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Differential equations on variable time scales

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Abstract

We introduce a class of differential equations on *variable* time scales with a transition condition between two consecutive parts of the scale. Conditions for existence and uniqueness of solutions are obtained. Periodicity, boundedness and stability of solutions are considered. The method of investigation is by means of two successive reductions: B-equivalence of the system [E. Akalín, M.U. Akhmet, The principles of B-smooth discontinuous flows, Computers and Mathematics with Applications 49 (2005) 981–995; M.U. Akhmet, Perturbations and Hopf bifurcation of the planar discontinuous dynamical system, Nonlinear Analysis 60 (2005) 163–178; M.U. Akhmet, N.A. Perestyuk, The comparison method for differential equations with impulse action, Differential Equations 26 (9) (1990) 1079–1086] on a variable time scale to a system on a time scale, a reduction to an impulsive differential equation [M.U. Akhmet, Perturbations and Hopf bifurcation of the planar discontinuous dynamical system, Nonlinear Analysis 60 (2005) 163–178; M.U. Akhmet, M. Turan, The differential equations on time scales through impulsive differential equations, Nonlinear Analysis 65 (2006) 2043–2060]. Appropriate examples are constructed to illustrate the theory.

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1. Introduction

In the last several decades, the theory of dynamic equations on time scales (DETS) has been developed very intensively. For the full description of the equations we refer to the nicely written books [7,16] and papers [17,20]. The equations have a very special transition condition for adjoint elements of time scales. To enlarge the field of applications of the DETS, and to have more theoretical opportunities we, in [5], proposed to generalize the transition operator, correspondingly to investigate *differential equations on time scales with the transition condition* (DETC).

In our recent investigations [2] it was found that the idea of the equations can be extended, if one: (1) involves in the discussion of certain union of separated sets in the (t, x) space such that intersection of each line $x =$ constant with the union is a time scale in the sense of Hilger (we call these separated sets altogether as the variable time scale); (2) introduces the differential equations, the domain of which are variable time scales. We call the systems as

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differential equations on variable time scales with transition condition (DETCV). The present paper is devoted to the development of methods to study these systems, and some theoretical results are obtained. To give an outline of the way of the study, we can shortly say that two consequent reductions are in the base: (a) reduction of DETCV to DETC, using B-equivalence method [1,3]; (b) the method of ψ- substitution [2,5] to reduce DETC to impulsive differential equations.

The paper is organized as follows. In this section, definitions related with non-variable time scales are considered. Section 2 has detailed description of variable time scales with examples. Section 3 describes the differential equations on variable time scales. The existence and uniqueness of solutions and B-equivalence and B-stability are considered in Sections 4 and 5. The description of the reduction process is given in Section 6. In the last two sections, we apply the procedure to investigate periodic solutions and stability of an equilibrium position.

We start with the following definition [16]. Any nonempty closed subset, $\mathbb{T}$, of $\mathbb{R}$ is called a *time scale*. For instance, $\mathbb{R}$ (real numbers), $\mathbb{Z}$ (integers), $\mathbb{N}$ (natural numbers) and $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ are examples of time scales while $\mathbb{Q}$ (rational numbers), $\mathbb{R} \setminus \mathbb{Q}$ (irrational numbers) and (0, 1) are not time scales [7].

On a time scale the functions $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ are called the forward and backward jump operators, respectively. The point $t \in \mathbb{T}$ is called right-scattered if $\sigma(t) > t$, and right-dense if $\sigma(t) = t$. Similarly, it is called left-scattered if $\rho(t) < t$, and left-dense if $\rho(t) = t$.

Let $\mathbb{T}$ be an arbitrary time scale. A function $\varphi : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if :

(i) it is continuous at each right-dense or maximal $t \in \mathbb{T}$;
(ii) the left-sided limit $\varphi(t-) = \lim_{\xi \to t^-} \varphi(\xi)$ exists at each left-dense t.

Similarly, a function $\varphi : \mathbb{T} \to \mathbb{R}$ is called ld-continuous if :

(i) it is continuous at each left-dense or minimal $t \in \mathbb{T}$;
(ii) the right-sided limit $\varphi(t+) = \lim_{\xi \to t^+} \varphi(\xi)$ exists at each right-dense t.

The Δ-derivative of a continuous function f, at a right-scattered point, is defined as

$$f^{\Delta}(t) := \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

and at a right-dense point, it is defined as

$$f^{\Delta}(t) := \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

if the limit exists.

A differential equation

$$y^{\Delta}(t) = f(t, y), \quad t \in \mathbb{T} \tag{1.1}$$

is said to be a differential equation on time scale, where the function $f(t, y) : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ in (1.1) is assumed to be rd-continuous for each $y \in \mathbb{R}^n$.

2. Description of a variable time scale

In this section, we give at first a general definition of a variable time scale, and next we describe a specific variable time scale, which is used to introduce DETCV.

Definition 1. A nonempty closed set $\mathbb{T}(x)$ in $\mathbb{R} \times \mathbb{R}^n$ is said to be a variable time scale if for any $x_0 \in \mathbb{R}^n$ the projection of $\mathbb{T}(x_0)$ on time axis, that is the set $\{t \in \mathbb{R} : (t, x_0) \in \mathbb{T}(x_0)\}$, is a time scale in Hilger's sense.

To illustrate the last definition let us consider the following example.

Example 2.1. Let $\{r_i\}_{i=1}^{\infty}$ be a positive increasing sequence of real numbers such that $\lim_{i\to\infty} r_i = \infty$ and

$$\mathcal{D}_i = \{(t, x) \in \mathbb{R} \times \mathbb{R} : r_{2i-1}^2 \le t^2 + x^2 \le r_{2i}^2\}.$$

Then, we define the variable time scale as $\mathbb{T}(x) = \bigcup_{i=1}^{\infty} \mathcal{D}_i$ (see Fig. 1).

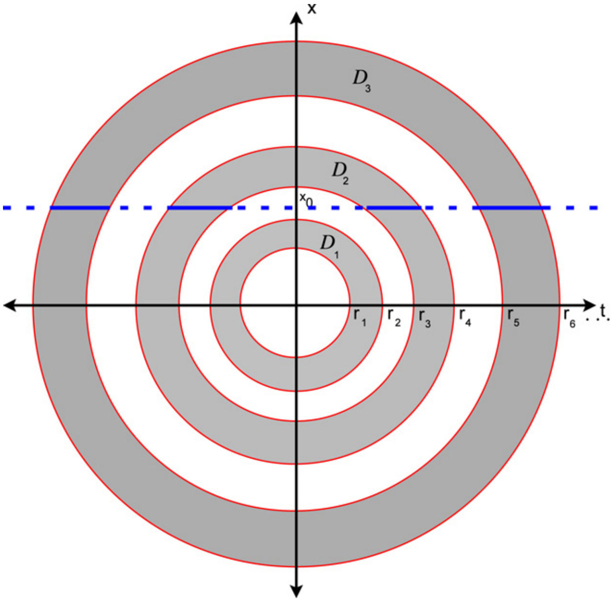

Fig. 1. An example of a variable time scale.

For a fixed $x_0 \in \mathbb{R}$, there exists a smallest k such that $r_{2k} \geq |x_0|$. Thus, we have

$$\mathbb{T}(x_0) = \bigcup_{i=k}^{\infty} \{(t, x_0) : t \in \mathbb{R}, r_{2i-1}^2 \leq t^2 + x_0^2 \leq r_{2i}^2\}.$$

The projection of $\mathbb{T}(x_0)$ on time axis is

$$\mathbb{T}_c = \bigcup_{i=k}^{\infty} \left(\left[-\sqrt{r_{2i}^2 - x_0^2}, -\sqrt{r_{2i-1}^2 - x_0^2} \right] \cup \left[\sqrt{r_{2i-1}^2 - x_0^2}, \sqrt{r_{2i}^2 - x_0^2} \right] \right),$$

which is a time scale in Hilger's sense. □

The following variable time scale may be considered as another example, too, but, a more important thing is that it is an essential element in the definition of differential equations with transition condition on a variable time scale, discussed in our paper.

Fix a sequence $\{t_i\} \in \mathbb{R}$ such that $t_i < t_{i+1}$ for all $i \in \mathbb{Z}$, and $|t_i| \to \infty$ as $|i| \to \infty$. Denote $\delta_i = t_{2i+1} - t_{2i}, \kappa_i = t_{2i} - t_{2i-1}$ and take a sequence of functions $\{\tau_i(x)\} \in C(\mathbb{R}^n, \mathbb{R})$. Assume that:

(C1) for some positive numbers $\theta', \theta \in \mathbb{R}, \theta' \leq t_{i+1} - t_i \leq \theta$;
(C2) there exists $\ell_0, 0 < 2\ell_0 < \theta'$, such that $\|\tau_i(x)\| \leq \ell_0$ for all $x \in \mathbb{R}^n, i \in \mathbb{Z}$.

Denote

$$l_i := \inf_{x \in \mathbb{R}^n} \{t_i + \tau_i(x)\}, \qquad r_i := \sup_{x \in \mathbb{R}^n} \{t_i + \tau_i(x)\}. \tag{2.2}$$

From (C1) and (C2) it follows that there exist positive numbers θ_l and θ_r such that

(C1′) $\theta_l \leq l_{i+1} - r_i \leq \theta_r, i \in \mathbb{Z}$.

We set

$$\begin{aligned} \mathcal{E}_i &= \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t_{2i} + \tau_{2i}(x) < t < t_{2i+1} + \tau_{2i+1}(x)\}, \\ \mathcal{S}_i &= \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t = t_i + \tau_i(x)\}, \\ \mathcal{D}_i &= \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t_{2i-1} + \tau_{2i-1}(x) \leq t \leq t_{2i} + \tau_{2i}(x)\}. \end{aligned} \tag{2.3}$$

Due to (C1'), $\mathcal{D}_i$ is not empty and we introduce the following set

$$\mathbb{T}_0(x) := \bigcup_{i=-\infty}^{\infty} \mathcal{D}_i. \tag{2.4}$$

In [5], we considered a special time scale $\mathbb{T}_c = \bigcup_{i=-\infty}^{\infty}[t_{2i-1}, t_{2i}]$. Comparing this time scale with the set $\mathbb{T}_0(x)$, it seems reasonable to call the latter as *the variable time scale*, and in our paper we are going to use, for sets of type $\mathbb{T}_c$, the term *non-variable time scales*. In fact, our results could be applied to the case when $\mathbb{T}_c$ contains some isolated points.

For the convenience of the reader let us consider the following example.

Example 2.2. Let $t_i = \pi i, \tau_i(x) = \frac{\sin(\|x\|)}{\|x\|^2+|i|+1}$ where $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ is the Euclidean norm of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then, we have

$$l_i = \pi i - \frac{1}{\sqrt{(c_i^2 + |i| + 1)^2 + 4c_i^2}}, \qquad r_i = \pi i + \frac{1}{\sqrt{(c_i^2 + |i| + 1)^2 + 4c_i^2}}$$

where the number $c_i > 0$ is the smallest real number which satisfies the equation $\tan(c_i) = (c_i^2 + |i| + 1)/(2c_i)$. Thus, for $\theta_l = \frac{\pi}{2}$ and $\theta_r = \pi$, we see that (C1) is satisfied and

$$\mathcal{D}_i = \left\{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t_{2i-1} + \tau_{2i-1}(x) \le t \le t_{2i} + \tau_{2i}(x)\right\}.$$

Then, the variable time scale could be established by (2.4).

3. Differential equations on a variable time scale

In what follows, we introduce a special operator which plays an important role in describing the differential equations on variable time scales as well as methods for investigation of these equations through the reduction to impulsive differential equations.

Let us consider a transition operator $\Pi_i : \mathcal{S}_{2i} \to \mathcal{S}_{2i+1}$, for all $i \in \mathbb{Z}$, such that $\Pi_i(t, y) = (\Pi_i^1(t, y), \Pi_i^2(t, y))$ where $\Pi_i^1 : \mathcal{S}_{2i} \to \mathbb{R}$ and $\Pi_i^2 : \mathcal{S}_{2i} \to \mathbb{R}^n$, and

$$\Pi_i^1(t, y) = t_{2i+1} + \tau_{2i+1}(\Pi_i^2(t, y)) \quad \text{and} \quad \Pi_i^2(t, y) = I_i(y) + y, \tag{3.5}$$

where $I_i : \mathbb{R}^n \to \mathbb{R}^n$ is a function. One can easily see that $\Pi_i^1(t, y)$ is the time coordinate of $(t^+, y^+) := \Pi_i(t, y)$, the image of $(t, y) \in \mathcal{S}_{2i}$ under the operator Π_i, and $\Pi_i^2(t, y)$ is the space coordinate of the image.

The differential equation which we are going to deal with is:

$$\begin{aligned} &y' = F(t, y), \quad (t, y) \in \mathbb{T}_0(y), \\ &t^+ = \Pi_i^1(t, y), \qquad y^+ = \Pi_i^2(t, y), \quad (t, y) \in \mathcal{S}_{2i}, \end{aligned} \tag{3.6}$$

where the derivative at the boundary points of the variable time scale in (3.6) is one-sided derivative and $F(t, y) : \mathbb{T}_0(y) \to \mathbb{R}^n$ is assumed to be continuous on its domain.

We call the last equation *a differential equation on a variable time scale with transition condition* and abbreviate it as DETCV.

To describe the solutions of differential equations with transition conditions on a variable time scale carefully, we begin the definition with *the graph* of a solution of (3.6). Accordingly, we start with the following construction. Consider a piece-wise curve $\mathcal{C}$ such that:

(1) $\mathcal{C}$ lies in $\mathbb{T}_0(y)$;
(2) the part of $\mathcal{C}$ in each $\mathcal{D}_i$, $i \in \mathbb{Z}$, is a continuous arc;
(3) if $\mathcal{C}$ has points in $\mathcal{D}_j$ and $\mathcal{D}_{j+1}$ for a fixed $j \in \mathbb{Z}$, then $\mathcal{C}$ intersects each of the surfaces $\mathcal{S}_{2j}$ and $\mathcal{S}_{2j+1}$ exactly once;
(4) $\mathcal{C}$ intersects each hyperplane $t = \theta$, $\theta \in \mathbb{R}$, at most at one point.

The curve can be viewed as the graph of a piece-wise function $y = \varphi(t)$. Let $t = \alpha_i$ and $t = \beta_i$ be the moments that the graph of $y = \varphi(t)$ intersects the surfaces $\mathcal{S}_{2i-1}$ and $\mathcal{S}_{2i}$, respectively, where the surfaces are defined previously.

From (C1) and (C2) or (C1') it is easily seen that $\alpha_i < \beta_i$ for all $i \in \mathbb{Z}$. Then, we set the non-variable time scale

$$\mathbb{T}_c^{\varphi} := \bigcup_{i=-\infty}^{\infty} [\alpha_i, \beta_i],$$

which is the domain of φ, and define the Δ-derivative as in the introduction. That is, for $t = \beta_i$, we have

$$\varphi^{\Delta}(\beta_i) = \frac{\varphi(\alpha_{i+1}) - \varphi(\beta_i)}{\alpha_{i+1} - \beta_i},$$

and

$$\varphi^{\Delta}(t) = \lim_{s \to t} \frac{\varphi(s) - \varphi(t)}{s - t},$$

for any other $t \in \mathbb{T}_c^{\varphi}$, whenever the limit exists.

Thus, to define a DETCV, we need:

(1) the variable time scale $\mathbb{T}_0(y) = \bigcup_{i=-\infty}^{\infty} \mathcal{D}_i$;
(2) the system of differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}t} = F(t, y), \quad (t, y) \in \mathbb{T}_0(y); \tag{3.7}$$

(3) the transition operator $\Pi_i : \mathcal{S}_{2i} \to \mathcal{S}_{2i+1}, i \in \mathbb{Z}$.

The class of equations is important as it can be reduced from the discontinuous dynamics [2]. Particularly, they are needed to develop the center manifold theory of these equations, and, consequently, the Hopf bifurcation theory [6].

Setting $\Delta t := t^+ - t$ and $\Delta y := y^+ - y$ we can rewrite (3.6) as

$$\begin{aligned} y' &= F(t, y), \quad (t, y) \in \mathbb{T}_0(y), \\ \Delta t|_{(t,y)\in\mathcal{S}_{2i}} &= \Pi_i^1(t, y) - t, \\ \Delta y|_{(t,y)\in\mathcal{S}_{2i}} &= \Pi_i^2(t, y) - y. \end{aligned} \tag{3.8}$$

Let us show how to construct a solution of (3.6), or equivalently of (3.8). Denote by $\phi(t, \kappa, \eta)$ a solution of the initial value problem $y(\kappa) = \eta$ for system

$$\frac{\mathrm{d}y}{\mathrm{d}t} = F(t, y), \tag{3.9}$$

and $y = y(t, t^0, y_0)$ a solution of the initial value problem $y(t^0) = y_0$ for the system (3.6). Assume that (t^0, y_0) is an interior point of $\mathcal{D}_k$ for some $k \in \mathbb{Z}$. We construct the solution for increasing t. The process of definition of $y(t)$ goes as follows: starting from (t^0, y_0), the solution is equal to $y(t) = \phi(t, t^0, y_0)$ up to a point $(\beta_k, y(\beta_k))$, where β_k is the first from the left solution of the equation $\beta = t_{2k} + \tau_{2k}(y(\beta))$, that is the first meeting point of the solution $\phi(t, t^0, y_0)$ with the surface $\mathcal{S}_{2k}$, and $y_k := y(\beta_k) = \phi(\beta_k, t^0, y_0)$. Then, applying the transition operator Π_k, we obtain $(\beta_k^+, y_k^+) = \left(\Pi_k^1(\beta_k, y_k), \Pi_k^2(\beta_k, y_k)\right)$. Denote $\alpha_{k+1} = \Pi_k^1(\beta_k, y_k)$. After α_{k+1}, there is no meeting of the solution with $\mathcal{S}_{2k+1}$. (A sufficient condition which ensures this fact will be given later.) The solution is not defined on the time interval (β_k, α_{k+1}). Next, on $\mathcal{D}_{k+1}$ the solution is equal to $y(t) = \phi(t, \alpha_{k+1}, y_k^+)$ and so on (see Fig. 2).

The way of investigation of DETCV has not been considered yet, except for the short episode in [2]. So, in what follows, we consider a quasilinear system as it is convenient to develop the methods of reductions proposed in [2,3,5]. That is, we shall assume $F(t, y)$ and $I_i(y)$ in a special form: $F(t, y) = A(t)y + f(t, y)$, $I_i(y) = B_i y + J_i(y)$ where $A(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is an $n \times n$ continuous real valued matrix-function, B_i is an $n \times n$ matrix, functions $f(t, y) : \mathbb{T}_0(y) \to \mathbb{R}^n$ and $J_i(y) : \mathbb{R}^n \to \mathbb{R}^n$ are continuous. Thus, the system which we will consider is:

$$\begin{aligned} y' &= A(t)y + f(t, y), \quad (t, y) \in \mathbb{T}_0(y), \\ \Delta t|_{(t,y)\in\mathcal{S}_{2i}} &= \Pi_i^1(t, y) - t, \\ \Delta y|_{(t,y)\in\mathcal{S}_{2i}} &= \Pi_i^2(t, y) - y, \end{aligned} \tag{3.10}$$

where $\Pi_i^1(t, y) = t_{2i+1} + \tau_{2i+1}(\Pi_i^2(t, y))$ and $\Pi_i^2(t, y) = B_i y + J_i(y) + y$.

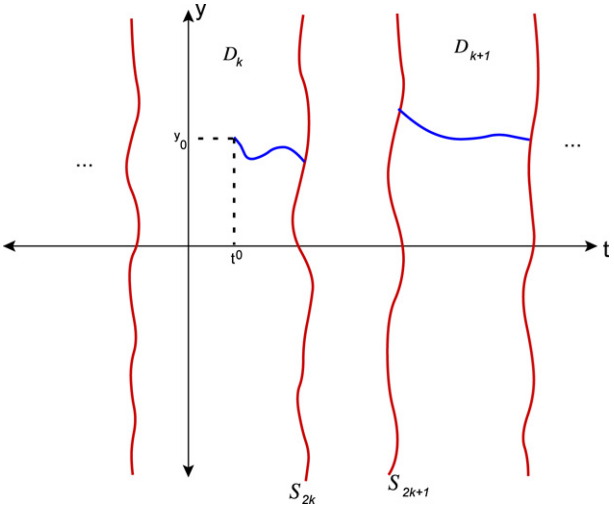

Fig. 2. A solution of differential equation on a variable time scale.

Example 3.1. The following planar system was considered in [2]:

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + f(x), \quad x \notin \Gamma,\\ &\Delta x|_{x\in\Gamma} = B(x)x, \end{aligned} \tag{3.11}$$

where $\Gamma = \cup_{i=1}^{p} l_i$ is a set of curves starting at the origin and which are defined by the equations $\langle a^i, x\rangle + \tau_i(x) = 0, i = \overline{1,p}$,

$$B(x) = (k + \kappa(x))Q \begin{pmatrix} \cos(\theta + \upsilon(x)) & -\sin(\theta + \upsilon(x)) \\ \sin(\theta + \upsilon(x)) & \cos(\theta + \upsilon(x)) \end{pmatrix} Q^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where the functions f, k, υ are smooth, $f(x) = o(\|x\|)$, $\kappa(x) = o(\|x\|)$, $\upsilon(x) = o(\|x\|)$, $\tau_i(x) = o(\|x\|^2)$, $i = \overline{1,p}$ and Q is some nonsingular matrix. By using polar coordinates, the system is written in the form:

$$\begin{aligned} &\frac{\mathrm{d}r}{\mathrm{d}\phi} = \lambda r + P(r,\phi), \quad (r,\phi) \notin \Gamma,\\ &\phi^{+}|_{(r,\phi)\in l_i} = \phi + \theta_i + \gamma(r,\phi),\\ &r^{+}|_{(r,\phi)\in l_i} = (1 + k_i)r + \omega(r,\phi), \end{aligned} \tag{3.12}$$

where Γ is presented as $l_i : \phi = \gamma_i + r\psi_i(r,\phi)$, $i = \overline{1,p}$.

Denote by ℓ_i' the image of ℓ_i under the transition operator $\Pi_i(\phi, r)$ where $\Pi_i^1(\phi, r) = \phi + \theta_i + \gamma(r,\phi)$, and $\Pi_i^2(\phi, r) = (1 + k_i)r + \omega(r,\phi)$, Let $\mathcal{D}_i$ be the set bounded by ℓ_i and ℓ_i'. In [2], it is shown that this set is nonempty and ℓ_i is wholly "above" ℓ_i' if the equation is considered in a small neighborhood of the origin.

Denoting $\mathbb{T}(r) = \bigcup_{i=1}^{p} \mathcal{D}_i$, we have that one deals with the following DETCV:

$$\begin{aligned} &\frac{\mathrm{d}r}{\mathrm{d}\phi} = \lambda r + P(r,\phi), \quad (\phi, r) \in \mathbb{T}(r),\\ &\phi^{+} = \Pi_i^1(\phi, r), \qquad r^{+} = \Pi_i^2(\phi, r), \quad (\phi, r) \in \ell_i. \quad \square \end{aligned} \tag{3.13}$$

Equations of the form (3.11) could be effectively applied as a model for the various mechanical processes with impacts [8,13,14,18,23]. That is why, the last example could be considered as a good **motivation** to investigate DETCV.

We are going to develop the theory starting with the present paper and discuss such problems as center manifold theorem, multidimensional Hopf bifurcation, and other types of bifurcations. We intend to investigate the problems using our approach to discontinuous dynamical systems [2].

Summarizing all the materials discussed above, we could say that there is a demand to develop the Hilger's differential equation on non-variable time scales to the differential equations on variable time scales of general type, as a particular case of DETCV. For this reason, let us specify the transition operator in the previous part, assuming $\Pi_k^2(\beta_k, y(\beta_k)) = F(\beta_k, y(\beta_k))(\alpha_{k+1} - \beta_k) + y(\beta_k)$, then (3.10) has a specified form

$$y^{\Delta} = F(t, y), \quad (t, y) \in \mathbb{T}_0(y). \tag{3.14}$$

The last system could be considered as *the differential equation on variable time scale* (DEVTS). We suppose that the theory of DEVTS should be developed as well as the theory of DETS has been [7]. One can wait that many interesting problems connected with topology of the variable time scale may appear. Some of these problems are going to be discussed in this paper.

4. Existence and uniqueness of solutions

Among some important properties of a differential equation, the problem of existence and uniqueness of solutions has great importance. In this section, we are going to investigate the problem for (3.10) for increasing t.

Remark 2. The continuation of the solution to the left cannot be considered yet, since the invertibility of the transition operator Π_i is not assumed.

Consider the following ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = A(t)y + f(t, y), \tag{4.15}$$

where the matrix $A(t)$ and the function $f(t, y)$ are the same as in (3.10). We will assume that the following Lipschitz condition holds uniformly with respect to $t \in \mathbb{R}$ and $i \in \mathbb{Z}$, for arbitrary $x, y \in \mathbb{R}^n$:

(C3) $\|\tau_i(x) - \tau_i(y)\| + \|J_i(x) - J_i(y)\| + \|f(t, x) - f(t, y)\| \le \ell\|x - y\|$.

Moreover, we assume that

(C4) $\sup_{t\in\mathbb{R}} \|f(t, 0)\| + \sup_{i\in\mathbb{Z}} \|J_i(0)\| = M < \infty$;
(C5) $\sup_{t\in\mathbb{R}} \|A(t)\| + \sup_{i\in\mathbb{Z}} \|B_i\| = N < \infty$;
(C6) $\tilde{M}\ell < 1$, where $\tilde{M} = \sup_{(t,y)\in\mathbb{T}_0(y)} \|A(t)y + f(t, y)\|$.

Then, we have the following theorem.

Theorem 3. *Assume that* (C1)–(C6) *hold and function f is continuous. Then for any* $(t^0, y_0) \in \mathbb{T}_0(y)$ *the system*

$$\begin{aligned} &y' = A(t)y + f(t, y), \quad (t, y) \in \mathbb{T}_0(y), \\ &t^+ = \Pi_i^1(t, y), \qquad y^+ = \Pi_i^2(t, y), \quad (t, y) \in \mathcal{S}_{2i}, \end{aligned} \tag{4.16}$$

with the initial condition $y(t^0) = y_0$, has a unique solution, $y(t, t^0, y_0)$, which can be continued to the right of t^0, to ∞.

Proof. For the following discussion, it is important that for each $(\gamma, y_\gamma) \in \mathbb{T}_0(y)$, if (γ, y_γ) is an interior point of $\mathcal{D}_i$, $i \in \mathbb{Z}$, then, because of (C3) and (C5), there exists a unique solution of the ordinary differential equation,

$$\begin{aligned} &y' = A(t)y + f(t, y), \\ &y(\gamma) = y_\gamma, \end{aligned} \tag{4.17}$$

which is continuable to $\mathcal{S}_{2i}$, the right boundary surface of $\mathcal{D}_i$ [9,12].

Assume that $(t^0, y_0) \in \mathcal{D}_k$ for some $k \in \mathbb{Z}$. On $\mathcal{D}_k$, we will consider (4.17) for $\gamma = t^0$, $y_\gamma = y_0$, which has the unique solution $y(t) = \phi(t, t^0, y_0)$ defined throughout $\mathcal{D}_k$. Let β_k be the first from left solution of $\beta = t_{2k} + \tau_{2k}(y(\beta))$. Then, by means of jump operators we obtain $\alpha_{k+1} := \Pi_k^1(\beta_k, y(\beta_k))$ and $y_k^+ := \Pi_k^2(\beta_k, y(\beta_k))$.

Next, on $\mathcal{D}_{k+1}$, we consider the ordinary differential equation (4.17) with the initial condition $y(\alpha_{k+1}) = y_k^+$, which has the unique solution $\phi(t, \alpha_{k+1}, y_k^+)$. Thus, the solution is not defined on the time interval (β_k, α_{k+1}).

Assume that the solution intersects the surface $\mathcal{S}_{2k+1}$ at any other point, say α^*_{k+1}, which is going to be a solution of the equation $\alpha^* = t_{2k+1} + \tau_{2k+1}(\phi(\alpha^*, \alpha_{k+1}, y_k^+))$. Clearly, we have $\alpha^*_{k+1} > \alpha_{k+1}$ and (C3) implies that

$$\left(\alpha^*_{k+1} - \alpha_{k+1}\right)\left(1 - \ell \sup_{t\in[\alpha_{k+1},\alpha^*_{k+1}]} \|A(t)\phi(t,\alpha_{k+1},y_k^+) + f(t,\phi(t,\alpha_{k+1},y_k^+))\|\right) \le 0$$

which yields a contradiction since $\tilde{M}\ell < 1$. Therefore, the solution does not have any other meeting point with the surface $\mathcal{S}_{2k+1}$. Hence, on $\mathcal{D}_{k+1}$, the unique solution is obtained as $\phi(t, \alpha_{k+1}, y_k^+)$. In this way, we can continue this solution to ∞. $\square$

5. *B*-equivalence and *B*-stability

A difficulty in investigating the system (3.10) is that the discontinuity moments of distinct solutions are not, in general, the same. To investigate the asymptotic properties of solutions of (3.10), we introduce the following concepts.

In what follows, we are going to adopt, for DETCV, the techniques of B-topology and B-equivalence which were introduced and developed in [2,3,11,15,21,22] for equations with impulses at variable moments of time.

Let $u(t) = y(t, t^0, y_0)$ be a solution of (3.10) such that the open neighborhood, $B((t^0, y_0), h)$, centered at (t^0, y_0) with sufficiently small radius $h > 0$ belongs to $\mathcal{D}_k$ for some $k \in \mathbb{Z}$. Let β_i^u be the moment when the solution $u(t)$ meets the surface $\mathcal{S}_{2i}$, and $\alpha_{i+1}^u = \Pi_i^1(\beta_i^u, y(\beta_i^u))$, for $i = k, k+1, \ldots$. We set the non-variable time scale

$$\mathbb{T}^u_{t^0} := [t^0, \beta_k^u] \cup \bigcup_{i=k+1}^{\infty} [\alpha_i^u, \beta_i^u].$$

Let $v(t) = y(t, t^1, y_1)$ be another solution of (3.10) with $(t^1, y_1) \in B((t^0, y_0), h)$ and let β_i^v be the moment when the solution $v(t)$ meets the surface $\mathcal{S}_{2i}$, and $\alpha_{i+1}^v = \Pi_i^1(\beta_i^v, y(\beta_i^v))$, for $i = k, k+1, \ldots$. We, similarly, define the non-variable time scale

$$\mathbb{T}^v_{t^1} := [t^1, \beta_k^v] \cup \bigcup_{i=k+1}^{\infty} [\alpha_i^v, \beta_i^v].$$

Define the distance between two non-variable time scales, $\mathbb{T}^u_{t^0}$ and $\mathbb{T}^v_{t^1}$, by

$$d(\mathbb{T}^u_{t^0}, \mathbb{T}^v_{t^1}) = \max\left\{\sup_{i\ge k+1} |\alpha_i^u - \alpha_i^v|, \sup_{i\ge k} |\beta_i^u - \beta_i^v|\right\}.$$

We say that two solutions u and v are in an ϵ-neighborhood of each other on $\mathbb{T}^u_{t^0}$ and $\mathbb{T}^v_{t^1}$ if:

(i) $d(\mathbb{T}^u_{t^0}, \mathbb{T}^v_{t^1}) < \epsilon$;
(ii) $|u(t) - v(t)| < \epsilon$ for all $t \in \mathbb{T}^u_{t^0} \cap \mathbb{T}^v_{t^1}$.

The topology defined by ϵ-neighborhoods of rd-continuous solutions will be called B-topology. It is easily seen that it is a Hausdorff topology. Topologies and metrics for spaces of discontinuous functions were introduced and developed in [2,3,15].

For any $\alpha, \beta \in \mathbb{R}$ we define the oriented interval $[\alpha, \hat{}\beta]$ as

$$[\alpha,\hat{}\beta] = \begin{cases}[\alpha,\beta], & \text{if } \alpha \le \beta\\ [\beta,\alpha], & \text{otherwise.}\end{cases} \tag{5.18}$$

Consider the non-variable time scale

$$\mathbb{T}^0_c = \bigcup_{i=-\infty}^{\infty} [l_{2i-1}, r_{2i}], \tag{5.19}$$

where l_i, r_i, $i \in \mathbb{Z}$, are as defined by (2.2) for the variable time scale $\mathbb{T}_0(y)$, and take a continuation $\tilde{f} : \mathbb{T}^0_c \times \mathbb{R}^n \to \mathbb{R}^n$ of $f : \mathbb{T}_0(y) \to \mathbb{R}^n$ which is Lipschitzian with the same Lipschitz constant ℓ.

Set $\mathbb{T}_c := \bigcup_{i=-\infty}^{\infty}[t_{2i-1}, t_{2i}]$. We start with proving the following lemma.

Lemma 5.1. *There are mappings $W_i(z) : \mathbb{R}^n \to \mathbb{R}^n$, $i \in \mathbb{Z}$, such that, corresponding to each solution $y(t)$ of (3.10), there is a solution $z(t)$ of the system*

$$\begin{aligned} z' &= A(t)z + \tilde{f}(t, z), \quad t \neq t_{2i}, \\ z(t_{2i+1}) &= B_i z(t_{2i}) + W_i(z(t_{2i})) + z(t_{2i}), \end{aligned} \tag{5.20}$$

such that $y(t) = z(t)$ for all $t \in \mathbb{T}_c$ except possibly on $[t_{2i-1}\widehat{,}\alpha_i]$ and $[\beta_i\widehat{,}t_{2i}]$ where α_i and β_i are the moments that $y(t)$ meets the surfaces $\mathcal{S}_{2i-1}$ and $\mathcal{S}_{2i}$, respectively.

The functions W_i satisfy the inequality

$$\|W_i(z) - W_i(y)\| \leq k(\ell)\ell\|z - y\|, \tag{5.21}$$

uniformly with respect to $i \in \mathbb{Z}$, for all $z, y \in \mathbb{R}^n$ such that $\|z\| \leq h$ and $\|y\| \leq h$; here $k(\ell) = k(\ell, h)$ is a bounded function.

Remark 4. We say that systems (3.10) and (5.20) are B-equivalent.

Proof. Fix $i \in \mathbb{Z}$. Let $z(t)$ be the solution of (3.10) such that $z(t_{2i}) = z$, and assume that α_i and β_i are solutions of $\alpha = t_{2i-1} + \tau_{2i-1}(z(\alpha))$, and $\beta = t_{2i} + \tau_{2i}(z(\beta))$, respectively. Let $z_1(t)$ be the solution of the system

$$\frac{\mathrm{d}z}{\mathrm{d}t} = A(t)z + \tilde{f}(t, z) \tag{5.22}$$

with the initial condition $z_1(\alpha_{i+1}) = \Pi_i^2(\beta_i, z(\beta_i))$.

We first note that $z_1(\alpha_{i+1}) = (I + B_i)z(\beta_i) + J_i(z(\beta_i))$. Moreover, for $t \in [t_{2i}\widehat{,}\beta_i]$,

$$z(t) = z(t_{2i}) + \int_{t_{2i}}^{t}\left[A(s)z(s) + \tilde{f}(s, z(s))\right]\mathrm{d}s, \tag{5.23}$$

and for $t \in [\alpha_{i+1}\widehat{,}t_{2i+1}]$,

$$\begin{aligned} z_1(t) &= z_1(\alpha_{i+1}) + \int_{\alpha_{i+1}}^{t}\left[A(s)z_1(s) + \tilde{f}(s, z_1(s))\right]\mathrm{d}s \\ &= (I + B_i)z(\beta_i) + J_i(z(\beta_i)) + \int_{\alpha_{i+1}}^{t}\left[A(s)z_1(s) + \tilde{f}(s, z_1(s))\right]\mathrm{d}s \\ &= (I + B_i)\left[z(t_{2i}) + \int_{t_{2i}}^{\beta_i}\left[A(s)z(s) + \tilde{f}(s, z(s))\right]\mathrm{d}s\right] \\ &\quad + J_i(z(\beta_i)) + \int_{\alpha_{i+1}}^{t}\left[A(s)z_1(s) + \tilde{f}(s, z_1(s))\right]\mathrm{d}s. \end{aligned} \tag{5.24}$$

Thus, we set

$$W_i(z) = (I + B_i)\int_{t_{2i}}^{\beta_i}\left[A(s)z(s) + \tilde{f}(s, z(s))\right]\mathrm{d}s + J_i(z(\beta_i)) + \int_{\alpha_{i+1}}^{t_{2i+1}}\left[A(s)z_1(s) + \tilde{f}(s, z_1(s))\right]\mathrm{d}s. \tag{5.25}$$

Substituting (5.25) in (5.20), we see that $W_i(z)$ satisfies the first conclusion of the lemma. Fig. 3 illustrates the procedure of the construction of $W_i(z)$.

We next prove (5.21). Let $\|z(t_{2i})\| \leq h$. By employing integrals (5.23) and (5.24), we find that the solutions $z(t)$ and $z_1(t)$ determined above satisfy the inequalities $\|z(t)\| \leq H$ and $\|z_1(t)\| \leq H$ on $[\beta_i\widehat{,}t_{2i}]$ and $[\alpha_{i+1}\widehat{,}t_{2i+1}]$, where

$$H = \left[M(1 + \ell) + (1 + N + \ell)(h + M\ell)\mathrm{e}^{N\ell + \ell^2}\right]\mathrm{e}^{N\ell + \ell^2}.$$

Let $y(t)$ be the solution of (3.10) such that $y(t_{2i}) = y$, and assume that $\bar{\alpha}_i$ and $\bar{\beta}_i$ are solutions of $\bar{\alpha} = t_{2i-1} + \tau_{2i-1}(y(\bar{\alpha}))$, and $\bar{\beta} = t_{2i} + \tau_{2i}(y(\bar{\beta}))$, respectively. Let $y_1(t)$ be the solution of (5.22) with the initial condition

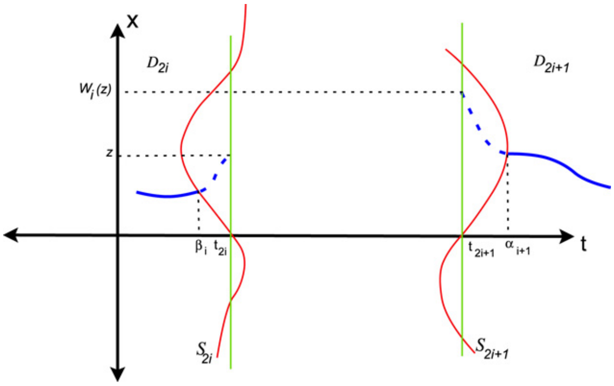

Fig. 3.

$y_1(\bar{\alpha}_{i+1}) = \Pi_i^2(\bar{\beta}_i, y(\bar{\beta}_i))$. Without loss of any generality, we assume that $\bar{\beta}_i \geq \beta_i$ and $\bar{\alpha}_{i+1} \leq \alpha_{i+1}$. Application of the Gronwall–Bellman lemma shows that, for $t \in [\beta_i, \hat{t}_{2i}]$,

$$\|z(t) - y(t)\| \leq \mathrm{e}^{(N+\ell)\ell}\|z - y\|. \tag{5.26}$$

The equation,

$$y(\bar{\beta}_i) = y(\beta_i) + \int_{\beta_i}^{\bar{\beta}_i} \left[A(s)y(s) + \tilde{f}(s, y(s))\right] \mathrm{d}s \tag{5.27}$$

gives us

$$\|y(\bar{\beta}_i) - y(\beta_i)\| \leq (NH + \ell H + M)(\bar{\beta}_i - \beta_i). \tag{5.28}$$

Thus, we obtain

$$\|z(\beta_i) - y(\bar{\beta}_i)\| \leq \mathrm{e}^{(N+\ell)\ell}\|z - y\| + (NH + \ell H + M)(\bar{\beta}_i - \beta_i). \tag{5.29}$$

Now, condition (C3) together with (5.29) leads to

$$\bar{\beta}_i - \beta_i \leq \frac{\ell \mathrm{e}^{(N+\ell)\ell}}{1 - \ell(NH + \ell H + M)}\|z - y\|. \tag{5.30}$$

Hence (5.29) becomes

$$\|z(\beta_i) - y(\bar{\beta}_i)\| \leq \frac{\mathrm{e}^{(N+\ell)\ell}}{1 - \ell(NH + \ell H + M)}\|z - y\|. \tag{5.31}$$

On the other hand,

$$y_1(\alpha_{i+1}) = y_1(\bar{\alpha}_{i+1}) + \int_{\bar{\alpha}_{i+1}}^{\alpha_{i+1}} \left[A(s)y_1(s) + \tilde{f}(s, y_1(s))\right] \mathrm{d}s \tag{5.32}$$

gives us

$$\|y_1(\alpha_{i+1}) - y_1(\bar{\alpha}_{i+1})\| \leq (NH + \ell H + M)(\alpha_{i+1} - \bar{\alpha}_{i+1}). \tag{5.33}$$

Using the transition operators and (5.31) we get,

$$\|z_1(\alpha_{i+1}) - y_1(\bar{\alpha}_{i+1})\| \leq \frac{(1 + N + \ell)\mathrm{e}^{(N+\ell)\ell}}{1 - \ell(NH + \ell H + M)}\|z - y\|. \tag{5.34}$$

Condition (C3) and (5.34) imply that

$$\alpha_{i+1} - \bar{\alpha}_{i+1} \leq \frac{\ell(1 + N + \ell)\mathrm{e}^{(N+\ell)\ell}}{1 - \ell(NH + \ell H + M)}\|z - y\|. \tag{5.35}$$

From (5.33)–(5.35) we obtain

$$\|z_1(\alpha_{i+1}) - y_1(\alpha_{i+1})\| \le H_1 \mathrm{e}^{(N+\ell)\ell}\|z - y\|. \tag{5.36}$$

where $H_1 = (1 + N + \ell)[1 + \ell(NH + \ell H + M)]/[1 - \ell(NH + \ell H + M)]$. Solutions $z_1(t)$ and $y_1(t)$ on $[\alpha_{i+1}, \hat{t}_{2i+1}]$ satisfy the inequality

$$\|z_1(t) - y_1(t)\| \le H_1 \mathrm{e}^{2(N+\ell)\ell}\|z - y\|. \tag{5.37}$$

Now, subtracting the expression

$$W_i(y) = (I + B_i)\int_{t_{2i}}^{\bar{\beta}_i}\Big[A(s)y(s) + \tilde{f}(s, y(s))\Big]\mathrm{d}s + J_i(y(\bar{\beta}_i)) + \int_{\bar{\alpha}_{i+1}}^{t_{2i+1}}\Big[A(s)y_1(s) + \tilde{f}(s, y_1(s))\Big]\mathrm{d}s \tag{5.38}$$

from Eq. (5.25), and using Eqs. (5.26), (5.30), (5.35) and (5.37), we conclude that Eq. (5.21) holds. This proves the lemma. □

Definition 5. A solution $y(t)$ is said to be B-stable, if for arbitrary $\epsilon > 0$, there is $\delta > 0$ such that a solution $\varphi(t)$ for which $\|\varphi(t^0) - y(t^0)\| < \delta$ is in the ϵ-neighborhood of $y(t)$ on $\mathbb{T}^y_{t^0}$ and $\mathbb{T}^\varphi_{t^0}$.

Definition 6. A B-stable solution $y(t)$ is called B-asymptotically stable, if there is $\delta > 0$ such that for arbitrary $\epsilon > 0$, there is $\theta > t^0$ such that a solution $\varphi(t)$ for which $\|\varphi(t^0) - y(t^0)\| < \delta$ is in the ϵ-neighborhood of $y(t)$ on $\mathbb{T}^y_\theta$ and $\mathbb{T}^\varphi_\theta$.

6. Reduction to an impulsive differential equation

Previously we have shown that a differential equation on a variable time scale is B-equivalent to a corresponding differential equation on a non-variable time scale. Now, we are going to reduce (5.20), which is B-equivalent to (3.10), into a system of impulsive differential equations. For this purpose, we need a special transformation called ψ-substitution [2,5], which is change of the independent variable and defined for $t \in \bigcup_{i=-\infty}^{\infty}(t_{2i-1}, t_{2i}]$ as,

$$\psi(t) = \begin{cases} t - \displaystyle\sum_{0<t_{2k}<t} \delta_k, & t \ge 0 \\ t + \displaystyle\sum_{t \le t_{2k} < 0} \delta_k, & t < 0, \end{cases} \tag{6.39}$$

where $\delta_k = t_{2k+1} - t_{2k}$. Setting $s_i = \psi(t_{2i})$, we see that this transformation has an inverse given by

$$\psi^{-1}(s) = \begin{cases} s + \displaystyle\sum_{0<s_k<s} \delta_k, & s \ge 0 \\ s - \displaystyle\sum_{s \le s_k < 0} \delta_k, & s < 0. \end{cases} \tag{6.40}$$

Lemma 6.1 (*[5]*). *$\psi'(t) = 1$ if $t \in \bigcup_{i=-\infty}^{\infty}(t_{2i-1}, t_{2i}]$.*

Now, using the substitution of the independent variable in (5.20) and letting $x(s) = z(\psi^{-1}(s))$, we obtain, for $t \ne t_{2i}$,

$$x' = A(\psi^{-1}(s))x + \tilde{f}(\psi^{-1}(s), x(s)),$$

and, for $t = t_{2i}$, we get

$$\begin{aligned} x(s_i^+) &= z(t_{2i+1}) \\ &= (I + B_i)z(t_{2i}) + W_i(z(t_{2i})) \\ &= (I + B_i)x(s_i) + W_i(x(s_i)). \end{aligned}$$

Thus, the second equation in (5.20) leads to,

$$\Delta x|_{s=s_i} = B_i x(s_i) + W_i(x(s_i)),$$

where $\Delta x|_{s=s_i} = x(s_i^+) - x(s_i)$. Hence, $x(s)$ is a solution of the impulsive differential equation:

$$\begin{aligned} x' &= A(\psi^{-1}(s))x + \tilde{f}(\psi^{-1}(s), x(s)), \quad s \neq s_i, \\ \Delta x|_{s=s_i} &= B_i x(s_i) + W_i(x(s_i)). \end{aligned} \tag{6.41}$$

It is known that, a solution of (6.41) satisfying $x(s^0) = x_0$, for $s \geq 0$ is given by

$$x(s) = X(s, s^0)x_0 + \int_{s^0}^{s} X(s, \xi)\tilde{f}(\psi^{-1}(\xi), x(\xi))\mathrm{d}\xi + \sum_{s^0 \leq s_i < s} X(s, s_i^+)W_i(x(s_i^+)), \tag{6.42}$$

where $X(s, s^0) = X(s)X^{-1}(s^0)$ and $X(s)$ is defined by

$$X(s) = \begin{cases} I, & s = 0, \\ \mathcal{X}(s, s_p)(I + B_p)\prod_{k=p}^{1}[\mathcal{X}(s_k, s_{k-1})(I + B_{k-1})]\mathcal{X}(s, 0), & s > 0, \end{cases}$$

in which $\mathcal{X}(s, r) = \mathcal{X}(s)\mathcal{X}^{-1}(r)$ is a transition matrix of $x' = A(\psi^{-1}(s))x$ and it is assumed that $0 < s^0 < \cdots < s_p < s < s_{p+1}$.

Now, using back substitution, we see that a solution $y(t)$, $y(t^0) = y_0$, of (5.20), for $t \geq t^0$, is given by,

$$y(t) = Y(t, t^0)t_0 + \int_{\mathbb{T}(t^0,t)} Y(t, \tau)f(\tau, y(\tau))\mathrm{d}\tau + \sum_{t^0 \leq t_{2i} < t} Y(t, t_{2i+1})W_i(t_{2i+1}), \tag{6.43}$$

where $Y(t, t^0) = Y(t)Y^{-1}(t^0)$ and $Y(t)$, for $0 < t^0 < \cdots < t_{2p+1} < t < t_{2p+2}$, is defined by

$$Y(t) = \begin{cases} I, & t = 0, \\ \mathcal{Y}(t, t_{2p+1})(I + B_p)\prod_{k=p}^{1}\left[\mathcal{Y}(t_{2k}, t_{2k-1})(I + B_{2k-1})\right]\mathcal{Y}(t, 0), & t > 0, \end{cases}$$

in which $\mathcal{Y}(t, \tau) = \mathcal{Y}(t)\mathcal{Y}^{-1}(\tau)$ is a transition matrix of $y' = A(t)y$. The notation $\int_{\mathbb{T}(a,b)} f(\tau)\mathrm{d}\tau$ was introduced in [5].

Thus, instead of investigating system (3.10), we are going to deal with (5.20) which turns out to an IDE, as in (6.41), after ψ-substitution.

On the basis of the discussion in Sections 5 and 6, one may conclude that the method of investigation of DETCV may be realized as consecutive reductions: (a) using a B-equivalence method to get a DETC; and (b) applying the ψ-substitution to DETC to obtain an IDE. We finalize the reductions with the interpretation of results for the issue DETCV. Fig. 4 illustrates this method.

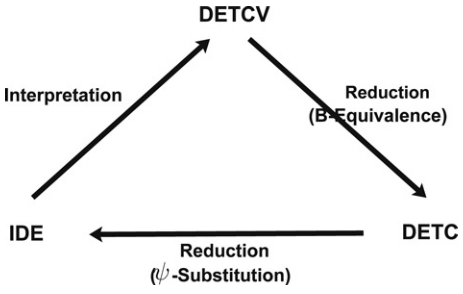

Fig. 4.

7. Periodic solutions

The variable time scale $\mathbb{T}_0(y)$ is said to satisfy an (ω, p)-property if $(t \pm \omega, y)$ is in $\mathbb{T}_0(y)$ whenever (t, y) is. In this case, one can easily see that, there exists $p \in \mathbb{N}$ such that the sequences $\{t_{2i-1}\}$ and $\{t_{2i}\}$ satisfy the (ω, p)-property, [5], and $\tau_{i+p}(y) = \tau_i(y)$ for all $i \in \mathbb{Z}$.

Suppose now that (3.10) is ω-periodic, i.e. $\mathbb{T}_0(y)$ satisfies the (ω, p)-property, $A(t)$ and $f(t, y)$ are ω-periodic functions of t, and $B_{i+p} = B_i$, $J_{i+p}(y) = J_i(y)$ uniformly with respect to $i \in \mathbb{Z}$.

Since (3.10) satisfies the conditions for the uniqueness of a solution, and is periodic it can be shown that the following result holds.

Lemma 7.1. *If (3.10) is periodic, then the sequence $W_i(z)$ is p-periodic uniformly with respect to $z \in \mathbb{R}^n$.*

Hence (5.20) is also periodic.

Lemma 7.2 (*[5]*). *If $\mathbb{T}_0(y)$ satisfies an (ω, p)-property, then $\psi(t + \omega) = \psi(t) + \psi(\omega)$.*

Lemma 7.3 (*[5]*). *A function $\phi(t)$ is an ω-periodic function on $\mathbb{T}_c$, if and only if $\phi(\psi^{-1}(s))$ is an $\tilde{\omega}$-periodic function on $\mathbb{R}$, where $\tilde{\omega} = \psi(\omega)$.*

Consider the ω-periodic system

$$\begin{aligned} &\frac{\mathrm{d}z}{\mathrm{d}t} = A(t)z + f(t), \quad t \neq t_{2i}, \\ &z(t_{2i+1}) = B_i z(t_{2i}) + W_i + z(t_{2i}), \end{aligned} \tag{7.44}$$

and let $Z(t)$ be the fundamental matrix of the corresponding homogeneous system,

$$\begin{aligned} &\frac{\mathrm{d}z}{\mathrm{d}t} = A(t)z, \quad t \neq t_{2i}, \\ &z(t_{2i+1}) = B_i z(t_{2i}) + z(t_{2i}). \end{aligned} \tag{7.45}$$

Using ψ-substitution, systems (7.44) and (7.45) reduce to

$$\begin{aligned} &\frac{\mathrm{d}u}{\mathrm{d}s} = A(\psi^{-1}(s))u + f(\psi^{-1}(s)), \quad s \neq s_i, \\ &\Delta u|_{s=s_i} = B_i u + W_i, \end{aligned} \tag{7.46}$$

and

$$\begin{aligned} &\frac{\mathrm{d}u}{\mathrm{d}s} = A(\psi^{-1}(s))u, \quad s \neq s_i, \\ &\Delta u|_{s=s_i} = B_i u, \end{aligned} \tag{7.47}$$

respectively, where $u(s) = z(\psi^{-1}(s))$.

According to [4], there is a piece-wise continuous Floquet–Lyapunov transformation $u = \Phi(s)v$ reducing (7.46) to a system with a constant matrix. There is, therefore, a constant nonsingular matrix S such that the transformation, $u = \Phi(s)Sw$, reduces (7.46) to

$$\begin{aligned} &\frac{\mathrm{d}w}{\mathrm{d}s} = \Lambda w + g(s), \quad s \neq s_i, \\ &\Delta w|_{s=s_i} = I_i, \end{aligned} \tag{7.48}$$

where $\Lambda = \mathrm{diag}(\Lambda_+, \Lambda_-)$ is a constant matrix with $\mathrm{Re}\,\lambda_j(\Lambda_+) > 0$ for $j = 1, 2, \ldots, m$, and $\mathrm{Re}\,\lambda_j(\Lambda_-) < 0$ for $j = m, m+1, \ldots, n$,

$$\begin{aligned} &\Lambda = S^{-1}\Phi^{-1}(s)\left[A(\psi^{-1}(s)) - \frac{\mathrm{d}\Phi(s)}{\mathrm{d}s}\Phi^{-1}(s)\right]\Phi(s)S, \\ &g(s) = S^{-1}\Phi^{-1}(s)f(\psi^{-1}(s)), \\ &I_i = S^{-1}\Phi^{-1}(s_i^+)W_i. \end{aligned}$$

It is natural to call $\lambda_j = \lambda_j(\Lambda)$ the characteristic indices and $\rho_j = e^{\lambda_j}$ the characteristic multipliers of (7.48), respectively [5,19]. Similarly, we call the numbers λ_j and ρ_j the characteristic indices and characteristic multipliers of (7.45).

Lemma 7.4. *If the real parts of the characteristic indices of (7.45) do not vanish, then (7.44) has a unique ω-periodic solution, which will be B-asymptotically stable when all characteristic indices of (7.45) have negative real parts.*

Proof. Let

$$G(s) = \begin{cases} \operatorname{diag}(\exp(\Lambda_+ s), 0), & \text{for } s < 0, \\ \operatorname{diag}(0, -\exp(\Lambda_- s)), & \text{for } s > 0, \end{cases}$$

and let $\alpha = \min_{1\le j\le n} |\mathrm{Re}\,\lambda_j(\Lambda)| + \epsilon$ where ϵ is an arbitrary positive number. In this case, it is known that there exists a number $K = K(\epsilon) > 1$, such that

$$\|G(s-r)\| \le K \exp(-\alpha|s-r|), \quad s, r \in \mathbb{R}.$$

By using this inequality, it was shown in [3] that

$$w_0(s) = \int_{-\infty}^{\infty} G(s-r)g(r)\mathrm{d}r + \sum_{-\infty}^{\infty} G(s-s_i)I_i$$

is an $\tilde{\omega}$-periodic solution of (7.48), for which

$$\|w_0(s)\| \le 2Km(\alpha) \max\left\{\max_s \|g(s)\|, \max_i \|I_i\|\right\},$$
$$m(\alpha) = \frac{1}{\alpha} + \frac{\exp(\alpha\theta)}{1-\exp(-\alpha\theta')}.$$

Hence, $u_0(s) = \Phi(s)Sw_0(s)$ is a periodic solution of (7.46) and, if $m_1 = \max_s \|\Phi(s)S\|$, then for $s \in \mathbb{R}$, we have

$$\|u_0(s)\| \le 2Km_1m(\alpha) \max\left\{\max_s \|f(\psi^{-1}(s))\|, \max_i \|I_i\|\right\}.$$

Therefore, $z_0(t) = u_0(\psi(t))$ is a periodic solution of (7.44) which satisfies the inequality,

$$\|z_0(t)\| \le 2Km_1m(\alpha) \max\left\{\max_{t\in\mathbb{T}_c} \|f(t)\|, \max_i \|I_i\|\right\},$$

for $t \in \mathbb{T}_c$. This proves the lemma. □

Now, let $C = 2Km_1m(\alpha)$ and fix $\gamma > 1$. Let $k(\ell) = k(\ell, h)$ be the function defined in Lemma 5.1, for $h = \gamma CM$. By applying Lemmas 7.1 and 7.4 and the successive-approximation method, exactly as it was done in [3], we can prove the following lemma.

Lemma 7.5. *Suppose that $\tilde{f}(t,z)$, $W_i(z)$ and $\tau_i(z)$ in (5.20) satisfy conditions* (C3), (C4) *and (5.21). If all characteristic indices of system (7.45) have non-vanishing real parts, then, when $\ell C \max\{1, k(\ell)\} < (\gamma-1)/\gamma$, system (5.20) has a unique ω-periodic solution $z_0(t)$ such that $\|z_0(t)\| \le h$ for $t \in \mathbb{T}_c$.*

The solution $z_0(t)$ is B-asymptotically stable, if the real parts of all characteristic indices of system (7.45) are negative.

On the basis of B-equivalence, Lemmas 5.1 and 7.5 and continuous dependence of solution on initial data for ordinary differential equation, one can prove the following theorem.

Theorem 7. *Suppose that system (3.10) satisfies conditions* (C1)–(C5) *and is ω-periodic. If the characteristic indices of system (7.45) have non-vanishing real parts, then for a sufficiently small Lipschitz constant ℓ, system (3.10) has a unique ω-periodic solution, which is B-asymptotically stable when all characteristic indices of system (7.45) have negative real parts.*

Example 7.1. Let us consider the variable time scale $\mathbb{T}_0(y)$ constructed by $t_i = i, \tau_i(y) = (-1)^i \ell \sin(y)$, where $y \in \mathbb{R}, 0 < \ell < \frac{1}{2}$, and consider the 2-periodic system

$$\begin{aligned} &y' = ky + \cos(\pi t), \quad (t, y) \in \mathbb{T}_0(y), \\ &y^+ = (p+1)y + I, \\ &t^+ = 2i + 1 - \ell \sin(y), \end{aligned} \tag{7.49}$$

with $k, p, I \in \mathbb{R}, I > 0$. The system which is B-equivalent to (7.49) is

$$\begin{aligned} &z' = kz + \cos(\pi t), \quad t \neq 2i, \\ &z(2i+1) = (p+1)z(2i) + W_i(z), \end{aligned} \tag{7.50}$$

where

$$\begin{aligned} W_i(z) &= (1+p)\int_{2i}^{\beta_i} [kz(s) + \cos(\pi s)]\mathrm{d}s + I + \int_{\alpha_{i+1}}^{2i+1} [kz(s) + \cos(\pi s)]\mathrm{d}s \\ &= (1+p)\int_{2i}^{\beta_i} kz(s)\mathrm{d}s + \int_{\alpha_{i+1}}^{2i+1} kz(s)\mathrm{d}s + I + \frac{\sin(\pi\beta_i) - \sin(\pi\alpha_{i+1})}{\pi} \end{aligned}$$

where $z(t)$ is a solution of (7.49) satisfying $z(2i) = z$ and α_i and β_i are solutions of $\alpha = 2i - 1 - \ell \sin(z(\alpha))$ and $\beta = 2i + \ell \sin(z(\beta))$, respectively.

The homogeneous system corresponding to (7.50) is

$$\begin{aligned} &z' = kz, \quad t \neq 2i, \\ &z(2i+1) = (p+1)z(2i). \end{aligned} \tag{7.51}$$

It is easily seen that for (7.49), the conditions (C1)–(C5) are satisfied and

$$Z(2) = [(p+1)\mathrm{e}^k]$$

is the matrix of monodromy and $\lambda = \ln(p+1) + k$ is the characteristic index of (7.51). By Theorem 7, if $\ln(p+1) + k \neq 0$, then system (7.49) has a unique 2-periodic solution which is B-asymptotically stable when $\ln(p+1) + k < 0$. □

8. Stability of an equilibrium position

In this part, we are again going to consider the quasilinear system

$$\begin{aligned} &y' = A(t)y + f(t, y), \quad (t, y) \in \mathbb{T}_0(y), \\ &t^+ = \Pi_i^1(t, y), \qquad y^+ = \Pi_i^2(t, y), \quad (t, y) \in \mathcal{S}_{2i}, \end{aligned} \tag{8.52}$$

on the variable time scale $\mathbb{T}_0(y)$. However, this time, the condition for existence of a Green's function is replaced by a more general one, namely, exponential dichotomy, [10]. Let

$$\begin{aligned} &y' = A(t)y, \quad t \neq t_{2i}, \\ &y(t_{2i+1}) = B_i y(t_{2i}) + y(t_{2i}), \end{aligned} \tag{8.53}$$

be the homogeneous system corresponding to (8.52). Moreover, suppose that the system which is B-equivalent to (8.52) is:

$$\begin{aligned} &z' = A(t)z + \tilde{f}(t, z), \quad t \neq t_{2i}, \\ &z(t_{2i+1}) = B_i z(t_{2i}) + W_i(z(t_{2i})) + z(t_{2i}). \end{aligned} \tag{8.54}$$

Suppose that there are m- and $(n-m)$-dimensional hyperplanes $Y_+(t)$ and $Y_-(t)$ in $\mathbb{T}_c \times \mathbb{R}^n$ such that if $y(t)$ is a solution of (8.53) and $y(t) \in Y_+(t)$, then $\|y(t)\| \leq a_1\|y(\tau)\| \exp(-\gamma_1(t-\tau)), -\infty < \tau \leq t < +\infty$ and, if $y(t) \in Y_-(t)$ then $\|y(t)\| \geq a_2\|y(\tau)\| \exp(\gamma_2(t-\tau)), -\infty < \tau \leq t < +\infty$. Here, $a_j, \gamma_j, j = 1, 2$ are positive constants. If (8.53) satisfies these conditions, then we say that (8.53) is *exponentially dichotomous*.

In this case, using the inequality $\psi(t) - \psi(\tau) \le t - \tau$ when $\tau \le t$, one can show that, for the reduced impulsive linear system

$$\begin{aligned} &x' = A(\psi^{-1}(s))x, \quad s \neq s_i, \\ &\Delta x|_{s=s_i} = B_i x(s_i), \end{aligned} \tag{8.55}$$

where $x(s) = y(\psi^{-1}(s))$, $s_i = \psi(t_{2i})$, there are m- and $(n-m)$-dimensional hyperplanes $X_+(s)$ and $X_-(s)$ in $\mathbb{R} \times \mathbb{R}^n$ such that if $x(s)$ is a solution of (8.55) and $x(s) \in X_+(s)$, then $\|x(s)\| \le a_1 \|x(r)\| \exp(-\gamma_1(s-r))$, $-\infty < r \le s < +\infty$ and, if $x(s) \in X_-(s)$, then $\|x(s)\| \ge a_2 \|x(r)\| \exp(\gamma_2(s-r))$, $-\infty < r \le s < +\infty$. Then, the linear system (8.55) with impulse action is said to be *exponentially dichotomous* (e.d.) [10].

If (8.55) is e.d., then by applying the orthogonalization method to a given set of linearly independent solutions $x_1(s), x_2(s), \ldots, x_n(s)$, we can construct a piecewise-continuous Lyapunov–Schmidt transformation $x = L(s)w$ reducing (8.55) to a block-diagonal system [3], i.e., a system splitting into two systems:

$$\frac{d\xi}{ds} = P_1(s)\xi, \quad s \neq s_i, \qquad \Delta\xi|_{s=s_i} = Q_i^1 \xi, \tag{8.56}$$

and

$$\frac{d\eta}{ds} = P_2(s)\eta, \quad s \neq s_i, \qquad \Delta\eta|_{s=s_i} = Q_i^2 \eta, \tag{8.57}$$

where $w = (\xi, \eta)$, with ξ an m-vector and η an $(n-m)$-vector. Corresponding to fundamental matrices $X_1(s,r)$ and $X_2(s,r)$ of (8.56) and (8.57), there are positive constants a and γ such that

$$\|X_1(s,r)\| \le a \exp(-\gamma(s-r)), \quad s \ge r,$$

and

$$\|X_2(s,r)\| \le a \exp(\gamma(s-r)), \quad s \le r.$$

Similarly, (8.52) can be reduced to the system

$$\begin{aligned} &\frac{d\xi}{ds} = P_1(s)\xi + \tilde{f}_1(s,w), \quad s \neq s_i, \\ &\frac{d\eta}{ds} = P_2(s)\eta + \tilde{f}_2(s,w), \quad s \neq s_i, \\ &\Delta\xi|_{s=s_i} = Q_i^1 \xi(s_i) + W_i^1(w(s_i)), \\ &\Delta\eta|_{s=s_i} = Q_i^2 \eta(s_i) + W_i^2(w(s_i)), \end{aligned} \tag{8.58}$$

after applying ψ-substitution and Lyapunov–Schmidt transformation, successively.

Besides the conditions imposed before, suppose that (8.52) satisfies

$$f(t,0) = J_i(0) = 0 \tag{8.59}$$

uniformly with respect to $t \in \mathbb{T}_c$ and $i \in \mathbb{Z}$.

We investigate the stability of an equilibrium position of (8.52), first noting that (8.59) implies $W_i(0) = 0$ for $i \in \mathbb{Z}$.

It follows from Lemma 5.1, B-equivalence, and the continuous dependence of solutions of (8.52) on initial data, that the following analog of the Lyapunov–Perron theorem holds.

Theorem 8. *Suppose that system (8.52) satisfies conditions* (C1)–(C4) *and (8.59), and system (8.53) is e.d. Then, for a sufficiently small Lipschitz constant ℓ, the equilibrium position of (8.52) is conditionally asymptotically stable with respect to an m-dimensional manifold of initial values containing the origin. If $m = n$, then the zero solution of (8.52) is asymptotically stable.*

Proof. By virtue of the reasoning given above, we consider the system (8.54) which can be reduced to the form (8.58). We assume that the functions on the right-hand side of (8.58) satisfy conditions analogous to (C3), (C4) and (5.21) with the same constants. Then, the integral-equation system

$$\begin{aligned}\xi &= X_1(s,s^0)c + \int_{s^0}^{s} X_1(s,r)\tilde{f}_1(r,w)\mathrm{d}r + \sum_{s_i<s} X_1(s,s_i)W_i^1(w),\\ \eta &= -\int_{s}^{\infty} X_2(s,r)\tilde{f}_2(r,w)\mathrm{d}r - \sum_{s_i>s} X_2(s,s_i)W_i^2(w),\end{aligned} \tag{8.60}$$

under the conditions

$$a\ell\|c\|(a+\epsilon)\left[\frac{2}{\gamma^2-\sigma^2} + \frac{2}{1-\exp(-\theta'(\gamma-\sigma))}\right] < \epsilon$$

and

$$2a\left[\frac{1}{\gamma-\sigma} + \frac{1}{1-\exp(-\theta'(\gamma-\sigma))}\right] < 1,$$

where ϵ and σ are arbitrary fixed constants such that $\epsilon > 0$ and $0 < \sigma < \gamma$, has a solution $w(s) = w(s, s^0, c)$ for which

$$\|w(s)\| \le (a+\epsilon)\|c\| \exp\left(-\sigma\left(s-s^0\right)\right). \tag{8.61}$$

If $s = s^0$ in (8.60), then

$$\begin{aligned}&\xi(s^0,s^0,c) = c\\ &\eta(s^0,s^0,c) = -\int_{s^0}^{\infty} X_2(s^0,r)\tilde{f}_2(r,w)\mathrm{d}r - \sum_{s_i>s^0} X_2(s^0,s_i)W_i^2(w(s_i)).\end{aligned} \tag{8.62}$$

By using the customary method [19], we can easily show that $w(s)$ is also a solution of (8.58). Hence, by virtue of (8.60) and (8.61), we conclude that (8.62) determines a set of initial values of solutions of (8.58) tending to an equilibrium state when $s \to \infty$. Since, B-equivalence and ψ-substitution do not change the dimension of the manifold, the theorem is proved.

9. Bounded solutions

Theorem 9. *If conditions* (C1)–(C5) *are satisfied for system* (8.52) *and* (8.53) *is e.d., then for a sufficiently small Lipschitz constant* ℓ, *system* (8.52) *has a unique solution, continuable to* $+\infty$ *and* $-\infty$, *uniformly bounded for all* t, $(t, y(t)) \in \mathbb{T}_0(y)$.

Proof. System (8.54) which is B-equivalent to (8.52) can be reduced to

$$\begin{aligned}x' &= A(\psi^{-1}(s))x + \tilde{f}(\psi^{-1}(s),x), \quad s \ne s_i,\\ \Delta x|_{s=s_i} &= B_i x(s_i) + W_i(x(s_i)),\end{aligned} \tag{9.63}$$

by means of ψ-substitution, where $x(s) = z(\psi^{-1}(s))$. In [3], it was shown that for $h = \nu a M\{1/\gamma + [\exp(\gamma\theta)/(1-\exp(-\gamma\theta'))]\}$, where $\nu > 1$ is fixed, under the condition

$$\ell a\left(\frac{1}{\gamma} + \frac{k(\ell)\exp(\gamma\theta)}{1-\exp(-\gamma\theta')}\right) < \frac{\nu-1}{\nu},$$

the system (9.63) has a unique bounded solution $x_0(s)$. Using the inverse substitution, we see that $y_0(t) = x_0(\psi(t))$ is a bounded solution of (8.54), and B-equivalence between (8.54) and (8.52) proves the theorem.

10. Conclusion

In the paper, we have introduced a new class of differential equations, differential equations on variable time scales with transition conditions. These systems naturally appear when we investigate discontinuous dynamics with non-fixed moments of impulses. Consequently, our results will be needed to develop methods of investigation of mechanical models with impacts. Particularly, interesting problems are related to bifurcations [8,13,14,23], chaos [13], etc. We are going to develop the theory of introduced equations according to these demands.

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