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Small gaps between primes: The GPY method and recent results

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Abstract

In the works with D. A. Goldston and J. Pintz (GPY), the use of short divisor sums has led to strong results concerning the existence of small gaps between primes. The results depend on the information about the distribution of primes in arithmetic progressions, specifically on the range where the estimate of the Bombieri-Vinogradov Theorem is taken to hold. Let p_n denote the n-th prime. We obtained, unconditionally, $\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = 0$. In fact, we have the stronger quantitative result $\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\sqrt{\log p_n}(\log\log p_n)^2} < \infty$. Furthermore for any fixed positive integer ν, it is shown that $\liminf_{n\to\infty} \frac{p_{n+\nu}-p_n}{\log p_n} \leq e^{-\gamma}(\sqrt{\nu}-1)^2$, along with a generalization for small differences between primes in arithmetic progressions where the modulus of the progression can be taken to be as large as $(\log\log p_n)^A$ with arbitrary $A > 0$. Assuming that the estimate of the Bombieri-Vinogradov Theorem holds with any level beyond the known level $\frac{1}{2}$, i.e. conditionally, the method establishes the existence of bounded gaps between consecutive primes. Another result is that given any arbitrarily small but fixed $\eta > 0$, a positive proportion of all gaps between consecutive primes are comprised of gaps which are smaller than η times the average gap (on this last matter a variety of quantitative results, some unconditional and some conditional, are obtained).

The corresponding situation for E_2-numbers q_n, numbers which are the product of two distinct primes, have been studied by S. W. Graham and the three mentioned researchers (GGPY) with the unconditional result that $\liminf_{n\to\infty}(q_{n+r}-q_n) \leq C(r)$ for certain constants $C(r)$, in particular $C(1) = 6$. The methods and results in this work also yielded stronger variants of the Erdös-Mirsky conjecture. For example, it is shown that there are infinitely many integers n which simultaneously satisfy $d(n) = d(n+1) = 24$, $\Omega(n) = \Omega(n+1) = 5$, $\omega(n) = \omega(n+1) = 4$, (here $d(n)$, $\Omega(n)$, $\omega(n)$ denote respectively the number of positive integer divisors of n, the number of primes dividing n counted with multiplicity, and the number of distinct prime divisors of n).

In 2013 there has been two further breakthroughs leading to unconditional proofs of the existence of bounded gaps betwen primes, both advancing from the GPY works. First, Zhang succeeded in combining the method of GPY with improved versions of special instances of the Bombieri-Vinogradov type of results of Bombieri, Friedlander, Iwaniec and Fouvry in the 1980's. This necessitated the use of deep results from the theory of Kloosterman sums, and bounds derived from Deligne's proof of the Riemann Hypothesis over finite fields. Zhang obtained that there exist infinitely many gaps between primes of size $< 7 \cdot 10^7$. Zhang didn't try to find the smallest bound possible, but a Polymath project initiated by T. Tao brought the bound from Zhang's method down to 4680. Maynard, upon modifying the weights used in our works (and also almost simultaneously T. Tao), brought the bound down to 600. Upon optimizations undertaken by another Polymath project the bound is now at 252. Maynard's method is considerably more elementary than Zhang's. Perhaps the most interesting aspect of Maynard's method is that it reveals that any fixed positive level of distribution of primes leads to the existence of bounded gaps between primes.

In my talk I shall try to give a presentation of the main ideas involved in these works.

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