EXERCISES AND SOLUTIONS
IN LINEAR ALGEBRA

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March 14, 2015
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Preface

I have given some linear algebra courses in various years. These problems are given to students from the books which I have followed that year. I have kept the solutions of exercises which I solved for the students. These notes are collection of those solutions of exercises.

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1.) Let \( A = \begin{bmatrix} 2 & 4 & -2 \\ 0 & -4 & 3 \\ -3 & -7 & 4 \end{bmatrix} \).

(a) Find the characteristic polynomial of \( A \).

Solution. The characteristic polynomial of \( A \) is \( f(x) = \det(xI - A) \). So,

\[
f(x) = \begin{vmatrix} x-2 & -4 & 2 \\ 0 & x+4 & -3 \\ 3 & 7 & x-4 \end{vmatrix} \\
= (x-2)(x+4 - 3) + 3 \begin{vmatrix} -4 & 2 \\ 7 & x-4 \end{vmatrix} \\
= (x-2)(x^2 - 16 + 21) + 3(12 - 2x - 8) \\
= (x-2)(x^2 + 5) + 3(4 - 2x) \\
= (x-2)(x-1)(x+1)
\]

(b) Find the minimal polynomial of \( A \).

Solution. We know that the minimal polynomial divides the characteristic polynomial and they same the same roots. Thus, the minimal polynomial for \( A \) is \( m_A(x) = f(x) = (x-2)(x-1)(x+1) \).

(c) Find the characteristic vectors and a basis \( B \) such that \( [A]_B \) is diagonal.

Solution. The characteristic values of \( A \) are \( c_1 = 2, c_2 = 1, c_3 = -1 \).

\[
A - 2I = \begin{bmatrix} 0 & 4 & -2 \\ 0 & -6 & 3 \\ -3 & -7 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & -7 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{cases} -3x - 7y + 2z = 0 \\ 2y - z = 0 \end{cases} \Rightarrow \begin{cases} z = 2y \\ x = -y \end{cases}
\]

Thus, \( \alpha_1 = (-1, 1, 2) \) is a characteristic vector associated with the characteristic value \( c_1 = 2 \).

\[
A - I = \begin{bmatrix} 1 & 4 & -2 \\ 0 & -5 & 3 \\ -3 & -7 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & -2 \\ 0 & -5 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{cases} x + 4y - 2z = 0 \\ -5y + 3z = 0 \end{cases} \Rightarrow \begin{cases} y = 3k \\ z = 5k \\ x = -2k \end{cases}
\]

Thus, \( \alpha_2 = (-2, 3, 5) \) is a characteristic vector associated with the characteristic value \( c_2 = 1 \).

\[
A + I = \begin{bmatrix} 3 & 4 & -2 \\ 0 & -3 & 3 \\ -3 & -7 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 4 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{cases} 3x + 4y - 2z = 0 \\ y - z = 0 \end{cases} \Rightarrow \begin{cases} x = -2t \\ y = 3t \\ z = 3t \end{cases}
\]
Thus, $\alpha_3 = (-2, 3, 3)$ is a characteristic vector associated with the characteristic value $c_3 = -1$.

Now, $B = \{\alpha_1, \alpha_2, \alpha_3\}$ is a basis and $[A]_B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is a diagonal matrix.

(d) Find $A$-conductor of the vector $\alpha = (1, 1, 1)$ into the invariant subspace spanned by $(-1, 1, 2)$.

**Solution.** Set $W = < (-1, 1, 2) >$ and denote the $A$-conductor of $\alpha$ into $W$ by $g(x)$. Since $m_A(A) = 0$ we have $m_A(A) \alpha \in W$. Thus, $g(x)$ divides $m_A(x)$. Hence, the possibilities for $g(x)$ are $x - 2, x - 1, x + 1, (x - 2)(x - 1), (x - 2)(x + 1), (x - 1)(x + 1)$. We will try these polynomials. (Actually, the answer could be given directly.) Now,

$$(A - 2I)\alpha = \begin{bmatrix} 0 & 4 & -2 \\ -3 & 7 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \end{bmatrix} \notin W \Rightarrow g(x) \neq x - 2,$$

$$(A - I)\alpha = \begin{bmatrix} 1 & 4 & -2 \\ -3 & 7 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix} \notin W \Rightarrow g(x) \neq x - 1,$$

$$(A + I)\alpha = \begin{bmatrix} 3 & 4 & -2 \\ -3 & 7 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} \notin W \Rightarrow g(x) \neq x + 1,$$

$$(A - 2I)(A - I)\alpha = \begin{bmatrix} 0 & 4 & -2 \\ -3 & 7 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ -9 \end{bmatrix} \notin W \Rightarrow g(x) \neq (x - 2)(x - 1),$$

$$(A - 2I)(A + I)\alpha = \begin{bmatrix} 0 & 4 & -2 \\ -3 & 7 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ -25 \end{bmatrix} \notin W \Rightarrow g(x) \neq (x - 2)(x + 1),$$

$$(A - I)(A + I)\alpha = \begin{bmatrix} 1 & 4 & -2 \\ -3 & 7 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ -30 \end{bmatrix} = -15\alpha_1 \in W \Rightarrow g(x) = x^2 - 1.$$

2.) Find a $3 \times 3$ matrix whose minimal polynomial is $x^2$.

**Solution.** For the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ we have $A \neq 0$ and $A^2 = 0$. Thus, $A$ is a $3 \times 3$ matrix whose minimal polynomial is $x^2$.

3.) Prove that similar matrices have the same minimal polynomial.

**Solution.** Let $A$ and $B$ be similar matrices, i.e., $B = P^{-1}AP$ for some invertible matrix $P$. For any $k > 0$ we have $B^k = (P^{-1}AP)^k = P^{-1}A^kP$ which implies that $f(B) = P^{-1}f(A)P$ for any polynomial $f(x)$. Let $f_A$ and $f_B$ be the minimal polynomials of $A$ and $B$, respectively. Then $f_A(B) = P^{-1}f_A(A)P = P^{-1}OP = O$ implies that $f_B$ divides $f_A$. On the other hand, $O = f_B(B) = P^{-1}f_B(A)P$ gives us $f_B(A) = O$. Hence, $f_A$ divides $f_B$. Therefore, we have $f_A = f_B$. 

1. Math 262 Exercises and Solutions

(1) Let $A$ be a $3 \times 3$ matrix with real entries. Prove that if $A$ is not similar over $\mathbb{R}$ to a triangular matrix then $A$ is similar over $\mathbb{C}$ to a diagonal matrix.

**Proof.** Since $A$ is a $3 \times 3$ matrix with real entries, the characteristic polynomial, $f(x)$, of $A$ is a polynomial of degree 3 with real coefficients. We know that every polynomial of degree 3 with real coefficients has a real root, say $c_1$.

On the other hand, since $A$ is not similar over $\mathbb{R}$ to a triangular matrix, the minimal polynomial of $A$ is not product of polynomials of degree one. So one of the irreducible factor, $h$, of the minimal polynomial of $A$ is degree 2. Then $h$ has two complex roots, one of which is the conjugate of the other. Thus, the characteristic polynomial has one real root and two complex roots, $c_1, \lambda$ and $\bar{\lambda}$.

The minimal polynomial over complex numbers is $(x - c_1)(x - \lambda)(x - \bar{\lambda})$ which implies that $A$ is diagonalizable over complex numbers.

(2) Let $T$ be a linear operator on a finite dimensional vector space over an algebraically closed field $\mathbb{F}$. Let $f$ be a polynomial over $\mathbb{F}$. Prove that $c$ is a characteristic value of $f(T)$ if and only if $f(t) = c$ where $t$ is a characteristic value of $T$.

**Proof.** Let $t$ be a characteristic value of $T$ and $\beta$ be a non-zero characteristic vector associated with the characteristic value $t$. Then, $T\beta = t\beta$, $T^2\beta = T(T\beta) = T(t\beta) = tT\beta = t^2\beta$, and inductively we can see that $T^k\beta = t^k\beta$ for any $k \geq 1$. Thus, for any polynomial $f(x)$ we have $f(T)\beta = f(t)\beta$ which means, since $\beta \neq 0$, that $f(t)$ is a characteristic value of the linear operator $f(T)$.

Assume that $c$ is a characteristic value of $f(T)$. Since $\mathbb{F}$ is algebraically closed, the minimal polynomial of $T$ is product of linear polynomials, that is, $T$ is similar to a triangular operator. If $[P^{-1}TP]_B$ is triangular matrix, then $[P^{-1}f(T)P]_B$ is
also triangular and on the diagonal of \([P^{-1} f(T)P]_B\) we have \(f(c_i)\), where \(c_i\) is a characteristic value of \(T\).

(3) Let \(c\) be a characteristic value of \(T\) and let \(W\) be the space of characteristic vectors associated with the characteristic value \(c\). What is the restriction operator \(T|_W\).

**Solution.** Every vector \(v \in W\) is a characteristic vector. Hence, \(Tv = cv\) for all \(v \in W\). Therefore, \(T|_W = cI\).

(4) Every matrix \(A\) satisfying \(A^2 = A\) is similar to a diagonal matrix.

**Solution.** \(A\) satisfies the polynomial \(x^2 - x\). Thus, the minimal polynomial, \(m_A(x)\), of \(A\) divides \(x^2 - x\), that is \(m_A(x) = x\) or \(m_A(x) = x - 1\) or \(m_A(x) = x(x - 1)\).

If \(m_A(x) = x\), then \(A = 0\).

If \(m_A(x) = x - 1\), then \(A = I\).

If \(m_A(x) = x(x - 1)\), then the minimal polynomial of \(A\) is product of distinct polynomials of degree one. Thus, by a theorem, the matrix \(A\) is similar to diagonal matrix with diagonal entries consisting of the characteristic values, 0 and 1.

(5) Let \(T\) be a linear operator on \(V\). If every subspace of \(V\) is invariant under \(T\) then it is a scalar multiple of the identity operator.

**Solution.** If \(\dim V = 1\) then for any \(0 \neq v \in V\), we have \(Tv = cv\), since \(V\) is invariant under \(T\). Hence, \(T = cI\).

Assume that \(\dim V > 1\) and let \(\mathcal{B} = \{v_1, v_2, \ldots, v_n\}\) be a basis for \(V\). Since \(W_1 = \langle v_1 \rangle\) is invariant under \(T\), we have \(Tv_1 = c_1 v_1\). Similarly, since \(W_2 = \langle v_2 \rangle\) is invariant under \(T\), we have \(Tv_2 = c_2 v_2\). Now, \(W_3 = \langle v_1 + v_2 \rangle\) is also invariant under \(T\). Hence, \(T(v_1 + v_2) = \lambda(v_1 + v_2)\) or \(c_1 v_1 + c_2 v_2 = \lambda(v_1 + v_2)\), which gives us \((c_1 - \lambda)v_1 + (c_2 - \lambda)v_2 = 0\). However, \(v_1\) and \(v_2\) are linearly independent and hence we should have \(c_1 = c_2 = \lambda\). Similarly, one can continue with the subspace \(\langle v_1 + v_2 + v_3 \rangle\).
and observe that \( T(v_3) = \lambda v_3 \). So for any \( v_i \in B \), we have \( Tv_i = \lambda v_i \). Thus, \( T = \lambda I \).

\( \text{(6)} \) Let \( V \) be the vector space of \( n \times n \) matrices over \( \mathbb{F} \). Let \( A \) be a fixed \( n \times n \) matrix. Let \( T \) be a linear operator on \( V \) defined by \( T(B) = AB \). Show that the minimal polynomial of \( T \) is the minimal polynomial of \( A \).

**Solution.** Let \( m_A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \) be the minimal polynomial of \( A \), so that \( m_A(A) = 0 \). It is easy to see that \( T^k(B) = A^kB \) for any \( k \geq 1 \). Then, for any \( B \in V \) we have

\[
m_A(T)B = (T^m + a_{n-1}T^{m-1} + \cdots + a_1T + a_0I)B
\]
\[
= T^m(B) + a_{n-1}T^{m-1}(B) + \cdots + a_1T(B) + a_0B
\]
\[
= A^mB + a_{n-1}A^{m-1}B + \cdots + a_1AB + a_0B
\]
\[
= (A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I)B
\]
\[
= m_A(A)B = 0.
\]

Thus, we obtain \( m_A(T) = 0 \), which means that \( m_T(x) \) divides \( m_A(x) \).

Now, let \( m_T(x) = x^m + c_{m-1}x^{m-1} + \cdots + c_1x + c_0 \) be the minimal polynomial of \( T \), so that \( m_T(T) = 0 \). Then, for any \( B \in V \) we have

\[
m_T(A)B = (A^m + c_{m-1}A^{m-1} + \cdots + c_1A + c_0I)B
\]
\[
= A^mB + c_{m-1}A^{m-1}B + \cdots + c_1AB + c_0B
\]
\[
= T^m(B) + c_{m-1}T^{m-1}(B) + \cdots + c_1T(B) + c_0B
\]
\[
= (T^m + c_{m-1}T^{m-1} + \cdots + c_1T + c_0I)B
\]
\[
= m_T(T)B = 0,
\]

which leads to \( m_T(A) = 0 \), meaning that \( m_A(x) \) divides \( m_T(x) \). Since, monic polynomials dividing each other are the same we have \( m_T(x) = m_A(x) \).
(7) If $E$ is a projection and $f$ is a polynomial, then show that $f(E) = aI + bE$. What are $a$ and $b$ in terms of the coefficients of $f$?

**Solution.** Let $f(x) = c_0 + c_1 x + \cdots + c_n x^n$. Then, $f(E) = c_0 I + c_1 E + \cdots + c_n E^n$. Since $E$ is a projection, $(E^2 = E)$, we have $E^k = E$ for any $k \geq 1$. Then,

$$f(E) = c_0 I + c_1 E + \cdots + c_n E^n = c_0 I + c_1 E + \cdots + c_n E = c_0 I + (c_1 + \cdots + c_n)E.$$ 

Thus, $a$ is the constant term of $f$ and $b$ is the sum of all other coefficients.

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(8) Let $V$ be a finite dimensional vector space and let $W_1$ be any subspace of $V$. Prove that there is a subspace $W_2$ of $V$ such that $V = W_1 \oplus W_2$.

**Proof.** Let $\mathcal{B}_{W_1} = \{\beta_1, \ldots, \beta_k\}$ be a basis for $W_1$. We may extend $\mathcal{B}_{W_1}$ to a basis $\mathcal{B}_V$ of $V$, say $\mathcal{B}_V = \{\beta_1, \ldots, \beta_k, \beta_{k+1}, \ldots, \beta_n\}$. Let $W_2$ be the subspace spanned by $\beta_{k+1}, \ldots, \beta_n$. Then, as they are linearly independent in $V$, we have $\mathcal{B}_{W_2} = \{\beta_{k+1}, \ldots, \beta_n\}$. Clearly $W_1 + W_2 = V$ as $W_1 + W_2$ contains a basis of $V$ and so spans $V$. Let $\beta \in W_1 \cap W_2$. Then, $\beta \in W_1$ implies that $\beta = c_1\beta_1 + \cdots + c_k\beta_k$, and $\beta \in W_2$ implies that $\beta = c_{k+1}\beta_{k+1} + \cdots + c_n\beta_n$. The last two equalities give us $c_1\beta_1 + \cdots + c_k\beta_k - c_{k+1}\beta_{k+1} - \cdots - c_n\beta_n = 0$, but since $\beta_i$'s are linearly independent, we obtain $c_i = 0$ for all $i = 1, \ldots, n$ which means that $\beta = 0$. That is $W_1 \cap W_2 = \{0\}$, and hence $V = W_1 \oplus W_2$.

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(9) Let $V$ be a real vector space and $E$ be an idempotent linear operator on $V$, that is a projection. Prove that $I + E$ is invertible. Find $(I + E)^{-1}$.

**Proof.** Since $E$ is an idempotent linear operator it is diagonalizable by Question 4. So there exists a basis of $V$ consisting of characteristics vectors of $E$ corresponding to the characteristic values 0 and 1. That is, there exists a basis
\[ \mathcal{B} = \{ \beta_1, \ldots, \beta_n \} \] such that \( E\beta_i = \beta_i \) for \( i = 1, \ldots, k \), and \( E\beta_i = 0 \) for \( i = k + 1, \ldots, n \). Then \((I + E)\beta_i = 2\beta_i \) for \( i = 1, \ldots, k \) and \((I + E)\beta_i = \beta_i \) for \( i = k + 1, \ldots, n \), that is,

\[
[I + E]_\mathcal{B} = \begin{bmatrix} 2I_1 & 0 \\ 0 & I_2 \end{bmatrix},
\]

where \( I_1 \) stands for \( k \times k \) identity matrix, \( I_2 \) is \( (n - k) \times (n - k) \) identity matrix and each 0 represents the zero matrix of appropriate dimension. It is now easy to see that \([I + E]_\mathcal{B} \) is invertible, since \( \det(I + E) = 2^k \neq 0 \).

To find the inverse of \((I + E)\), we note that

\[
([I + E]_\mathcal{B})^{-1} = \begin{bmatrix} \frac{1}{2}I_1 & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}I_1 & 0 \\ 0 & 0 \end{bmatrix} = I - \frac{1}{2}[E]_\mathcal{B}.
\]

Therefore, \((I + E)^{-1} = I - \frac{1}{2}E\). (You may verify that really this is the inverse, by showing that \((I + E)(I - \frac{1}{2}E) = (I - \frac{1}{2}E)(I + E) = I\).)

(10) Let \( T \) be a linear operator on \( V \) which commutes with every projection operator on \( V \). What can you say about \( T \)?

**Solution.** Let \( \mathcal{B} \) be a basis for \( V \) and \( \beta_i \in \mathcal{B}, i \in I \) where \( I \) is some index set. We can write \( V \) as a direct sum \( V = W_i \oplus U \) where \( W_i = \langle \beta_i \rangle \). Then there exists a projection \( E_i \) of \( V \) onto the subspace \( W_i \) for each \( i \in I \). Note that \( E_i v \in W_i \) for all \( v \in V \), and \( E_i \beta_i = \beta_i \). Now, by assumption, the linear operator \( T \) commutes with \( E_i \) for all \( i \in I \), that is, \( TE_i = E_i T \). Then, for \( \beta_i \in W_i \), we have \( TE_i \beta_i = E_i T \beta_i \in W_i \) implies that \( T \beta_i = T(E_i \beta_i) = c_i \beta_i \) for some constant \( c_i \in \mathbb{F} \). Thus, \( \beta_i \) is a characteristic vector of \( T \). Hence, \( V \) has a basis consisting of characteristic vectors of \( T \). It follows that \( T \) is a diagonalizable linear operator on \( V \).

(11) Let \( V \) be the vector space of continuous real valued functions on the interval \([-1, 1]\) of the real line. Let \( W_c \) be the space
of even functions, \( f(-x) = f(x) \), and \( W_o \) be the space of odd functions, \( f(-x) = -f(x) \).

a) Show that \( V = W_e \oplus W_o \).

b) If \( T \) is the indefinite integral operator \( (Tf)(x) = \int_0^x f(t) dt \), are \( W_e \) and \( W_o \) invariant under \( T \)?

Solution. a) Let \( f \in V \). Then, we may write

\[
f(x) = \frac{f(x) + f(-x) + f(x) - f(-x)}{2} = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}.
\]

Observe that \( f_e(x) = \frac{f(x) + f(-x)}{2} \) is a continuous even function and \( f_o(x) = \frac{f(x) - f(-x)}{2} \) is a continuous odd function. Hence, \( f = f_e + f_o \), that is \( V = W_e + W_o \). To show that \( V = W_e \oplus W_o \), we need to show that \( W_e \cap W_o = \{0\} \). To see this, let \( g \in W_e \cap W_o \). Then, \( g \in W_e \) implies that \( g(-x) = g(x) \), and \( g \in W_o \) implies that \( g(-x) = -g(x) \). Thus, we have \( g(x) = -g(x) \) or \( g(x) = 0 \) for all \( x \in [-1, 1] \), which means that \( g = 0 \).

b) For \( f(x) = x \in W_o \), we have \( (Tf)(x) = x^2/2 \notin W_o \), and for \( g(x) = x^2 \in W_e \), we have \( (Tg)(x) = x^3/3 \notin W_e \). Thus, neither \( W_e \) nor \( W_o \) are invariant under \( T \).

(12) Let \( V \) be a finite dimensional vector space over the field \( \mathbb{F} \), and let \( T \) be a linear operator on \( V \), such that \( \text{rank}(T) = 1 \). Prove that either \( T \) is diagonalizable or \( T \) is nilpotent, but not both.

Proof. Since \( \text{rank}(T) = \dim(\text{Im}(T)) = 1 \), we have \( \dim(\text{Ker}(T)) = n - 1 \). Let \( 0 \neq \beta \in \text{Im}(T) \). So, \( \text{Im}(T) = \langle \beta \rangle \). Since \( \beta \in \text{Im}(T) \), there exists a vector \( \alpha_0 \in V \) such that \( T\alpha_0 = \beta \). Let \( \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} \) be a basis for \( \text{Ker}(T) \). Then, \( B = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} \) is a basis for \( V \).

We have \( T\alpha_i = 0 \) for all \( i = 1, 2, \ldots, n-1 \).
If \( T\alpha_0 \in \text{Ker}(T) \), then \( T\alpha_0 = c_1\alpha_1 + \cdots + c_{n-1}\alpha_{n-1} \) and

\[
[T]_B = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
c_1 & 0 & 0 & \cdots & 0 \\
c_2 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
c_{n-1} & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

and it is easily seen that \( T^2 = 0 \) meaning that \( T \) is nilpotent. Note that at least one of \( c_i \)'s is nonzero, since otherwise, \( \alpha_0 \) would be in \( \text{Ker}(T) \) which contradicts with the choice of \( B \).

If \( T\alpha_0 \not\in \text{Ker}(T) \), then \( T\beta \in \text{Im}(T) \) and \( T\beta = c_0\beta \). In this case we construct a new basis \( B' = \{\beta, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} \) and

\[
[T]_{B'} = \begin{bmatrix}
c_0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]

which means that \( T \) is diagonalizable.

(13) Let \( T \) be a linear operator on the finite dimensional vector space \( V \). Suppose \( T \) has a cyclic vector. Prove that if \( U \) is any linear operator which commutes with \( T \), then \( U \) is a polynomial in \( T \).

**Proof.** Let \( B = \{\alpha, T\alpha, \ldots, T^{n-1}\alpha\} \) be a basis for \( V \) containing the cyclic vector \( \alpha \) and let \( m(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \) be the minimal polynomial of \( T \). Since \( U\alpha \) is in \( V \), it can be written as a linear combination of basis vectors. Then, \( U\alpha = b_0\alpha + b_1T\alpha + \cdots + b_{n-1}T^{n-1}\alpha \) where \( b_0, b_1, \ldots, b_{n-1} \) are elements of the field \( \mathbb{F} \). That is, \((b_0I + b_1T + \cdots + b_{n-1}T^{n-1} - U)\alpha = 0 \). Now, since \( U \) and \( T \) commute, we have

\[
UT(\alpha) = TU(\alpha) = T(b_0\alpha + b_1T\alpha + \cdots + b_{n-1}T^{n-1}\alpha) = b_0T\alpha + b_1T^2\alpha + \cdots + b_{n-1}T^n\alpha = (b_0I + b_1T + \cdots + b_{n-1}T^{n-1})T\alpha
\]
which means that
\[(b_0I + b_1T + \cdots + b_{n-1}T^{n-1} - U)T\alpha = 0.\]

Similarly, we can show that \((b_0I + b_1T + \cdots + b_{n-1}T^{n-1} - U)T^i\alpha = 0\) for all \(i = 2, 3, \ldots, n-1\). Since the transformation \(b_0I + b_1T + \cdots + b_{n-1}T^{n-1} - U\) maps each basis vector to the zero vector, it is identically equal to zero on the whole space. Thus, we obtain

\[U = b_0I + b_1T + \cdots + b_{n-1}T^{n-1}.\]

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(14) Give an example of two 4 \(\times\) 4 nilpotent matrices which have the same minimal polynomial but which are not similar.

**Solution.** Let \(A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\) and \(B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\).

It is easy to see that \(m_A(x) = m_B(x) = x^2\) but they are not similar since, \(A\) has 3 distinct characteristic vectors corresponding to the characteristic value zero, but \(B\) has only two characteristic vectors corresponding to the characteristic value zero.

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(15) Show that if \(N\) is a nilpotent linear operator on an \(n\)–dimensional vector space \(V\), then the characteristic polynomial for \(N\) is \(x^n\).

**Solution.** Recall that \(N\) is nilpotent, if \(N^k = 0\) for some \(k \in \mathbb{N}^+\). Since, \(N\) is a nilpotent linear operator on \(V\), the minimal polynomial for \(N\) is of the form \(x^m\) for some \(m \leq n\). Then, all characteristic values of \(N\) are zero. Since the minimal polynomial is a product of linear polynomials, \(N\) is a triangulable operator. It follows that there exists a basis \(\mathcal{B}\) of
V such that

$$[N]_B = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ * & 0 & \cdots & 0 \\ * & * & \cdots & \vdots \\ * & * & \cdots & 0 \end{bmatrix}.$$ 

since, similar matrices have the characteristic polynomial, it follows that the characteristic polynomial of $N$ is $x^n$ where $n = \dim V$.

(16) Let $T$ be a linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$ 

Prove that $T$ has no cyclic vector. What is the $T$ cyclic subspace generated by the vector $\beta = (1, -1, 3)$?

**Solution.** Assume that $T$ has a cyclic vector $\alpha = (a_1, a_2, a_3)$. Then $B = \{\alpha, T\alpha, T^2\alpha\}$ will be a basis for $\mathbb{R}^3$. That is, the vectors $\alpha = (a_1, a_2, a_3), T\alpha = (2a_1, 2a_2, -a_3), T^2\alpha = (4a_1, 4a_2, a_3)$ must be linearly independent, or the matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ 2a_1 & 2a_2 & -a_3 \\ 4a_1 & 4a_2 & a_3 \end{bmatrix}$$

must be invertible. Applying elementary row operations, we obtain

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ 2a_1 & 2a_2 & -a_3 \\ 4a_1 & 4a_2 & a_3 \end{bmatrix} \rightarrow \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & -3a_3 \\ 0 & 0 & -3a_3 \end{bmatrix} \rightarrow \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{bmatrix}$$

which is not invertible. Hence, $T$ has no cyclic vector.

To find the cyclic subspace generated by $\beta$, it is enough to check if $\beta$ and $T\beta$ are independent since we have already shown that the set $\{\alpha, T\alpha, T^2\alpha\}$ can not be linearly independent for any $\alpha \in \mathbb{R}^3$. Clearly, $\beta = (1, -1, 3)$ and $T\beta = (2, -2, -3)$ are
linearly independent since, otherwise, one of them would be a multiple of the other one which is not the case here. Thus, the cyclic subspace generated by $\beta$ is

$$Z(\beta; T) = \langle (1, -1, 3), (2, -2, -3) \rangle = \{ \lambda(1, -1, 3) + \mu(2, -2, -3) : \lambda, \mu \in \mathbb{R} \}.$$  

(17) Find the minimal polynomial and rational form of the matrix

$$T = \begin{bmatrix} c & 0 & -1 \\ 0 & c & 1 \\ -1 & 1 & c \end{bmatrix}.$$  

**Solution.** The characteristic polynomial of $T$ is

$$f_T(x) = \det(xI - T) = \begin{vmatrix} x - c & 0 & 1 \\ 0 & x - c & -1 \\ 1 & -1 & x - c \end{vmatrix}$$

$$= (x - c)(x - c)(x - c - 1) - (x - c)$$

$$= (x - c)(x - c - 2)$$

$$= (x - c)(x - c - \sqrt{2})(x - c + \sqrt{2}).$$

Since the characteristic polynomial and the minimal polynomial have the same roots and the minimal polynomial divides the characteristic polynomial we have $m_T(x) = f_T(x) = (x - c)(x - c - 2) = (x - c)^3 - 2(x - c) = x^3 + (-3c)x^2 + (3c^2 - 2)x + (-c^3 + 2c).$ Thus the rational form of $T$ is

$$R = \begin{bmatrix} 0 & 0 & c^3 - 2c \\ 1 & 0 & -3c^2 + 2 \\ 0 & 1 & 3c \end{bmatrix}.$$