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Impulsive Hopfield-type neural network system with piecewise constant argument

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ABSTRACT

In this paper we introduce an impulsive Hopfield-type neural network system with piecewise constant argument of generalized type. Sufficient conditions for the existence of the unique equilibrium are obtained. Existence and uniqueness of solutions of such systems are established. Stability criterion based on linear approximation is proposed. Some sufficient conditions for the existence and stability of periodic solutions are derived. An example with numerical simulations is given to illustrate our results.

1. Introduction and preliminaries

Scientists often are interested in systems, which are either continuous-time or discrete-time. They are widely studied in the theory of neural networks, but there is a somewhat new category of dynamical system, which is neither continuous-time nor purely discrete-time; among them are dynamical systems with impulses, and systems with piecewise constant arguments [1–5]. It is obvious that processes of 'integrate-and-fire' type in neural networks [6–9] request the systems as a mathematical modeling instrument. Significant parts of pioneer results for impulsive differential equations (IDE) and differential equations with piecewise constant argument (EPCA) can be found in [1,10,11].

In recent years, dynamics of Hopfield-type neural networks have been studied and developed by many authors by using IDE [2,5,18–21] and EPCA [12]. To the best of our knowledge, there have been no results on the dynamical behavior of impulsive Hopfield-type neural networks with piecewise constant arguments. Our paper contains an attempt to fill the gap by considering differential equations with piecewise constant arguments of generalized type (EPCAG) [13–15].

Denote by $\mathbb{N}$ and $\mathbb{R}^+ = [0, \infty)$ the sets of natural and nonnegative real numbers, respectively, and denote a norm on $\mathbb{R}^m$ by $\|\cdot\|$ where $\|u\| = \sum_{j=1}^m |u_i|$. The main subject under investigation in this paper is the following impulsive Hopfield-type neural network system with piecewise constant argument

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^m b_{ij} f_j(x_j(t)) + \sum_{j=1}^m c_{ij} g_j(x_j(\beta(t))) + d_i, \quad t \neq \theta_k \\ \Delta x_i|_{t=\theta_k} &= I_k(x_i(\theta_k^-)), \quad i = 1, 2, \ldots, m, \ k \in \mathbb{N}, \end{aligned} \tag{1.1}$$

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where $\beta(t) = \theta_{k-1}$ if $\theta_{k-1} \le t < \theta_k, k \in \mathbb{N}, t \in \mathbb{R}^+$, is an identification function, $\theta_k > 0, k \in \mathbb{N}$, is a sequence of real numbers such that there exist two positive real numbers $\underline{\theta}, \bar{\theta}$ such that $\underline{\theta} \le \theta_{k+1} - \theta_k < \bar{\theta}, k \in \mathbb{N}$, $\Delta x_i(\theta_k)$ denotes $x_i(\theta_k) - x_i(\theta_k^-)$, where $x_i(\theta_k^-) = \lim_{h\to 0^-} x_i(\theta_k + h)$. Moreover, $a_i > 0, i = 1, 2, \ldots, m$ are constants, m denotes the number of neurons in the network, $x_i(t)$ corresponds to the state of the ith unit at time t, $f_j(x_j(t))$ and $g_j(x_j(\beta(t)))$ denote, respectively, the measures of activation to its incoming potentials of the unit j at time $t \in [\theta_{k-1}, \theta_k), k = 1, 2, \ldots,$ and θ_{k-1}; b_{ij}, c_{ij}, d_i are constants; b_{ij} denotes the synaptic connection weight of the unit j on the unit i at time t, c_{ij} denotes the synaptic connection weight of the unit j on the unit i at time θ_{k-1}, d_i is the input from outside the network to the unit i.

We denote $PC(J, \mathbb{R}^m)$, where $J \subset \mathbb{R}^+$ is an interval, the set of all right continuous functions $\varphi : J \to \mathbb{R}^m$ with possible points of discontinuity of the first kind at $\theta_k \in J, k \in \mathbb{N}$.

Moreover, we introduce a set $PC^{(1)}(J, \mathbb{R}^m)$ of functions $\varphi : J \to \mathbb{R}^m$ such that $\varphi, \varphi' \in PC(J, \mathbb{R}^m)$, where the derivative at points θ_k is assumed to be the right derivative.

Throughout this paper, we assume that the functions $I_k : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, the parameters b_{ij}, c_{ij}, d_i are real, the activation functions $f_j, g_j \in C(\mathbb{R}^m)$ with $f_j(0) = 0, g_j(0) = 0$, and they satisfy the following conditions:

(C1) there exist Lipschitz constants $L_j, \bar{L}_j > 0$ such that

$$|f_j(u) - f_j(v)| \le L_j|u - v|,$$
$$|g_j(u) - g_j(v)| \le \bar{L}_j|u - v|$$

for all $u, v \in \mathbb{R}^m, j = 1, 2, \ldots, m$;

(C2) the impulsive operator I_i satisfies

$$|I_i(u) - I_i(v)| \le \ell|u - v|$$

for all $u, v \in \mathbb{R}^m, i = 1, 2, \ldots, m$, where l is a positive Lipschitz constant.

For the sake of convenience, we adopt the following notations:

$$\alpha_1 = \sum_{i=1}^{m}\sum_{j=1}^{m} |b_{ji}|L_i, \qquad \alpha_2 = \sum_{i=1}^{m}\sum_{j=1}^{m} |c_{ji}|\bar{L}_i, \qquad \alpha_3 = \sum_{i=1}^{m} a_i + \alpha_1.$$

Furthermore, the following assumptions will be needed throughout the paper:

(C3) $\bar{\theta}\,[\alpha_3 + \alpha_2] < 1$;

(C4) $\bar{\theta}\left[\alpha_2 + \alpha_3\left(1 + \bar{\theta}\alpha_2\right) e^{\bar{\theta}\alpha_3}\right] < 1$.

Taking into account the definition of solutions for EPCAG [13] and IDE [16], we understand a solution of (1.1) as a function from $PC^{(1)}(J, \mathbb{R}^m), J \subset \mathbb{R}^+$, which satisfies the differential equation and the impulsive condition of (1.1). The differential equation is satisfied for all $t \in J$, except possibly at the moments of discontinuity θ_k, where the right side derivative exists and it satisfies the differential equation as well.

Let us denote an equilibrium solution of (1.1) as the constant vector $x^* = (x_1^*, \ldots, x_m^*)^{\mathrm{T}} \in \mathbb{R}^m$, where each x_i^* satisfies

$$a_i x_i^* = \sum_{j=1}^{m} b_{ij} f_j(x_j^*) + \sum_{j=1}^{m} c_{ij} g_j(x_j^*) + d_i, \tag{1.2}$$

where impulsive part $I_k(\cdot)$ of (1.1) is assumed to satisfy $I_k(x_i^*) = 0$ for all $i = 1, \ldots, m, k \in \mathbb{N}$. Particularly, if $c_{ij} = 0$, the system (1.1) reduces to the system in [2].

The proof of following theorem is almost identical to the verification of Lemma 2.2 in [2] with slight changes which are caused by the piecewise constant argument.

Theorem 1.1. *Assume that the neural parameters a_i, b_{ij}, c_{ij} and Lipschitz constants $L_j, \bar{L}_j$ satisfy*

$$a_i > L_i \sum_{j=1}^{m} |b_{ji}| + \bar{L}_i \sum_{j=1}^{m} |c_{ji}|, \quad i = 1, \ldots, m.$$

Then, (1.1) has a unique equilibrium.

In the theory of EPCAG [13], we take the function $\beta(t) = \theta_k$ if $\theta_k \le t < \theta_{k+1}$, that is, $\beta(t)$ is right continuous. However, as it is usually done in the theory of IDE, at the points of discontinuity θ_k of the solution, solutions are left continuous. One should say that for the following investigation the right continuity is more convenient assumption if one considers equations with piecewise constant arguments.

The rest of this paper is organized as follows: In the next section, we obtain sufficient conditions for the existence and uniqueness of the solutions and the equivalence lemma for (1.1). We get a criteria for the global asymptotic stability of the impulsive Hopfield-type neural networks with piecewise constant arguments of generalized type by using linearization in Section 3. Section 4 is devoted to the existence and stability of a unique ω-periodic solution of (1.1). An illustrative example is given in Section 5.

2. Existence and uniqueness of solutions

Consider the following system

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\theta_{r-1})) + d_i, \quad a_i > 0,\ i = 1, 2, \ldots, m \tag{2.3}$$

for $\theta_{r-1} \le t \le \theta_r$.

Now, we continue with the following lemma which provides the conditions for the existence and uniqueness of solutions for arbitrary initial moment ξ.

Lemma 2.1. *Let* (C1), (C3), (C4) *be satisfied. Then for each* $x^0 \in \mathbb{R}^m$, *and* $\xi, \theta_{r-1} \le \xi < \theta_r, r \in \mathbb{N}$, *there exists a unique solution* $x(t) = x(t, \xi, x^0) = (x_1(t), \ldots, x_m(t))^{\mathrm{T}}$, *of* (2.3), $\theta_{r-1} \le t \le \theta_r$, *such that* $x(\xi) = x^0 = (x_1^0, \ldots, x_m^0)^{\mathrm{T}}$.

Proof. *Existence*: It is enough to show that the equivalent integral equation

$$z_i(t) = x_i^0 + \int_{\xi}^{t} \left[-a_i z_i(s) + \sum_{j=1}^{m} b_{ij} f_j(z_j(s)) + \sum_{j=1}^{m} c_{ij} g_j(z_j(\theta_{r-1})) + d_i \right] \mathrm{d}s$$

has a unique solution $z(t) = (z_1(t), \ldots, z_m(t))^{\mathrm{T}}$.

Define a norm $\|z(t)\|_0 = \max_{[\theta_{r-1},\theta_r]} \|z(t)\|$ and construct the following sequences $z_i^n(t),\ z_i^0(t) \equiv x_i^0,\ i = 1, \ldots, m$, $n \ge 0$ such that

$$z_i^{n+1}(t) = x_i^0 + \int_{\xi}^{t} \left[-a_i z_i^n(s) + \sum_{j=1}^{m} b_{ij} f_j(z_j^n(s)) + \sum_{j=1}^{m} c_{ij} g_j(z_j^n(\theta_{r-1})) + d_i \right] \mathrm{d}s.$$

One can find that

$$\|z^{n+1}(t) - z^n(t)\|_0 = \max_{[\theta_{r-1},\theta_r]} \|z^{n+1}(t) - z^n(t)\| \le \left[\bar{\theta}\left[\alpha_3 + \alpha_2\right]\right]^n \kappa,$$

where

$$\kappa = \bar{\theta} \left[(\alpha_3 + \alpha_2) \|x^0\| + \sum_{i=1}^{m} d_i \right].$$

Hence, the sequences $z_i^n(t)$ are convergent and their limits satisfy the integral equation on $[\theta_{r-1}, \theta_r]$. The existence is proved. *Uniqueness*: It is sufficient to check that for each $t \in [\theta_{r-1}, \theta_r]$, and $x^2 = (x_1^2, \ldots, x_m^2)^{\mathrm{T}}, x^1 = (x_1^1, \ldots, x_m^1)^{\mathrm{T}} \in \mathbb{R}^m, x^2 \ne x^1$, the condition $x(t, \theta_{r-1}, x^1) \ne x(t, \theta_{r-1}, x^2)$ is valid. Let us denote the solutions of (1.1) by $x^1(t) = x(t, \theta_{r-1}, x^1), x^2(t) = x(t, \theta_{r-1}, x^2)$. Assume on the contrary that there exists $t^* \in [\theta_{r-1}, \theta_r]$ such that $x^1(t^*) = x^2(t^*)$. Then, we have

$$x_i^1 - x_i^2 = \int_{\theta_{r-1}}^{t^*} \left[-a_i \left(x_i^2(s) - x_i^1(s)\right) + \sum_{j=1}^{m} b_{ij} [f_j(x_j^2(s)) - f_j(x_j^1(s))] + \sum_{j=1}^{m} c_{ij} [g_j(x_j^2(\theta_{r-1})) - g_j(x_j^1(\theta_{r-1}))] \right] \mathrm{d}s,$$
$$i = 1, \ldots, m. \tag{2.4}$$

Taking the absolute value of both sides for each $i = 1, \ldots, m$ and adding all equalities, we obtain

$$\begin{aligned}
\|x^2 - x^1\| &= \sum_{i=1}^{m} \left| \int_{\theta_{r-1}}^{t^*} \left[-a_i \left(x_i^2(s) - x_i^1(s)\right) + \sum_{j=1}^{m} b_{ij} \left[f_j(x_j^2(s)) - f_j(x_j^1(s))\right] \right.\right. \\
&\quad \left.\left. + \sum_{j=1}^{m} c_{ij} \left[g_j(x_j^2(\theta_{r-1})) - g_j(x_j^1(\theta_{r-1}))\right] \right] \mathrm{d}s \right| \\
&\le \sum_{i=1}^{m} \left\{ \int_{\theta_{r-1}}^{t^*} \left[a_i |x_i^2(s) - x_i^1(s)| + \sum_{j=1}^{m} L_i |b_{ji}| |x_i^2(s) - x_i^1(s)| + \sum_{j=1}^{m} \bar{L}_i |c_{ji}| |x_i^2 - x_i^1| \right] \mathrm{d}s \right\} \\
&\le \int_{\theta_{r-1}}^{t^*} \alpha_3 \|x^1(s) - x^2(s)\| \mathrm{d}s + \bar{\theta}\alpha_2 \|x^1 - x^2\|.
\end{aligned} \tag{2.5}$$

Furthermore, for $t \in [\theta_{r-1}, \theta_r]$, the following is valid:

$$\begin{aligned}
\|x^1(t) - x^2(t)\| &\le \|x^1 - x^2\| + \sum_{i=1}^{m} \left\{ \int_{\theta_{r-1}}^{t} \left[a_i |x_i^2(s) - x_i^1(s)| + \sum_{j=1}^{m} L_i |b_{ji}| |x_i^2(s) - x_i^1(s)| + \sum_{j=1}^{m} \bar{L}_i |c_{ji}| |x_i^2 - x_i^1| \right] \mathrm{d}s \right\} \\
&\le \left(1 + \bar{\theta}\alpha_2\right) \|x^1 - x^2\| + \int_{\theta_{r-1}}^{t} \alpha_3 \|x^1(s) - x^2(s)\| \mathrm{d}s.
\end{aligned}$$

Using Gronwall–Bellman inequality, it follows that

$$\|x^1(t) - x^2(t)\| \leq \left(1 + \bar{\theta}\alpha_2\right) e^{\bar{\theta}\alpha_3} \|x^1 - x^2\|. \tag{2.6}$$

Consequently, substituting (2.6) in (2.5), we obtain

$$\|x^1 - x^2\| \leq \left[\bar{\theta}\alpha_3 \left(1 + \bar{\theta}\alpha_2\right) e^{\bar{\theta}\alpha_3} + \bar{\theta}\alpha_2\right] \|x^1 - x^2\|. \tag{2.7}$$

Thus, one can see that (C4) contradicts with (2.7). The lemma is proved. □

Theorem 2.1. *Assume that conditions* (C1), (C3), (C4) *are fulfilled. Then, for every* $(t_0, x^0) \in \mathbb{R}^+ \times \mathbb{R}^m$, *there exists a unique solution* $x(t) = x(t, t_0, x^0) = (x_1(t), \ldots, x_m(t))^{\mathrm{T}},\ t \geq t_0$, *of* (1.1), *such that* $x(t_0) = x^0$.

Proof. Fix $t_0 \in \mathbb{R}^+$. There exists $r \in \mathbb{N}$ such that $t_0 \in [\theta_{r-1}, \theta_r)$. Use Lemma 2.1 with $\xi = t_0$ to obtain the unique solution $x(t, t_0, x^0)$ on $[\xi, \theta_r]$. Then apply the impulse condition to evaluate uniquely $x(\theta_r, t_0, x^0) = x(\theta_r^-, t_0, x^0) + I(x(\theta_r^-, t_0, x^0))$. Next, on the interval $[\theta_r, \theta_{r+1})$ the solution satisfies the ordinary differential equation

$$y_i'(t) = -a_i y_i(t) + \sum_{j=1}^{m} b_{ij} f_j(y_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(y_j(\theta_r)) + d_i \quad a_i > 0,\ i = 1, 2, \ldots, m.$$

The system has a unique solution $y(t, \theta_r, x(\theta_r, t_0, x^0))$. By definition of the solution of (1.1), $x(t, t_0, x^0) = y(t, \theta_r, x(\theta_r, t_0, x^0))$ on $[\theta_r, \theta_{r+1}]$. The mathematical induction completes the proof. □

Let us introduce the following two lemmas. We will prove just the second one, the proof for the first one is similar.

Lemma 2.2. *A function* $x(t) = x(t, t_0, x^0) = (x_1(t), \ldots, x_m(t))^{\mathrm{T}}$, *where* t_0 *is a fixed real number, is a solution of* (1.1) *on* $\mathbb{R}^+$ *if and only if it is a solution, on* $\mathbb{R}^+$, *of the following integral equation:*

$$\begin{aligned} x_i(t) = {} & e^{-a_i(t-t_0)} x_i^0 + \int_{t_0}^{t} e^{-a_i(t-s)} \left[\sum_{j=1}^{m} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(s))) + d_i\right] ds \\ & + \sum_{t_0 \leq \theta_k < t} e^{-a_i(t-\theta_k)} I_k(x_i(\theta_k^-)), \quad \text{for } i = 1, \ldots, m,\ t \geq t_0. \end{aligned} \tag{2.8}$$

Lemma 2.3. *A function* $x(t) = x(t, t_0, x^0) = (x_1(t), \ldots, x_m(t))^{\mathrm{T}}$, *where* t_0 *is a fixed real number, is a solution of* (1.1) *on* $\mathbb{R}^+$ *if and only if it is a solution, on* $\mathbb{R}^+$, *of the following integral equation:*

$$\begin{aligned} x_i(t) = {} & x_i^0 + \int_{t_0}^{t} \left[-a_i x_i(s) + \sum_{j=1}^{m} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(s))) + d_i\right] ds \\ & + \sum_{t_0 \leq \theta_k < t} I_k(x_i(\theta_k^-)), \quad \text{for } i = 1, \ldots, m,\ t \geq t_0. \end{aligned} \tag{2.9}$$

Proof. Sufficient part of this lemma can be easily proved. Therefore, we only prove the necessity of this lemma. Fix $i = 1, \ldots, m$. Assume that $x(t) = (x_1(t), \ldots, x_m(t))^{\mathrm{T}}$ is a solution of (1.1) on $\mathbb{R}^+$. Denote

$$\varphi_i(t) = x_i^0 + \int_{t_0}^{t} \left[-a_i x_i(s) + \sum_{j=1}^{m} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(s))) + d_i\right] ds + \sum_{t_0 \leq \theta_r < t} I_r(x_i(\theta_r^-)). \tag{2.10}$$

It is clear that the expression in the right side exists for all t.

Assume that $t_0 \in (\theta_{r-1}, \theta_r)$, then differentiating the last expression, we get

$$\varphi_i'(t) = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(t))) + d_i.$$

We also have

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(x_j(\beta(t))) + d_i.$$

Hence, for $t \neq \theta_r,\ r \in \mathbb{N}$, we obtain

$$[\varphi_i(t) - x_i(t)]' = 0. \tag{2.11}$$

Moreover, it follows from Eq. (2.10) that

$$\Delta\varphi_i(\theta_r) = \varphi_i(\theta_r) - \varphi_i(\theta_r^-) = I_r(\varphi_i(\theta_r^-)). \tag{2.12}$$

One can see that $\varphi_i(t_0) = x_i^0$. Then, by (2.11), we have that $\varphi_i(t) = x_i(t)$ on $[t_0, \theta_r)$, which implies $\varphi_i(\theta_r^-) = x_i(\theta_r^-)$. Next, using Eq. (2.12) and the last equation, we obtain

$$\varphi_i(\theta_r) = \varphi_i(\theta_r^-) + I_r(\varphi_i(\theta_r^-)) = x_i(\theta_r^-) + I_r(x_i(\theta_r^-)) = x_i(\theta_r).$$

Therefore, one can conclude that $\varphi_i(\theta_r) = x_i(\theta_r)$ for $t \in [t_0, \theta_r)$. Similarly, in the light of above discussion, one can also obtain that $\varphi_i(t) = x_i(t)$ on $[\theta_r, \theta_{r+1})$. We can complete the proof by using mathematical induction. □

3. Stability of equilibrium

In this section, we will give sufficient conditions for the global asymptotic stability of the equilibrium, x^*, of (1.1) based on linearization [10]. The system (1.1) can be simplified as follows. Let $y_i = x_i - x_i^*$, for each $i = 1, \ldots, m$. Then,

$$\begin{aligned} y_i'(t) &= -a_i y_i(t) + \sum_{j=1}^{m} b_{ij}\phi_j(y_j(t)) + \sum_{j=1}^{m} c_{ij}\psi_j(y_j(\beta(t))), \quad t \neq \theta_k \\ \Delta y_i|_{t=\theta_k} &= \bar{I}_k(y_i(\theta_k^-)), \quad i = 1, 2, \ldots, m, \ k \in \mathbb{N}, \end{aligned} \tag{3.13}$$

where $\phi_j(y_j(t)) = f_j(y_j(t) + x_j^*) - f_j(x_j^*)$, $\psi_j(y_j(t)) = g_j(y_j(t) + x_j^*) - g_j(x_j^*)$ and $\bar{I}_k(y_j(\theta_k^-)) = I_k(y_j(\theta_k^-) + x_j^*)$. For each $j = 1, \ldots, m$, and $k \in \mathbb{N}$, $\phi_j(\cdot), \psi_j(\cdot)$, and $\bar{I}_k$ are Lipschitzian since $f_j(\cdot), g_j(\cdot)$ and I_k are Lipschitzian with L_j, $\bar{L}_j$ and l respectively, and $\phi_j(0) = 0$, $\psi_j(0) = 0$; furthermore, $\bar{I}_k(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous with $\bar{I}_k(0) = 0$.

It is clear that the stability of the zero solution of (3.13) is equivalent to the stability of the equilibrium x^* of (1.1). Therefore, in what follows, we discuss the stability of the zero solution of (1.1).

Let us denote

$$\bar{B} = \left\{1 - \bar{\theta}\left[\alpha_2 + \alpha_3\left(1 + \bar{\theta}\alpha_2\right)e^{\bar{\theta}\alpha_3}\right]\right\}^{-1}.$$

The following lemma is an important auxiliary result of the paper (see, also, [17]).

Lemma 3.1. *Let $y(t) = (y_1(t), \ldots, y_m(t))^{\mathrm{T}}$ be a solution of (3.13) and (C1), (C3), (C4) be satisfied. Then, the following inequality*

$$\|y(\beta(t))\| \le \bar{B}\|y(t)\| \tag{3.14}$$

holds for all $t \in \mathbb{R}^+$.

Proof. Fix $t \in \mathbb{R}^+$, there exists $k \in \mathbb{N}$ such that $t \in [\theta_{k-1}, \theta_k)$. Then, from Lemma 2.3, we have

$$\begin{aligned} \|y(t)\| &= \sum_{i=1}^{m} |y_i(t)| \\ &\le \|y(\theta_{k-1})\| + \sum_{i=1}^{m}\left\{\int_{\theta_{k-1}}^{t}\left[a_i|y_i(s)| + \sum_{j=1}^{m} L_i|b_{ji}||y_i(s)| + \sum_{j=1}^{m}\bar{L}_i|c_{ji}||y_i(\theta_{k-1})|\right]ds\right\} \\ &\le (1 + \bar{\theta}\alpha_2)\|y(\theta_{k-1})\| + \int_{\theta_{k-1}}^{t}\alpha_3\|y(s)\|ds. \end{aligned}$$

By using Gronwall–Bellman Lemma, we get

$$\|y(t)\| \le \left(1 + \bar{\theta}\alpha_2\right)e^{\bar{\theta}\alpha_3}\|y(\theta_{k-1})\|. \tag{3.15}$$

Moreover, for $t \in [\theta_{k-1}, \theta_k)$, we have

$$\begin{aligned} \|y(\theta_{k-1})\| &= \|y(t)\| + \sum_{i=1}^{m}\left\{\int_{\theta_{k-1}}^{t}\left[a_i|y_i(s)| + \sum_{j=1}^{m} L_i|b_{ji}\|y_i(s)| + \sum_{j=1}^{m}\bar{L}_i|c_{ji}\|y_i(\theta_{k-1})|\right]ds\right\} \\ &\le \|y(t)\| + \bar{\theta}\alpha_2\|y(\theta_{k-1})\| + \int_{\theta_{k-1}}^{t}\alpha_3\|y(s)\|ds. \end{aligned}$$

It follows from (3.15) that

$$\|y(\theta_{k-1})\| \le \|y(t)\| + \bar{\theta}\alpha_2\|y(\theta_{k-1})\| + \bar{\theta}\alpha_3\left(1 + \bar{\theta}\alpha_2\right)e^{\bar{\theta}\alpha_3}\|y(\theta_{k-1})\|.$$

Then, we have from condition (C4) that

$$\|y(\theta_{k-1})\| \leq \bar{B}\|y(t)\|, \quad t \in [\theta_{k-1}, \theta_k).$$

Thus, (3.14) holds for all $t \in \mathbb{R}^+$. This proves the lemma. □

Now, we are ready to give sufficient conditions for the global asymptotic stability of (1.1). Let us denote the solution of linear homogeneous system of (3.13) as $\bar{y} = \mathrm{diag}(y_1, \ldots, y_m)$.

From now on we need the following assumption:

(C5) $\gamma - \alpha_1 - \bar{B}\alpha_2 - \frac{\ln(1+l)}{\underline{\theta}} > 0$, where $\gamma = \min_{1\leq i\leq m} a_i$ is positive.

The following theorem is a modified version of the theorem in [10], for our system.

Theorem 3.1. *Assume that* (C1)–(C5) *are fulfilled. Then, the zero solution of* (3.13) *is globally asymptotically stable.*

Proof. Let $y(t) = (y_1(t), \ldots, y_m(t))^{\mathrm{T}}$ be an arbitrary solution of (3.13). From Lemma 2.2, we have

$$\begin{aligned}\|y(t)\| &\leq \mathrm{e}^{-\gamma(t-t_0)}\|y_0\| + \sum_{i=1}^{m}\left\{\int_{t_0}^{t}\mathrm{e}^{-\gamma(t-s)}\left[\sum_{j=1}^{m}L_i|b_{ji}||y_i(s)| + \sum_{j=1}^{m}\bar{L}_i|c_{ji}||y_i(\beta(s))|\right]\mathrm{d}s + \sum_{t_0\leq\theta_k<t}\mathrm{e}^{-\gamma(t-\theta_k)}l|y_i(\theta_k^-)|\right\}\\ &\leq \mathrm{e}^{-\gamma(t-t_0)}\|y_0\| + \left(\alpha_1 + \bar{B}\alpha_2\right)\int_{t_0}^{t}\mathrm{e}^{-\gamma(t-s)}\|y(s)\|\mathrm{d}s + l\sum_{t_0\leq\theta_k<t}\mathrm{e}^{-\gamma(t-\theta_k)}|y(\theta_k^-)|.\end{aligned}$$

Also, previous inequality can be written as,

$$\mathrm{e}^{\gamma(t-t_0)}\|y(t)\| \leq \|y_0\| + \left(\alpha_1 + \bar{B}\alpha_2\right)\int_{t_0}^{t}\mathrm{e}^{\gamma(s-t_0)}\|y(s)\|\mathrm{d}s + l\sum_{t_0\leq\theta_k<t}\mathrm{e}^{\gamma(\theta_k-t_0)}\|y(\theta_k^-)\|.$$

By applying Gronwall–Bellman inequality [10], we obtain

$$\mathrm{e}^{\gamma(t-t_0)}\|y(t)\| \leq \mathrm{e}^{(\alpha_1+\bar{B}\alpha_2)(t-t_0)}[1+l]^{i(t_0,t)}\|y_0\|,$$

where $i(t_0, t)$ is the number of points θ_k in $[t_0, t)$. Then, we have that

$$\|y(t)\| \leq \mathrm{e}^{-(\gamma-\alpha_1-\bar{B}\alpha_2-\frac{\ln(1+l)}{\underline{\theta}})(t-t_0)}\|y_0\|.$$

So, using (C5), we see that $\|y(t)\| \mapsto 0$ as $t \mapsto \infty$. That is, the zero solution of system (3.13) is globally asymptotically stable. □

4. Existence and stability of periodic solutions

In this part, we will establish some sufficient conditions for the existence of periodic solutions of (1.1). Then, we will study the stability of these solutions. Firstly, we shall need the following assumptions:

(C6) the sequence θ_k satisfies $\theta_{k+p} = \theta_k + w$, $k \in \mathbb{N}$ and $I_{k+p} = I_k$ for a fixed positive real period w and some positive integer p.

(C7) $\beta^* = \mathcal{K}\left[\omega\left(\alpha_1 + \bar{B}\alpha_2\right) + lp\right] < 1$, where $\mathcal{K} = \frac{1}{1-\mathrm{e}^{-\gamma\omega}}$.

For θ_k, $k \in \mathbb{N}$, let $[0, \omega] \cap \{\theta_k\}_{k\in\mathbb{N}} = \{\theta_1, \ldots, \theta_p\}$.

Here, we will give the following version of the Poincaré criterion for system (1.1). One can easily prove the following lemma (see, also, [10]).

Lemma 4.1. *Suppose that conditions* (C1), (C3), (C4), (C6) *are valid. Then, solution* $x(t) = x(t, t_0, x^0) = (x_1, \ldots, x_m)^{\mathrm{T}}$ *of* (1.1) *with* $x(t_0) = x^0$ *is ω-periodic if and only if* $x(\omega) = x(0)$.

Theorem 4.1. *Assume that conditions* (C1)–(C4), (C6), (C7) *are valid. Then system* (1.1) *has a unique ω-periodic solution.*

Proof. Let $PC_\omega = \{\varphi \in PC^{(1)}(\mathbb{R}^+, \mathbb{R}^m) \mid \varphi(t+\omega) = \varphi(t),\ t \geq 0\}$ be a Banach space of periodic functions with the norm $\|\varphi\|_0 = \max_{0\leq t\leq\omega}\|\varphi(t)\|$.

Let $\varphi(t) = (\varphi_1(t), \ldots, \varphi_m(t))^{\mathrm{T}} \in PC_\omega$. Using Lemma 2.2, similarly to the proof in [10], one can show that if $\varphi \in PC_\omega$ then the system

$$\begin{aligned}&x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{m}b_{ij}f_j(\varphi_j(t)) + \sum_{j=1}^{m}c_{ij}g_j(\varphi_j(\beta(t))) + d_i, \quad t \neq \theta_k,\\ &\Delta x_i|_{t=\theta_k} = I_k(\varphi_i(\theta_k^-)), \quad i = 1, \ldots, m,\ k = 1, 2, \ldots, p\end{aligned}$$

has the unique ω-periodic solution

$$x_i^*(t) = \int_0^\omega \mathcal{H}_i(t,s)\left[\sum_{j=1}^m b_{ij}f_j(\varphi_j(s)) + \sum_{j=1}^m c_{ij}g_j(\varphi_j(\beta(s))) + d_i\right]\mathrm{d}s + \sum_{k=1}^p \mathcal{H}_i(t,\theta_k)I_k(\varphi_i(\theta_k^-)),$$

where

$$\mathcal{H}_i(t,s) = (1-\mathrm{e}^{-a_i\omega})^{-1}\begin{cases}\mathrm{e}^{-a_i(t-s)}, & 0\le s\le t\le\omega\\ \mathrm{e}^{-a_i(\omega+t-s)}, & 0\le t<s\le\omega.\end{cases}$$

The function $\{\mathcal{H}_i(t,s)\}_{i=1,\ldots,m}$ is a Green's function. One can find that

$$\max_{t,s\in\mathcal{D}}\left|\{\mathcal{H}_i(t,s)\}_{i=1,\ldots,m}\right| = \frac{1}{1-\mathrm{e}^{-a_i\omega}},$$

where $\mathcal{D} = [0,\omega]\times[0,\omega]$.

Define the operator $\mathcal{E}$ in PC_ω by

$$\mathcal{E} : PC_\omega \to PC_\omega$$

such that if $\varphi\in PC_\omega$, then

$$(\mathcal{E}\varphi)_i(t) = \int_0^\omega \mathcal{H}_i(t,s)\left[\sum_{j=1}^m b_{ij}f_j(\varphi_j(s)) + \sum_{j=1}^m c_{ij}g_j(\varphi_j(\beta(s))) + d_i\right]\mathrm{d}s + \sum_{k=1}^p \mathcal{H}_i(t,\theta_k)I_k(\varphi_i(\theta_k^-)), \quad i=1,\ldots,m.$$

Let $PC_\omega^* = \{\varphi \mid \varphi\in PC_\omega, \|\varphi-\varphi_0\|_0 \le \frac{\beta^*\mathcal{C}}{1-\beta^*}\}$, where $\mathcal{C} = \mathcal{K}\omega\sum_{i=1}^m d_i$ and $(\varphi_0)_i(t) = \int_0^\omega \mathcal{H}_i(t,s)d_i\mathrm{d}s,\ i=1,\ldots,m.$ Then it is easy to see that PC_ω^* is a closed convex subset of PC_ω. According to the definition of the norm of Banach space PC_ω, we have

$$\|\varphi_0(t)\| = \sum_{i=1}^m\left|\int_0^\omega \mathcal{H}_i(t,s)d_i\mathrm{d}s\right| \le \frac{1}{1-\mathrm{e}^{-a_i\omega}}\sum_{i=1}^m\left[\int_0^\omega d_i\mathrm{d}s\right] \le \mathcal{C} < \infty.$$

So,

$$\|\varphi_0\|_0 \le \mathcal{C}.$$

Then, for an arbitrary $\varphi\in PC_\omega^*$, we have

$$\|\varphi\|_0 \le \|\varphi-\varphi_0\|_0 + \|\varphi_0\|_0 \le \frac{\beta^*\mathcal{C}}{1-\beta^*} + \mathcal{C} = \frac{\mathcal{C}}{1-\beta^*}.$$

Now, we need to prove that $\mathcal{E}$ maps PC_ω^* into itself. That is, we shall show that $\mathcal{E}\varphi\in PC_\omega^*$ for any $\varphi\in PC_\omega^*$. One can easily verify that $(\mathcal{E}\varphi)(t) = ((\mathcal{E}\varphi)_1,\ldots,(\mathcal{E}\varphi)_m)^{\mathrm{T}}$ is ω-periodic function. Now, if $\varphi\in PC_\omega^*$, then

$$\begin{aligned}\|\mathcal{E}\varphi-\varphi_0\| &= \sum_{i=1}^m\left|\int_0^\omega \mathcal{H}_i(t,s)\left[\sum_{j=1}^m b_{ij}f_j(\varphi_j(s)) + \sum_{j=1}^m c_{ij}g_j(\varphi_j(\beta(s)))\right]\mathrm{d}s + \sum_{k=1}^p \mathcal{H}_i(t,\theta_k)I_k(\varphi_i(\theta_k^-))\right|\\ &\le \sum_{i=1}^m\frac{1}{1-\mathrm{e}^{-a_i\omega}}\left\{\int_0^\omega\left[\sum_{j=1}^m L_j|b_{ij}||\varphi_j(s)| + \sum_{j=1}^m \bar{L}_j|c_{ij}||\varphi_j(\beta(s))|\right]\mathrm{d}s + \sum_{k=1}^p|I_k(\varphi_i(\theta_k^-))|\right\}\\ &\le \mathcal{K}\sum_{i=1}^m\left\{\int_0^\omega\left[\sum_{j=1}^m L_i|b_{ji}||\varphi_i(s)| + \sum_{j=1}^m \bar{L}_i|c_{ji}||\varphi_i(\beta(s))|\right]\mathrm{d}s + \sum_{k=1}^p l|\varphi_i(\theta_k^-)|\right\}\\ &\le \mathcal{K}\left(\int_0^\omega[\alpha_1\|\varphi(s)\| + \alpha_2\|\varphi(\beta(s))\|]\,\mathrm{d}s + l\sum_{k=1}^p\|\varphi(\theta_k^-)\|\right).\end{aligned}$$

Thus, it follows that

$$\begin{aligned}\|\mathcal{E}\varphi-\varphi_0\|_0 &\le \mathcal{K}\left(\omega\left(\alpha_1+\bar{B}\alpha_2\right)+lp\right)\|\varphi\|_0\\ &\le \beta^*\frac{\mathcal{C}}{1-\beta^*} = \frac{\beta^*\mathcal{C}}{1-\beta^*}.\end{aligned}$$

In the view of (C7), $\mathcal{E}\varphi\in PC_\omega^*$.

Finally, we shall show that $\mathcal{E}$ is a contraction mapping. If $\varphi^1, \varphi^2 \in PC^*_\omega$, then

$$\begin{aligned}
\|\mathcal{E}\varphi^1(t) - \mathcal{E}\varphi^2(t)\| &= \sum_{i=1}^m \left|(\mathcal{E}\varphi^1)_i(t) - (\mathcal{E}\varphi^2)_i(t)\right| \\
&\le \sum_{i=1}^m \Bigg\{ \int_0^\omega |\mathcal{H}_i(t,s)| \left[\sum_{j=1}^m L_j|b_{ij}||\varphi_j^1(s) - \varphi_j^2(s)| + \bar{B}\sum_{j=1}^m \bar{L}_j|c_{ij}||\varphi_j^1(s) - \varphi_j^2(s)| \right] \mathrm{d}s \\
&\quad + \sum_{k=1}^p |\mathcal{H}_i(t,\theta_k)| l |\varphi_i^1(\theta_k^-) - \varphi_i^2(\theta_k^-)| \Bigg\} \\
&\le \mathcal{K} \sum_{i=1}^m \Bigg\{ \int_0^\omega \left[\sum_{j=1}^m L_i|b_{ji}||\varphi_i^1(s) - \varphi_i^2(s)| + \sum_{j=1}^m \bar{L}_i|c_{ji}||\varphi_i^1(\beta(s)) - \varphi_i^2(\beta(s))| \right] \mathrm{d}s \\
&\quad + \sum_{k=1}^p l|\varphi_i^1(\theta_k^-) - \varphi_i^2(\theta_k^-)| \Bigg\} \\
&\le \mathcal{K} \left(\int_0^\omega \left[\alpha_1\|\varphi^1(s) - \varphi^2(s)\| + \alpha_2\|\varphi^1(\beta(s)) - \varphi^2(\beta(s))\|\right] \mathrm{d}s + l\sum_{k=1}^p \|\varphi^1(\theta_k^-) - \varphi^2(\theta_k^-)\| \right).
\end{aligned}$$

Hence,

$$\|\mathcal{E}\varphi^1 - \mathcal{E}\varphi^2\|_0 \le \mathcal{K}\left(\omega\left(\alpha_1 + \bar{B}\alpha_2\right) + lp\right)\|\varphi^1 - \varphi^2\|_0.$$

Noting (C7), it can be seen that $\mathcal{E}$ is a contraction mapping in PC^*_ω. Consequently, by using Banach fixed point theorem, $\mathcal{E}$ has a unique fixed point $\varphi^* \in PC^*_\omega$, such that $\mathcal{E}\varphi^* = \varphi^*$, which implies that (1.1) has a unique ω-periodic solution. □

We are now in a position to give and prove the stability of the periodic solution of (1.1).

Theorem 4.2. *Assume that conditions* (C1)–(C7) *are valid. Then the periodic solution of* (1.1) *is globally asymptotically stable.*

Proof. By Theorem 4.1, we know that (1.1) has an ω-periodic solution $x^*(t) = (x_1^*, \ldots, x_m^*)^{\mathrm{T}}$. Suppose that $x(t) = (x_1, \ldots, x_m)^{\mathrm{T}}$ is an arbitrary solution of (1.1) and let $y(t) = x(t) - x^*(t) = (x_1 - x_1^*, \ldots, x_m - x_m^*)^{\mathrm{T}}$. Then, similar to the proof of Theorem 3.1, one can show that

$$\|y(t)\| \le \mathrm{e}^{-\gamma(t-t_0)}\|y_0\| + \sum_{i=1}^m \left\{ \int_{t_0}^t \mathrm{e}^{-\gamma(t-s)} \left[\sum_{j=1}^m L_i|b_{ji}||y_i(s)| + \sum_{j=1}^m \bar{L}_i|c_{ji}||y_i(\beta(t))| \right] \mathrm{d}s + \sum_{t_0 \le \theta_k < t} \mathrm{e}^{-\gamma(t-\theta_k)} l|y_i(\theta_k^-)| \right\}$$

and hence

$$\|y(t)\| \le \mathrm{e}^{-(\gamma - \alpha_1 - \bar{B}\alpha_2 - \frac{\ln(1+l)}{\underline{\theta}})(t-t_0)}\|y_0\|.$$

Thus, the periodic solution of (1.1) is globally asymptotically stable. □

5. An illustrative example

Consider the following impulsive Hopfield-type neural network system with piecewise constant argument

$$\begin{aligned}
x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^2 b_{ij} f_j(x_j(t)) + \sum_{j=1}^2 c_{ij} g_j(x_j(\beta(t))) + d_i, \quad t \ne \theta_k \\
\Delta x_i|_{t=\theta_k} &= I_k(x_i(\theta_k^-)), \quad i = 1, 2,\ k = 1, 2, \ldots,
\end{aligned} \tag{5.16}$$

where $\beta(t) = \theta_k$ if $\theta_k \le t < \theta_{k+1}$, $k \in \mathbb{N}$, $\theta_k = k + (-1)^k/12$. The distance $\theta_{k+1} - \theta_k$, $k \in \mathbb{N}$, is equal to either $\underline{\theta} = 5/6$, or $\bar{\theta} = 7/6$. The output functions are $f_i(x) = \tanh(x/2)$, $g_i(x) = (|x+1| - |x-1|)/8$. Obviously, $L_i = 1/2$ and $\bar{L}_i = 1/4$. Taking $b_{ij} = c_{ij} = 1/64$ for $i, j = 1, 2$, $I_k = (-1)^k x/32 + 1/12$ with $l = 1/32$, and $d_1 = 1/6$, $d_2 = 1/7$, $a_1 = 0.18$, $a_2 = 0.19$, we get $p = 2$, $\omega = 2$, $\gamma = 0.18$, $\bar{B} = 4.53370$, $\mathcal{K} = 3.30786$ and $\beta^* = 0.88194 < 1$. It is easily checked that the system (5.16) satisfies Theorems 1.1, 3.1, 4.1 and 4.2. Consequently, the system (5.16) has a unique 2-periodic solution which is globally asymptotically stable. Since it is globally asymptotically stable, any other solution is eventually 2-periodic. The fact can be seen by simulation in Fig. 1a, b and Fig. 2.

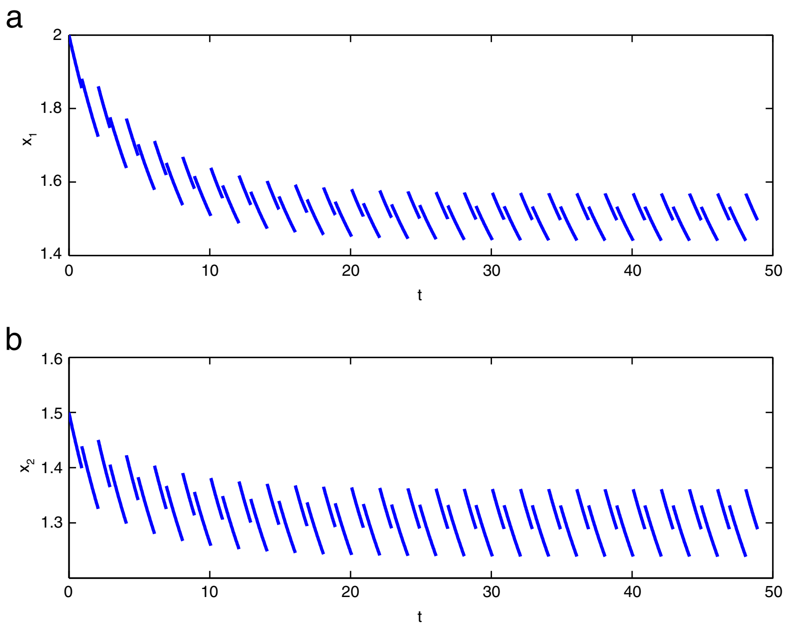

Fig. 1. (a) 2-periodic solution of $x_1(t)$ of system (5.16) for $t \in [0, 50]$ with the initial value $x_1(t_0) = 2$. (b) 2-periodic solution of $x_2(t)$ of system (5.16) for $t \in [0, 50]$ with the initial value $x_2(t_0) = 1.5$.

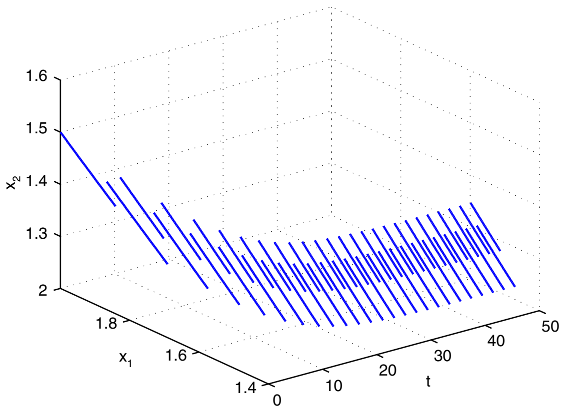

Fig. 2. Eventually 2-periodic solutions of system (5.16).

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