 4 \\
x + 2y + 2z + w &= 5 \\
4x + 6y + 10z + 7w &= 13 \\
5x + 6y + 8z + 5w &= 14
\end{align*}
\]

We apply row reduction to the corresponding augmented matrix as

\[
\begin{bmatrix}
1 & 2 & 4 & 3 & 4 \\
2 & 2 & 2 & 1 & 5 \\
4 & 6 & 10 & 7 & 13 \\
5 & 6 & 8 & 5 & 14
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 4 & 3 & 4 \\
0 & -2 & -6 & -5 & -3 \\
0 & -2 & -6 & -5 & -3 \\
0 & -4 & -12 & -10 & -6
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 4 & 3 & 4 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Let \( w = t \), \( z = u \). Then \(-2y - 6u - 5t = -3\) implies \( y = -6u + 5t - 3 \), and \( x - 6u - 5t + 3 + 4u + 3t = 4 \) implies \( x = 2u + 2t + 1 \). Hence the solution set is

\[ S = \{ (2u + 2t + 1, -6u + 5t - 3, u, t) : u, t \in \mathbb{R} \}. \]
7) You are given the system of linear equations
\[
\begin{align*}
\lambda x + y + \lambda z &= 0 \\
x + 2y + 5z &= 0 \\
x + 5y + 3z &= 0
\end{align*}
\]
where \( \lambda \in \mathbb{R} \) is a constant.
What is the value of \( \lambda \), if this system has infinitely many solutions?

Let \( A = \begin{bmatrix} \lambda & 1 & \lambda \\ 1 & 2 & 5 \\ 1 & 5 & 3 \end{bmatrix} \). Then \( |A| = \begin{vmatrix} \lambda & 1 & \lambda \\ 1 & 2 & 5 \\ 1 & 5 & 3 \end{vmatrix} = -16\lambda + 2 \).

This system has infinitely many solutions if and only if \( |A| = 0 \), or equivalently \( \lambda = \frac{1}{8} \).

8) The elements of the sequence \( \{a_n\}_{n=0}^{\infty} \) satisfy the recurrence relation \( a_{n+2} = 5a_{n+1} - 6a_n \). Given \( a_0 = 6 \) and \( a_1 = 17 \), express the general term of this sequence explicitly.

The characteristic equation of the recurrence relation is \( x^2 - 5x + 6 = (x - 2)(x - 3) = 0 \). Since 2 and 3 are the solutions of this equation, the general term is of the form \( a_k = A \cdot 2^k + B \cdot 3^k \). The initial conditions \( a_0 = 6 \) and \( a_1 = 17 \) yields the system of equations
\[
\begin{align*}
A + B &= 6 \\
2A + 3B &= 17
\end{align*}
\]
So \( A = 1 \) and \( B = 5 \). Hence \( a_k = 2^k + 5 \cdot 3^k \).

9) What must be the value(s) of \( a \), if the system
\[
\begin{align*}
x + ay + w &= 1 \\
2x + (3a + 2)y + z + 3w &= 4 \\
3x + 3ay + z + 4w &= 6 \\
6x + (7a + 2)y + z + (a + 8)w &= 12
\end{align*}
\]
is consistent (has at least one solution)?

We apply row reduction on the corresponding augmented matrix and get
\[
\begin{bmatrix} 1 & a & 0 & 1 & 1 \\ 2 & 3a + 2 & 1 & 3 & 4 \\ 3 & 3a & 1 & 4 & 6 \\ 6 & 7a + 2 & 1 & a + 8 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & a & 0 & 1 & 1 \\ -2R_1 + R_2 \rightarrow R_2 & 0 & a + 2 & 1 & 2 \\ -3R_1 + R_3 \rightarrow R_3 & 0 & 0 & 1 & 1 \\ -6R_1 + R_4 \rightarrow R_4 & 0 & a + 2 & 1 & a + 2 \\ -R_2 + R_4 \rightarrow R_4 & 0 & 0 & 0 & a + 1 \\ & 1 & a & 0 & 1 & 1 \\ & 0 & a + 2 & 1 & 1 & 2 \\ & 0 & 0 & 1 & 1 & 3 \\ & 0 & 0 & 0 & a + 1 & 4 \end{bmatrix}
\]
Therefore, if \( a \neq -1 \) and \( a \neq -2 \), then the given system is consistent.
10) Given \[ \begin{align*}
2x - y + z &= 3 \\
x - 2y + 4z &= 1 \\
5x - 3y + z &= 11
\end{align*} \]
determine \( x \) using Cramer’s rule.

Since \( D = \begin{vmatrix} 2 & -1 & 1 \\ 1 & -2 & 4 \\ 5 & -3 & 1 \end{vmatrix} = 8 \) and \( D_x = \begin{vmatrix} 3 & -1 & 1 \\ 1 & -2 & 4 \\ 11 & -3 & 1 \end{vmatrix} = 6 \), we find \( x = \frac{D_x}{D} = \frac{3}{4} \).

11) There are 62 USB discs in Ateş’s backpack. 12 of them have 2 GB, 15 of them have 4 GB, 21 of them have 8 GB and 14 of them have 16 GB capacity. At least how many USB discs must Ateş pick in order to guarantee that he gets at least one disc with 4 GB capacity and at least one disc with 16 GB capacity?

If Ateş picks 48 discs, it may happen that 12 of them have 2 GB, 15 of them have 4 GB and 21 of them have 8 GB capacity. So he may fail to have a USB disc with 4 GB capacity. However, if he picks 49 discs, then, by the pigeonhole principle, at least two of the discs have capacities 4 GB and 16 GB.