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Attraction of Li–Yorke chaos by retarded SICNNs

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ABSTRACT

In the present study, dynamics of retarded shunting inhibitory cellular neural networks (SICNNs) is investigated with Li–Yorke chaotic external inputs and outputs. Within the scope of our results, we prove the presence of generalized synchronization in coupled retarded SICNNs, and confirm it by means of the auxiliary system approach. We have obtained more than just synchronization, as it is proved that the Li–Yorke chaos is extended with its ingredients, proximality and frequent separation, which have not been considered in the theory of synchronization at all. Our procedure is used to synchronize chains of unidirectionally coupled neural networks. The results may explain the high performance of brain functioning and can be extended by specific stability analysis methods. Illustrations supporting the results are depicted. For the first time in the literature, proximality and frequent separation features are demonstrated numerically for continuous-time dynamics.

1. Introduction

Cellular neural networks (CNNs) have been paid much attention due to their local connectivity and easy hardware implementation. Time delays occur during the hardware implementation of neural networks because of the finite switching speed of the amplifiers. The occurrence of time delays may lead to an oscillation and instability of the networks [1,2]. Moreover, the introduction of delay in the signals transmitted among the cells of CNNs is required by the process of moving images [3]. Therefore, the consideration of neural networks with time delay is important for applications.

Chaotic dynamics has been widely investigated in neural networks [4–24]. In their study, King et al. [4] observed chaotic behavior in a model of the central dopaminergic neuronal system. It is shown in paper [5] that chaos can be expected in mathematical models of neural systems possessing time delays. In order to study the dynamical properties of a neural network in chaotic wandering state, Kuroiwa et al. [10] utilized a model which was proposed by Aihara et al. [7]. The same model was also used in [8] to investigate the synchronization characteristics in response to external inputs in a coupled lattice based on a Newman–Watts model. The existence of a period-doubling cascade was demonstrated by Wang [14] in a discrete-time neural network. Ke and Oommen [15] considered the chaotic and pattern recognition properties of a neural network, which is based on the logistic map. In the paper [16], the existence of chaos was demonstrated in the dynamics of fractional-order Hopfield type neural networks. The presence of chaos in the Hodgkin–Huxley model with its original parameters was revealed in the paper [17], where the solutions were found by displaying rectangles in a cross-section whose images under the return map produce a Smale horseshoe. Moreover, the verification of chaotic behavior in Hopfield neural networks was provided by virtue of the horseshoes in the studies [18,19]. The problem of creating a robust chaotic neural network was studied by Potapov and Ali [20]. Furthermore, chaotic dynamics in CNNs was studied in the papers [21–24].

The presence of chaos in neural networks is useful for separating image segments [8], information processing [12,13] and synchronization of neural networks [25–30]. Besides, the synchronization phenomenon is also observable in the dynamics of coupled chaotic CNNs [31,32]. The detection and characterization of synchronization in neural networks is of great interest, since they may provide the opportunity to understand how the brain and nervous system works [33]. Chaotic dynamics can improve the performance of CNNs on problems that have local minima in energy (cost) functions, since chaotic behavior of CNNs can help the network avoid local minima and reach the global optimum [34]. Moreover, chaotic dynamics in CNNs is an important tool for the studies of chaotic communication [35–37] and combinatorial optimization problems [38].

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The term chaos, as a mathematical notion, has first been used by Li and Yorke [39] for one dimensional difference equations. The concept of snap-back repellers for high dimensional maps was introduced by Marotto [40]. According to the results of the paper [40], if a multidimensional continuously differentiable map has a snap-back repeller, then it is Li–Yorke chaotic. Li et al. [41] used Marotto's Theorem to prove the existence of Li–Yorke chaos in a spatiotemporal chaotic system. Li–Yorke sensitivity, which links the Li–Yorke chaos with the notion of sensitivity, was studied in [42], and generalizations of Li–Yorke chaos to mappings in Banach spaces and complete metric spaces were provided in [43]. In the present paper, we develop the concept of Li–Yorke chaos to the multidimensional dynamics of retarded shunting inhibitory cellular neural networks, and prove its existence rigorously.

Marotto's Theorem is also useful in the theory of neural networks to prove the presence of chaos rigorously. It was used by Lin and Ruan [44] to determine chaotic dynamics in a pacemaker neuron type integrate-and-fire circuit having two states with a periodic pulse-train input. Moreover, in the paper [45], the chaos was approved by virtue of Marotto's Theorem in discrete time delayed Hopfield neural networks.

A class of CNNs which was introduced by Bouzerdoum and Pinter [46] is shunting inhibitory cellular neural networks (SICNNs). SICNNs have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision and image processing [47–54].

The model of SICNNs in the most original formulation [46] is as follows. Consider a two-dimensional grid of processing cells, and let $C_{ij}, i=1,2,\dots,m, j=1,2,\dots,n$, denote the cell at the (i,j) position of the lattice. The r-neighborhood of C_{ij} is defined as

$$N_r(i,j)=\{C_{kl}: \max\{|k-i|,|l-j|\}\le r, 1\le k\le m, 1\le l\le n\}.$$

In SICNNs, neighboring cells exert mutual inhibitory interactions of the shunting type. The dynamics of the cell C_{ij} is described by the nonlinear ordinary differential equation

$$\frac{dx_{ij}(t)}{dt}=-a_{ij}x_{ij}(t)-\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(t))x_{ij}(t)+L_{ij}(t),$$

where x_{ij} is the activity of the cell C_{ij}; $L_{ij}(t)$ is the external input to C_{ij}; the constant $a_{ij}>0$ represents the passive decay rate of the cell activity; $C_{ij}^{kl}\ge 0$ is the connection or coupling strength of the postsynaptic activity of the cell C_{kl} transmitted to the cell C_{ij}; and the activation function $f(x_{kl})$ is a positive continuous function representing the output or firing rate of the cell C_{kl}.

In the present study, we consider SICNNs with delay in the form

$$\frac{dx_{ij}(t)}{dt}=-a_{ij}x_{ij}(t)-\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(t-\tau))x_{ij}(t)+L_{ij}(t), \tag{1.1}$$

where τ is a positive number.

To give an explanation of the title of the present study, let us start with the chaos to be attracted. It is a set of bounded functions, $\mathcal{L}$, chaotic in the Li–Yorke sense. In the next section, the set $\mathcal{L}$ will be described in a detailed form. We apply the elements of the chaotic set, $L(t)=\{L_{ij}(t)\}$, as external inputs in the SICNN (1.1). Next, we verify that the network (1.1) outputs a set of solutions of the same nature as the set $\mathcal{L}$, which are bounded functions chaotic in the Li–Yorke sense. We denote the set of the outputs of (1.1) by $\tilde{\mathcal{L}}$. Thus, we say that the SICNN (1.1) "attracts" a chaotic set $\mathcal{L}$ if it produces the chaotic output $\tilde{\mathcal{L}}$. It is worth noting that under the conditions that will be introduced in the next section, the SICNN (1.1) does not possess chaos provided that the external inputs are not chaotic, but regular or absent. In the papers [55,56], retarded SICNNs were considered with periodic/almost periodic inputs, and it was demonstrated that the same regular dynamics appear in the outputs.

Li–Yorke chaos is chosen in our study mainly for two reasons. Firstly, the presence of Li–Yorke chaos can be obtained through the reduction to scalar discrete equations, for instance, to the logistic map. This reduction can be done in the multidimensional case. Secondly, Marotto's Theorem allows us to study the chaos by reduction to *multidimensional* discrete equations. In the parametric sense, the chaos is generic, for example, the logistic map $x_{n+1}=\mu x_n(1-x_n)$ is chaotic for the parameter value μ between 3.84 and 4 [39].

In their studies, Freeman and his collaborators [57–62] revealed that chaotic dynamics is an inevitable attribution of brain activities. Considering the brain as a collection of neural networks, one may suppose that the chaos appearance can happen in two ways. The first one is the "endogenous chaos", which is generated by an individual neural network itself without an influence from outside. This type of chaos appearance was widely investigated in the literature [14–24]. The second way is the extension of chaos from one network to another. One can consider the synchronization of chaos [63–74] within the scope of the latter way. However, synchronization of chaos relies deeply on its description as well as on the verification of asymptotic closeness between the outputs. Therefore, this type of chaos extension brings us far from the effectiveness of chaos for the brain activities. It brings us to the comprehension of schizophrenia, insomnia and epilepsy [5] rather than regular brain functioning. Nevertheless, Breakspear and Terry [75] reported that synchronization plays an important role for activities of healthy brain. That is why it is important to find mathematical methods for the chaos extension between neural networks, where the asymptotic closeness is fully removed or its presence is weakened in some sense.

In the case of identical synchronization [66], one requires the condition $\lim_{t\to\infty}\|y(t)-x(t)\|=0$, where x and y denote the states of the drive and response systems, respectively. This type of asymptotic relation is strong and to weaken it, one should consider the theory of generalized synchronization [33,67–72]. In this theory, the previous relation is replaced by $\lim_{t\to\infty}\|y(t)-\phi(x(t))\|=0$, where ϕ is a transformation. The presence of the synchronization manifold $y=\phi(x)$ in the drive-response systems is mostly investigated by numerical analyses [33,67,70]. The concept of generalized synchronization for coupled systems with delay was considered in [72]. In the present study, we suggest an easy theoretical approach to verify the presence of synchronization based on the exponential convergence of outputs. Moreover, by the traditional simulation methods [33,70], we will check that generalized synchronization takes place in the attraction of chaos. It is worth noting that, in our study, we verify the ingredients of Li–Yorke chaos, which cannot be realized by the concept of synchronization at all, and this is one of the principal novelties of our results. The ingredients, *proximality and frequent separation, may play an essential role in the brain dynamics*. This idea can be supported if one follows the experimental analyses of Freeman and his collaborators [57–62], and develop researches in this direction.

Investigations of neural networks will not be adequate for application problems unless delay is not introduced in the models. Therefore, a large number of papers paid special attention to the presence of delay in SICNNs [55,56,76–86]. In these papers, the existence and stability of periodic, almost periodic and anti-periodic solutions of SICNNs were studied. Despite the fact that SICNNs and chaos are important in neoroscience, there are still very few papers which consider chaos in this type of neural networks. As far as we know, the subject was considered only in the studies [87,88], and the analyses were made only *numerically* without a theoretical support. That is the reason why even *a type of chaos* was not indicated in these studies. The theoretical approach for SICNNs based on the rigorous definition of chaos presented by Li and Yorke [39] was started in our paper [89], where we discussed the chaotification of SICNNs without delay. The way of chaos expansion in continuous-time dynamics was also considered in the paper [90] without time delay by taking into account chaos in the sense of both Li–Yorke [39] and Devaney [91] as well as for period-doubling cascade [92,93] and intermittency [94].

The novelty of the present paper is the discussion of chaotic dynamics in SICNNs with time delay. The investigation of chaotic outputs in retarded SICNNs is much more sophisticated than the one without delay [89]. The method of adaptation of the Li–Yorke chaos for differential equations with retardation considered in the present study is new not only for neural networks, but also for the theory as a whole. This also provides a contribution to the chaos theory. Moreover, we analyze the relation between generalized synchronization [33,67–72] and our approach about the chaotification of neural systems in a detailed form, and such discussions have never been reported before for SICNNs in the literature. In Section 4, we take into account retarded SICNNs with external inputs in the form of relay functions. However, in the studies [95–98], relay systems were considered without time delay. Our results are also applicable to other kinds of recurrent networks such as Hopfield and Cohen–Grossberg neural networks [99–106].

Motivated by the deficiency of mathematical methods for chaos recognition in neural networks and the importance of irregular behavior for effective brain activities, we suggest the results of the present paper. It is the first time in the literature that rigorous mathematical methods are used to prove not only the presence of chaos in retarded SICNNs, but also how chaos can be exported between neural networks. Another novelty is the precise achievement of chaos in the sense of Li–Yorke with its ingredients, proximality and frequent separation, which may play an important role for the working principle of a nervous system. To the best of our knowledge, the numerical demonstration of the proximality and frequent separation features for continuous-time dynamics have never been reported before (see Fig. 5 and the related text). Our results can provide further research areas in neuroscience, in particular, by the consideration of experiments of Freeman and other neurobiologists [5,6,57–62].

The primary contributions of the present study are summarized below:

(i) We give a mathematical description of the Li–Yorke chaos for continuous-time neural networks with delay. Moreover, simulations of the ingredients of Li–Yorke chaos, proximality and frequent separation, have been performed for continuous-time dynamics for the first time.
(ii) By means of external inputs, we theoretically prove the presence of chaos in retarded SICNNs with arbitrary high number of cells, and we provide a way of chaos extension among coupled neural networks with delay.
(iii) We make use of the exponential convergence of solutions (see Lemma 2.2) to prove the presence of generalized synchronization in coupled retarded SICNNs, and confirm its presence by means of the auxiliary system approach [33,70]. Our procedure can be easily extended to synchronize chains of unidirectionally coupled neural networks with delay. This may be important in neuroscience to explain the high performance of brain functioning [6,62]. The proposed approach cannot be reduced to generalized synchronization, since we have obtained more than just synchronization. We prove that the Li–Yorke chaos is extended with its ingredients, proximality and frequent separation, which have not been considered in the theory of synchronization at all.
(iv) Our results can be extended in neuroscience by specific stability analysis methods, for example, by the linear matrix inequality technique [25,107–110].

The rest of the paper is organized as follows. In Section 2, the description of Li–Yorke chaos is presented and two lemmas about the existence of unique bounded on $\mathbb{R}$ solutions of SICNNs and their stability are provided. In Section 3, the presence of Li–Yorke chaos is theoretically proved for retarded SICNNs of the form (1.1). Section 4 is devoted for an example. In this part, a chain of SICNNs is used to show the effectiveness of the proposed results. Moreover, the ingredients of Li–Yorke chaos are demonstrated numerically. We compared our method with generalized synchronization both theoretically and numerically in Section 5. Finally, a conclusion is given in Section 6.

2. Preliminaries

Throughout the paper, $\mathbb{R}$ and $\mathbb{N}$ will stand for the sets of real numbers and natural numbers, respectively, and the norm $\|u\| = \max_{(i,j)} |u_{ij}|$ will be used, where $u = \{u_{ij}\} = (u_{11}, \dots, u_{1n}, \dots, u_{m1} \dots, u_{mn}) \in \mathbb{R}^{m \times n}$.

The description of Li–Yorke chaos that will be utilized in the paper is as follows. Suppose that $\mathcal{L}$ is a collection of continuous functions $L(t) = \{L_{ij}(t)\}, i = 1, 2, \dots, m, j = 1, 2, \dots, n$, such that $\sup_{t \in \mathbb{R}} \|L(t)\| \le M$, where M is a positive number.

We say that a couple $(L(t), \overline{L}(t)) \in \mathcal{L} \times \mathcal{L}$ is proximal if for arbitrary small $\epsilon > 0$ and arbitrary large $E > 0$, there exists an interval J with a length no less than E such that $\|L(t) - \overline{L}(t)\| < \epsilon$ for $t \in J$. On the other hand, a couple $(L(t), \overline{L}(t)) \in \mathcal{L} \times \mathcal{L}$ is called frequently $(\epsilon_0, \Delta)-$separated if there exist positive numbers ϵ_0, Δ and infinitely many intervals $J_q = [\alpha_q, \beta_q], q \in \mathbb{N}$, each with a length no less than Δ, such that $\beta_q \to \infty$ as $q \to \infty$ and $\|L(t) - \overline{L}(t)\| > \epsilon_0$ for each t from these intervals. It is worth noting that the numbers ϵ_0 and Δ depend on the functions $L(t)$ and $\overline{L}(t)$.

A couple $(L(t), \overline{L}(t)) \in \mathcal{L} \times \mathcal{L}$ is a Li–Yorke pair if it is proximal and frequently $(\epsilon_0, \Delta)-$separated for some positive numbers ϵ_0 and Δ. Moreover, an uncountable set $\mathcal{L}_S \subset \mathcal{L}$ is called a scrambled set if $\mathcal{L}_S$ does not contain any periodic functions and each couple of different functions inside $\mathcal{L}_S \times \mathcal{L}_S$ is a Li–Yorke pair.

The collection $\mathcal{L}$ is called a Li–Yorke chaotic set if: (i) there exists a positive number T_0 such that $\mathcal{L}$ possesses a periodic function of period kT_0 for any $k \in \mathbb{N}$; (ii) $\mathcal{L}$ possesses a scrambled set $\mathcal{L}_S$; (iii) for any function $L(t) \in \mathcal{L}_S$ and any periodic function $\overline{L}(t) \in \mathcal{L}$, the couple $(L(t), \overline{L}(t))$ is frequently $(\epsilon_0, \Delta)-$separated for some positive numbers ϵ_0 and Δ.

The following conditions are required:

(C1) There exist positive numbers M_{ij} such that $\sup_{t \in \mathbb{R}} |L_{ij}(t)| \le M_{ij}$.
(C2) There exists a positive number M_f such that $\sup_{s \in \mathbb{R}} |f(s)| \le M_f$.
(C3) There exists a positive number L_f such that $|f(s_1) - f(s_2)| \le L_f |s_1 - s_2|$ for all $s_1, s_2 \in \mathbb{R}$.
(C4) $M_f \delta < 1$, where $\delta = \max_{(i,j)} \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} / a_{ij}$.
(C5) $(M_f + K_0 L_f)\delta < 1$, where $K_0 = \overline{M}/(1 - M_f \delta)$ and $\overline{M} = \max_{(i,j)} M_{ij}/a_{ij}$.

One can confirm that a bounded on $\mathbb{R}$ function $x(t) = \{x_{ij}(t)\}$, $i = 1, 2, \dots, m, j = 1, 2, \dots, n$, is a solution of the network (1.1) if and only if the following integral equation is satisfied:

$$x_{ij}(t) = -\int_{-\infty}^{t} e^{-a_{ij}(t-s)} \left[\sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(s-\tau)) x_{ij}(s) - L_{ij}(s) \right] ds. \tag{2.2}$$

The following assertion is about the existence and uniqueness of bounded on $\mathbb{R}$ solutions of system (1.1).

Lemma 2.1. *Suppose that the conditions (C1)–(C5) are valid. Then, for any $L(t) = \{L_{ij}(t)\}, i = 1, 2, \dots, m, j = 1, 2, \dots, n$, there exists a unique* *bounded on $\mathbb{R}$ solution $\phi_L(t)$ of the network (1.1) such that $\sup_{t\in\mathbb{R}}\|\phi_L(t)\|\le K_0$.*

Proof. Consider the set C_0 of continuous functions $u(t)=\{u_{ij}(t)\}, i=1,2,\ldots,m, j=1,2,\ldots,n$, which are defined on $\mathbb{R}$, such that $\|u\|_0\le K_0$, where $\|u\|_0=\sup_{t\in\mathbb{R}}\|u(t)\|$. Define the operator Π on C_0 as

$$(\Pi u(t))_{ij}=-\int_{-\infty}^{t}\mathrm{e}^{-a_{ij}(t-s)}\left[\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}f(u_{kl}(s-\tau))u_{ij}(s)-L_{ij}(s)\right]ds,$$

where $\Pi u(t)=\{(\Pi u)_{ij}(t)\}$. If $u(t)$ belongs to C_0, then we have

$$|(\Pi u(t))_{ij}|\le\int_{-\infty}^{t}e^{-a_{ij}(t-s)}\left[\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}|f(u_{kl}(s-\tau))||u_{ij}(s)|+|L_{ij}(s)|\right]ds$$

$$\le\frac{1}{a_{ij}}\left(M_{ij}+M_fK_0\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}\right).$$

Accordingly, the inequality $\|\Pi u\|_0\le\overline{M}+M_fK_0\delta=K_0$ holds. Therefore, $\Pi(C_0)\subseteq C_0$.

On the other hand, for any $u(t),v(t)\in C_0$, one can verify that

$$|(\Pi u(t))_{ij}-(\Pi v(t))_{ij}|\le\int_{-\infty}^{t}e^{-a_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}|f(u_{kl}(s-\tau))u_{ij}(s)$$
$$-f(u_{kl}(s-\tau))v_{ij}(s)|\,ds$$
$$+\int_{-\infty}^{t}e^{-a_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}|f(u_{kl}(s-\tau))v_{ij}(s)-f(v_{kl}(s-\tau))v_{ij}(s)|\,ds$$

$$\le(M_f+K_0L_f)\frac{\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}}{a_{ij}}\|u-v\|_0.$$

Thus, $\|\Pi u-\Pi v\|_0\le(M_f+K_0L_f)\delta\|u-v\|_0$, and the operator Π is contractive according to the condition (C5). Consequently, for any $L(t)$, there exists a unique bounded on $\mathbb{R}$ solution $\phi_L(t)$ of system (1.1) such that $\sup_{t\in\mathbb{R}}\|\phi_L(t)\|\le K_0$. □

Consider the collection $\mathcal{L}$ whose elements are functions of the form $L(t)=\{L_{ij}(t)\}, i=1,2,\ldots,m, j=1,2,\ldots,n$, such that $\sup_{t\in\mathbb{R}}|L_{ij}(t)|\le M_{ij}$ for each i and j. Suppose that $\tilde{\mathcal{L}}$ denotes the set of bounded on $\mathbb{R}$ solutions $\phi_L(t)$ of the network (1.1), where $L(t)=\{L_{ij}(t)\}$ belongs to $\mathcal{L}$. In the present paper, we assume that $\mathcal{L}$ is an equicontinuous family on $\mathbb{R}$.

Making use of the technique indicated in the proof of Theorem 2 [56], one can prove the following assertion, which confirms the attractiveness of the set $\tilde{\mathcal{L}}$. A similar result for systems without delay was obtained in the paper [90].

Lemma 2.2. *If the conditions (C1)–(C5) are fulfilled, then for a fixed $L(t)=\{L_{ij}(t)\}, i=1,2,\ldots,m, j=1,2,\ldots,n$, all solutions of system (1.1) converge exponentially to the unique bounded on $\mathbb{R}$ solution $\phi_L(t)$.*

3. Li–Yorke chaos

Our purpose in the present section is to demonstrate that the network (1.1) behaves chaotically provided that the external inputs are chaotic. In the following lemmas, we will take advantage of the sets $\mathcal{L}$ and $\tilde{\mathcal{L}}$, which are defined in Section 2. The main result will be mentioned in Theorem 3.1.

Let us denote $K_1=2\overline{M}/(1-(M_f+K_0L_f)\delta)$, $\gamma=\min_{(i,j)}a_{ij}$ and $\overline{\delta}=\max_{(i,j)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}/(2a_{ij}-\gamma)$. We note that the number γ is positive since each $a_{ij}, i=1,2,\ldots,m, j=1,2,\ldots,n$, are positive.

The following conditions are needed:

(C6) $[M_f+(K_0+K_1)L_f]\delta<1$;
(C7) $2(M_f+K_0L_fe^{\gamma\tau/2})\overline{\delta}<1$.

The next lemma is about the proximality feature of system (1.1).

Lemma 3.1. *Under the conditions (C1)–(C7), if a pair $(L(t),\overline{L}(t))\in\mathcal{L}\times\mathcal{L}$ is proximal, then the same is true for the pair $(\phi_L(t),\phi_{\overline{L}}(t))\in\tilde{\mathcal{L}}\times\tilde{\mathcal{L}}$.*

Proof. Set $R_0=2K_0/(1-2(M_f+K_0L_fe^{\gamma\tau/2})\overline{\delta})$, $R_1=1/\gamma[1-\delta(M_f+K_0L_f)]$ and take a positive number η such that $\eta\le1/(R_0+R_1)$. Fix an arbitrary small number $\epsilon>0$ and a positive number E such that $E>(4/\gamma)\ln(1/\eta\epsilon)$. Because the pair $(L(t),\overline{L}(t))\in\mathcal{L}\times\mathcal{L}$ is proximal, there exists an interval $J=[\sigma,\sigma+E_0]$, where $E_0\ge E$, such that $\|L(t)-\overline{L}(t)\|<\eta\epsilon$ for $t\in J$.

The bounded on $\mathbb{R}$ solutions $\phi_L(t)=\{\phi_L^{ij}(t)\}$ and $\phi_{\overline{L}}(t)=\{\phi_{\overline{L}}^{ij}(t)\}$, $i=1,2,\ldots,m, j=1,2,\ldots,n$, satisfy the relation

$$\phi_L^{ij}(t)-\phi_{\overline{L}}^{ij}(t)=e^{-a_{ij}(t-\sigma)}(\phi_L^{ij}(\sigma)-\phi_{\overline{L}}^{ij}(\sigma))+\int_{\sigma}^{t}e^{-a_{ij}(t-s)}(L_{ij}(s)-\overline{L}_{ij}(s))\,ds$$
$$-\int_{\sigma}^{t}e^{-a_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}[f(\phi_L^{kl}(s-\tau))\phi_L^{ij}(s)-f(\phi_{\overline{L}}^{kl}(s-\tau))\phi_{\overline{L}}^{ij}(s)]\,ds.$$

Denote by $w(t)=\{w_{ij}(t)\}$ the difference $\phi_L(t)-\phi_{\overline{L}}(t)$. Then for each i and j, we have that

$$w_{ij}(t)=e^{-a_{ij}(t-\sigma)}(\phi_L^{ij}(\sigma)-\phi_{\overline{L}}^{ij}(\sigma))+\int_{\sigma}^{t}e^{-a_{ij}(t-s)}(L_{ij}(s)-\overline{L}_{ij}(s))\,ds$$
$$-\int_{\sigma}^{t}e^{-a_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}[f(w_{kl}(s-\tau)+\phi_{\overline{L}}^{kl}(s-\tau))(w_{ij}(s)$$
$$+\phi_{\overline{L}}^{ij}(s))-f(\phi_{\overline{L}}^{kl}(s-\tau))\phi_{\overline{L}}^{ij}(s)]\,ds.$$

Let Ψ be the set of continuous functions $w(t)=\{w_{ij}(t)\}, i=1,2,\ldots,m, j=1,2,\ldots,n$, which are defined on $\mathbb{R}$, such that $\|w(t)\|\le R_0e^{-\gamma(t-\sigma)/2}+R_1\eta\epsilon$ for $\sigma-\tau\le t\le\sigma+E_0$ and $\|w\|_0\le K_1$, where $\|w\|_0=\sup_{t\in\mathbb{R}}\|w(t)\|$.

Define on Ψ the operator $\tilde{\Pi}$ as follows:

$$(\tilde{\Pi}w(t))_{ij}=\begin{cases}\phi_L^{ij}(t)-\phi_{\overline{L}}^{ij}(t), \quad t<\sigma,\\ e^{-a_{ij}(t-\sigma)}(\phi_L^{ij}(\sigma)-\phi_{\overline{L}}^{ij}(\sigma))+\int_{\sigma}^{t}e^{-a_{ij}(t-s)}(L_{ij}(s)-\overline{L}_{ij}(s))\,ds\\ \quad-\int_{\sigma}^{t}e^{-a_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}[f(w_{kl}(s-\tau)+\phi_{\overline{L}}^{kl}(s-\tau))(w_{ij}(s)+\phi_{\overline{L}}^{ij}(s))\\ \quad-f(\phi_{\overline{L}}^{kl}(s-\tau))\phi_{\overline{L}}^{ij}(s)]\,ds, \quad t\ge\sigma.\end{cases}$$

First, we will show that $\tilde{\Pi}:\Psi\to\Psi$. Indeed, if $w(t)$ belongs to Ψ, then for $t\in[\sigma,\sigma+E_0]$ it is true that

$$|(\tilde{\Pi}w(t))_{ij}|\le e^{-a_{ij}(t-\sigma)}|\phi_L^{ij}(\sigma)-\phi_{\overline{L}}^{ij}(\sigma)|+\int_{\sigma}^{t}e^{-a_{ij}(t-s)}|L_{ij}(s)-\overline{L}_{ij}(s)|\,ds$$
$$+\int_{\sigma}^{t}e^{-a_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}|f(w_{kl}(s-\tau)+\phi_{\overline{L}}^{kl}(s-\tau))||w_{ij}(s)|\,ds$$
$$+\int_{\sigma}^{t}e^{-a_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}|f(w_{kl}(s-\tau)+\phi_{\overline{L}}^{kl}(s-\tau))$$
$$-f(\phi_{\overline{L}}^{kl}(s-\tau))||\phi_{\overline{L}}^{kl}(s)|\,ds$$
$$\le2K_0e^{-\gamma(t-\sigma)}+\int_{\sigma}^{t}e^{-a_{ij}(t-s)}\eta\epsilon\,ds$$
$$+\int_{\sigma}^{t}e^{-a_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}M_f(R_0e^{-\gamma(s-\sigma)/2}+R_1\eta\epsilon)\,ds$$
$$+\int_{\sigma}^{t}e^{-a_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}K_0L_f(R_0e^{-\gamma(s-\tau-\sigma)/2}+R_1\eta\epsilon)\,ds$$
$$=2K_0e^{-\gamma(t-\sigma)}+\frac{\eta\epsilon}{a_{ij}}(1-e^{-a_{ij}(t-\sigma)})$$
$$+2R_0(M_f+K_0L_fe^{\gamma\tau/2})\frac{\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}}{2a_{ij}-\gamma}e^{-\gamma(t-\sigma)/2}(1-e^{-(a_{ij}-\gamma/2)(t-\sigma)})$$
$$+R_1\eta\epsilon(M_f+K_0L_f)\frac{\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}}{a_{ij}}(1-e^{-a_{ij}(t-\sigma)}).$$

Hence, for $t \in [\sigma, \sigma+E_0]$, it can be verified that

$$\begin{aligned}\|\tilde{\Pi} w(t)\| &\leq 2K_0 e^{-\gamma(t-\sigma)} + \frac{\eta\epsilon}{\gamma} + 2R_0(M_f + K_0 L_f e^{\gamma\tau/2})\overline{\delta} e^{-\gamma(t-\sigma)/2} \\ &\quad + R_1 \eta\epsilon (M_f + K_0 L_f)\delta \\ &\leq 2\left[K_0 + R_0(M_f + K_0 L_f e^{\gamma\tau/2})\overline{\delta}\right] e^{-\gamma(t-\sigma)/2} + \eta\epsilon\left[\frac{1}{\gamma} + R_1(M_f + K_0 L_f)\delta\right] \\ &= R_0 e^{-\gamma(t-\sigma)/2} + R_1\eta\epsilon.\end{aligned}$$

Since $R_0 > 2K_0$, the inequality $\|\tilde{\Pi} w(t)\| \leq R_0 e^{-\gamma(t-\sigma)/2} + R_1\eta\epsilon$ holds also for $\sigma - \tau \leq t < \sigma$.

On the other hand, if $w(t)$ belongs to Ψ, then making benefit of the inequality $K_1 \geq 2K_0$ one can confirm for $t \geq \sigma$ that

$$\begin{aligned}|(\tilde{\Pi} w(t))_{ij}| &\leq e^{-a_{ij}(t-\sigma)}|\phi_L^{ij}(\sigma) - \phi_{\overline{L}}^{ij}(\sigma)| + \int_\sigma^t 2M_{ij} e^{-a_{ij}(t-s)}\, ds \\ &\quad + \int_\sigma^t e^{-a_{ij}(t-s)} \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} |f(w_{kl}(s-\tau) + \phi_{\overline{L}}^{kl}(s-\tau))||w_{ij}(s)|\, ds \\ &\quad + \int_\sigma^t e^{-a_{ij}(t-s)} \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} L_f |w_{kl}(s-\tau)||\phi_{\overline{L}}^{ij}(s)|\, ds \\ &\leq 2K_0 e^{-a_{ij}(t-\sigma)} + \left(\frac{2M_{ij}}{a_{ij}} + K_1(M_f + K_0 L_f)\frac{\sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}}{a_{ij}}\right)(1 - e^{-a_{ij}(t-\sigma)}) \\ &\leq e^{-a_{ij}(t-\sigma)}[2K_0 - 2\overline{M} - K_1(M_f + K_0 L_f)\delta] + 2\overline{M} + K_1(M_f + K_0 L_f)\delta \leq K_1.\end{aligned}$$

Therefore, the inequality $\|\tilde{\Pi} w\|_0 \leq K_1$ is valid. Thus, $\tilde{\Pi}(\Psi) \subseteq \Psi$.

Now, we shall verify that the operator $\tilde{\Pi}$ is a contraction. Suppose that $w(t), \overline{w}(t) \in \Psi$. For $t \geq \sigma$, we have that

$$\begin{aligned}|(\tilde{\Pi} w(t))_{ij} - (\tilde{\Pi}\overline{w}(t))_{ij}| &\leq \int_\sigma^t e^{-a_{ij}(t-s)} \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} |f(w_{kl}(s-\tau) \\ &\qquad + \phi_{\overline{L}}^{kl}(s-\tau))(w_{ij}(s) + \phi_{\overline{L}}^{ij}(s)) \\ &\qquad - f(\overline{w}_{kl}(s-\tau) + \phi_{\overline{L}}^{kl}(s-\tau))(\overline{w}_{ij}(s) + \phi_{\overline{L}}^{ij}(s))|\, ds \\ &\leq \int_\sigma^t e^{-a_{ij}(t-s)} \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} |f(w_{kl}(s-\tau) + \phi_{\overline{L}}^{kl}(s-\tau))||w_{ij}(s) - \overline{w}_{ij}(s)|\, ds \\ &\quad + \int_\sigma^t e^{-a_{ij}(t-s)} \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} L_f |w_{kl}(s-\tau) - \overline{w}_{kl}(s-\tau)||\overline{w}_{ij}(s) + \phi_{\overline{L}}^{ij}(s)|\, ds \\ &\leq [M_f + (K_0 + K_1)L_f]\frac{\sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}}{a_{ij}} \sup_{t \geq \sigma - \tau} \|w(t) - \overline{w}(t)\|.\end{aligned}$$

In view of the equation $\|\tilde{\Pi} w(t) - \tilde{\Pi}\overline{w}(t)\| = 0$ for $t < \sigma$, the last inequality implies that

$$\|\tilde{\Pi} w - \tilde{\Pi}\overline{w}\|_0 \leq [M_f + (K_0 + K_1)L_f]\delta \|w - \overline{w}\|_0,$$

and the operator $\tilde{\Pi}$ is contractive according to the condition (C6).

By means of the uniqueness of solutions one can conclude that $w(t) = \phi_L(t) - \phi_{\overline{L}}(t)$ is the unique fixed point of the operator $\tilde{\Pi}$.

Since the number E satisfies the inequality $E > (4/\gamma)\ln(1/\eta\epsilon)$, we have $e^{-\gamma(t-\sigma)/2} < \eta\epsilon$, provided that $t \geq \sigma + E/2$. Therefore, the inequality $\|\phi_L(t) - \phi_{\overline{L}}(t)\| < (R_0 + R_1)\eta\epsilon \leq \epsilon$ holds for $t \in [\sigma + E/2, \sigma + E_0]$. Consequently, the pair $(\phi_L(t), \phi_{\overline{L}}(t)) \in \tilde{\mathcal{L}} \times \tilde{\mathcal{L}}$ is proximal. □

Lemma 3.2. *Suppose that the conditions (C1)–(C5) are fulfilled. If a pair* $(L(t), \overline{L}(t)) \in \mathcal{L} \times \mathcal{L}$ *is frequently* (ϵ_0, Δ)*–separated for some positive numbers* ϵ_0 *and* Δ*, then there exist positive numbers* ϵ_1 *and* $\overline{\Delta}$ *such that the pair* $(\phi_L(t), \phi_{\overline{L}}(t)) \in \tilde{\mathcal{L}} \times \tilde{\mathcal{L}}$ *is frequently* $(\epsilon_1, \overline{\Delta})$*–separated.*

Proof. Since the pair $(L(t), \overline{L}(t)) \in \mathcal{L} \times \mathcal{L}$ is frequently (ϵ_0, Δ) separated for some numbers $\epsilon_0 > 0, \Delta > 0$, there exist infinitely many intervals $J_q = [\alpha_q, \beta_q], q \in \mathbb{N}$, each with a length no less than Δ, such that $\beta_q \to \infty$ as $q \to \infty$, and $\|L(t) - \overline{L}(t)\| > \epsilon_0$ for each t from these intervals. The essence of the proof is to determine numbers $\epsilon_1 > 0, \overline{\Delta} > 0$ and infinitely many intervals $\overline{J}_q = [\overline{\alpha}_q, \overline{\beta}_q], q \in \mathbb{N}$, each with length $\overline{\Delta}$, such that $\overline{\beta}_q \to \infty$ as $q \to \infty$, and $\|\phi_L(t) - \phi_{\overline{L}}(t)\| > \epsilon_1$ for each t from the intervals $\overline{J}_q$, $q \in \mathbb{N}$.

Since $\mathcal{L}$ is an equicontinuous family on $\mathbb{R}$, there exists a positive number κ such that for any $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \kappa$, the inequality

$$|(L_{ij}(t_1) - \overline{L}_{ij}(t_1)) - (L_{ij}(t_2) - \overline{L}_{ij}(t_2))| < \frac{\epsilon_0}{2} \tag{3.3}$$

holds for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. For each $q \in \mathbb{N}$, set $\theta_q = \beta_q - \kappa/2$.

Let us fix $q \in \mathbb{N}$. There exist integers i_0, j_0 such that

$$|L_{i_0 j_0}(\beta_q) - \overline{L}_{i_0 j_0}(\beta_q)| = \|L(\beta_q) - \overline{L}(\beta_q)\| > \epsilon_0. \tag{3.4}$$

By virtue of the inequality (3.3), it can be verified for each $t \in [\theta_q, \theta_q + \kappa]$ that

$$\begin{aligned}|L_{i_0 j_0}(\beta_q) - \overline{L}_{i_0 j_0}(\beta_q)| - |L_{i_0 j_0}(t) - \overline{L}_{i_0 j_0}(t)| &\leq |(L_{i_0 j_0}(t) - \overline{L}_{i_0 j_0}(t)) \\ &\quad - (L_{i_0 j_0}(\beta_q) - \overline{L}_{i_0 j_0}(\beta_q))| < \frac{\epsilon_0}{2}.\end{aligned}$$

Therefore, making use of (3.4), one can confirm that

$$|L_{i_0 j_0}(t) - \overline{L}_{i_0 j_0}(t)| > |L_{i_0 j_0}(\beta_q) - \overline{L}_{i_0 j_0}(\beta_q)| - \frac{\epsilon_0}{2} > \frac{\epsilon_0}{2}, \quad \theta_q \leq t \leq \theta_q + \kappa. \tag{3.5}$$

For each i and j, there exist numbers $\zeta_{ij}^q \in [\theta_q, \theta_q + \kappa]$ such that

$$\int_{\theta_q}^{\theta_q + \kappa} (L(s) - \overline{L}(s))\, ds = \kappa(L_{11}(\zeta_{11}^q) - \overline{L}_{11}(\zeta_{11}^q), \ldots, L_{mn}(\zeta_{mn}^q) - \overline{L}_{mn}(\zeta_{mn}^q)).$$

Thus, by means of the inequality (3.5), we obtain that

$$\left\|\int_{\theta_q}^{\theta_q + \kappa} (L(s) - \overline{L}(s))\, ds\right\| \geq \kappa|L_{i_0 j_0}(\zeta_{i_0 j_0}^q) - \overline{L}_{i_0 j_0}(\zeta_{i_0 j_0}^q)| > \frac{\kappa\epsilon_0}{2}. \tag{3.6}$$

For $t \in [\theta_q, \theta_q + \kappa]$, by the help of the relations

$$\begin{aligned}\phi_L^{ij}(t) &= \phi_L^{ij}(\theta_q) - \int_{\theta_q}^t \left[a_{ij} + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(\phi_L^{kl}(s-\tau))\right]\phi_L^{ij}(s)\, ds \\ &\quad + \int_{\theta_q}^t L_{ij}(s)\, ds\end{aligned}$$

and

$$\phi_{\overline{L}}^{ij}(t) = \phi_{\overline{L}}^{ij}(\theta_q) - \int_{\theta_q}^t \left[a_{ij} + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(\phi_{\overline{L}}^{kl}(s-\tau))\right]\phi_{\overline{L}}^{ij}(s)\, ds + \int_{\theta_q}^t \overline{L}_{ij}(s)\, ds,$$

we attain that

$$\begin{aligned}\phi_L^{ij}(\theta_q + \kappa) - \phi_{\overline{L}}^{ij}(\theta_q + \kappa) &= \int_{\theta_q}^{\theta_q + \kappa} (L_{ij}(s) - \overline{L}_{ij}(s))\, ds + (\phi_L^{ij}(\theta_q) - \phi_{\overline{L}}^{ij}(\theta_q)) \\ &\quad - \int_{\theta_q}^{\theta_q + \kappa} a_{ij}(\phi_L^{ij}(s) - \phi_{\overline{L}}^{ij}(s))\, ds \\ &\quad - \int_{\theta_q}^{\theta_q + \kappa} \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} [f(\phi_L^{kl}(s-\tau))\phi_L^{ij}(s) \\ &\quad - f(\phi_{\overline{L}}^{kl}(s-\tau))\phi_{\overline{L}}^{ij}(s)]\, ds.\end{aligned}$$

Hence, we have that

$$\begin{aligned}\|\phi_L(\theta_q + \kappa) - \phi_{\overline{L}}(\theta_q + \kappa)\| &\geq \left\|\int_{\theta_q}^{\theta_q + \kappa} (L(s) - \overline{L}(s))\, ds\right\| \\ &\quad - \|\phi_L(\theta_q) - \phi_{\overline{L}}(\theta_q)\| - \max_{(i,j)}\left|\int_{\theta_q}^{\theta_q + \kappa} a_{ij}(\phi_L^{ij}(s) - \phi_{\overline{L}}^{ij}(s))\, ds\right| \\ &\quad - \max_{(i,j)}\left|\int_{\theta_q}^{\theta_q + \kappa} \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} [f(\phi_L^{kl}(s-\tau))\phi_L^{ij}(s) \right. \\ &\quad \left. - f(\phi_{\overline{L}}^{kl}(s-\tau))\phi_{\overline{L}}^{ij}(s)]\, ds\right|.\end{aligned} \tag{3.7}$$

Set $\overline{a}=\max_{(i,j)}a_{ij}, \overline{c}=\max_{(i,j)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}$ and $H_0=\max_{(i,j)}M_{ij}$. The inequalities (3.6) and (3.7) together imply that

$$\max_{t\in[\theta_q-\tau,\theta_q+\kappa]}\|\phi_L(t)-\phi_{\overline{L}}(t)\| \geq \|\phi_L(\theta_q+\kappa)-\phi_{\overline{L}}(\theta_q+\kappa)\|$$
$$>\frac{\kappa\epsilon_0}{2}-[1+\kappa\overline{a}+\kappa\overline{c}(L_fK_0+M_f)]\max_{t\in[\theta_q-\tau,\theta_q+\kappa]}\|\phi_L(t)-\phi_{\overline{L}}(t)\|.$$

Therefore, $\max_{t\in[\theta_q-\tau,\theta_q+\kappa]}\|\phi_L(t)-\phi_{\overline{L}}(t)\|>\overline{\epsilon}$, where $\overline{\epsilon}=\kappa\epsilon_0/2[2+\kappa\overline{a}+\kappa\overline{c}(L_fK_0+M_f)]$.

Now, suppose that $\max_{t\in[\theta_q-\tau,\theta_q+\kappa]}\|\phi_L(t)-\phi_{\overline{L}}(t)\|=\|\phi_L(\xi_q)-\phi_{\overline{L}}(\xi_q)\|$, for some $\xi_q\in[\theta_q-\tau,\theta_q+\kappa]$. Take a positive number Δ_0 such that $\Delta_0\leq\frac{\overline{\epsilon}}{4(H_0+K_0\overline{a}+M_fK_0\overline{c})}$. For $t\in[\xi_q-\Delta_0,\xi_q+\Delta_0]$, with the aid of the relation

$$\phi_L^{ij}(t)-\phi_{\overline{L}}^{ij}(t)=(\phi_L^{ij}(\xi_q)-\phi_{\overline{L}}^{ij}(\xi_q))$$
$$+\int_{\xi_q}^{t}(L_{ij}(s)-\overline{L}_{ij}(s))\,ds-\int_{\xi_q}^{t}a_{ij}(\phi_L^{ij}(s)-\phi_{\overline{L}}^{ij}(s))\,ds$$
$$-\int_{\xi_q}^{t}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}[f(\phi_L^{kl}(s-\tau))\phi_L^{ij}(s)-f(\phi_{\overline{L}}^{kl}(s-\tau))\phi_{\overline{L}}^{ij}(s)]\,ds,$$

we obtain the inequality

$$\|\phi_L(t)-\phi_{\overline{L}}(t)\|\geq\|\phi_L(\xi_q)-\phi_{\overline{L}}(\xi_q)\|$$
$$-\max_{(i,j)}\left|\int_{\xi_q}^{t}(L_{ij}(s)-\overline{L}_{ij}(s))\,ds\right|-\max_{(i,j)}\left|\int_{\xi_q}^{t}a_{ij}(\phi_L^{ij}(s)-\phi_{\overline{L}}^{ij}(s))\,ds\right|$$
$$-\max_{(i,j)}\left|\int_{\xi_q}^{t}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}[f(\phi_L^{kl}(s-\tau))\phi_L^{ij}(s)-f(\phi_{\overline{L}}^{kl}(s-\tau))\phi_{\overline{L}}^{ij}(s)]\,ds\right|$$
$$>\overline{\epsilon}-2\Delta_0(H_0+K_0\overline{a}+M_fK_0\overline{c})$$
$$\geq\frac{\overline{\epsilon}}{2}.$$

Hence, we have $\|\phi_L(t)-\phi_{\overline{L}}(t)\|>\overline{\epsilon}/2$ for each t from the intervals $\overline{J}_q=[\overline{\alpha}_q,\overline{\beta}_q], q\in\mathbb{N}$, where $\overline{\alpha}_q=\xi_q-\Delta_0$ and $\overline{\beta}_q=\xi_q+\Delta_0$. One can confirm that $\overline{\beta}_q\to\infty$ as $q\to\infty$. Consequently, the couple $(\phi_L(t),\phi_{\overline{L}}(t))\in\tilde{\mathcal{L}}\times\tilde{\mathcal{L}}$ is frequently $(\epsilon_1,\overline{\Delta})$–separated, where $\epsilon_1=\overline{\epsilon}/2$ and $\overline{\Delta}=2\Delta_0$. □

The main result of the present study is as follows.

Theorem 3.1. *Under the conditions (C1)–(C7), the set $\tilde{\mathcal{L}}$ is Li–Yorke chaotic, provided that the same is true for the set $\mathcal{L}$.*

Proof. Since the set $\mathcal{L}$ is Li–Yorke chaotic, there exists a positive number T_0 such that for any $k\in\mathbb{N}$, $\mathcal{L}$ possesses a periodic function with period kT_0. One can use the integral equation (2.2) together with condition (C5) to verify that $L(t)\in\mathcal{L}$ is kT_0-periodic if and only if $\phi_L(t)\in\tilde{\mathcal{L}}$ is kT_0-periodic. Thus, for each $k\in\mathbb{N}$, the set $\tilde{\mathcal{L}}$ contains a kT_0-periodic function.

Suppose that $\mathcal{L}_S$ is a scrambled set inside $\mathcal{L}$. Consider the collection $\tilde{\mathcal{L}}_S$ with elements of the form $\phi_L(t)$, where $L(t)\in\mathcal{L}_S$. Because of the one-to-one correspondence between the elements of $\mathcal{L}_S$ and $\tilde{\mathcal{L}}_S$, the set $\tilde{\mathcal{L}}_S$ is uncountable. Moreover, no periodic functions exist inside $\tilde{\mathcal{L}}_S$, since no such functions take place inside $\mathcal{L}_S$.

Lemmas 3.1 and 3.2 together ensure that $\tilde{\mathcal{L}}_S$ is a scrambled set. Additionally, Lemma 3.2 implies that any pair of functions inside $\tilde{\mathcal{L}}_S\times\tilde{\mathcal{L}}_P$ is frequently $(\epsilon_1,\overline{\Delta})$–separated for some positive numbers ϵ_1 and $\overline{\Delta}$, where $\tilde{\mathcal{L}}_P$ denotes the set of all periodic functions inside $\tilde{\mathcal{L}}$. As a consequence, the set $\tilde{\mathcal{L}}$ is Li–Yorke chaotic. □

Since time delay is an inevitable feature of neural networks, the result presented in Theorem 3.1 is much more realistic than the one obtained in [89]. That is, unless retardation is not introduced in the models, investigations of neural networks will not be adequate for application problems. Introducing delay requests a more sophisticated mathematical analysis, and this is the first time in the literature that the approach developed in [89,90] is applied to functional differential equations.

Suppose that $F:\mathbb{R}^{m\times n}\to\mathbb{R}^{\overline{m}\times\overline{n}}$ is a function such that for all $s_1,s_2\in\mathbb{R}^{m\times n}$ the inequality

$$L_1\|s_1-s_2\|\leq\|F(s_1)-F(s_2)\|\leq L_2\|s_1-s_2\| \tag{3.8}$$

is valid, where L_1 and L_2 are positive numbers. One can verify that if a collection $\mathcal{L}$ of functions is Li–Yorke chaotic, then the collection with elements of the form $F(L(t))$, where $L(t)\in\mathcal{L}$, is also Li–Yorke chaotic.

In the next section, we will focus on a neural system consisting of three layers such that each layer is a SICNN, and the connections between the layers are provided through nonlinear functions that satisfy the inequality (3.8).

4. An example

In the theory of neural networks, one can consider interconnected collections of neurons, called layers. Additionally, a neural system is a collection of neural networks, which can be considered as single layers. Each neuron in a neural network is capable of receiving input signals, processing them and sending an output signal. Neural signals consist of short electrical pulses, which are called action potentials or spikes. That is why the discontinuity phenomena is a natural property of neural networks. A chain of action potentials emitted by a single neuron is called a spike train. Action potentials in a spike train are usually well separated, and it is impossible to excite a second spike during or immediately after the first one [111]. In this section, we take into account an example of a neural system consisting of three layers, where each layer is a SICNN. Discontinuous external inputs are used in the first layer to provide the chaos.

Consider the retarded SICNNs

$$\frac{dx_{ij}}{dt}=-a_{ij}x_{ij}-\sum_{C_{kl}\in N_1(i,j)}C_{ij}^{kl}f(x_{kl}(t-\tau_1))x_{ij}+L_{ij}(t), \tag{4.9}$$

$$\frac{dy_{ij}}{dt}=-b_{ij}y_{ij}-\sum_{\overline{C}_{kl}\in N_1(i,j)}\overline{C}_{ij}^{kl}g(y_{kl}(t-\tau_2))y_{ij}+\overline{L}_{ij}(t), \tag{4.10}$$

$$\frac{dz_{ij}}{dt}=-c_{ij}z_{ij}-\sum_{\overline{\overline{C}}_{kl}\in N_1(i,j)}\overline{\overline{C}}_{ij}^{kl}h(z_{kl}(t-\tau_3))z_{ij}+\overline{\overline{L}}_{ij}(t), \tag{4.11}$$

in which $i,j=1,2,3$,

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}=\begin{pmatrix}3&7&2\\9&4&5\\1&3&6\end{pmatrix},$$
$$\begin{pmatrix}C_{11}&C_{12}&C_{13}\\C_{21}&C_{22}&C_{23}\\C_{31}&C_{32}&C_{33}\end{pmatrix}=\begin{pmatrix}0&0.008&0.002\\0.001&0.003&0.007\\0.004&0&0.006\end{pmatrix},$$

$$\begin{pmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{pmatrix}=\begin{pmatrix}4&7&5\\3&6&8\\10&9&4\end{pmatrix},$$
$$\begin{pmatrix}\overline{C}_{11}&\overline{C}_{12}&\overline{C}_{13}\\\overline{C}_{21}&\overline{C}_{22}&\overline{C}_{23}\\\overline{C}_{31}&\overline{C}_{32}&\overline{C}_{33}\end{pmatrix}=\begin{pmatrix}0.004&0.007&0.002\\0&0.006&0.003\\0.005&0.009&0.008\end{pmatrix},$$

$$\begin{pmatrix}c_{11}&c_{12}&c_{13}\\c_{21}&c_{22}&c_{23}\\c_{31}&c_{32}&c_{33}\end{pmatrix}=\begin{pmatrix}2&8&4\\1&1&3\\6&2&5\end{pmatrix},$$

$$\begin{pmatrix} \overline{\overline{C}}_{11} & \overline{\overline{C}}_{12} & \overline{\overline{C}}_{13} \\ \overline{\overline{C}}_{21} & \overline{\overline{C}}_{22} & \overline{\overline{C}}_{23} \\ \overline{\overline{C}}_{31} & \overline{\overline{C}}_{32} & \overline{\overline{C}}_{33} \end{pmatrix} = \begin{pmatrix} 0.006 & 0 & 0.002 \\ 0.004 & 0.001 & 0.008 \\ 0 & 0.007 & 0.002 \end{pmatrix},$$

$f(s)=\frac{1}{2}s^2$, $g(s)=\frac{1}{3}s^3$, $h(s)=\sqrt{s}$, $\tau_1=0.5$, $\tau_2=3$ and $\tau_3=2.5$.

To obtain chaotic SICNNs with delay by means of the presented method, one needs a collection of external inputs which are known to be chaotic in the sense of Li–Yorke. For that reason, in system (4.9), the external inputs, $L_{ij}(t)$, will be considered as relay functions with chaotically changing switching moments [95–98]. More precisely, we set $L_{ij}(t)=\nu_{ij}(t,\zeta)$, where

$$\nu_{ij}(t,\zeta)=\begin{cases} \alpha_{ij} & \text{if } \zeta_{2q}<t\le\zeta_{2q+1}, \\ \beta_{ij} & \text{if } \zeta_{2q-1}<t\le\zeta_{2q}. \end{cases} \tag{4.12}$$

In the relay function (4.12), $\alpha=\{\alpha_{ij}\}$ and $\beta=\{\beta_{ij}\}$ are different from each other and the sequence $\zeta=\{\zeta_q\}$, $q\in\mathbb{Z}$, of switching moments is the same for each i and j. The sequence ζ is defined by the formula $\zeta_q=q+\kappa_q$, $q\in\mathbb{Z}$, where the sequence $\{\kappa_q\}, \kappa_0\in[0,1]$, is generated through the equation $\kappa_{q+1}=\lambda(\kappa_q)$, and $\lambda(s)=3.9s(1-s)$ is the logistic map, which is chaotic in the Li–Yorke sense [39]. The interval $[0,1]$ is invariant under the iterations of the map $\lambda(s)$ [112]. The presence of chaos in the SICNN (4.9) can be proved in a similar manner to those mentioned in Section 3. It is worth noting that external inputs of the form (4.12) are Li–Yorke chaotic and this type of inputs has never been considered before in the literature for SICNNs with delay.

Let us use $\alpha_{ij}=1.2$ and $\beta_{ij}=2.5$ in (4.9). Clearly, $\sum_{C_{kl}\in N_1(1,1)} C_{11}^{kl}=0.012$, $\sum_{C_{kl}\in N_1(1,2)} C_{12}^{kl}=0.021$, $\sum_{C_{kl}\in N_1(1,3)} C_{13}^{kl}=0.020$, $\sum_{C_{kl}\in N_1(2,1)} C_{21}^{kl}=0.016$, $\sum_{C_{kl}\in N_1(2,2)} C_{22}^{kl}=0.031$, $\sum_{C_{kl}\in N_1(2,3)} C_{23}^{kl}=0.026$, $\sum_{C_{kl}\in N_1(3,1)} C_{31}^{kl}=0.008$, $\sum_{C_{kl}\in N_1(3,2)} C_{32}^{kl}=0.021$, $\sum_{C_{kl}\in N_1(3,3)} C_{33}^{kl}=0.016$. One can confirm that the conditions (C1)–(C7) hold for system (4.9) with $\gamma=1$, $L_f=2.5$, $M_f=3.125$, $\overline{M}=2.5$, $\delta=0.01$, $\overline{\delta}=0.008$, $K_0=2.581$ and $K_1=5.53$. Therefore, the collection $\mathcal{L}_x$ of bounded on $\mathbb{R}$ solutions of (4.9) with different ζ is a Li–Yorke chaotic set.

Consider the constant function $u_1(t)=\{u_1^{ij}(t)\}$ with $u_1^{11}(t)=0.652$, $u_1^{12}(t)=0.263$, $u_1^{13}(t)=0.942$, $u_1^{21}(t)=0.215$, $u_1^{22}(t)=0.517$, $u_1^{23}(t)=0.364$, $u_1^{31}(t)=1.846$, $u_1^{32}(t)=0.658$ and $u_1^{33}(t)=0.361$. We use the sequence ζ with $\zeta_0=0.38$, and represent in Fig. 1 the solution $x(t)=\{x_{ij}(t)\}$ of (4.9) satisfying $x(t)=u_1(t)$ for $t_0-\tau_1\le t\le t_0$, where $t_0=0.38$. Fig. 1 reveals that each coordinate of the solution behaves chaotically.

Now, we shall focus on the SICNN (4.10). Consider the function $\varphi(v)=\{\varphi_{ij}(v)\}$, where $v=\{v_{ij}\}$ and $\varphi_{11}(v)=0.5\tanh(v_{11})$, $\varphi_{12}(v)=2v_{12}+\arctan v_{12}$, $\varphi_{13}(v)=2v_{13}^2$, $\varphi_{21}(v)=\sqrt{v_{31}}$, $\varphi_{22}(v)=0.4e^{v_{32}}$, $\varphi_{23}(v)=v_{33}+0.7\cos v_{33}$, $\varphi_{31}(v)=1/(v_{21}^2+1)$, $\varphi_{32}(v)=(v_{22}^2+2v_{22}+2)/(v_{22}+1)$, $\varphi_{33}(v)=(0.9+v_{23})^3$. In system (4.10), we set $\overline{L}_{ij}(t)=\varphi_{ij}(x(t))$ for each $i,j=1,2,3$. That is, the external inputs $\overline{L}_{ij}(t)$ of the network (4.10) are provided through the outputs of (4.9).

The function φ satisfies the inequality (3.8) inside the compact region where the chaotic attractor of system (4.9) takes place. Therefore, the collection which consists of elements of the form $\varphi(x(t))$, $x(t)\in\mathcal{L}_x$, is a Li–Yorke chaotic set.

One can evaluate that $\sum_{\overline{C}_{kl}\in N_1(1,1)} \overline{C}_{11}^{kl}=0.017$, $\sum_{\overline{C}_{kl}\in N_1(1,2)} \overline{C}_{12}^{kl}=0.022$, $\sum_{\overline{C}_{kl}\in N_1(1,3)} \overline{C}_{13}^{kl}=0.018$, $\sum_{\overline{C}_{kl}\in N_1(2,1)} \overline{C}_{21}^{kl}=0.031$, $\sum_{\overline{C}_{kl}\in N_1(2,2)} \overline{C}_{22}^{kl}=0.044$, $\sum_{\overline{C}_{kl}\in N_1(2,3)} \overline{C}_{23}^{kl}=0.035$, $\sum_{\overline{C}_{kl}\in N_1(3,1)} \overline{C}_{31}^{kl}=0.020$, $\sum_{\overline{C}_{kl}\in N_1(3,2)} \overline{C}_{32}^{kl}=0.031$, $\sum_{\overline{C}_{kl}\in N_1(3,3)} \overline{C}_{33}^{kl}=0.026$, and the conditions (C1)–(C7) hold for system (4.10) with $\gamma=3$, $L_g=0.49$, $M_g=0.1145$, $\overline{M}=0.762175$, $\delta=\overline{\delta}=0.031/3$, $K_0=0.7631$ and $K_1=1.5322$. Consequently, the set $\mathcal{L}_y$ of bounded on $\mathbb{R}$ solutions of system (4.10) is Li–Yorke chaotic in accordance with Theorem 3.1.

We represent in Fig. 2 the solution of system (4.10) with $y(t)=u_2(t)$ for $t_0-\tau_2\le t\le t_0$, where $u_2(t)=\{u_2^{ij}\}$ is a constant function defined as $u_2^{11}(t)=0.071$, $u_2^{12}(t)=0.125$, $u_2^{13}(t)=0.412$, $u_2^{21}(t)=0.454$, $u_2^{22}(t)=0.132$, $u_2^{23}(t)=0.127$, $u_2^{31}(t)=0.094$, $u_2^{32}(t)=0.245$, $u_2^{33}(t)=0.442$ and $t_0=0.38$. Fig. 2 supports the theoretical results such that the dynamics of the SICNN (4.10) is chaotic.

In a similar way, in system (4.11), we take $\overline{\overline{L}}_{ij}(t)=\psi_{ij}(y(t))$, $i,j=1,2,3$, where the function $\psi(v)=\{\psi_{ij}(v)\}$ is defined through the equations $\psi_{11}(v)=\frac{1}{3}v_{33}$, $\psi_{12}(v)=2v_{32}+\sin v_{32}$, $\psi_{13}(v)=10v_{31}^3$, $\psi_{21}(v)=(1+v_{11})^{1/3}$, $\psi_{22}(v)=0.5\arctan v_{12}$, $\psi_{23}(v)=2v_{13}$, $\psi_{31}(v)=\tanh v_{21}$, $\psi_{32}(v)=1/(v_{22}+2)$ and $\psi_{33}(v)=1.5\sqrt{1+v_{23}}$. It can be verified that the inequality (3.8) holds for the function ψ,

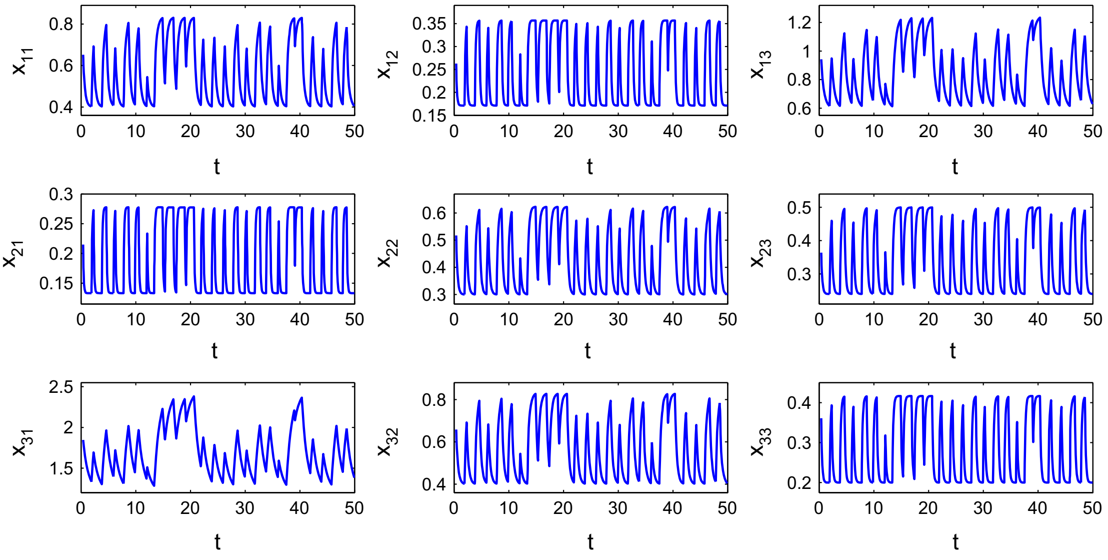

Fig. 1. The irregular behavior in each cell of the SICNN (4.9).

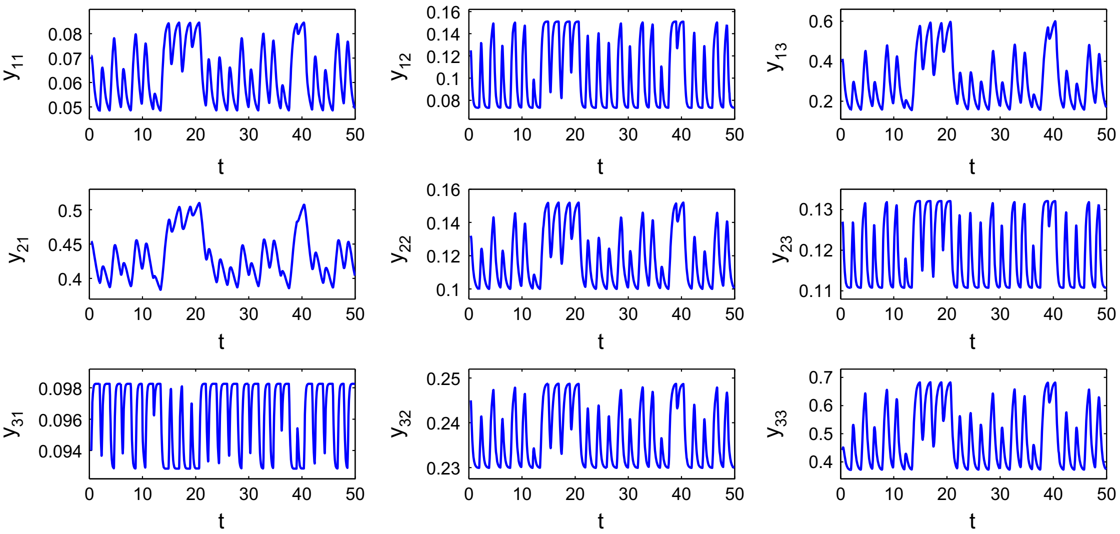

Fig. 2. The motions that appear in the cells of the SICNN (4.10). Our theoretical discussions are supported such that each cell behaves chaotically.

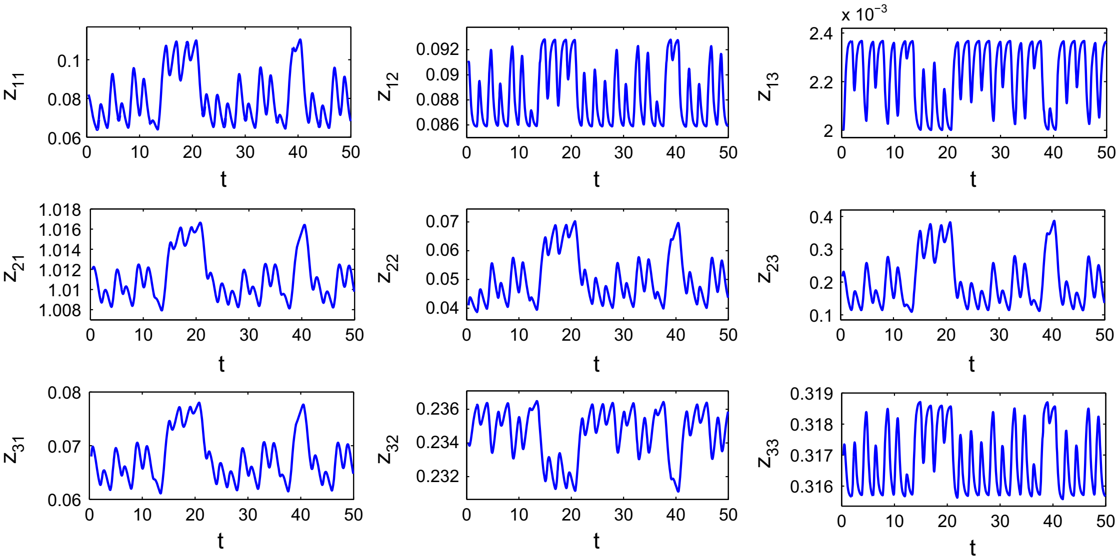

Fig. 3. The chaotic motions in each cell of the SICNN (4.11).

and the collection with elements of the form $\psi(y(t))$, $y(t) \in \mathcal{L}_y$, is Li–Yorke chaotic.

In system (4.11), we have that $\sum_{\overline{\overline{C}}_{kl} \in N_1(1,1)} \overline{\overline{C}}^{kl}_{11} = 0.011$, $\sum_{\overline{\overline{C}}_{kl} \in N_1(1,2)} \overline{\overline{C}}^{kl}_{12} = 0.021$, $\sum_{\overline{\overline{C}}_{kl} \in N_1(1,3)} \overline{\overline{C}}^{kl}_{13} = 0.011$, $\sum_{\overline{\overline{C}}_{kl} \in N_1(2,1)} \overline{\overline{C}}^{kl}_{21} = 0.018$, $\sum_{\overline{\overline{C}}_{kl} \in N_1(2,2)} \overline{\overline{C}}^{kl}_{22} = 0.03$, $\sum_{\overline{\overline{C}}_{kl} \in N_1(2,3)} \overline{\overline{C}}^{kl}_{23} = 0.02$, $\sum_{\overline{\overline{C}}_{kl} \in N_1(3,1)} \overline{\overline{C}}^{kl}_{31} = 0.012$, $\sum_{\overline{\overline{C}}_{kl} \in N_1(3,2)} \overline{\overline{C}}^{kl}_{32} = 0.022$, $\sum_{\overline{\overline{C}}_{kl} \in N_1(3,3)} \overline{\overline{C}}^{kl}_{33} = 0.018$. Moreover, the conditions (C1)–(C7) are valid with $\gamma = 1$, $L_h = 0.4951$, $M_h = 1.01$, $\overline{M} = 1.0292$, $\delta = \overline{\delta} = 0.03$, $K_0 = 1.0614$ and $K_1 = 2.1579$. Thus, in compliance with Theorem 3.1, the dynamics of system (4.11) is Li–Yorke chaotic. That is, the set $\mathcal{L}_z$ of bounded on $\mathbb{R}$ solutions of (4.11) is chaotic in the sense of Li–Yorke.

The behavior of the SICNN (4.11) is observable in Fig. 3, which depicts the solution with $z(t) = u_3(t)$ for $t_0 - \tau_3 \le t \le t_0$, where $t_0 = 0.38$ and the constant function $u_3(t) = \{u_3^{ij}\}$ is defined as $u_3^{11}(t) = 0.082$, $u_3^{12}(t) = 0.091$, $u_3^{13}(t) = 0.002$, $u_3^{21}(t) = 1.012$, $u_3^{22}(t) = 0.041$, $u_3^{23}(t) = 0.217$, $u_3^{31}(t) = 0.068$, $u_3^{32}(t) = 0.234$, $u_3^{33}\ (t) = 0.317$. The illustration supports our results such that the SICNN (4.11) exhibits chaos.

Even if we consider constant initial functions in the simulations, the illustrated outputs in Figs. 1–3 converge to bounded on $\mathbb{R}$ solutions, which are known to be chaotic, and that is the reason why chaotic behavior is observable. Moreover, it is possible to use other values of the delays τ_1, τ_2 and τ_3 in the neural system (4.9)–(4.10)–(4.11) provided that the condition $(C7)$ is fulfilled.

To confirm one more time that the neural system (4.9)–(4.10)–(4.11) exhibits chaotic motions, we illustrate in Fig. 4 the projection of the trajectory with $x(t) = u_1(t)$, $t_0 - \tau_1 \le t \le t_0$, $y(t) = u_2(t)$, $t_0 - \tau_2 \le t \le t_0$, $z(t) = u_3(t)$, $t_0 - \tau_3 \le t \le t_0$, on the $x_{22} - y_{21} - z_{33}$ space, where $t_0 = 0.38$. Fig. 4 supports our results such that a chaotic attractor takes place in the dynamics of the neural system. The obtained chaos for the neural system (4.9)–(4.10)–(4.11) is in the sense of Li–Yorke, and it is remarkable that the presence of chaos with a precise type in neural systems consisting of retarded SICNNs has never been reported before.

In order to illustrate the proximality and frequent separation features in the neural system (4.9)–(4.10)–(4.11), we represent in Fig. 5 the x_{22}, y_{22} and z_{22} coordinates of the solutions corresponding to the sequence ζ with $\zeta_0 = 0.38$ and $\zeta_0 = 0.39$ in blue and red colors, respectively. In the former, we use the initial conditions $x(t) = u_1(t)$, $t_0 - \tau_1 \le t \le t_0$, $y(t) = u_2(t)$, $t_0 - \tau_2 \le t \le t_0$, $z(t) = u_3(t)$, $t_0 - \tau_3 \le t \le t_0$, where $t_0 = 0.38$. For the solution shown in red color the initial conditions $x(t) = \overline{u}_1(t)$, $t_1 - \tau_1 \le t \le t_1$, $y(t) = \overline{u}_2(t)$, $t_1 - \tau_2 \le t \le t_1$, $z(t) = \overline{u}_3(t)$, $t_1 - \tau_3 \le t \le t_1$, where $t_1 = 0.39$, are used. Here, $\overline{u}_1(t) = \{u_1^{ij}(t)\}$, $\overline{u}_2(t) = \{u_2^{ij}(t)\}$ and $\overline{u}_3(t) = \{u_3^{ij}(t)\}$ are constant functions defined as $\overline{u}_1^{11}(t) = 0.428$, $\overline{u}_1^{12}(t) = 0.351$, $\overline{u}_1^{13}(t) = 0.745$, $\overline{u}_1^{21}(t) = 0.623$, $\overline{u}_1^{22}(t) = 0.553$, $\overline{u}_1^{23}(t) = 0.254$, $\overline{u}_1^{31}(t) = 1.725$, $\overline{u}_1^{32}(t) = 0.742$, $\overline{u}_1^{33}(t) = 0.249$, $\overline{u}_2^{11}(t) = 0.086$, $\overline{u}_2^{12}(t) = 0.234$, $\overline{u}_2^{13}(t) = 0.321$, $\overline{u}_2^{21}(t) = 0.253$, $\overline{u}_2^{22}(t) = 0.201$, $\overline{u}_2^{23}(t) = 0.113$, $\overline{u}_2^{31}(t) = 0.105$, $\overline{u}_2^{32}(t) = 0.194$, $\overline{u}_2^{33}(t) = 0.454$, $\overline{u}_3^{11}(t) = 0.095$, $\overline{u}_3^{12}(t) = 0.094$, $\overline{u}_3^{13}(t) = 0.001$, $\overline{u}_3^{21}(t) = 1.145$, $\overline{u}_3^{22}(t) = 0.038$, $\overline{u}_3^{23}(t) = 0.332$, $\overline{u}_3^{31}(t) = 0.089$, $\overline{u}_3^{32}(t) = 0.251$ and $\overline{u}_3^{33}(t) = 0.212$.

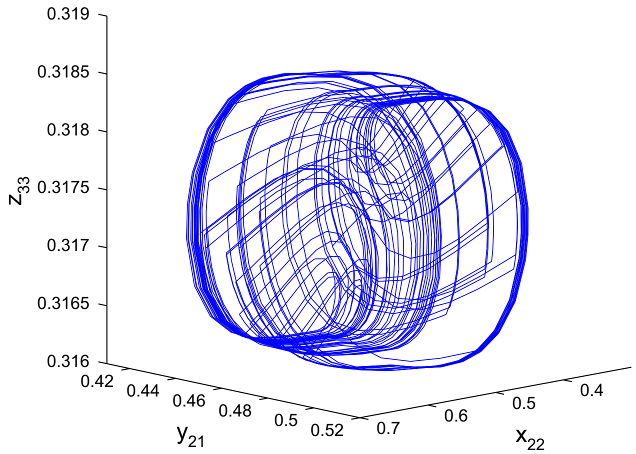

Fig. 4. The projection of the chaotic trajectory of the neural system (4.9)–(4.10)–(4.11) on the $x_{22} - y_{21} - z_{33}$ space.

It is seen in Fig. 5 that the x_{22}, y_{22} and z_{22} coordinates of the represented outputs are separated from each other by a positive number approximately for $1105 \le t \le 1110.3$, $1105 \le t \le 1111.4$ and $1105 \le t \le 1112.3$, respectively. On the other hand, one can observe the presence of the proximality feature in each of the coordinates such that the solutions are almost identical approximately for $1120 \le t \le 1133$. In addition to this, the solutions are again separated from each other by a positive number approximately for $1133 \le t \le 1140$.

5. Discussions

In the example presented in Section 4, a neural system consisting of three layers is considered. Each layer of the neural system (4.9)–(4.10)–(4.11) is, in fact, a retarded SICNN. The layers are connected in a unidirectional way such that between the layers we have feed-forward connections. The schematic diagram of the neural system is shown in Fig. 6, where the unidirectional connections between the cells of different layers are presented in blue and red colors. It is worth noting that feed-backward connections exist within the layers, and black color is used to depict them in the figure. The first layer admits the chaos due to the external inputs in the form of chaotic relay functions. The chaotic outputs of the first layer are used as external inputs for the second one, and therefore, the latter also possesses chaotic motions in accordance with our theoretical results. Besides, being affected by the outputs of the second layer, the SICNN (4.11) exhibits chaos too. As a result, the system (4.9)–(4.10)–(4.11) admits chaotic motions, and we call this process as the *chaotification* of the neural system.

The first notions of chaotic synchronization were introduced and developed in the papers [63–66]. Afraimovich et al. [65] proposed the synchronization of chaotic systems that are different and not restricted in coupling. To realize this proposal, Rulkov et al. [67] considered the concept of generalized synchronization (GS) for unidirectionally coupled systems with a skew product structure in the form

$$x'(t) = F(x(t)) \tag{5.13}$$

and

$$y'(t) = G(x(t), y(t)). \tag{5.14}$$

The systems (5.13) and (5.14) are called the drive and response systems, respectively. GS [33,67–71] is said to occur if there exist sets B_x, B_y of initial conditions and a transformation ϕ, defined on the chaotic attractor of (5.13), such that for all $x(0) \in B_x$, $y(0) \in B_y$ the relation $\lim_{t \to \infty} \| y(t) - \phi(x(t)) \| = 0$ holds. In the case of GS, a motion

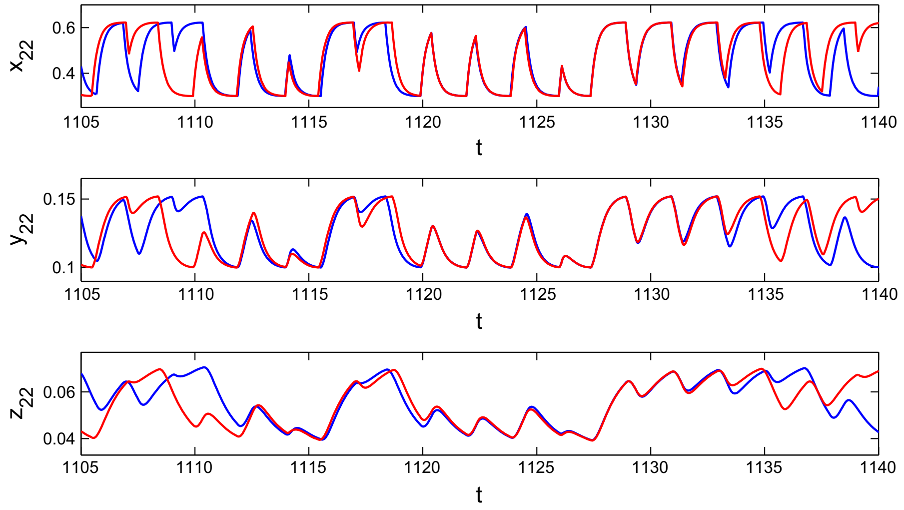

Fig. 5. The presence of the proximality and frequent separation features in the neural system (4.9)–(4.10)–(4.11). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

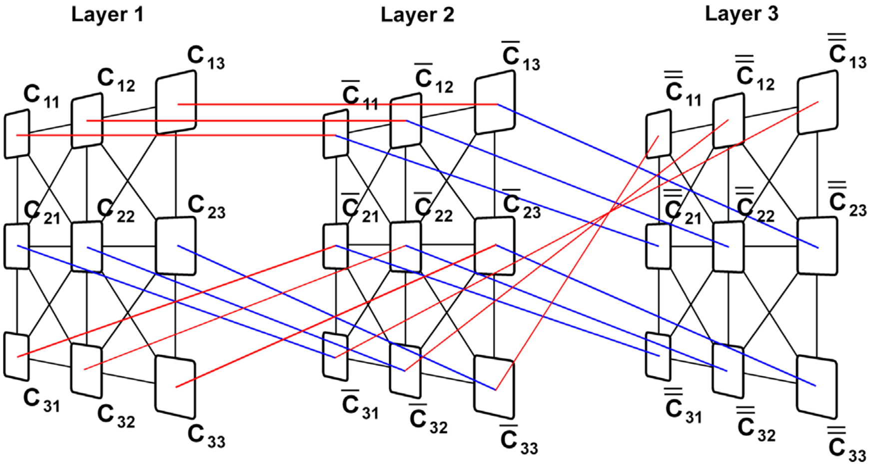

Fig. 6. The schematic diagram of the neural system (4.9)–(4.10)–(4.11). The layers are unidirectionally coupled, and each layer of the neural system is a SICNN. The couplings between the cells of different layers are presented in blue and red colors, while the connections within each SICNN are shown in black color. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

starting on $B_x \times B_y$ collapses onto a manifold $M \subset B_x \times B_y$ of synchronized motions. The transformation ϕ is not required to exist for the transient trajectories. If ϕ is the identity transformation, then identical synchronization takes place [66].

It is formulated in paper [68] that GS occurs in the coupled system (5.13)–(5.14) if and only if for all $x_0 \in B_x$, $y_1, y_2 \in B_y$, the asymptotic stability criterion

$$\lim_{t\to\infty} \| y(t,x_0,y_1) - y(t,x_0,y_2) \| = 0$$

holds, where $y(t,x_0,y_1)$ and $y(t,x_0,y_2)$ are the solutions of (5.14) with the same $x(t)$ such that $y(0,x_0,y_1)=y_1$, $y(0,x_0,y_2)=y_2$ and $x(0)=x_0$.

Now, let us discuss the concept of GS for the neural system (4.9)–(4.10)–(4.11). Lemma 2.2 implies that for a fixed output $x(t)=\{x_{ij}(t)\}$ of (4.9), the criterion

$$\lim_{t\to\infty} \| y(t,x(t),\varphi_1(t)) - y(t,x(t),\varphi_2(t)) \| = 0$$

holds for any initial functions $\varphi_1(t)$ and $\varphi_2(t)$, where $y(t,x(t),\varphi_1(t))$ and $y(t,x(t),\varphi_2(t))$ denote the solutions of the network (4.10) with $y(t,x(t),\varphi_1(t))=\varphi_1(t)$ and $y(t,x(t),\varphi_2(t))=\varphi_2(t)$ for $t\in[-\tau_2,0]$. Therefore, one can conclude that GS occurs in the dynamics of the coupled SICNNs (4.9) and (4.10). It is worth noting that a similar discussion is valid for the SICNNs (4.10) and (4.11), and they are also synchronized in the generalized sense. Since different coefficients and different external inputs are used in the networks (4.10) and (4.11), one can confirm that different synchronization manifolds take place for the couples (4.9)–(4.10), (4.10)–(4.11) and (4.9)–(4.11).

We apply Lemma 2.2 to prove the presence of GS and we need just to verify the conditions (C1)–(C7) to confirm the presence of synchronization in a couple of neural networks. It is shown for the first time in the literature that the technique is used to synchronize a chain of SICNNs. Since the synchronization manifolds are not the same for different pairs of SICNNs from the chain, the complexity of chaos in the family increases. The results may be used to explain the high performance of brain functioning [6,62].

A numerical method that can be used to investigate coupled systems for GS is the auxiliary system approach [33,70]. We will use this approach to support the theoretical discussions about the presence of GS in the coupled SICNNs (4.9)–(4.10).

Let us take into account the auxiliary system

$$\frac{d\zeta_{ij}}{dt} = -b_{ij}\zeta_{ij} - \sum_{\overline{C}_{kl}\in N_1(i,j)} \overline{C}_{ij}^{kl} g(\zeta_{kl}(t-\tau_2))\zeta_{ij} + \overline{L}_{ij}(t), \tag{5.15}$$

which is an identical copy of (4.10).

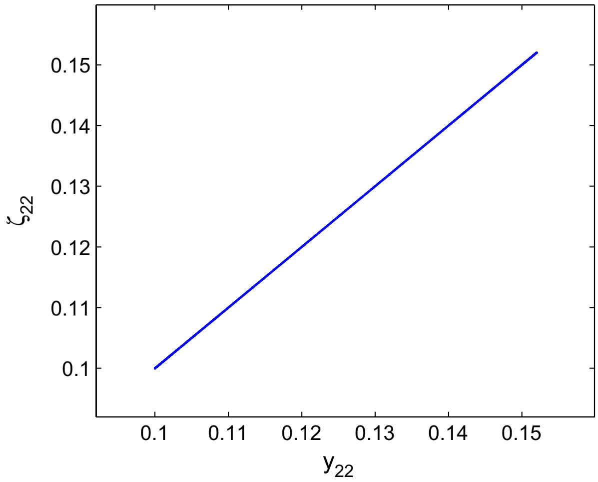

Fig. 7. Application of the auxiliary system approach to the coupled SICNNs (4.9)–(4.10) indicates that GS exists for the couple.

In the networks (4.10) and (5.15), we use the external inputs $x_{ij}(t)$, $i=1, 2, 3$, $j = 1, 2, 3$, which are depicted in Fig. 1, and represent in Fig. 7 the projection of the stroboscopic plot of the network (4.9)–(4.10)–(5.15) on the y_{22}–ζ_{22} plane using the initial functions $y(t)=u_2(t)$, which was described in the previous section, and $\zeta(t)=v(t)$ for $t_0-\tau_2 \le t \le t_0$, where $t_0=0.38$ and $v(t)=\{v_{ij}\}$ is the constant function defined as $v_{11}(t)=0.114$, $v_{12}(t)=0.191$, $v_{13}(t)=0.302$, $v_{21}(t)=0.512$, $v_{22}(t)=0.041$, $v_{23}(t)=0.215$, $v_{31}(t)=0.287$, $v_{32}(t)=0.158$, $v_{33}(t)=0.294$. In the simulation the first 50 iterations are omitted. One can see in Fig. 7 that the plot is on the line $\zeta_{22}=y_{22}$, and this result supports our theoretical discussions about the presence of GS for the coupled SICNNs (4.9)–(4.10). A similar simulation can be performed for the coupled SICNNs (4.10)–(4.11).

6. Conclusions

Delayed neural networks have applications in many areas such as signal and image processing, associative memories, combinatorial optimization and automatic control. Because of the finite switching speed of the amplifiers, time delays occur during the hardware implementation of neural networks. Therefore, it is of prime importance to study neural networks with time delays.

Chaotic dynamics is useful in neural networks for separating image segments and information processing. The presence of synchronization in neural networks provides a criterion for the existence of a dynamical correspondence between the systems, and helps for a better understanding of neural processes. Moreover, chaos can improve the performance of CNNs on problems that have local minima in energy (cost) functions and it is an important tool for the studies of chaotic communication and combinatorial optimization problems.

In the present study, SICNNs with delay are considered with chaotic external inputs, and this is the first time that a theoretically approved chaos is obtained in such networks. As an example, we have considered a neural system consisting of three layers such that each layer is a retarded SICNN. Piecewise constant external inputs are utilized in the first layer of this neural system to ensure the presence of chaos in the sense of Li–Yorke. The results of the obtained chaotification process are discussed through the generalized synchronization point of view, and the proximality and frequent separation features are demonstrated numerically. The results of the present paper can be extended easily if the delay is variable and also for the case of advanced argument. Our approach can be applied to other types of chaos such as the one analyzed through period-doubling cascade.

Freeman and his collaborators [57–62] achieved remarkable observations and conclusions that reveal the essentialness of deterministic chaos for the brain functioning. Another hypothesis is that chaos is undesirable and it occurs in brains subject to pathological malfunctions [5]. This also provides an interesting and considerable direction to the analysis of neural network problems in the chaos theory. We suppose that the present study can give some contributions in both directions. The proposed chaotification procedure indicates not only the advantage of the deterministic chaos over random noise for the analysis, but also significant properties of self-organization [113,114]. Our results may be useful for the investigation of environmental inputs of the brain both on low and high levels of organization as well as learning by considering it as the creation of new structures (motions) in neural networks.

The brain comprises functionally specialized areas, which perform specific tasks and have differentiated parts or structures within. These different structures have to work together for a cerebral activity to occur. In the papers [115–117], the authors proposed the presence of synchronization as the underlying reason for such processes. Breakspear and Terry [75] reported the detection of generalized synchronization between different brain regions by means of electroencephalogram signals. In the present paper, we have demonstrated the presence of generalized synchronization by means of interconnected SICNNs with delay, and our results may provide an opportunity to understand the complex structure of the brain and the rest of the nervous system.

Acknowledgments

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