EXERCISES AND SOLUTIONS IN LINEAR ALGEBRA

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CHAPTERS

Preface

I have given some linear algebra courses in various years. These problems are given to students from the books which I have followed that year. I have kept the solutions of exercises which I solved for the students. These notes are collection of those solutions of exercises.

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M E T U
DEPARTMENT OF MATHEMATICS

Math 262 Quiz I
(05.03.2008)

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Name : Answer Key
ID Number : 2360262
Signature : 0000000
Duration : 60 minutes

Show all your work. Unsupported answers will not be graded.

1.) Let $A = \begin{bmatrix} 2 & 4 & -2 \\ 0 & -4 & 3 \\ -3 & -7 & 4 \end{bmatrix}$.

(a) Find the characteristic polynomial of A.

Solution. The characteristic polynomial of A is $f(x) = \det(xI - A)$. So,

$$\begin{aligned} f(x) &= \begin{vmatrix} x-2 & -4 & 2 \\ 0 & x+4 & -3 \\ 3 & 7 & x-4 \end{vmatrix} \\ &= (x-2)\begin{vmatrix} x+4 & -3 \\ 7 & x-4 \end{vmatrix} + 3\begin{vmatrix} -4 & 2 \\ x+4 & -3 \end{vmatrix} \\ &= (x-2)(x^2-16+21)+3(12-2x-8) \\ &= (x-2)(x^2+5)+3(4-2x) \\ &= (x-2)(x^2+5-6) \\ &= (x-2)(x-1)(x+1) \end{aligned}$$

(b) Find the minimal polynomial of A.

Solution. We know that the minimal polynomial divides the characteristic polynomial and they same the same roots. Thus, the minimal polynomial for A is $m_A(x) = f(x) = (x-2)(x-1)(x+1)$.

(c) Find the characteristic vectors and a basis $\mathcal{B}$ such that $[A]_{\mathcal{B}}$ is diagonal.

Solution. The characteristic values of A are $c_1 = 2, c_2 = 1, c_3 = -1$.

$$A - 2I = \begin{bmatrix} 0 & 4 & -2 \\ 0 & -6 & 3 \\ -3 & -7 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & -7 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{aligned} -3x-7y+2z &= 0 \\ 2y - z &= 0 \end{aligned} \Rightarrow \begin{aligned} z &= 2y \\ x &= -y \end{aligned}$$

Thus, $\alpha_1 = (-1, 1, 2)$ is a characteristic vector associated with the characteristic value $c_1 = 2$.

$$A - I = \begin{bmatrix} 1 & 4 & -2 \\ 0 & -5 & 3 \\ -3 & -7 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & -2 \\ 0 & -5 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{aligned} x+4y-2z &= 0 \\ -5y+3z &= 0 \end{aligned} \Rightarrow \begin{aligned} y &= 3k \\ z &= 5k \\ x &= -2k \end{aligned}$$

Thus, $\alpha_2 = (-2, 3, 5)$ is a characteristic vector associated with the characteristic value $c_2 = 1$.

$$A + I = \begin{bmatrix} 3 & 4 & -2 \\ 0 & -3 & 3 \\ -3 & -7 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 4 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{aligned} 3x+4y-2z &= 0 \\ y - z &= 0 \end{aligned} \Rightarrow \begin{aligned} x &= -2t \\ y &= 3t \\ z &= 3t \end{aligned}$$

Thus, $\alpha_3 = (-2, 3, 3)$ is a characteristic vector associated with the characteristic value $c_3 = -1$.

Now, $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ is a basis and $[A]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is a diagonal matrix.

(d) Find A-conductor of the vector $\alpha = (1, 1, 1)$ into the invariant subspace spanned by $(-1, 1, 2)$.

Solution. Set $W = < (-1, 1, 2) >$ and denote the A-conductor of α into W by $g(x)$. Since $m_A(A) = 0$ we have $m_A(A)\alpha \in W$. Thus, $g(x)$ divides $m_A(x)$. Hence, the possibilities for $g(x)$ are $x-2, x-1, x+1, (x-2)(x-1), (x-2)(x+1), (x-1)(x+1)$. We will try these polynomials. (Actually, the answer could be given directly.) Now,

$$(A-2I)\alpha = \begin{bmatrix} 0 & 4 & -2 \\ 0 & -6 & 3 \\ -3 & -7 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -8 \end{bmatrix} \notin W \Rightarrow g(x) \neq x-2,$$

$$(A-I)\alpha = \begin{bmatrix} 1 & 4 & -2 \\ 0 & -5 & 3 \\ -3 & -7 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -7 \end{bmatrix} \notin W \Rightarrow g(x) \neq x-1,$$

$$(A+I)\alpha = \begin{bmatrix} 3 & 4 & -2 \\ 0 & -3 & 3 \\ -3 & -7 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -5 \end{bmatrix} \notin W \Rightarrow g(x) \neq x+1,$$

$$(A-2I)(A-I)\alpha = \begin{bmatrix} 0 & 4 & -2 \\ 0 & -6 & 3 \\ -3 & -7 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \\ -7 \end{bmatrix} = \begin{bmatrix} 6 \\ -9 \\ -9 \end{bmatrix} \notin W \Rightarrow g(x) \neq (x-2)(x-1),$$

$$(A-2I)(A+I)\alpha = \begin{bmatrix} 0 & 4 & -2 \\ 0 & -6 & 3 \\ -3 & -7 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 10 \\ -15 \\ -25 \end{bmatrix} \notin W \Rightarrow g(x) \neq (x-2)(x+1),$$

$$(A-I)(A+I)\alpha = \begin{bmatrix} 1 & 4 & -2 \\ 0 & -5 & 3 \\ -3 & -7 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 15 \\ -15 \\ -30 \end{bmatrix} = -15\alpha_1 \in W \Rightarrow g(x) = x^2 - 1.$$

2.) Find a 3×3 matrix whose minimal polynomial is x^2.

Solution. For the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ we have $A \neq 0$ and $A^2 = 0$. Thus, A is a 3×3 matrix whose minimal polynomial is x^2.

3.) Prove that similar matrices have the same **minimal** polynomial.

Solution. Let A and B be similar matrices, i.e., $B = P^{-1}AP$ for some invertible matrix P. For any $k > 0$ we have $B^k = (P^{-1}AP)^k = P^{-1}A^kP$ which implies that $f(B) = P^{-1}f(A)P$ for any polynomial $f(x)$. Let f_A and f_B be the minimal polynomials of A and B, respectively. Then $f_A(B) = P^{-1}f_A(A)P = P^{-1}OP = O$ implies that f_B divides f_A. On the other hand, $O = f_B(B) = P^{-1}f_B(A)P$ gives us $f_B(A) = O$. Hence, f_A divides f_B. Therefore, we have $f_A = f_B$.

1. Math 262 Exercises and Solutions

(1) Let A be a 3×3 matrix with real entries. Prove that if A is not similar over $\mathbb{R}$ to a triangular matrix then A is similar over $\mathbb{C}$ to a diagonal matrix.

Proof. Since A is a 3×3 matrix with real entries, the characteristic polynomial, $f(x)$, of A is a polynomial of degree 3 with real coefficients. We know that every polynomial of degree 3 with real coefficients has a real root, say c_1.

On the other hand, since A is not similar over $\mathbb{R}$ to a triangular matrix, the minimal polynomial of A is not product of polynomials of degree one. So one of the irreducible factor, h, of the minimal polynomial of A is degree 2. Then h has two complex roots, one of which is the conjugate of the other. Thus, the characteristic polynomial has one real root and two complex roots, c_1, λ and $\bar{\lambda}$.

The minimal polynomial over complex numbers is $(x - c_1)(x - \lambda)(x - \bar{\lambda})$ which implies that A is diagonalizable over complex numbers.

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(2) Let T be a linear operator on a finite dimensional vector space over an algebraically closed field $\mathbb{F}$. Let f be a polynomial over $\mathbb{F}$. Prove that c is a characteristic value of $f(T)$ if and only if $f(t) = c$ where t is a characteristic value of T.

Proof. Let t be a characteristic value of T and β be a non-zero characteristic vector associated with the characteristic value t. Then, $T\beta = t\beta$, $T^2\beta = T(T\beta) = T(t\beta) = tT\beta = t^2\beta$, and inductively we can see that $T^k\beta = t^k\beta$ for any $k \geq 1$. Thus, for any polynomial $f(x)$ we have $f(T)\beta = f(t)\beta$ which means, since $\beta \neq 0$, that $f(t)$ is a characteristic value of the linear operator $f(T)$.

Assume that c is a characteristic value of $f(T)$. Since $\mathbb{F}$ is algebraically closed, the minimal polynomial of T is product of linear polynomials, that is, T is similar to a triangular operator. If $[P^{-1}TP]_{\mathcal{B}}$ is triangular matrix, then $[P^{-1}f(T)P]_{\mathcal{B}}$ is also triangular and on the diagonal of $[P^{-1}f(T)P]_{\mathcal{B}}$ we have $f(c_i)$, where c_i is a characteristic value of T.

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(3) Let c be a characteristic value of T and let W be the space of characteristic vectors associated with the characteristic value c. What is the restriction operator $T|_W$.

Solution. Every vector $v \in W$ is a characteristic vector. Hence, $Tv = cv$ for all $v \in W$. Therefore, $T|_W = cI$.

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(4) Every matrix A satisfying $A^2 = A$ is similar to a diagonal matrix.

Solution. A satisfies the polynomial $x^2 - x$. Thus, the minimal polynomial, $m_A(x)$, of A divides $x^2 - x$, that is $m_A(x) = x$ or $m_A(x) = x - 1$ or $m_A(x) = x(x-1)$.

If $m_A(x) = x$, then $A = 0$.

If $m_A(x) = x - 1$, then $A = I$.

If $m_A(x) = x(x-1)$, then the minimal polynomial of A is product of distinct polynomials of degree one. Thus, by a Theorem, the matrix A is similar to diagonal matrix with diagonal entries consisting of the characteristic values, 0 and 1.

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(5) Let T be a linear operator on V. If every subspace of V is invariant under T then it is a scalar multiple of the identity operator.

Solution. If $\dim V = 1$ then for any $0 \neq v \in V$, we have $Tv = cv$, since V is invariant under T. Hence, $T = cI$.

Assume that $\dim V > 1$ and let $\mathcal{B} = \{v_1, v_2, \cdots, v_n\}$ be a basis for V. Since $W_1 = \langle v_1 \rangle$ is invariant under T, we have $Tv_1 = c_1 v_1$. Similarly, since $W_2 = \langle v_2 \rangle$ is invariant under T, we have $Tv_2 = c_2 v_2$. Now, $W_3 = \langle v_1 + v_2 \rangle$ is also invariant under T. Hence, $T(v_1 + v_2) = \lambda(v_1 + v_2)$ or $c_1 v_1 + c_2 v_2 = \lambda(v_1 + v_2)$, which gives us $(c_1 - \lambda)v_1 + (c_2 - \lambda)v_2 = 0$. However, v_1 and v_2 are linearly independent and hence we should have $c_1 = c_2 = \lambda$. Similarly, one can continue with the subspace $\langle v_1 + v_2 + v_3 \rangle$

and observe that $T(v_3) = \lambda v_3$. So for any $v_i \in \mathcal{B}$, we have $Tv_i = \lambda v_i$. Thus, $T = \lambda I$.

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(6) Let V be the vector space of $n \times n$ matrices over $\mathbb{F}$. Let A be a fixed $n \times n$ matrix. Let T be a linear operator on V defined by $T(B) = AB$. Show that the minimal polynomial of T is the minimal polynomial of A.

Solution. Let $m_A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be the minimal polynomial of A, so that $m_A(A) = 0$. It is easy to see that $T^k(B) = A^k B$ for any $k \geq 1$. Then, for any $B \in V$ we have

$$\begin{aligned} m_A(T)B &= (T^n + a_{n-1}T^{n-1} + \cdots + a_1 T + a_0 I)B \\ &= T^n(B) + a_{n-1}T^{n-1}(B) + \cdots + a_1 T(B) + a_0 B \\ &= A^n B + a_{n-1}A^{n-1}B + \cdots + a_1 AB + a_0 B \\ &= (A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I)B \\ &= m_A(A)B = 0. \end{aligned}$$

Thus, we obtain $m_A(T) = 0$, which means that $m_T(x)$ divides $m_A(x)$.

Now, let $m_T(x) = x^m + c_{m-1}x^{m-1} + \cdots + c_1 x + c_0$ be the minimal polynomial of T, so that $m_T(T) = 0$. Then, for any $B \in V$ we have

$$\begin{aligned} m_T(A)B &= (A^m + c_{m-1}A^{m-1} + \cdots + c_1 A + c_0 I)B \\ &= A^m B + c_{m-1}A^{m-1}B + \cdots + c_1 AB + c_0 B \\ &= T^m(B) + c_{m-1}T^{m-1}(B) + \cdots + c_1 T(B) + c_0 B \\ &= (T^m + c_{m-1}T^{m-1} + \cdots + c_1 T + c_0 I)B \\ &= m_T(T)B = 0, \end{aligned}$$

which leads to $m_T(A) = 0$, meaning that $m_A(x)$ divides $m_T(x)$. Since, monic polynomials dividing each other are the same we have $m_T(x) = m_A(x)$.

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(7) If E is a projection and f is a polynomial, then show that $f(E) = aI + bE$. What are a and b in terms of the coefficients of f?

Solution. Let $f(x) = c_0 + c_1x + \cdots + c_nx^n$. Then, $f(E) = c_0I + c_1E + \cdots + c_nE^n$. Since E is a projection, ($E^2 = E$), we have $E^k = E$ for any $k \geq 1$. Then,

$$\begin{aligned} f(E) &= c_0I + c_1E + \cdots + c_nE^n \\ &= c_0I + c_1E + \cdots + c_nE \\ &= c_0I + (c_1 + \cdots + c_n)E. \end{aligned}$$

Thus, a is the constant term of f and b is the sum of all other coefficients.

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(8) Let V be a finite dimensional vector space and let W_1 be any subspace of V. Prove that there is a subspace W_2 of V such that $V = W_1 \oplus W_2$.

Proof. Let $\mathcal{B}_{W_1} = \{\beta_1, \cdots, \beta_k\}$ be a basis for W_1. We may extend $\mathcal{B}_{W_1}$ to a basis $\mathcal{B}_V$ of V, say $\mathcal{B}_V = \{\beta_1, \cdots, \beta_k, \beta_{k+1}, \cdots, \beta_n\}$. Let W_2 be the subspace spanned by $\beta_{k+1}, \cdots, \beta_n$. Then, as they are linearly independent in V, we have $\mathcal{B}_{W_2} = \{\beta_{k+1}, \cdots, \beta_n\}$. Clearly $W_1 + W_2 = V$ as $W_1 + W_2$ contains a basis of V and so spans V. Let $\beta \in W_1 \cap W_2$. Then, $\beta \in W_1$ implies that $\beta = c_1\beta_1 + \cdots + c_k\beta_k$, and $\beta \in W_2$ implies that $\beta = c_{k+1}\beta_{k+1} + \cdots + c_n\beta_n$. The last two equalities give us $c_1\beta_1 + \cdots + c_k\beta_k - c_{k+1}\beta_{k+1} - \cdots + c_n\beta_n = 0$, but since β_i's are linearly independent, we obtain $c_i = 0$ for all $i = 1, \cdots, n$ which means that $\beta = 0$. That is $W_1 \cap W_2 = \{0\}$, and hence $V = W_1 \oplus W_2$.

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(9) Let V be a real vector space and E be an idempotent linear operator on V, that is a projection. Prove that $I + E$ is invertible. Find $(I + E)^{-1}$.

Proof. Since E is an idempotent linear operator it is diagonalizable by Question 4. So there exists a basis of V consisting of characteristics vectors of E corresponding to the characteristic values 0 and 1. That is, there exists a basis

$\mathcal{B} = \{\beta_1, \cdots, \beta_n\}$ such that $E\beta_i = \beta_i$ for $i = 1, \cdots, k$, and $E\beta_i = 0$ for $i = k+1, \cdots, n$. Then $(I+E)\beta_i = 2\beta_i$ for $i = 1, \cdots, k$ and $(I+E)\beta_i = \beta_i$ for $i = k+1, \cdots, n$, that is,

$$[I+E]_{\mathcal{B}} = \begin{bmatrix} 2I_1 & 0 \\ 0 & I_2 \end{bmatrix},$$

where I_1 stands for $k \times k$ identity matrix, I_2 is $(n-k)\times(n-k)$ identity matrix and each 0 represents the zero matrix of appropriate dimension. It is now easy to see that $[I+E]_{\mathcal{B}}$ is invertible, since $\det(I+E) = 2^k \neq 0$.

To find the inverse of $(I+E)$, we note that

$$([I+E]_{\mathcal{B}})^{-1} = \begin{bmatrix} \frac{1}{2}I_1 & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}I_1 & 0 \\ 0 & 0 \end{bmatrix} = I - \frac{1}{2}[E]_{\mathcal{B}}.$$

Therefore, $(I+E)^{-1} = I - \frac{1}{2}E$. (You may verify that really this is the inverse, by showing that $(I+E)(I-\frac{1}{2}E) = (I-\frac{1}{2}E)(I+E) = I$.)

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(10) Let T be a linear operator on V which commutes with every projection operator on V. What can you say about T?

Solution. Let $\mathcal{B}$ be a basis for V and $\beta_i \in \mathcal{B}, i \in I$ where I is some index set. We can write V as a direct sum $V = W_i \oplus U$ where $W_i = \langle \beta_i \rangle$. Then there exists a projection E_i of V onto the subspace W_i for each $i \in I$. Note that $E_i v \in W_i$ for all $v \in V$, and $E_i\beta_i = \beta_i$. Now, by assumption, the linear operator T commutes with E_i for all $i \in I$, that is, $TE_i = E_iT$. Then, for $\beta_i \in W_i$, we have $TE_i\beta_i = E_iT\beta_i \in W_i$ implies that $T\beta_i = T(E_i\beta_i) = c_i\beta_i$ for some constant $c_i \in \mathbb{F}$. Thus, β_i is a characteristic vector of T. Hence, V has a basis consisting of characteristic vectors of T. It follows that T is a diagonalizable linear operator on V.

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(11) Let V be the vector space of continuous real valued functions on the interval $[-1,1]$ of the real line. Let W_e be the space

of even functions, $f(-x) = f(x)$, and W_o be the space of odd functions, $f(-x) = -f(x)$.

a) Show that $V = W_e \oplus W_o$.

b) If T is the indefinite integral operator $(Tf)(x) = \int_0^x f(t)dt$, are W_e and W_o invariant under T?

Solution. a) Let $f \in V$. Then, we may write

$$f(x) = \frac{f(x) + f(-x) + f(x) - f(-x)}{2} = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}.$$

Observe that $f_e(x) = \dfrac{f(x) + f(-x)}{2}$ is a continuous even function and $f_o(x) = \dfrac{f(x) - f(-x)}{2}$ is a continuous odd function. Hence, $f = f_e + f_o$, that is $V = W_e + W_o$. To show that $V = W_e \oplus W_o$, we need to show that $W_e \cap W_o = \{0\}$. To see this, let $g \in W_e \cap W_o$. Then, $g \in W_e$ implies that $g(-x) = g(x)$, and $g \in W_o$ implies that $g(-x) = -g(x)$. Thus, we have $g(x) = -g(x)$ or $g(x) = 0$ for all $x \in [-1, 1]$, which means that $g = 0$.

b) For $f(x) = x \in W_o$, we have $(Tf)(x) = x^2/2 \notin W_o$, and for $g(x) = x^2 \in W_e$, we have $(Tg)(x) = x^3/3 \notin W_e$. Thus, neither W_e nor W_o are invariant under T.

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(12) Let V be a finite dimensional vector space over the field $\mathbb{F}$, and let T be a linear operator on V, such that $rank(T) = 1$. Prove that either T is diagonalizable or T is nilpotent, but not both.

Proof. Since $rank(T) = dim(Im(T)) = 1$, we have $dim(Ker(T)) = n - 1$. Let $0 \neq \beta \in Im(T)$. So, $Im(T) = \langle \beta \rangle$. Since $\beta \in Im(T)$, there exists a vector $\alpha_0 \in V$ such that $T\alpha_0 = \beta$. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$ be a basis for $Ker(T)$. Then, $\mathcal{B} = \{\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$ is a basis for V.

We have $T\alpha_i = 0 \quad$ for all $i = 1, 2, \cdots, n-1$.

If $T\alpha_0 \in Ker(T)$, then $T\alpha_0 = c_1\alpha_1 + \cdots + c_{n-1}\alpha_{n-1}$ and

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ c_1 & 0 & 0 & \cdots & 0 \\ c_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & 0 & 0 & \cdots & \vdots \end{bmatrix}$$

and it is easily seen that $T^2 = 0$ meaning that T is nilpotent. Note that at least one of c_i's is nonzero, since otherwise, α_0 would be in $Ker(T)$ which contradicts with the choice of $\mathcal{B}$.

If $T\alpha_0 \notin Ker(T)$, then $T\beta \in Im(T)$ and $T\beta = c_0\beta$. In this case we construct a new basis $\mathcal{B}' = \{\beta, \alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$ and

$$[T]_{\mathcal{B}'} = \begin{bmatrix} c_0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

which means that T is diagonalizable.

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(13) Let T be a linear operator on the finite dimensional vector space V. Suppose T has a cyclic vector. Prove that if U is any linear operator which commutes with T, then U is a polynomial in T.

Proof. Let $\mathcal{B} = \{\alpha, T\alpha, \cdots, T^{n-1}\alpha\}$ be a basis for V containing the cyclic vector α and let $m(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be the minimal polynomial of T. Since $U\alpha$ is in V, it can be written as a linear combination of basis vectors. Then, $U\alpha = b_0\alpha + b_1T\alpha + \cdots + b_{n-1}T^{n-1}\alpha$ where $b_0, b_1, \cdots, b_{n-1}$ are elements of the field $\mathbb{F}$. That is, $(b_0I + b_1T + \cdots + b_{n-1}T^{n-1} - U)\alpha = 0$. Now, since U and T commute, we have

$$\begin{aligned} UT(\alpha) = TU(\alpha) &= T(b_0\alpha + b_1T\alpha + \cdots + b_{n-1}T^{n-1}\alpha) \\ &= b_0T\alpha + b_1T^2\alpha + \cdots + b_{n-1}T^n\alpha \\ &= (b_0I + b_1T + \cdots + b_{n-1}T^{n-1})T\alpha \end{aligned}$$

which means that

$$(b_0 I + b_1 T + \cdots + b_{n-1} T^{n-1} - U) T\alpha = 0.$$

Similarly, we can show that $(b_0 I + b_1 T + \cdots + b_{n-1} T^{n-1} - U) T^i \alpha = 0$ for all $i = 2, 3, \cdots, n-1$. Since the transformation $b_0 I + b_1 T + \cdots + b_{n-1} T^{n-1} - U$ maps each basis vector to the zero vector, it is identically equal to zero on the whole space. Thus, we obtain

$$U = b_0 I + b_1 T + \cdots + b_{n-1} T^{n-1}.$$

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(14) Give an example of two 4×4 nilpotent matrices which have the same minimal polynomial but which are not similar.

Solution. Let $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

It is easy to see that $m_A(x) = m_B(x) = x^2$ but they are not similar since, A has 3 distinct characteristic vectors corresponding to the characteristic value zero, but B has only two characteristic vectors corresponding to the characteristic value zero.

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(15) Show that if N is a nilpotent linear operator on an $n-$dimensional vector space V, then the characteristic polynomial for N is x^n.

Solution. Recall that N is nilpotent, if $N^k = 0$ for some $k \in \mathbb{N}^+$. Since, N is a nilpotent linear operator on V, the minimal polynomial for N is of the form x^m for some $m \leq n$. Then, all characteristic values of N are zero. Since the minimal polynomial is a product of linear polynomials, N is a triangulable operator. It follows that there exists a basis $\mathcal{B}$ of

V such that

$$[N]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \star & 0 & \cdots & 0 \\ \star & \star & \ddots & \vdots \\ \star & \star & \cdots & 0 \end{bmatrix}.$$

since, similar matrices have the characteristic polynomial, it follows that the characteristic polynomial of N is x^n where $n = dimV$.

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(16) Let T be a linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Prove that T has no cyclic vector. What is the T cyclic subspace generated by the vector $\beta = (1, -1, 3)$?

Solution. Assume that T has a cyclic vector $\alpha = (a_1, a_2, a_3)$. Then $\mathcal{B} = \{\alpha, T\alpha, T^2\alpha\}$ will be a basis for $\mathbb{R}^3$. That is, the vectors $\alpha = (a_1, a_2, a_3), T\alpha = (2a_1, 2a_2, -a_3), T^2\alpha = (4a_1, 4a_2, a_3)$ must be linearly independent, or the matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ 2a_1 & 2a_2 & -a_3 \\ 4a_1 & 4a_2 & a_3 \end{bmatrix}$$

must be invertible. Applying elementary row operations, we obtain

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ 2a_1 & 2a_2 & -a_3 \\ 4a_1 & 4a_2 & a_3 \end{bmatrix} \overset{-2R_1+R_2}{\underset{-4R_1+R_3}{\longrightarrow}} \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & -3a_3 \\ 0 & 0 & -3a_3 \end{bmatrix} \overset{-2R_2+R_3}{\underset{-\frac{1}{3}R_2}{\longrightarrow}} \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{bmatrix}$$

which is not invertible. Hence, T has no cyclic vector.

To find the cyclic subspace generated by β, it is enough to check if β and $T\beta$ are independent since we have already shown that the set $\{\alpha, T\alpha, T^2\alpha\}$ can not be linearly independent for any $\alpha \in \mathbb{R}^3$. Clearly, $\beta = (1, -1, 3)$ and $T\beta = (2, -2, -3)$ are linearly independent since, otherwise, one of them would be a multiple of the other one which is not the case here. Thus, the cyclic subspace generated by β is

$$Z(\beta;T) = \langle(1,-1,3),(2,-2,-3)\rangle = \{\lambda(1,-1,3)+\mu(2,-2,-3) : \lambda,\mu \in \mathbb{R}\}.$$

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(17) Find the minimal polynomial and rational form of the matrix

$$T = \begin{bmatrix} c & 0 & -1 \\ 0 & c & 1 \\ -1 & 1 & c \end{bmatrix}.$$

Solution. The characteristic polynomial of T is

$$\begin{aligned} f_T(x) = \det(xI - T) &= \begin{vmatrix} x-c & 0 & 1 \\ 0 & x-c & -1 \\ 1 & -1 & x-c \end{vmatrix} \\ &= (x-c)\begin{vmatrix} x-c & -1 \\ -1 & x-c \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ x-c & -1 \end{vmatrix} \\ &= (x-c)((x-c)^2-1)-(x-c) \\ &= (x-c)((x-c)^2-2) \\ &= (x-c)(x-c-\sqrt{2})(x-c+\sqrt{2}). \end{aligned}$$

Since the characteristic polynomial and the minimal polynomial have the same roots and the minimal polynomial divides the characteristic polynomial we have $m_T(x) = f_T(x) = (x-c)((x-c)^2-2) = (x-c)^3 - 2(x-c) = x^3 + (-3c)x^2 + (3c^2-2)x + (-c^3+2c)$. Thus the rational form of T is

$$R = \begin{bmatrix} 0 & 0 & c^3-2c \\ 1 & 0 & -3c^2+2 \\ 0 & 1 & 3c \end{bmatrix}.$$
