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Bifurcation of three-dimensional discontinuous cycles

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ABSTRACT

We consider three-dimensional discontinuous dynamical systems with non-fixed moments of impacts. Existence of the center manifold is proved for the system. The result is applied for the extension of the planar Hopf bifurcation theorem [M.U. Akhmet, Perturbations and Hopf bifurcation of the planar discontinuous dynamical system, Nonlinear Analysis 60 (2005) 163–178]. Illustrative examples are constructed for the theory.

1. Introduction

Dynamical systems are used to describe real world motions using differential (continuous time) or difference (discrete time) equations. In the last several decades, the need for discontinuous dynamical systems has been increased because they, often, describe the model better when the discontinuous and continuous motions are mingled. This need has made scientists to improve and develop the theory of these systems. Many new results have arised. One must mention that namely, systems with nonprescribed time of discontinuities were apparently first introduced for investigation of the real world [1,2], and this fact emphasizes very much the practical sense of the theory. The problem is one of the most difficult and interesting subjects of investigations [3–9]. It was emphasized in the early stage of theory's development, [10].

In [11], the Hopf bifurcation for the planar discontinuous dynamical system has been studied. Here, we extend this result to three-dimensional space based on the center manifold. The advantage is that we use the method of B-equivalence [11–13] as well as the results of time scales which are developed in [11,13].

This paper is organized as follows. In the Section 2, we start to analyze the non-perturbed system. Section 3 describes the perturbed system. The center manifold is given in Section 4. In Section 5, the bifurcation of periodic solutions is studied. Section 6 is devoted to examples in order to illustrate the theory. In Section 7 a brief conclusion is given.

2. The non-perturbed system

Let $\mathbb{N}$, $\mathbb{R}$ be the sets of all natural and real numbers, respectively, $\mathbb{R}^2$ be a real euclidean space. Denote by $\langle x, y\rangle$ the dot product of vectors $x, y \in \mathbb{R}^2$. Let $\|x\| = \langle x, x\rangle^{1/2}$ be the norm of a vector $x \in \mathbb{R}^2$, and $\mathbb{R}^{2\times 2}$ be the set of real-valued constant 2×2 matrices, $I \in \mathbb{R}^{2\times 2}$ be the identity matrix. We shall consider in $\mathbb{R}^3$ the following dynamical system:

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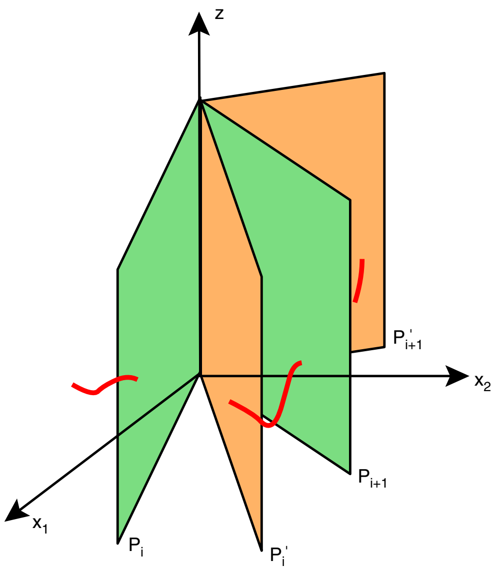

Fig. 1. The discontinuity set and a trajectory of (1).

$$\begin{aligned}
&\frac{\mathrm{d}x}{\mathrm{d}t} = Ax,\\
&\frac{\mathrm{d}z}{\mathrm{d}t} = \hat{b}z, \quad (x,z) \notin \Gamma_0,\\
&\Delta x\,|_{(x,z)\in\Gamma_0} = B_0 x,\\
&\Delta z\,|_{(x,z)\in\Gamma_0} = c_0 z,
\end{aligned} \tag{1}$$

where $A, B_0 \in \mathbb{R}^{2\times 2}$, $\hat{b}, c_0 \in \mathbb{R}$, Γ_0 is a subset of $\mathbb{R}^3$ and will be described below. The phase point of (1) moves between two consecutive intersections with the set Γ_0 along one of the trajectories of the system $x' = Ax, z' = \hat{b}z$. When the solution meets the set Γ_0 at the moment τ, the point $x(t)$ has a jump $\Delta x\,|_\tau := x(\tau+) - x(\tau)$ and the point $z(t)$ has a jump $\Delta z\,|_\tau := z(\tau+) - z(\tau)$. Thus, we suppose that the solutions are left continuous functions.

From now on, G denotes a neighborhood of the origin.

The following assumptions will be needed throughout the paper:

(C1) $\Gamma_0 = \bigcup_{i=1}^{p} \mathcal{P}_i, p \in \mathbb{N}$, where $\mathcal{P}_i = \ell_i \times \mathbb{R}$, ℓ_i are half-lines starting at the origin defined by $\langle a^i, x\rangle = 0$ for $i = 1, \ldots, p$, $a^i = (a^i_1, a^i_2) \in \mathbb{R}^2$ are constant vectors;

(C2) $A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, where $\beta \neq 0$;

(C3) there exists a regular matrix $Q \in \mathbb{R}^{2\times 2}$ and nonnegative real numbers k and θ such that

$$B_0 = kQ\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} Q^{-1} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For the sake of brevity, in what follows, every angle for a point or a line is considered with respect to the half-line of the first coordinate axis in x-plane. Denote $\ell'_i = (I + B_0)\ell_i$, $i = 1, \ldots, p$. Let γ_i and ζ_i be the angles of ℓ_i and ℓ'_i for $i = 1, \ldots, p$, respectively, and $B_0 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$;

(C4) $0 < \gamma_1 < \zeta_1 < \gamma_2 < \cdots < \gamma_p < \zeta_p < 2\pi$, and $(b_{11} + 1)\cos\gamma_i + b_{12}\sin\gamma_i \neq 0$ for $i = 1, \ldots, p$.

In Fig. 1, the discontinuity set and a trajectory of the system (1) are shown. The planes $\mathcal{P}_i$ form the set Γ_0 and each $\mathcal{P}'_i$ is the image of $\mathcal{P}_i$ under the transformation $(I + B)x$.

The system (1) is said to be a D_0-*system* if conditions (C1)–(C4) hold. It is easy to see that the origin is a unique singular point of D_0-*system* and (1) is not linear.

Let us subject (1) to the transformation $x_1 = r\cos\phi, x_2 = r\sin\phi, z = z$ and exclude the time variable t. The solution $(r(\phi, r_0, z_0), z(\phi, r_0, z_0))$ which starts at the point $(0, r_0, z_0)$ satisfies the following system in cylindrical coordinates:

$$\begin{aligned}
&\frac{\mathrm{d}r}{\mathrm{d}\phi} = \lambda r,\\
&\frac{\mathrm{d}z}{\mathrm{d}\phi} = bz, \quad \phi \neq \gamma_i (\mathrm{mod}\ 2\pi),\\
&\Delta r\,|_{\phi=\gamma_i(\mathrm{mod}\ 2\pi)} = k_i r,\\
&\Delta z\,|_{\phi=\gamma_i(\mathrm{mod}\ 2\pi)} = c_0 z,
\end{aligned} \tag{2}$$

where $\lambda = \alpha/\beta$, $b = \hat{b}/\beta$, the variable ϕ is ranged over the time scale

$$\mathbb{R}_\phi = \mathbb{R} \setminus \bigcup_{i=-\infty}^{\infty} \bigcup_{j=1}^{p} (2\pi i + \gamma_j, 2\pi i + \zeta_j]$$

and

$$k_i = \left[((b_{11}+1)\cos\gamma_i + b_{12}\sin\gamma_i)^2 + (b_{21}\cos\gamma_i + (b_{22}+1)\sin\gamma_i)^2\right]^{1/2} - 1.$$

Eq. (2) is 2π-periodic, so, in what follows we shall consider just the section $[0, 2\pi]$. That is, the system

$$\begin{aligned}
&\frac{\mathrm{d}r}{\mathrm{d}\phi} = \lambda r,\\
&\frac{\mathrm{d}z}{\mathrm{d}\phi} = bz, \quad \phi \neq \gamma_i,\\
&\Delta r\,|_{\phi=\gamma_i} = k_i r,\\
&\Delta z\,|_{\phi=\gamma_i} = c_0 z,
\end{aligned} \tag{3}$$

is provided for discussion, where $\phi \in [0, 2\pi]_\phi = [0, 2\pi] \setminus \cup_{i=1}^{p} (\gamma_i, \zeta_i]$. System (3) is a sample of time-scale differential equation. Let us use the *ψ-substitution*, $\varphi = \psi(\phi) = \phi - \sum_{0<\gamma_j<\phi} \theta_j$, $\theta_j = \zeta_j - \gamma_j$, which was introduced and developed in [11,13]. The range of this new variable is $[0, 2\pi - \sum_{i=1}^{p} \theta_i]$.

It is easy to check that upon ψ-substitution (3) reduces to the following impulsive equations:

$$\begin{aligned}
&\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \lambda r,\\
&\frac{\mathrm{d}z}{\mathrm{d}\varphi} = bz, \quad \varphi \neq \varphi_i,\\
&\Delta r\,|_{\varphi=\varphi_i} = k_i r,\\
&\Delta z\,|_{\varphi=\varphi_i} = c_0 z,
\end{aligned} \tag{4}$$

where $\varphi_i = \psi(\gamma_i)$. Solving (4) as an impulsive system [14,15] and using ψ-substitution one can obtain that a solution of (3) is of the form

$$r(\phi) = \exp\left(\lambda\left(\phi - \sum_{0<\gamma_i<\phi} \theta_i\right)\right) \left[\prod_{0<\gamma_i<\phi} (1+k_i)\right] r_0, \tag{5}$$

$$z(\phi) = \exp\left(b\left(\phi - \sum_{0<\gamma_i<\phi} \theta_i\right)\right) \left[\prod_{0<\gamma_i<\phi} (1+c_0)\right] z_0, \tag{6}$$

for $\phi \in [0, 2\pi]_\phi$. Denote

$$q_1 = \exp\left(\lambda\left(2\pi - \sum_{i=1}^{p} \theta_i\right)\right) \prod_{i=1}^{p} (1+k_i), \tag{7}$$

$$q_2 = \exp\left(b\left(2\pi - \sum_{i=1}^{p} \theta_i\right)\right) \prod_{i=1}^{p} (1+c_0). \tag{8}$$

Depending on q_1 and q_2 we may see that the following lemmas are valid.

Lemma 2.1. *Assume that $q_1 = 1$. Then, if*

(i) $q_2 = 1$ *then all solutions are periodic with period* $\omega = \left(2\pi - \sum_{i=1}^{p} \theta_i\right) \beta^{-1}$;
(ii) $q_2 = -1$ *then a solution that starts to its motion on x_1x_2-plane is ω-periodic and all other solutions are 2ω-periodic;*
(iii) $|q_2| > 1$ *then a solution that starts to its motion on x_1x_2-plane is ω-periodic and all other solutions lie on the surface of a cylinder and they move away from the origin (i.e. zero solution is unstable);*
(iv) $|q_2| < 1$ *then a solution that starts to its motion on x_1x_2-plane is ω-periodic and all other solutions lie on the surface of a cylinder and they move toward the x_1x_2-plane (i.e. zero solution is stable).*

Lemma 2.2. *Assume that $q_1 < 1$. Then, if*

(i) $|q_2| < 1$ *all solutions will spiral toward the origin, i.e., origin is an asymptotically stable fixed point;*
(ii) $|q_2| > 1$ *a solution that starts to its motion on x-plane spirals toward the origin and a solution that starts to its motion on z-axis will move away from the origin. In this case the origin is half-stable (or conditionally stable);*
(iii) $q_2 = 1$ $(q_2 = -1)$ *then a solution that starts to its motion on z-axis is periodic with period ω (2ω) and all other solutions will approach to z-axis.*

Lemma 2.3. *Assume that $q_1 > 1$. Then, if*

(i) $|q_2| < 1$ *then origin is a stable focus;*
(ii) $|q_2| > 1$ *then origin is an unstable focus;*
(iii) $q_2 = 1$ $(q_2 = -1)$ *then a solution that starts to its motion on z-axis is periodic with period ω (2ω) and all other solutions will approach to z-axis.*

We note that when $q_2 = -1$, (this means z may be negative, too) the solutions starting their motion out of x_1x_2-plane, will move above and below the x_1x_2-plane. More explicitly, if a solution starts to its motion above the x-plane, then after the time corresponding to ω, it will be below the x-plane, and in the next duration corresponding to ω, it will try to move above x-plane and at the end of that duration it will be above the x-plane, and so on.

From now on, we assume that $q_1 = 1$ and $|q_2| < 1$.

3. The perturbed system

Let G denote a sufficiently small neighborhood of the origin and consider the system

$$\begin{aligned}
\frac{dx}{dt} &= Ax + f(x, z),\\
\frac{dz}{dt} &= \hat{b}z + g(x, z), \quad (x, z) \notin \Gamma,\\
\Delta x\,|_{(x,z)\in\Gamma} &= B(x)x,\\
\Delta z\,|_{(x,z)\in\Gamma} &= c(z)z,
\end{aligned} \tag{9}$$

where the following assumptions are assumed to be true:

(C5) $\Gamma = \bigcup_{i=1}^{p} \mathcal{S}_i$, where $\mathcal{S}_i = s_i \times \mathbb{R}$ and the equation of s_i is given by $s_i : \langle a^i, x\rangle + \tau_i(x) = 0$, for $i = 1, \ldots, p$;
(C6)

$$B(x) = (k + \kappa(x))Q \begin{bmatrix} \cos(\theta + \Theta(x)) & -\sin(\theta + \Theta(x)) \\ \sin(\theta + \Theta(x)) & \cos(\theta + \Theta(x)) \end{bmatrix} Q^{-1} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $c(z) = c_0 + \tilde{c}(z)$;
(C7) functions $f, g, \kappa, \tilde{c}$ and Θ are in C^1 and τ_i is in C^2;
(C8) $f(x, z) = \mathcal{O}(\|(x, z)\|^2)$, $g(x, z) = \mathcal{O}(\|(x, z)\|^2)$, $\kappa(x) = \mathcal{O}(\|x\|)$, $\Theta(x) = \mathcal{O}(\|x\|)$, $\tilde{c}(z) = \mathcal{O}(z)$, $\tau_i(x) = \mathcal{O}(\|x\|^2)$, $i = 1, \ldots, p$, and $f(0, z) = 0$, $g(0, z) = 0$ for all $z \in \mathbb{R}$.

Moreover, it is supposed that the matrices A, Q, the vectors a^i, $i = 1, \ldots, p$, constants k, θ are the same as for (1), i.e.,

(C9) the one associated with (9) is D_0 system.

Remark 3.1. Conditions (C5) and (C6) imply that surfaces $\mathcal{S}_i$ do not intersect each other except on z-axis and neither of them intersects itself.

The system (9) is said to be a *D-system* if the conditions (C1)–(C8) hold.

In what follows we assume without loss of generality that $\gamma_i \neq \frac{\pi}{2}j, j = 1, 2, 3$. Then one can transform the equation in (C5) to the polar coordinates so that $s_i : a_i^1 r\cos\phi + a_i^2 r\sin\phi + \tau_i(r\cos\phi, r\sin\phi) = 0$ and, hence

$$\phi = \tan^{-1}\left(\tan\gamma_i - \frac{\tau_i(r\cos\phi, r\sin\phi)}{a_i^2 r\cos\phi}\right).$$

Using Taylor expansion gives that the previous equation can be written, for sufficiently small r, as

$$s_i : \phi = \gamma_i + \Psi_i(r,\phi), \quad i = 1,\ldots,p$$

where functions Ψ_i are 2π-periodic in ϕ, continuously differentiable and $\Psi_i = \mathcal{O}(r)$.

If the phase point $(x_1(t), x_2(t), z(t))$ meets the discontinuity surface $\mathcal{S}_i$ at the angle θ, then after the jump, the point $(x_1(\theta+), x_2(\theta+), z(\theta+))$ will be on the surface $\mathcal{S}_i' = \{(u,v) \in \mathbb{R}^3 : u = (I + B(x))x, v = (1+c_0)z + c(z), (x,z) \in \mathcal{S}_i\}$. For the remaining part of the paper the following assertion is very important and the proof can be found in [11].

Lemma 3.1. *If the conditions* (C7) *and* (C8) *are valid then the surface $\mathcal{S}_i'$ is placed between the surfaces $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$ for every i if G is sufficiently small.*

Using the cylindrical coordinates $x_1 = r\cos\phi, x_2 = r\sin\phi, z = z$, one can find that the differential part of (9) has the following form:

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}\phi} &= \lambda r + P(r,\phi,z), \\ \frac{\mathrm{d}z}{\mathrm{d}\phi} &= bz + Q(r,\phi,z), \end{aligned} \tag{10}$$

where, as is known [16], the functions $P(r,\phi,z)$ and $Q(r,\phi,z)$ are 2π-periodic in ϕ, continuously differentiable in all variables and $P = \mathcal{O}(r,z), Q = \mathcal{O}(r,z)$, with $P(0,\phi,z) = 0, Q(0,\phi,z) = 0$, for all $\phi, z \in \mathbb{R}$. Denote $x^+ = (x_1^+, x_2^+) = (I + B(x))x, x^+ = r^+(\cos\phi^+, \sin\phi^+), \tilde{x}^+ = (\tilde{x}_1^+, \tilde{x}_2^+) = (I + B(0))x$, where $x = (x_1, x_2) \in s_i, i = 1,\ldots,p$. The inequality $\|x^+ - \tilde{x}^+\| \leq \|B(x) - B(0)\| \cdot \|x\|$ implies that $r^+ = (1+k_i)r + \omega(r,\phi)$. Moreover, using the relation $\frac{x_2^+}{x_1^+}$ and $\frac{\tilde{x}_2^+}{\tilde{x}_1^+}$ and condition (C5) one can conclude that $\phi^+ = \phi + \theta_i + \gamma(r,\phi)$. Functions ω and γ are 2π-periodic in ϕ and $\omega = \mathcal{O}(r^2), \gamma = \mathcal{O}(r)$. Finally, the transformed system is of the following form:

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}\phi} &= \lambda r + P(r,\phi,z), \\ \frac{\mathrm{d}z}{\mathrm{d}\phi} &= bz + Q(r,\phi,z), \quad (r,\phi,z) \notin \Gamma, \\ \Delta r\,|_{(r,\phi)\in s_i} &= k_i r + \omega(r,\phi), \\ \Delta\phi\,|_{(r,\phi)\in s_i} &= \theta_i + \gamma(r,\phi), \\ \Delta z\,|_{(r,\phi)\in s_i} &= c_0 z + \tilde{c}(z). \end{aligned} \tag{11}$$

Let us introduce the following system besides (11):

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}\phi} &= \lambda\rho + P(\rho,\phi,z), \\ \frac{\mathrm{d}z}{\mathrm{d}\phi} &= bz + Q(\rho,\phi,z), \quad \phi \neq \gamma_i, \\ \Delta\rho\,|_{\phi=\gamma_i} &= k_i\rho + W_i^1(\rho,z), \\ \Delta\phi\,|_{\phi=\gamma_i} &= \theta_i, \\ \Delta z\,|_{\phi=\gamma_i} &= c_0 z + W_i^2(\rho,z), \end{aligned} \tag{12}$$

where all elements, except for $W_i = (W_i^1, W_i^2), i = 1,\ldots,p$, are the same as in (11) and the domain of (12) is $[0, 2\pi]_\phi$. We shall define the functions W_i below.

Let $(r(\phi, r_0, z_0), z(\phi, r_0, z_0))$ be a solution of (11) ϕ_i be the angle where the phase point intersects $\mathcal{S}_i$. Denote also by $\chi_i = \phi_i + \theta_i + \gamma(r(\phi_i, r_0, z_0), \phi_i)$ the angle where the phase point has to be after the jump.

Further $(\alpha \hat{,} \beta], \{\alpha, \beta\} \subset \mathbb{R}$ denotes the oriented interval, that is

$$(\alpha \hat{,} \beta] = \begin{cases} (\alpha, \beta] & \text{if } \alpha \leq \beta, \\ (\beta, \alpha] & \text{otherwise.} \end{cases}$$

Definition 3.1. We shall say that systems (11) and (12) are B-equivalent in G if for every solution $(r(\phi, r_0, z_0), z(\phi, r_0, z_0))$ of (11) whose trajectory is in G for all $\phi \in [0, 2\pi]_\phi$ there exists a solution $(\rho(\phi, r_0, z_0), z(\phi, r_0, z_0))$ of (12) which satisfies the relation

$$r(\phi, r_0, z_0) = \rho(\phi, r_0, z_0), \quad \phi \in [0, 2\pi]_\phi \setminus \bigcup_{i=1}^{p} \{(\phi_i, \hat{\gamma}_i] \cup (\zeta_i, \hat{\chi}_i]\}, \tag{13}$$

and, conversely, for every solution $(\rho(\phi, r_0, z_0), z(\phi, r_0, z_0))$ of (12) whose trajectory is in G, there exists a solution $(r(\phi, r_0, z_0), z(\phi, r_0, z_0))$ of (11) which satisfies (13).

Fix $i = 1, \ldots, p$. Let $(r_1(\phi), z_1(\phi))$, $(r_1(\gamma_i), z_1(\gamma_i)) = (\rho, z)$, be a solution of

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}\phi} &= \lambda r + P(r, \phi, z), \\ \frac{\mathrm{d}z}{\mathrm{d}\phi} &= bz + Q(r, \phi, z), \end{aligned} \tag{14}$$

and let $\phi = \eta_i$ be the meeting angle of the solution with $\mathscr{P}_i$. Then

$$\begin{aligned} r_1(\eta_i) &= \mathrm{e}^{\lambda(\eta_i - \gamma_i)}\rho + \int_{\gamma_i}^{\eta_i} \mathrm{e}^{\lambda(\eta_i - s)} P(r_1(s), s, z_1(s))\mathrm{d}s, \\ z_1(\eta_i) &= \mathrm{e}^{b(\eta_i - \gamma_i)}z + \int_{\gamma_i}^{\eta_i} \mathrm{e}^{b(\eta_i - s)} Q(r_1(s), s, z_1(s))\mathrm{d}s. \end{aligned}$$

Set $\eta_i' = \eta_i + \theta_i + \gamma(r_1(\eta_i), \eta_i)$ and $(\rho', z') = ((1 + k_i)r_1(\eta_i) + \omega(r_1(\eta_i), \eta_i), (1 + c_0)z_1(\eta_i) + c(z_1(\eta_i)))$. Let $(r_2(\phi), z_2(\phi))$, $(r_2(\eta_i'), z_2(\eta_i')) = (\rho', z')$, be a solution of (14). Then,

$$\begin{aligned} r_2(\zeta_i) &= \mathrm{e}^{\lambda(\zeta_i - \eta_i')}\rho' + \int_{\eta_i'}^{\zeta_i} \mathrm{e}^{\lambda(\zeta_i - s)} P(r_2(s), s, z_2(s))\mathrm{d}s, \\ z_2(\zeta_i) &= \mathrm{e}^{b(\zeta_i - \eta_i')}z' + \int_{\eta_i'}^{\zeta_i} \mathrm{e}^{b(\zeta_i - s)} Q(r_2(s), s, z_2(s))\mathrm{d}s. \end{aligned}$$

We define that

$$\begin{aligned} W_i^1(\rho, z) &= r_2(\zeta_i) - (1 + k_i)\rho \\ &= \mathrm{e}^{\lambda(\zeta_i - \eta_i')}\left[(1 + k_i)\left(\mathrm{e}^{\lambda(\eta_i - \gamma_i)}\rho + \int_{\gamma_i}^{\eta_i} \mathrm{e}^{\lambda(\eta_i - s)} P(r_1(s), s, z_1(s))\mathrm{d}s\right) + \omega(r_1(\eta_i), \eta_i)\right] \\ &\quad + \int_{\eta_i'}^{\zeta_i} \mathrm{e}^{\lambda(\zeta_i - s)} P(r_1(s), s, z_1(s))\mathrm{d}s - (1 + k_i)\rho, \end{aligned}$$

or, if simplified

$$\begin{aligned} W_i^1(\rho, z) &= (1 + k_i)(\mathrm{e}^{-\lambda\gamma(r_1(\eta_i), \eta_i)} - 1)\rho + (1 + k_i)\int_{\gamma_i}^{\eta_i} \mathrm{e}^{\lambda(\zeta_i - \theta_i - s - \gamma(r_1(\eta_i), \eta_i))} P(r_1(s), s, z_1(s))\mathrm{d}s \\ &\quad + \int_{\eta_i'}^{\zeta_i} \mathrm{e}^{\lambda(\zeta_i - s)} P(r_2(s), s, z_2(s))\mathrm{d}s + \mathrm{e}^{\lambda(\zeta_i - \eta_i')}\omega(r_1(\eta_i), \eta_i). \end{aligned} \tag{15}$$

We, similarly, define

$$\begin{aligned} W_i^2(\rho, z) &= z_2(\zeta_i) - (1 + c_0)z \\ &= \mathrm{e}^{b(\zeta_i - \eta_i')}\left[(1 + c_0)\left(\mathrm{e}^{b(\eta_i - \gamma_i)}z + \int_{\gamma_i}^{\eta_i} \mathrm{e}^{b(\eta_i - s)} Q(r_1(s), s, z_1(s))\mathrm{d}s\right) + \tilde{c}(z_1(\eta_i))\right] \\ &\quad + \int_{\eta_i'}^{\zeta_i} \mathrm{e}^{b(\zeta_i - s)} Q(r_1(s), s, z_1(s))\mathrm{d}s - (1 + c_0)z, \end{aligned}$$

or,

$$\begin{aligned} W_i^2(\rho, z) &= (1 + k_i)(\mathrm{e}^{-b\gamma(r_1(\eta_i), \eta_i)} - 1)z + (1 + c_0)\int_{\gamma_i}^{\eta_i} \mathrm{e}^{(\zeta_i - \theta_i - s - \gamma(r_1(\eta_i), \eta_i))} Q(r_1(s), s, z_1(s))\mathrm{d}s \\ &\quad + \int_{\eta_i'}^{\zeta_i} \mathrm{e}^{b(\zeta_i - s)} Q(r_2(s), s, z_2(s))\mathrm{d}s + \mathrm{e}^{b(\zeta_i - \eta_i')}\tilde{c}(z_1(\eta_i)). \end{aligned} \tag{16}$$

We note that there exists a Lipschitz constant ℓ and a bounded function $m(\ell)$ such that

$$\|W_i^j(\rho_1, z_1) - W_i^j(\rho_2, z_2)\| \le m(\ell)\ell(\|\rho_1 - \rho_2\| + \|z_1 - z_2\|), \tag{17}$$

for all $\rho_1, \rho_2, z_1, z_2 \in \mathbb{R}, j = 1, 2$. For detailed proof and explanation about (17) we refer to [11,13].

4. Center manifold

Now, using ψ-substitution (12) becomes:

$$\begin{aligned}
\frac{d\rho}{d\varphi} &= \lambda\rho + F(\rho, \varphi, z),\\
\frac{dz}{d\varphi} &= bz + G(\rho, \varphi, z), \quad \varphi \ne \varphi_i,\\
\Delta\rho\,|_{\varphi=\varphi_i} &= k_i\rho + W_i^1(\rho, z),\\
\Delta z\,|_{\varphi=\varphi_i} &= c_0 z + W_i^2(\rho, z),
\end{aligned} \tag{18}$$

where $\varphi = \psi(\phi), \varphi_i = \psi(\gamma_i), F(\rho, \varphi, z) = P(\rho, \psi^{-1}(\varphi), z)$ and $G(\rho, \varphi, z) = Q(\rho, \psi^{-1}(\varphi), z)$. Functions F and G are T-periodic in φ, with $T = \psi(2\pi)$, and satisfy

$$\|F(\rho, \varphi, z) - F(\rho', \varphi, z')\| \le L(\|\rho - \rho'\| + \|z - z'\|), \tag{19}$$

$$\|G(\rho, \varphi, z) - G(\rho', \varphi, z')\| \le L(\|\rho - \rho'\| + \|z - z'\|), \tag{20}$$

for some Lipschitz constant L.

Following the methods given in [12], one can see that system (18) has two integral manifolds whose equations are given by:

$$\Phi_0(\varphi, \rho) = \int_{-\infty}^{\varphi} \pi_0(\varphi, s)G(\rho(s, \varphi, \rho), s, z(s, \varphi, \rho))ds + \sum_{\varphi_i<\varphi} \pi_0(\varphi, \varphi_i^+)W_i^2(\rho(\varphi_i^+, \varphi, \rho), z(\varphi_i^+, \varphi, \rho)), \tag{21}$$

and

$$\Phi_-(\varphi, z) = -\int_{\varphi}^{\infty} \pi_-(\varphi, s)F(\rho(s, \varphi, z), s, z(s, \varphi, z))ds + \sum_{\varphi_i<\varphi} \pi_-(\varphi, \varphi_i^+)W_i^1(\rho(\varphi_i^+, \varphi, z), z(\varphi_i^+, \varphi, z)), \tag{22}$$

where

$$\pi_0(\varphi, s) = e^{b(\varphi-s)} \prod_{s\le\varphi_j<\varphi} (1 + c_0)$$

and

$$\pi_-(\varphi, s) = e^{\lambda(\varphi-s)} \prod_{s\le\varphi_j<\varphi} (1 + k_j).$$

In (21), the pair $(\rho(s, \varphi, \rho), z(s, \varphi, \rho))$ denotes a solution of (18) satisfying $\rho(\varphi, \varphi, \rho) = \rho$. Similarly, $(\rho(s, \varphi, z), z(s, \varphi, z))$, in (22), is solution of (18) with $z(\varphi, \varphi, z) = z$.

In [12], it was shown that there exist constants K_0, M_0, σ_0 such that Φ_0 satisfies:

$$\Phi_0(\varphi, 0) = 0, \tag{23}$$

$$\|\Phi_0(\varphi, \rho_1) - \Phi_0(\varphi, \rho_2)\| \le K_0\ell\|\rho_1 - \rho_2\|, \tag{24}$$

for all ρ_1, ρ_2 such that a solution $w(\varphi) = (\rho(\varphi), z(\varphi))$ of (18) with $w(\varphi_0) = (\rho_0, \Phi_0(\varphi_0, \rho_0)), \rho_0 \ge 0$, is defined on $\mathbb{R}$ and satisfies

$$\|w(\varphi)\| \le M_0\rho_0 e^{-\sigma_0(\varphi-\varphi_0)}, \quad \varphi \ge \varphi_0. \tag{25}$$

Similarly, it was shown that there exist constants K_-, M_-, σ_- such that Φ_- satisfies:

$$\Phi_-(\varphi, 0) = 0, \tag{26}$$

$$\|\Phi_-(\varphi, z_1) - \Phi_-(\varphi, z_2)\| \le K_-\ell\|z_1 - z_2\|, \tag{27}$$

for all z_1, z_2 such that a solution $w(\varphi) = (\rho(\varphi), z(\varphi))$ of (18) with $w(\varphi_0) = (\Phi_-(\varphi_0, z_0), z_0), z_0 \in \mathbb{R}$, is defined on $\mathbb{R}$ and satisfies

$$\|w(\varphi)\| \le M_-\|z_0\|e^{-\sigma_-(\varphi-\varphi_0)}, \quad \varphi \le \varphi_0. \tag{28}$$

Set $S_0 = \{(\rho, \varphi, z) : z = \Phi_0(\varphi, \rho)\}$ and $S_- = \{(\rho, \varphi, z) : \rho = \Phi_-(\varphi, z)\}$. Here, S_0 is called the *center manifold* and S_- is called the *stable manifold*. A sketch of an arbitrary center manifold is shown in Fig. 2.

The analogues of the following two Lemma's together with their proofs can be found in [12].

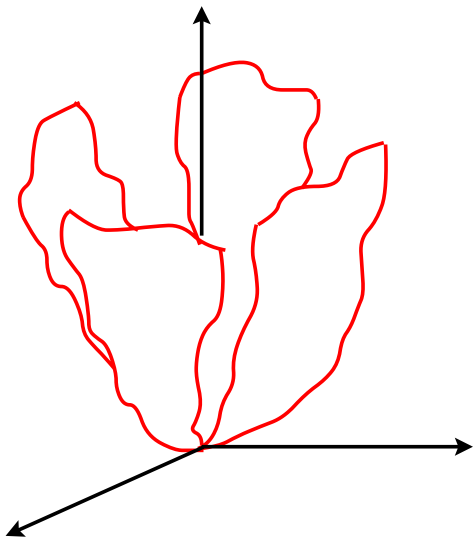

Fig. 2. The center manifold.

Lemma 4.1. *If the Lipschitz constant ℓ is sufficiently small, then for every solution $w(\varphi) = (\rho(\varphi), z(\varphi))$ of (18) there exists a solution $\mu(\varphi) = (u(\varphi), v(\varphi))$ on the center manifold, S_0, such that*

$$\begin{aligned} &\|\rho(\varphi) - u(\varphi)\| \leq 2M_0\|\rho(\varphi_0) - u(\varphi_0)\|\mathrm{e}^{-\sigma_0(\varphi-\varphi_0)}, \\ &\|z(\varphi) - v(\varphi)\| \leq M_0\|z(\varphi_0) - v(\varphi_0)\|\mathrm{e}^{-\sigma_0(\varphi-\varphi_0)}, \quad \varphi \geq \varphi_0, \end{aligned} \tag{29}$$

where M_0 and σ_0 are the constants used in (25).

Lemma 4.2. *For sufficiently small Lipschitz constant ℓ the surface S_0 is stable in large.*

On the local center manifold S_0, the first coordinate of the solutions of (18) satisfies the following system:

$$\begin{aligned} &\frac{\mathrm{d}\rho}{\mathrm{d}\varphi} = \lambda\rho + F(\rho, \varphi, \Phi_0(\varphi, \rho)), \quad \varphi \neq \varphi_i, \\ &\Delta\rho\,|_{\varphi=\varphi_i} = k_i\rho + W_i^1(\rho, \Phi_0(\varphi, \rho)). \end{aligned} \tag{30}$$

Now, it is time to consider the reduction principle:

Theorem 4.1. *Assume that conditions* (C1)–(C10) *are fulfilled. Then the trivial solution of (18) is stable, asymptotically stable or unstable if the trivial solution of (30) is stable, asymptotically stable or unstable, respectively.*

Using inverse of ψ-substitution and B-equivalence, one can see that the following theorem holds:

Theorem 4.2. *Assume that conditions* (C1)–(C10) *are fulfilled. Then the trivial solution of (9) is stable, asymptotically stable or unstable if the trivial solution of (30) is stable, asymptotically stable or unstable, respectively.*

5. Bifurcation of periodic solutions

This section is devoted to the bifurcation of a periodic solution for the discontinuous dynamical system. Let us consider the system,

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + f(x, z) + \mu\tilde{f}(x, z, \mu), \\ &\frac{\mathrm{d}z}{\mathrm{d}t} = \hat{b}z + g(x, z) + \mu\tilde{g}(x, z, \mu), \quad (x, z) \notin \Gamma(\mu), \\ &\Delta x\,|_{(x,z)\in\Gamma(\mu)} = B(x, \mu)x, \\ &\Delta z\,|_{(x,z)\in\Gamma(\mu)} = c(z, \mu)z. \end{aligned} \tag{31}$$

Assume that the following conditions are satisfied:

(A1) the set $\Gamma(\mu) = \bigcup_{i=1}^{p} \mathcal{S}_i(\mu)$, where $\mathcal{S}_i(\mu) = s_i(\mu) \times \mathbb{R}$ and the equation of $s_i(\mu)$ is given by $s_i(\mu) : \langle a^i, x\rangle + \tau_i(x) + \mu\nu(x, \mu) = 0$, for $i = 1, \ldots, p$;

(A2) there exists a matrix $Q(\mu) \in \mathbb{R}^{2\times 2}, Q(0) = Q$, analytic in $(-\mu_0, \mu_0)$, and real numbers γ, χ such that $Q^{-1}(\mu)B(x, \mu)Q(\mu) =$

$$(k + \mu\gamma + \kappa(x)) \begin{bmatrix} \cos(\theta + \mu\chi + \Theta(x)) & -\sin(\theta + \mu\chi + \Theta(x)) \\ \sin(\theta + \mu\chi + \Theta(x)) & \cos(\theta + \mu\chi + \Theta(x)) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $c(z, \mu) = c_0 + \tilde{c}(z) + \mu d(z, \mu)$;

(A3) associated with (31) systems

$$\begin{aligned} \frac{dx}{dt} &= Ax, \\ \frac{dz}{dt} &= \hat{b}z, \quad (x, z) \notin \Gamma_0, \\ \Delta x\,|_{(x,z)\in\Gamma_0} &= B_0 x, \\ \Delta z\,|_{(x,z)\in\Gamma_0} &= c_0 z \end{aligned} \tag{32}$$

and

$$\begin{aligned} \frac{dx}{dt} &= Ax + f(x, z), \\ \frac{dz}{dt} &= \hat{b}z + g(x, z), \quad (x, z) \notin \Gamma(0), \\ \Delta x\,|_{(x,z)\in\Gamma(0)} &= B(x, 0)x, \\ \Delta z\,|_{(x,z)\in\Gamma(0)} &= c(z, 0)z \end{aligned} \tag{33}$$

are D_0-*system* and *D-system* respectively;

(A4) functions $\tilde{f}$ and ν are analytic in their all arguments;

(A5) $\tilde{f}(0, 0, \mu) = 0$, $\nu(0, \mu) = 0$, uniformly for $\mu \in (-\mu_0, \mu_0)$.

We, first of all, linearize system (31) around origin. Note that the eigenvalues of the linearized system are continuously depend on μ, and hence for sufficiently small values of μ, the eigenvalues of the coefficient matrix in the linearized system will be in a similar form to the eigenvalues of the coefficient matrix in (1). Thus, by means of a regular transformation, one can show that the right-hand side of (31) is like the right-hand side of (9) with the only difference that all coefficients depend on μ. This is why, without loss of any generality, we assume that (31) is in linearized form.

Using polar coordinates one can write system (31) in the following form:

$$\begin{aligned} \frac{dr}{d\phi} &= \lambda(\mu)r + P(r, \phi, z, \mu), \\ \frac{dz}{d\phi} &= b(\mu)z + Q(r, \phi, z, \mu), \quad (r, \phi, z) \notin \Gamma(\mu), \\ \Delta r\,|_{(r,\phi)\in\ell_i(\mu)} &= k_i(\mu)r + \omega(r, \phi, \mu), \\ \Delta\phi\,|_{(r,\phi)\in\ell_i(\mu)} &= \theta_i(\mu) + \gamma(r, \phi, \mu), \\ \Delta z\,|_{(r,\phi)\in\ell_i(\mu)} &= c_0(\mu)z + \tilde{c}(z, \mu). \end{aligned} \tag{34}$$

Let the system

$$\begin{aligned} \frac{d\rho}{d\phi} &= \lambda(\mu)\rho + P(\rho, \phi, z, \mu), \\ \frac{dz}{d\phi} &= b(\mu)z + Q(\rho, \phi, z, \mu), \quad \phi \neq \gamma_i(\mu), \\ \Delta\rho\,|_{\phi=\gamma_i(\mu)} &= k_i(\mu)\rho + W_i^1(\rho, z, \mu), \\ \Delta\phi\,|_{\phi=\gamma_i(\mu)} &= \theta_i(\mu), \\ \Delta z\,|_{\phi=\gamma_i(\mu)} &= c_0(\mu)z + W_i^2(\rho, z, \mu), \end{aligned} \tag{35}$$

where $\gamma_i(\mu), i = 1, \ldots, p$, are angles of $m_i(\mu)$, be *B*-equivalent to (34). Here, for each *i*, the line $m_i(\mu)$ is obtained by linearizing $s_i(\mu)$ around the origin. That is, we have $m_i(\mu) : \langle a^i, x\rangle + \mu\frac{\partial\nu(0,\mu)}{\partial x} = 0$. The functions $W_i^1(\rho, z, \mu)$ and

$W_i^2(\rho, z, \mu)$ can be defined in the same manner as in (15) and (16), respectively. Applying ψ-substitution to (35) we get,

$$\begin{aligned} \frac{d\rho}{d\varphi} &= \lambda(\mu)\rho + F(\rho, \varphi, z, \mu), \\ \frac{dz}{d\varphi} &= b(\mu)z + G(\rho, \varphi, z, \mu), \quad \varphi \neq \varphi_i(\mu), \\ \Delta\rho\,|_{\varphi=\varphi_i(\mu)} &= k_i(\mu)\rho + W_i^1(\rho, z, \mu), \\ \Delta z\,|_{\varphi=\varphi_i(\mu)} &= c_0(\mu)z + W_i^2(\rho, z, \mu). \end{aligned} \tag{36}$$

Following the methods, as we did to obtain (21) and (22) one can see that system (36) has two integral manifolds whose equations are given by:

$$\begin{aligned} \Phi_0(\varphi, \rho, \mu) = & \int_{-\infty}^{\varphi} \pi_0(\varphi, s, \mu) G(\rho(s, \varphi, \rho, \mu), s, z(s, \varphi, \rho, \mu), \mu) ds \\ & + \sum_{\varphi_i(\mu)<\varphi} \pi_0(\varphi, \varphi_i^+, \mu) W_i^2(\rho(\varphi_i^+, \varphi, \rho, \mu), z(\varphi_i^+, \varphi, \rho, \mu)), \end{aligned} \tag{37}$$

and

$$\begin{aligned} \Phi_-(\varphi, z, \mu) = & -\int_{\varphi}^{\infty} \pi_-(\varphi, s, \mu) F(\rho(s, \varphi, z, \mu), s, z(s, \varphi, z, \mu), \mu) ds \\ & + \sum_{\varphi_i(\mu)<\varphi} \pi_-(\varphi, \varphi_i^+, \mu) W_i^1(\rho(\varphi_i^+, \varphi, z, \mu), z(\varphi_i^+, \varphi, z, \mu)), \end{aligned} \tag{38}$$

where

$$\pi_0(\varphi, s, \mu) = e^{b(\varphi-s)} \prod_{s\leq\varphi_j(\mu)<\varphi} (1 + c_0(\mu)),$$

and

$$\pi_-(\varphi, s, \mu) = e^{\lambda(\varphi-s)} \prod_{s\leq\varphi_j(\mu)<\varphi} (1 + k_j(\mu)).$$

In (37), the pair $(\rho(s, \varphi, \rho, \mu), z(s, \varphi, \rho, \mu))$ denotes a solution of (36) satisfying $\rho(\varphi, \varphi, \rho, \mu) = \rho$. Similarly, $(\rho(s, \varphi, z, \mu), z(s, \varphi, z, \mu))$, in (38), is a solution of (36) with $z(\varphi, \varphi, z, \mu) = z$.

Set $S_0(\mu) = \{(\rho, \varphi, z) : z = \Phi_0(\varphi, \rho, \mu)\}$ and $S_-(\mu) = \{(\rho, \varphi, z) : \rho = \Phi_-(\varphi, z, \mu)\}$.

On the local center manifold, $S_0(\mu)$, the first coordinate of the solutions of (36) satisfies the following system:

$$\begin{aligned} \frac{d\rho}{d\varphi} &= \lambda(\mu)\rho + F(\rho, \varphi, \Phi_0(\varphi, \rho, \mu)), \quad \varphi \neq \varphi_i(\mu), \\ \Delta\rho\,|_{\varphi=\varphi_i(\mu)} &= k_i(\mu)\rho + W_i^1(\rho, \Phi_0(\varphi, \rho, \mu)). \end{aligned} \tag{39}$$

Similar to (7) and (8) one can define the functions

$$q_1(\mu) = \exp\left(\lambda(\mu)\left(2\pi - \sum_{i=1}^{p} \theta_i(\mu)\right)\right) \prod_{i=1}^{p} (1 + k_i(\mu)), \tag{40}$$

and

$$q_2(\mu) = \exp\left(b(\mu)\left(2\pi - \sum_{i=1}^{p} \theta_i(\mu)\right)\right) \prod_{i=1}^{p} (1 + c_0(\mu)). \tag{41}$$

System (39) is the system studied in [11] and there it was shown that this system, for sufficiently small μ, has a periodic solution with period $T = \psi(2\pi)$. Here we will show that if the first coordinate of a solution of (36) is T-periodic, then so is the second coordinate.

Now, since

$$\begin{aligned} &\pi_0(\varphi + T, s + T, \mu) = \pi_0(\varphi, s, \mu), \\ &\rho(s + T, \varphi + T, \rho, \mu) = \rho(s, \varphi, \rho, \mu), \\ &z(s + T, \varphi + T, \rho, \mu) = z(s, \varphi, \rho, \mu), \end{aligned}$$

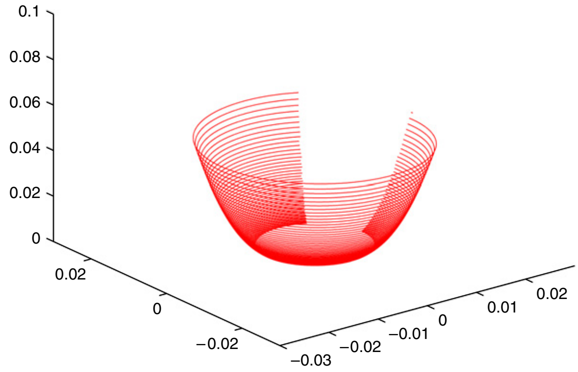

Fig. 3. A trajectory of (42).

and G is T-periodic in φ, we have,

$$\begin{aligned}\Phi_0(\varphi+T,\rho,\mu) &= \int_{-\infty}^{\varphi+T}\pi_0(\varphi+T,s,\mu)G(\rho(s,\varphi+T,\rho,\mu),s,z(s,\varphi+T,\rho,\mu),\mu)\mathrm{d}s\\ &\quad+\sum_{\varphi_i(\mu)<\varphi+T}\pi_0(\varphi+T,\varphi_i^+,\mu)W_i^2(\rho(\varphi_i^+,\varphi+T,\rho,\mu),z(\varphi_i^+,\varphi+T,\rho,\mu))\\ &= \int_{-\infty}^{\varphi}\pi_+(\varphi,t,\mu)G(\rho(t,\varphi,\rho,\mu),t,z(t,\varphi,\rho,\mu),\mu)\mathrm{d}t\\ &\quad+\sum_{\bar\varphi_i(\mu)<\varphi}\pi_+(\varphi,\bar\varphi_i{}^+,\mu)W_i^2(\rho(\bar\varphi_i{}^+,\varphi,\rho,\mu),z(\bar\varphi_i{}^+,\varphi,\rho,\mu))\\ &= \Phi_0(\varphi,\rho,\mu),\end{aligned}$$

where in the second equation we have used the substitutions $s=t+T$ and $\varphi_i(\mu)=\bar\varphi_i(\mu)+T$. Now, we have the following theorem which, in case of two dimension, can be found in [11].

Theorem 5.1. *Assume that $q_1(0)=1, q_1'(0)\neq 0, |q_2(0)|<1$, and the origin is a focus for (33). Then, for sufficiently small r_0 and z_0, there exists a function $\mu=\delta(r_0,z_0)$ such that the solution $(r(\phi,\delta(r_0,z_0)),z(\phi,\delta(r_0,z_0)))$ of (34), with the initial condition $r(0,\delta(r_0,z_0))=r_0, z(0,\delta(r_0,z_0))=z_0$, is periodic with a period, $T'=\left(2\pi-\sum_{i=1}^{p}\theta_i\right)\beta^{-1}+o(|\mu|)$.*

6. Examples

Example 6.1. Consider the following dynamical system:

$$\begin{aligned}x_1' &= (0.1-\mu)x_1-20x_2+2x_1x_2,\\ x_2' &= 20x_1+(0.1-\mu)x_2+3x_1^2z,\\ z' &= (-0.3+\mu)z+\mu^2x_1z,\quad (x_1,x_2,z)\notin\mathcal{S},\\ \Delta x_1|_{(x_1,x_2,z)\in\mathcal{S}} &= \left((\kappa_1+\mu^3)\cos\left(\frac{\pi}{3}\right)-1\right)x_1-(\kappa_1+\mu^3)\sin\left(\frac{\pi}{3}\right)x_2,\\ \Delta x_2|_{(x_1,x_2,z)\in\mathcal{S}} &= (\kappa_1+\mu^3)\sin\left(\frac{\pi}{3}\right)x_1+\left((\kappa_1+\mu^3)\cos\left(\frac{\pi}{3}\right)-1\right)x_2,\\ \Delta z|_{(x_1,x_2,z)\in\mathcal{S}} &= (\kappa_2+\mu-1)z,\end{aligned}\tag{42}$$

where $\kappa_1=\exp(-\frac{\pi}{120})$, $\kappa_2=\exp(-\frac{\pi}{400})$, $\mathcal{S}=s\times\mathbb{R}$, the curve s is given by the equation $x_2=x_1^2+\mu x_1^3$, $x_1>0$. Using (40) and (41), one can define

$$q_1(\mu)=(\kappa_1+\mu^3)\exp\left((0.1-\mu)\frac{5\pi}{60}\right),$$

and

$$q_2(\mu)=(\kappa_2+\mu)\exp\left((-0.3+\mu)\frac{5\pi}{60}\right).$$

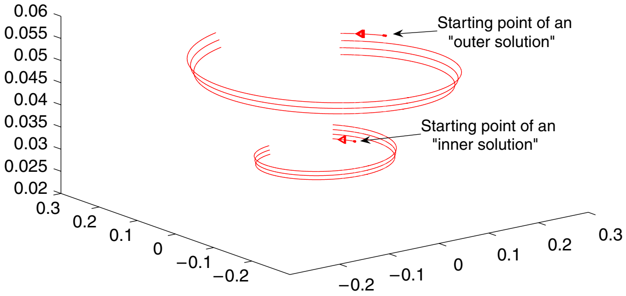

Fig. 4. There must exist a discontinuous limit cycle of (42).

It is easily seen that $q_1(0) = \kappa_1 \exp(\frac{\pi}{120}) = 1, q_1'(0) = -\frac{\pi}{12} \neq 0, q_2(0) = \exp(-\frac{11\pi}{200}) < 1$. Therefore, by Theorem 5.1, system (42) has a periodic solution with period $\approx \frac{5\pi}{60}$ if $|\mu|$ is sufficiently small.

Fig. 3 shows the trajectory of (42) with the parameter $\mu = 0.05$ and the initial value $(x_{10}, x_{20}, z_0) = (0.02, 0, 0.05)$. Since there is an asymptotically stable center manifold, no matter which initial condition is taken, the trajectory will get closer and closer to the center manifold as time increases.

In Fig. 4, the existence of a discontinuous limit cycle is illustrated. There an outer and an inner solution are shown which spiral to a trajectory lying between these two. Since the exact value of the initial point for the periodic solution is not known we have shown two trajectories of (42).

Example 6.2. Consider the following dynamical system:

$$\begin{aligned} x_1' &= (-2+\mu)x_1 - x_2 + \mu z^2, \\ x_2' &= x_1 + (-2+\mu)x_2, \\ z' &= (-1+\mu)z + \mu^2 x_1 z, \quad (x_1,x_2,z) \notin \mathcal{S}, \\ \Delta x_1 |_{(x_1,x_2,z)\in\mathcal{S}} &= \left((\kappa_1 - x_1^2 - x_2^2)\cos\left(\frac{\pi}{3}\right) - 1\right)x_1 - (\kappa_1 - x_1^2 - x_2^2)\sin\left(\frac{\pi}{3}\right)x_2, \\ \Delta x_2 |_{(x_1,x_2,z)\in\mathcal{S}} &= (\kappa_1 - x_1^2 - x_2^2)\sin\left(\frac{\pi}{3}\right)x_1 + \left((\kappa_1 - x_1^2 - x_2^2)\cos\left(\frac{\pi}{3}\right) - 1\right)x_2, \\ \Delta z |_{(x_1,x_2,z)\in\mathcal{S}} &= (\kappa_2 - 1 - z^2)z, \end{aligned} \tag{43}$$

where $\kappa_1 = \exp(\frac{10\pi}{3})$, $\kappa_2 = \exp(\frac{5\pi}{6})$, $\mathcal{S} = s \times \mathbb{R}$, s is a curve given by the equation $x_2 = x_1 + \mu^2 x_1^3, x_1 > 0$. Using (40) and (41), one can define

$$q_1(\mu) = \kappa_1 \exp\left((-2+\mu)\frac{5\pi}{3}\right),$$

and

$$q_2(\mu) = \kappa_2 \exp\left((-1+\mu)\frac{5\pi}{3}\right).$$

Now, $q_1(0) = \kappa_1 \exp(-\frac{10\pi}{3}) = 1, q_1'(0) = \frac{5\pi}{3} \neq 0, q_2(0) = \kappa_2 \exp(\frac{5\pi}{3}) = \exp(-\frac{5\pi}{6})$. Moreover, associated D-system is:

$$\begin{aligned} x_1' &= -2x_1 - x_2, \\ x_2' &= x_1 - 2x_2, \\ z' &= -z, \quad (x_1,x_2,z) \notin \mathcal{P}, \\ \Delta x_1 |_{(x_1,x_2,z)\in\mathcal{P}} &= \left((\kappa_1 - x_1^2 - x_2^2)\cos\left(\frac{\pi}{3}\right) - 1\right)x_1 - (\kappa_1 - x_1^2 - x_2^2)\sin\left(\frac{\pi}{3}\right)x_2, \\ \Delta x_2 |_{(x_1,x_2,z)\in\mathcal{P}} &= (\kappa_1 - x_1^2 - x_2^2)\sin\left(\frac{\pi}{3}\right)x_1 + \left((\kappa_1 - x_1^2 - x_2^2)\cos\left(\frac{\pi}{3}\right) - 1\right)x_2, \\ \Delta z |_{(x_1,x_2,z)\in\mathcal{P}} &= (\kappa_2 - 1 - z^2)z, \end{aligned} \tag{44}$$

where $\mathcal{P} = \ell \times \mathbb{R}$, ℓ is given by the equation $x_2 = x_1, x_1 > 0$, and the origin is stable focus. Indeed, using cylindrical coordinates, denote the solution of (44) starting at the angle $\phi = \frac{\pi}{4}$ by $(r(\phi, r_0, z_0), z(\phi, r_0, z_0))$.

We obtain

$$r_n = (\kappa_1 - r_{n-1}^2)r_{n-1} \exp\left(-\frac{10\pi}{3}\right),$$

and

$$z_n = (\kappa_2 - z_{n-1}^2)z_{n-1} \exp\left(-\frac{5\pi}{3}\right),$$

where $r_n = r(\frac{\pi}{4} + 2\pi n, r_0, z_0)$ and $z_n = z(\frac{\pi}{4} + 2\pi n, r_0, z_0)$. It is easily seen that the sequences r_n and z_n are monotonically decreasing for sufficiently small (r_0, z_0), and there exists a limit of (r_n, z_n). Assume that this limit is $(\xi, \eta) \neq (0, 0)$. Then it implies that there exists a periodic solution of (44) and $\xi = (\kappa_1 - \xi^2)\xi \exp(-\frac{10\pi}{3})$ and $\eta = (\kappa_2 - \eta^2)\eta \exp(-\frac{5\pi}{3})$ which give us a contradiction. Thus, $(\xi, \eta) = (0, 0)$. Consequently, the origin is a stable focus of (44) and by Theorem 5.1 the system (43) has a limit cycle with period $\approx \frac{5\pi}{3}$ if $|\mu|$ is sufficiently small.

7. Conclusion

In this paper, we have studied the existence of a center manifold and the Hopf bifurcation for a certain three-dimensional discontinuous dynamical system. The bifurcation of discontinuous cycle is observed by means of the B-equivalence method and its consequences. These results will be extended to arbitrary dimension for a more general type of equation.

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