Please show your work in all questions.

1. (5+5+5+5=20 points) Evaluate the following integrals

(a) \( \int (x - \frac{1}{x} \sqrt{x + \frac{1}{x}}) \, dx = \int (x^{\frac{3}{2}} + x^{\frac{1}{2}} - x^{-\frac{1}{2}} - x^{-\frac{3}{2}}) \, dx = \)

\[ = \frac{9}{5} x^{\frac{5}{2}} + \frac{2}{3} x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} + C \]

(b) \( \int \sin^2 t \cos t \, dt = \left| u = \sin(t) \right| \frac{du}{\cos(t)dt} = \int u^2 \, du = \frac{u^3}{3} + C = \)

\[ = \frac{\sin^3(t)}{3} + C \]

(c) \( \int \frac{4x^3}{\sqrt{x^2 + 1}} \, dx = \left| u = x^2 + 1 \right| \frac{du}{2x \, dx} = 2 \int \frac{u - 1}{\sqrt{u}} \, du = 2 \int (\sqrt{u} - u^{-\frac{1}{2}}) \, du \)

\[ = 2 \left( \frac{2}{3} u^{\frac{3}{2}} - 2 u^{\frac{1}{2}} \right) + C = \frac{4}{3} (x^2 + 1)^{\frac{3}{2}} - 4 (x^2 + 1)^{\frac{1}{2}} + C \]

(d) \( \int_{-1}^{1} x^4 \tan(x) \, dx = \int_{-1}^{1} dx + \int_{-1}^{1} \frac{x^4 \tan(x)}{1+x^2} \, dx = 2 + \int_{-1}^{1} \frac{x^4 \tan(x)}{1+x^2} \, dx \)

The function \( f(x) = \frac{x^4 \tan(x)}{1+x^2} \) is odd and we deal with the symmetric interval \([-1,1]\). Using Symmetry Property for the definite integral, we conclude \( \int_{-1}^{1} f(x) \, dx = 0 \).
2. \((1+1+4+3+2+4+4=20\) points) Follow the outline below to sketch the graph of the function \(f(x) = \frac{x^3}{9-x^2}\).

(a) Find the domain of \(f(x)\).
\[
\text{dom}(f) = \mathbb{R} \setminus \{\pm 3\}.
\]

(b) Find the intercepts.
\[x = 0 \implies y = 0.\] So, we have only \((0, 0)\).

(c) Find the symmetries of \(f(x)\), if any.
\[f(-x) = \frac{(-x)^3}{9-(-x)^2} = -f(x), \text{ that is, } f(x) \text{ is odd.}\]

(d) Determine all asymptotes.
\[
\frac{x^3}{9-x^2} = -x + \frac{9x}{9-x^2} \implies y = -x \text{ is a slant asymptote.}
\]
\[x = -3 \text{ and } x = 3 \text{ are vertical asymptotes, for}
\]
\[
\lim_{{x \to -3^-}} f(x) = +\infty, \lim_{{x \to -3^+}} f(x) = -\infty, \lim_{{x \to 3^-}} f(x) = +\infty, \lim_{{x \to 3^+}} f(x) = -\infty.
\]

(e) Find the intervals of increase and decrease, and the critical points.
\[
f'(x) = \frac{x^2(9x^2 - 2x^2)}{(9-x^2)^2}
\]
\[C.P.(f) = \left\{0, \pm \sqrt{3} \right\}\]
\[
\begin{array}{c|cccccc}
\text{Sign } f(x) & - & + & + & + & + & - \\
\text{Sign } f''(x) & + & + & - & - & - & - \\
\text{Behavior} & \text{decr.} & \text{incr.} & \text{cond.} & \text{incr.} & \text{cond.} & \text{cond.}
\end{array}
\]

(f) Find all local maxima and minima.
\[
\text{Using F.D.T., we conclude that } x = -3\sqrt{3} \text{ is a local max,}
\]
\[
x = 3\sqrt{3} \text{ is a local min. points.}
\]
(g) Determine the concavity of \( f(x) \), and its inflection points.

\[
\frac{d^2f}{dx^2} = \frac{18x(37+x^4)}{(3-x^2)^3}
\]

C.P. \( (f') \) = \{ 0 \}  \; \text{The point} \; x=0 \; \text{is an inflection point (see to the table)}

(h) Sketch the graph of \( f(x) \).
3. (17 points) We want to form a right angled trapezoid with one corner at the origin, two of the sides along the positive x and y axes, and the other two sides having lengths of 1m and 2m (see the figure below). Find the maximum possible area of such a trapezoid.

If \( A \) is the area then
\[
A = \frac{1}{2} xy + y
\]
Since \( x^2 + y^2 = 4 \), it follows that \( y = \sqrt{4 - x^2} \), 0 \leq x \leq 2. Then
\[
A = A(x) = \left(\frac{1}{2} x + 1\right) \sqrt{4 - x^2}, \quad 0 \leq x \leq 2.
\]
Take the derivative
\[
A'(x) = -\frac{x^2 + x - 2}{\sqrt{4 - x^2}}
\]
C.P. \( A \) = \{1, -2\}
\[
A(1) = \frac{3\sqrt{3}}{2}
\]
Since \( A(0) = 2 \), \( A(2) = 0 \), we derive that \( A(x) \) takes abs. max. at \( x = 1 \).

Finally,
\[
\frac{3\sqrt{3}}{2} > \left(\frac{1}{2}\sqrt{2} + \frac{3}{2}\right) \sqrt{6 - \frac{3}{2}}
\]

Similarly, \( x^2 + y^2 = 1 \) \( \Rightarrow y = \sqrt{1 - x^2} \), \( 0 \leq x \leq 1 \). Then
\[
A = A(x) = \left(\frac{1}{2} x + 2\right) \sqrt{1 - x^2}, \quad 0 \leq x \leq 1.
\]
Similarly,
\[
A'(x) = -\frac{x^2 + 2x - \frac{1}{2}}{\sqrt{1 - x^2}}
\]
C.P. \( A \) = \{\frac{\pm \sqrt{6} - 2 \pm \frac{1}{2}}{2}\} = \{\pm \sqrt{-3/2} - 1\}
\[
A\left(\frac{\sqrt{3}}{2} - 1\right) = \left(\frac{1}{2}\sqrt{2} + \frac{3}{2}\right) \left(\sqrt{6 - \frac{3}{2}}\right)
\]
Since \( A(0) = 2 \), \( A(1) = 0 \) and \( A\left(\frac{\sqrt{3}}{2} - 1\right) > 2 \), we conclude that \( A(x) \) takes abs. max. at \( x = \frac{\sqrt{3}}{2} - 1 \).
3. (17 points) We want to form a right angled trapezoid with one corner at the origin, two of the sides along the positive $x$ and $y$ axes, and the other two sides having lengths of $1m$ and $2m$ (see the figure below). Find the maximum possible area of such a trapezoid.

**Alternate solution**

If $A$ is the area then

$$A = 2\sin \theta + \frac{4 \sin \theta \cos \theta}{2}$$

$$A = 2 \sin \theta + 2 \sin \theta \cos \theta$$

for $0 \leq \theta \leq \frac{\pi}{2}$.

**Critical points:**

$$0 = A' = 2 \cos \theta + 2 \cos^2 \theta - 2 \sin^2 \theta$$

$$0 = 4 \cos^2 \theta + 2 \cos \theta - 2$$

So $\cos \theta = -1 \pm \sqrt{1 + 8} = -1 \pm \frac{3}{2}$

$$\cos \theta = -1$$

$\theta = \pi$

$$\cos \theta = \frac{1}{2}$$

$\theta = \frac{\pi}{3}$

$$A\left(\frac{\pi}{3}\right) = 2 \frac{\sqrt{3}}{2} + 2 \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$= \frac{3\sqrt{3}}{2}$$

$A(0) = 0 \neq A\left(\frac{\pi}{3}\right) = 2$

so this is a max.

$$A(x) = 2 \sqrt{6 - \frac{3}{2}} + \frac{1}{2} \sqrt{6 - \frac{3}{2}} \left(\frac{3}{2} - 1\right)$$

$$= \sqrt{6 - \frac{3}{2}} \left(\frac{3}{2} \frac{\sqrt{2}}{2} + \frac{3}{2}\right)$$

This is clearly bigger than 2.

so it is also a max.

$$\frac{3\sqrt{3}}{2} > \sqrt{6 - \frac{3}{2}} \left(\frac{3}{2} \frac{\sqrt{2}}{2} + \frac{3}{2}\right)$$
4. \((6+(2+2+2)+6)=18\) points

(a) Find the derivative of \(\int_0^{x^2} \tan(\cos(t))dt\) with respect to \(x\).

Using the Chain Rule, we obtain that

\[
\frac{d}{dx} \int_0^{x^2} \tan(\cos(t))dt = \tan(\cos(x^2)) \cdot 2x
\]

(b) Suppose that \(f'(x)\) has the graph shown in the figure to the right. Fill in the blanks in the following three questions accordingly:

(i) If \(f(1) = 4\), then \(f(5) = \_\_\_\_\_.\) Indeed,

\[
f(5) = f(1) + \int_1^5 f'(x)dx = 4 + A_1 - A_2 = 4 + 4 - 2 = 6.
\]

(ii) \(f(x)\) has a local maximum (or maxima) at \(\_\_\_\_.\) and a local minimum (or minima) at \(\_\_\_\_.\)

The points where \(f'(x)\) changes its sign.

(iii) \(f(x)\) has an inflection point (or points) at \(\_\_\_.\) and \(\_\_.\)

The critical points for \(f'(x)\), where we have local max and local min, respectively.

(c) Express \(\lim_{n \to \infty} \sum_{i=1}^{n} \frac{\sin(i/n)}{n}\) as a definite integral.

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \sin \left( \frac{i}{n} \right) = \int_0^1 \sin(x)dx
\]

thanks to the definition of Riemann sums.
5. (15 points) Sketch and find the area between the curves \( y = \sin x \) and \( y = \frac{1}{2} \) between \( x = 0 \) and \( x = \pi \). (Note that the region has three pieces.)

Note that \( \sin(x) = \frac{1}{2} \) in \([0, \pi]\) iff \( x = \frac{\pi}{6} \) or \( x = \frac{5\pi}{6} \).

Using the symmetry, we conclude that

\[
\frac{1}{2} A = \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (\sin(x) - \frac{1}{2}) \, dx + \int_{\frac{5\pi}{6}}^{\pi} (\frac{1}{2} - \sin(x)) \, dx =
\]

\[
= -\cos(x) \bigg|_{\frac{\pi}{2}}^{\frac{5\pi}{6}} - \frac{1}{2} x \bigg|_{\frac{\pi}{2}}^{\frac{5\pi}{6}} + \frac{1}{2} x \bigg|_{\frac{5\pi}{6}}^{\pi} + \cos(x) \bigg|_{\frac{5\pi}{6}}^{\pi}
\]

\[
= \frac{\sqrt{3}}{2} - \frac{\pi}{6} + \frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2} = \sqrt{3} - 1 - \frac{\pi}{12}
\]

Hence \( A = 2\sqrt{3} - 2 - \frac{\pi}{6} \).
6. (10 points) Suppose that a twice differentiable function $f(x)$ satisfies the equation $f''(x) = (f(x))^2 + 1$ for all $x$. Show that $f(x)$ has at most one critical point.

Assume that $f(x)$ has two different critical points, say $a$ and $b$ (a < b). So, $f'(a) = f'(b) = 0$. Using M. V. T., we derive that

$$\frac{f'(b) - f'(a)}{b - a} = f''(c)$$

for some point $c \in (a, b)$. Then $f''(c) = 0$. But $f''(c) = f(c)^2 + 1 \geq 1$, a contradiction. Hence $a = b$. 