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# Entrainment by Chaos

**M. U. Akhmet · M. O. Fen**

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**Abstract** A new phenomenon, the entrainment of limit cycles by chaos, which results from the appearance of cyclic irregular behavior, is discussed. In this study, sensitivity is considered as the main ingredient of chaos to be captured, and the period-doubling cascade is chosen for extension. Theoretical results are supported by simulations and discussions regarding Chua's oscillators, entrainment of toroidal attractors by chaos, synchronization, and controlling problems. It is demonstrated that the entrainment cannot be considered as generalized synchronization of chaotic systems.

**Keywords** Limit cycle · Sensitivity · Period-doubling cascade · Hopf bifurcation · Toroidal attractor · Chua's oscillator · Chaos control

**Mathematics Subject Classification** 34C28

## 1 Introduction

Christiaan Huygens was the first to introduce the concept of entrainment when he observed that two pendulum clocks mounted next to each other on the same support often become synchronized (Huygens 1665). One could also mention the practice of fine-tuning brainwaves to a desired frequency, that is, brainwave entrainment (Oster 1973; Walter and Walter 1949), or the idea of entrainment in biomusicology, which is understood to be the synchronization of organisms to an external rhythm (Clayton et al. 2004). The entrainment phenomenon is also known in hydrodynamics as the movement of one fluid induced by another (Dombrowski et al. 2005). In the present article, we discuss the entrainment of limit cycles by chaos and demonstrate that entrainment,

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in mathematical theory, is not confined to the notion of frequency, period, or phase (Pikovsky et al. 2001; Minorsky 1947; Sendiña-Nadal et al. 2009) but extends to the concept of chaos as well. Unidirectional coupling, which has been extensively studied in physics (Sendiña-Nadal et al. 2009; Caneco et al. 2009; Anishchenko et al. 1994; Wu and Jiao 2007; Keller and Zweimüller 2002), is investigated in this work. The results presented can be used to generate entrainment by chaos in business cycle models (Lorenz 1993), chaotic cycles in electrical circuits, such as those obtained via the Van der Pol equations (Hassard et al. 1981), and chaotic oscillations in Belousov–Zhabotinsky reactions (Field and Györgyi 1993). Indeed, the results can be applied and developed in any field in which limit cycles are observed.

Entrainment by chaos is understood in this paper as the deformation of limit cycles to chaotic cycles. The presence of chaotic behavior is confirmed by checking for the existence of infinitely many unstable periodic solutions and sensitivity, which is the main ingredient of chaos (Robinson 1995; Guckenheimer and Holmes 1940; Lorenz 1963; Wiggins 1988).

In the studies Akhmet and Fen 2012a,b, 2013a,b; Akhmet 2008, 2009a,b,c,d, 2010a,b, 2011, we considered systems with asymptotically stable and hyperbolic equilibria, and perturbed them chaotically. It was found that the solutions admit the same type of chaos that perturbations do. Unlike the mechanism discussed in Akhmet and Fen 2013a, where we considered chaos near fixed points, in the present article, we take into account systems with orbitally stable limit cycles and perturb them with chaos. As a result, we obtain *chaotic cycles*, that is, motions that behave cyclically and chaotically at the same time.

To the best of our knowledge, the generation of chaos is considered in synchronization theory (Pecora and Carroll 1990; Kapitaniak 1994; Macau et al. 2002; Rulkov et al. 1995; Kocarev and Parlitz 1996; Hunt et al. 1997; Abarbanel et al. 1996; Gonzales-Miranda 2004). However, the chaos of the response system must be asymptotically close to that of the driver, and this property is used for the verification of chaos in the literature. Since we do not use this proximity, our method is of a different type in the generation of chaos than that used in synchronization theory. Moreover, we have shown that our results are not reducible to synchronization in general.

In Sect. 7, we present the results of simulations of a chaotic torus and a Chua oscillator, though these entrainment phenomena must be investigated further. It is also of great interest to prove entrainment by chaos around hyperbolic periodic solutions (Aulbach 1981; Hale and Stokes 1960; Mitropolskij and Lykova 1973).

To illustrate the main idea of our study, we present the example of an oscillating chemical reaction. The paper Lengyel et al. 1990 considers the chlorine dioxide-iodine-malonic acid ($ClO_2 - I_2 - MA$) chemical reaction, which arises from the following three component reactions:

$$
\begin{aligned}
&MA + I_2 \longrightarrow IMA + I^- + H^+,\\
&ClO_2 + I^- \longrightarrow ClO_2^- + \frac{1}{2} I_2,\\
&ClO_2^- + 4I^- + 4H^+ \longrightarrow Cl^- + 2I_2 + 2H_2O.
\end{aligned}
\tag{1.1}
$$

After making reasonable simplifications and nondimensionalizations, Lengyel et al. (1990) reduced the rate equations to the system

$$
\begin{aligned}
u_1' &= a - u_1 - \frac{4u_1u_2}{1+u_1^2},\\
u_2' &= bu_1\left(1 - \frac{u_2}{1+u_1^2}\right),
\end{aligned}
\tag{1.2}
$$

where $u_1$ and $u_2$ represent the dimensionless concentrations of $I^-$ and $ClO_2^-$ ions, respectively, and the parameters $a > 0$ and $b > 0$ depend on the empirical rate constants and the concentrations of the slow reactants.

For a given value of parameter $a$, system (1.2) undergoes a Hopf bifurcation at the parameter value $b = b_0 \equiv 3a/5 - 25/a$ such that when $b > b_0$, all trajectories spiral into the stable fixed point $(u_1^*, u_2^*) = (a/5, 1+a^2/25)$, whereas for $b < b_0$ trajectories are attracted to an orbitally stable limit cycle. If we consider system (1.2) with the coefficient $a = 11$, Hopf bifurcation occurs for $b_0 = 238/55$, and an orbitally stable limit cycle takes place for $b = 2.1$ (Strogatz 1994).

Next, we take into account the Birkhoff–Shaw chaotic attractor (Thompson and Stewart 2002; Shaw 1981), which is generated by the following system of differential equations:

$$
\begin{aligned}
x_1' &= 0.7x_2 + 10x_1(0.1 - x_2^2),\\
x_2' &= -x_1 + 0.25\sin(1.57t).
\end{aligned}
\tag{1.3}
$$

The following system is obtained by perturbing system (1.2) using solutions of (1.3):

$$
\begin{aligned}
y_1' &= 11 - y_1 - \frac{4y_1y_2}{1+y_1^2} + 0.5\tan\left(\frac{x_1(t)}{2}\right),\\
y_2' &= 2.1y_1\left(1 - \frac{y_2}{1+y_1^2}\right) + 0.4x_2(t).
\end{aligned}
\tag{1.4}
$$

The present paper rigorously demonstrates that system (1.4) displays chaotic motions around the orbitally stable limit cycle of system (1.2). Figure 1a depicts the chaotic trajectory, $x(t)$, of (1.3), with $x_1(0) = 0.2$, $x_2(0) = 0.3$. If one substitutes $x(t)$ into (1.4), then the system admits a chaotic trajectory, $y(t)$, with $y_1(0) = 0.75$, $y_2(0) = 4.82$, as shown in Fig. 1b. That is, an entrainment by chaos is observed. Moreover, the irregular behavior of the $y_2$ coordinate over time is illustrated in Fig. 2.

Now let us use the auxiliary system approach (Abarbanel et al. 1996; Gonzales-Miranda 2004) to investigate the couple (1.3) + (1.4) for generalized synchronization.

Consider the auxiliary system

$$
\begin{aligned}
z_1' &= 11 - z_1 - \frac{4z_1z_2}{1+z_1^2} + 0.5\tan\left(\frac{x_1(t)}{2}\right),\\
z_2' &= 2.1z_1\left(1 - \frac{z_2}{1+z_1^2}\right) + 0.4x_2(t).
\end{aligned}
\tag{1.5}
$$

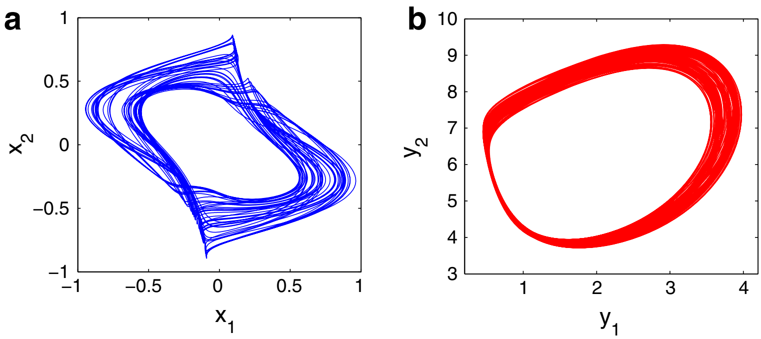



**Fig. 1** The chaotic trajectory in **a** corresponds to system (1.3), and the irregular structure around the limit cycle in **b** is a manifestation of entrainment by chaos

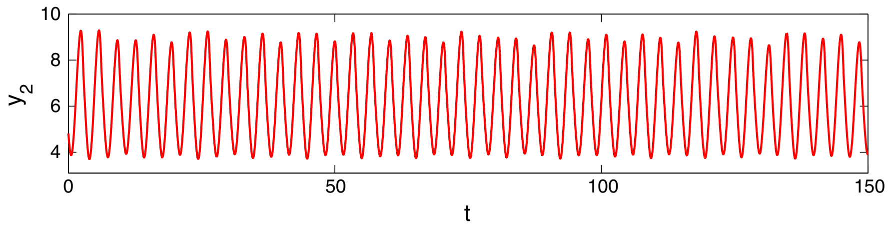



**Fig. 2** The graph presents the irregular behavior of the $y_2$ coordinate and supports the existence of entrainment by chaos

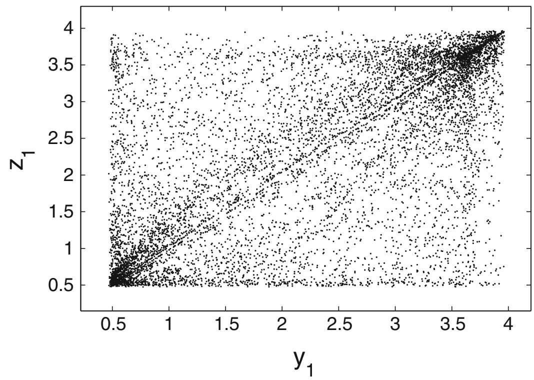



**Fig. 3** The auxiliary system approach shows that systems (1.3) and (1.4) are unsynchronized

By marking the trajectory of system (1.3) + (1.4) + (1.5) with initial data $x_1(0) = 0.2$, $x_2(0) = 0.3$, $y_1(0) = 0.75$, $y_2(0) = 4.82$, $z_1(0) = 2.92$, $z_2(0) = 8.78$ at times $t$, which are integer multiples of $2\pi/1.57$, and omitting the first 200 iterations, we obtain the stroboscopic plot whose projection on the $y_1 - z_1$ plane is shown in Fig. 3. Since the plot is not placed on the line $z_1 = y_1$, we conclude that generalized synchronization does not occur in the couple (1.3) + (1.4).

## 2 Preliminaries

Throughout the paper, $\mathbb{R}$ and $\mathbb{R}_+$ will denote the set of real numbers and the interval $[0, \infty)$, respectively. We will make use of the usual Euclidean norm for vectors and the norm induced by the Euclidean norm for square matrices (Horn and Johnson 1992).

Let us introduce the system

$$x' = F(t, x), \tag{2.1}$$

where $F : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function in all its arguments. We treat system (2.1) as a source of chaos and thus call it a *generator* system.

Now consider the system

$$u' = f(u), \tag{2.2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function. We subject system (2.2) to the entrainment mechanism in the following way:

$$y' = f(y) + \mu g(x) \tag{2.3}$$

such that we now consider the system

$$y' = f(y) + \mu g(x(t)), \tag{2.4}$$

where $x(t)$ are solutions of system (2.1), $\mu$ is a nonzero constant, and $g : \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function. Here, the couple (2.1) + (2.3) is a system with a skew product structure.

*Remark 2.1* The results presented in the remaining sections are valid even if we replace the nonautonomous system (2.1) with the autonomous equation

$$x' = \overline{F}(x), \tag{2.5}$$

where $\overline{F} : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function.

The following conditions are required:

(A1) There exists a positive number $L_f$ such that

$$\|f(y_1) - f(y_2)\| \leq L_f \|y_1 - y_2\|$$

for all $y_1, y_2 \in \mathbb{R}^n$;

(A2) There exists a positive number $L_g$ such that

$$\|g(x_1) - g(x_2)\| \geq L_g \|x_1 - x_2\|$$

for all $x_1, x_2 \in \mathbb{R}^m$;

(A3) There exist positive numbers $M_F$, $M_f$, and $M_g$ such that $\sup_{t\in\mathbb{R}_+, x\in\mathbb{R}^m} \|F(t, x)\| \leq M_F$, $\sup_{y\in\mathbb{R}^n} \|f(y)\| \leq M_f$, and $\sup_{x\in\mathbb{R}^m} \|g(x)\| \leq M_g$.

For a given $x(t)$ the existence of solutions of system (2.4), as well as their continuation to $+\infty$, follow from the Lipschitz condition for the function $f$ because the perturbation $\mu g(x(t))$ depends only on $t$ and from the fact that the domain of the equation is the entire space $\mathbb{R}^n$ (Hartman 1964).

We mainly assume that system (2.1) (2.5) admits a chaotic attractor, let us say a set in $\mathbb{R}^m$ for (2.5). Fix $x_0$ from the attractor and take a solution $x(t)$ of (2.5) with $x(0) = x_0$. Since we use the solution $x(t)$ as a perturbation in system (2.3), we call it a *chaotic function*. Chaotic functions may be irregular and regular (periodic and unstable) (Devaney 1987; Palmer 2000; Sander and Yorke 2011, 2012; Alligood et al. 1996; Kovacic and Brennan 2011; Feigenbaum 1980).

We also assume that the nonlinear autonomous system (2.2) possesses a nonconstant $T$-periodic solution $p(t)$ for some positive number $T$ and consider system (2.3) in a neighborhood of the orbit

$$\gamma = \left\{\sigma \in \mathbb{R}^n : \sigma = p(t),\ t \in [0, T]\right\}. \tag{2.6}$$

It is clear that $p'(t)$ is a nontrivial $T$-periodic solution of the variational system

$$v' = A(t)v, \tag{2.7}$$

where $A(t) = \frac{\partial f(p(t))}{\partial u}$ is an $n \times n$ real, continuous, $T$-periodic matrix function, and, consequently, 1 is a characteristic multiplier of system (2.7).

In what follows, we assume that 1 is a simple characteristic multiplier of the variational system (2.7) and the remaining $n - 1$ characteristic multipliers are less than 1 in modulus. Under this assumption, according to the Andronov–Witt theorem (Farkas 2010), the periodic solution $p(t)$ of system (2.2) is asymptotically orbitally stable, with the asymptotic phase property.

In what follows, we will understand chaos in terms of sensitivity and the existence of infinitely many unstable periodic solutions in a bounded region.

## 3 Replication of Sensitivity

In this section, we will give the definition of sensitivity for Eq. (2.1) and then continue with its replication in (2.3).

In what follows, for a given chaotic solution $x(t)$ of system (2.1), the function $\eta_{x(t)}(t, \eta_0)$ will represent the solution of (2.4), with $\eta_{x(t)}(0, \eta_0) = \eta_0$.

System (2.1) is called sensitive if there exist positive numbers $\epsilon_0$ and $\Delta$ such that for an arbitrary positive number $\delta_0$ and for each chaotic solution $x(t)$ of system (2.1) there exist a chaotic solution $\overline{x}(t)$ of the same system and an interval $J \subset \mathbb{R}_+$, with a length no less than $\Delta$, such that $\|x(0) - \overline{x}(0)\| < \delta_0$ and $\|x(t) - \overline{x}(t)\| > \epsilon_0$ for all $t \in J$.

We say that system (2.3) replicates the sensitivity of (2.1) if there exist positive numbers $\epsilon_1$ and $\overline{\Delta}$ such that for an arbitrary positive number $\delta_1$ and for each solution $\eta_{x(t)}(t, \eta_0)$ there exist an interval $J^1 \subset \mathbb{R}_+$, with a length no less than $\overline{\Delta}$, and a solution $\eta_{\overline{x}(t)}(t, \eta_1)$ such that $\|\eta_0 - \eta_1\| < \delta_1$ and $\left\|\eta_{x(t)}(t, \eta_0) - \eta_{\overline{x}(t)}(t, \eta_1)\right\| > \epsilon_1$ for all $t \in J^1$.

**Theorem 3.1** *If conditions* $(A1)$–$(A3)$ *hold, then system* (2.3) *replicates the sensitivity of system* (2.1).

*Proof* Fix an arbitrary positive number $\delta_1$ and a solution $\eta_{x(t)}(t, \eta_0)$ of (2.4). Since system (2.1) is sensitive, there exist positive numbers $\epsilon_0$ and $\Delta$ such that for arbitrary $\delta_0 > 0$ the inequalities $\|x(0) - \overline{x}(0)\| < \delta_0$ and $\|x(t) - \overline{x}(t)\| > \epsilon_0$, $t \in J$, hold for some chaotic solution $\overline{x}(t)$ of (2.1) and for some interval $J \subset \mathbb{R}_+$ with a length no less than $\Delta$.

Now, let us fix arbitrary $\eta_1 \in \mathbb{R}^n$ such that $\|\eta_0 - \eta_1\| < \delta_1$. Our aim is to determine the positive numbers $\epsilon_1$, $\overline{\Delta}$ and an interval $J^1$ with length $\overline{\Delta}$ such that $\left\|\eta_{x(t)}(t, \eta_0) - \eta_{\overline{x}(t)}(t, \eta_1)\right\| > \epsilon_1$ for all $t \in J^1$.

Suppose that $g(x) = (g_1(x), g_2(x), \ldots, g_n(x))$, where each $g_j$, $1 \leq j \leq n$, is a real-valued function.

Since for each chaotic solution $x(t)$ of (2.1) the inequality $\sup_{t \in \mathbb{R}_+} \left\|x'(t)\right\| \leq M_F$ holds, one can conclude that the collection of chaotic solutions of Eq. (2.1) constitutes an equicontinuous family on $\mathbb{R}_+$. According to our assumption that system (2.1) possesses a chaotic attractor, there exists a positive number $M$ such that $\sup_{t \in \mathbb{R}_+} \|x(t)\| \leq M$ for each chaotic solution $x(t)$ of (2.1). Making use of the uniform continuity of the function $\overline{g} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$, defined as $\overline{g}(x_1, x_2) = g(x_1) - g(x_2)$, on the compact region $\{(x_1, x_2) \in \mathbb{R}^m \times \mathbb{R}^m : \|x_1\| \leq M, \|x_2\| \leq M\}$, together with the equicontinuity of the collection of chaotic solutions of system (2.1), one can verify that the set consisting of the elements of the form $g_j(x(t)) - g_j(\overline{x}(t))$, $1 \leq j \leq n$, where $x(t)$ and $\overline{x}(t)$ are chaotic solutions of (2.1), is an equicontinuous family on $\mathbb{R}_+$. Therefore, there exists a positive number $\tau < \Delta$, independent of the functions $x(t)$ and $\overline{x}(t)$, such that for any $t_1, t_2 \in \mathbb{R}_+$, with $|t_1 - t_2| < \tau$, the inequality

$$\left|\left(g_j\left(x(t_1)\right) - g_j\left(\overline{x}(t_1)\right)\right) - \left(g_j\left(x(t_2)\right) - g_j\left(\overline{x}(t_2)\right)\right)\right| < \frac{L_g \epsilon_0}{2n}$$

holds for all $1 \leq j \leq n$.

Condition $(A2)$ implies for all $t \in J$ that $\|g(x(t)) - g(\overline{x}(t))\| \geq L_g \|x(t) - \overline{x}(t)\|$. Thus, for each $t \in J$ there exists an integer $j_0$, $1 \leq j_0 \leq n$, that possibly depends on $t$ such that

$$\left|g_{j_0}(x(t)) - g_{j_0}(\overline{x}(t))\right| \geq \frac{L_g}{n} \|x(t) - \overline{x}(t)\|.$$

Let $s_0$ be the midpoint of the interval $J$ and $\theta = s_0 - \tau/2$. One can find an integer $j_0 = j_0(s_0)$, $1 \leq j_0 \leq n$, such that

$$\left|g_{j_0}(x(s_0)) - g_{j_0}(\overline{x}(s_0))\right| > \frac{L_g \epsilon_0}{n}. \tag{3.1}$$

On the other hand, for all $t \in [\theta, \theta + \tau]$ we have

$$\left|g_{j_0}\left(x(s_0)\right) - g_{j_0}\left(\overline{x}(s_0)\right)\right| - \left|g_{j_0}\left(x(t)\right) - g_{j_0}\left(\overline{x}(t)\right)\right| < \frac{L_g \epsilon_0}{2n},$$

and by means of (3.1) we obtain

$$\left|g_{j_0}(x(t)) - g_{j_0}(\overline{x}(t))\right| > \frac{L_g \epsilon_0}{2n},\ t \in [\theta, \theta + \tau].$$

The last inequality implies that

$$\left\| \int_{\theta}^{\theta+\tau} [g(x(s)) - g(\overline{x}(s))]\, \mathrm{d}s \right\| > \frac{\tau L_g \epsilon_0}{2n}. \tag{3.2}$$

Using the inequality

$$\left\|\eta_{x(t)}(\theta+\tau, \eta_0) - \eta_{\overline{x}(t)}(\theta+\tau, \eta_1)\right\| \geq |\mu| \left\| \int_{\theta}^{\theta+\tau} [g(x(s)) - g(\overline{x}(s))] \mathrm{d}s \right\|$$
$$- \left\|\eta_{x(t)}(\theta, \eta_0) - \eta_{\overline{x}(t)}(\theta, \eta_1)\right\| - \int_{\theta}^{\theta+\tau} L_f \left\|\eta_{x(t)}(s, \eta_0) - \eta_{\overline{x}(t)}(s, \eta_1)\right\| \mathrm{d}s,$$

together with (3.2), one can verify that

$$\max_{t \in [\theta, \theta+\tau]} \left\|\eta_{x(t)}(t, \eta_0) - \eta_{\overline{x}(t)}(t, \eta_1)\right\| > \frac{|\mu|\, \tau L_g \epsilon_0}{2n(2 + \tau L_f)}.$$

Suppose that the function $\left\|\eta_{x(t)}(t, \eta_0) - \eta_{\overline{x}(t)}(t, \eta_1)\right\|$ takes its maximum on the interval $[\theta, \theta + \tau]$ at the point $\xi$.

Let us define

$$\overline{\Delta} = \min\left\{\frac{\tau}{2}, \frac{|\mu|\, \tau L_g \epsilon_0}{8n(M_f + M_g\, |\mu|)(2 + \tau L_f)}\right\}$$

and

$$\theta^1 = \begin{cases} \xi, & \text{if } \xi \leq \theta + \tau/2 \\ \xi - \overline{\Delta}, & \text{if } \xi > \theta + \tau/2 \end{cases}.$$

We note that the interval $J^1 = [\theta^1, \theta^1 + \overline{\Delta}]$ is a subset of $J$. For $t \in J^1$ it can be verified that

$$\left\|\eta_{x(t)}(t, \eta_0) - \eta_{\overline{x}(t)}(t, \eta_1)\right\| > \epsilon_1, \tag{3.3}$$

where $\epsilon_1 = \frac{|\mu| \tau L_g \epsilon_0}{4n(2+\tau L_f)}$, and the length $\overline{\Delta}$ of the interval $J^1$ does not depend on $x(t)$ and $\overline{x}(t)$. Consequently, system (2.3) replicates the sensitivity of system (2.1). □

## 4 Replication of Unstable Periodic Solutions

We begin this section by describing the period-doubling cascade for system (2.1) and continue with its extension to system (2.4) through system (2.3).

In this section, assume that system (2.1) admits a period-doubling cascade. That is, there exists an equation

$$x' = G(t, x, \lambda), \tag{4.1}$$

where $\lambda$ is a parameter and the function $G : \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ is such that for some finite number $\lambda_\infty$, $G(t, x, \lambda_\infty)$ is equal to the function $F(t, x)$ on the right-hand side of system (2.1).

The following condition is required.

(A4) There exists a positive number $\omega$ such that the periodicity property $G(t + \omega, x, \lambda) = G(t, x, \lambda)$ holds for all $t \in \mathbb{R}_+$, $x \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$.

System (2.1) is said to admit a period-doubling cascade (Sander and Yorke 2011, 2012; Alligood et al. 1996; Kovacic and Brennan 2011; Feigenbaum 1980) if there exist a natural number $k_0$ and a sequence of period-doubling bifurcation values $\{\lambda_j\}$, $\lambda_j \to \lambda_\infty$ as $j \to \infty$ such that for each natural number $j$ a periodic solution with period $k_0 2^j \omega$ appears, and as the parameter $\lambda$ increases or decreases through $\lambda_j$, system (4.1) undergoes a period-doubling bifurcation. As a consequence, at the parameter value $\lambda = \lambda_\infty$, there exist infinitely many unstable periodic solutions of system (4.1) and, hence, of system (2.1), all lying in a bounded region.

Now let us introduce the following definition (Yoshizawa 1975). We say that the solutions of the nonautonomous system (2.4), with a fixed $x(t)$, are ultimately bounded if there exists a number $B > 0$ such that for every solution $y(t)$, $y(t_0) = y_0$, of system (2.4) there exists a positive number $R$ such that the inequality $\|y(t)\| < B$ holds for all $t \geq t_0 + R$.

The following condition is required in the next theorem, which can be verified using Theorem 15.8 (Yoshizawa 1975).

(A5) Solutions of system (2.4) are ultimately bounded by a bound common for all $x(t)$.

We say that system (2.3) replicates the period-doubling cascade of system (2.1) if for each periodic solution $x(t)$ of (2.1) system (2.4) admits a periodic solution with the same period.

**Theorem 4.1** *If conditions* $(A1)$–$(A5)$ *hold, then system* (2.3) *replicates the period-doubling cascade of system* (2.1).

We emphasize that the instability of the infinite number of periodic solutions of system (2.3) is ensured by Theorem 3.1. Condition $(A5)$ can be verified directly, for example, using Lyapunov functions, as in the case of system (6.6) presented in Sect. 6.

## 5 Main Result

Let $H$ be the set of all solutions $\eta_{x(t)}(t, \eta_0)$ of (2.4). Based on the previous results, one can say that solutions in $H$ are sensitive and there are infinitely many unstable periodic solutions in the set; that is, $H$ is chaotic.

We say that the entrainment of the limit cycle by chaos is observed in system (2.3) if there exists a neighborhood $N$ of $\gamma$ where the chaos is developed. Moreover, there exists an open ball in $\mathbb{R}^n$ centered at $p(0)$ such that each chaotic solution $\eta_{x(t)}(t, \eta_0)$ that starts inside the ball remains in $N$ for all $t \geq 0$.

**Theorem 5.1** *Suppose that conditions* $(A1)$–$(A5)$ *hold. If* $|\mu|$ *is sufficiently small, then there is an entrainment of system* (2.3) *by chaos.*

*Proof* Assume, without loss of generality, that $p(0) = 0$ and $p'(0) = (\bar{p}_1, 0, 0, \ldots, 0)$ for some positive number $\bar{p}_1$. First, we will show that for sufficiently small $|\mu|$ the solutions of system (2.4) remain and rotate in a neighborhood of the limit cycle. That is, $\eta_{x(t)}(\theta_i, \eta_0)$ belongs to a neighborhood of the origin, if $\|\eta_0\|$ is sufficiently small, for a sequence $\theta_i \to \infty$ as $i \to \infty$ with uniformly bounded $\theta_{i+1} - \theta_i$.

Let us denote by $\zeta(t, \zeta_0)$ the solution of Eq. (2.2) with $\zeta(0, \zeta_0) = \zeta_0$. There exists a hypersurface $S$ such that the orbit $\gamma$ of the periodic solution $p(t)$ intersects this surface transversally, as shown in the appendix. Therefore, there exists a number $\epsilon_1 > 0$ such that if $\|\zeta(t, \zeta_0) - p(t)\| < \epsilon_1$ for each $t \in [0, 2T]$, then $\zeta(t, \zeta_0)$ intersects $S$ at some moment $t_1 \in [0, 2T]$.

Suppose that a positive number $\delta = \delta(\epsilon_1)$ is chosen such that $\delta \leq \epsilon_1 e^{-2L_f T}$. Throughout the proof, $B_\delta$ will stand for the open ball in $\mathbb{R}^n$ centered at the origin with radius $\delta$. Let an arbitrary $\zeta_0 \in B_\delta$ be given.

The solutions $\zeta(t, \zeta_0)$ and $p(t) = \zeta(t, 0)$ satisfy the relation

$$\|\zeta(t, \zeta_0) - p(t)\| \leq \|\zeta_0\| + \int_0^t L_f \, \|\zeta(s, \zeta_0) - p(s)\| \, \mathrm{d}s.$$

Using $\|\zeta_0\| < \delta$, one can verify that if $0 \leq t \leq 2T$, then $\|\zeta(t, \zeta_0) - p(t)\| < \delta e^{2L_f T}$. Therefore,

$$\|\zeta(t, \zeta_0) - p(t)\| < \epsilon_1$$

for $t \in [0, 2T]$, and $\zeta_1 = \zeta(t_1(\zeta_0), \zeta_0)$ belongs to $S$ for some $t_1(\zeta_0) \in [0, 2T]$. It is clear that $\|\zeta_1\| < R$, where $R = \epsilon_1 + \rho$ and $\rho = \max_{t \in [0,T]} \|p(t)\|$.

Now let us fix an arbitrary number $l \in (0, 1)$. In accordance with inequality (9.6), presented in the appendix, there exists a natural number $n_0 = n_0\left(\frac{l\delta}{R}\right)$, independent of $\zeta_0$, such that

$$\|\zeta(n_0 T + t_1(\zeta_0), \zeta_0)\| < l\delta. \tag{5.1}$$

Let $\epsilon = \delta(\frac{1-l}{2})$, and suppose that the nonzero number $|\mu|$ is sufficiently small so that $|\mu| < \frac{\epsilon L_f}{M_g\left[e^{L_f(n_0+2)T} - 1\right]}$.

Take a solution $\eta_{x(t)}(t, \eta_0)$, with $\eta_0 \in B_\delta$. By the previous discussions, there exists a number $t_1(\eta_0) \in [0, 2T]$ such that $\zeta(t_1(\eta_0), \eta_0)$ belongs to $S$.

It can be verified that $\left\|\eta_{x(t)}(t, \eta_0) - \zeta(t, \eta_0)\right\| \le |\mu| M_g t + \int_0^t L_f \left\|\eta_{x(t)}(s, \eta_0) - \zeta(s, \eta_0)\right\| \mathrm{d}s$, and therefore we have

$$\left\|\eta_{x(t)}(t, \eta_0) - \zeta(t, \eta_0)\right\| \le \frac{|\mu| M_g}{L_f}\left(e^{L_f t} - 1\right),\ t \le (n_0 + 2)T.$$

In this case, we obtain the inequality

$$\left\|\eta_{x(t)}(n_0 T + t_1(\eta_0), \eta_0) - \zeta(n_0 T + t_1(\eta_0), \eta_0)\right\| < \epsilon,$$

and by means of (5.1) we have $\eta_1 = \eta_{x(t)}(\theta_1, \eta_0) \in B_\delta$, where $\theta_1 = n_0 T + t_1(\eta_0)$. We note that the point $\eta_1$ depends on both $\eta_0$ and $x(t)$.

Similarly to the preceding discussion, one can find that the inequality

$$\left\|\eta_{x(t)}(t, \eta_0) - \zeta(t - \theta_1, \eta_1)\right\| \le \frac{|\mu| M_g}{L_f}\left(e^{L_f (n_0+2)T} - 1\right)$$

holds for all $t \in [\theta_1, \theta_1 + (n_0 + 2)T]$. Additionally, the existence of a number $t_2(\eta_1) \in [0, 2T]$, such that $\zeta(t_2(\eta_1), \eta_1) \in S$, can be verified. Therefore, we have

$$\left\|\eta_{x(t)}(2n_0 T + t_1(\eta_0) + t_2(\eta_1), \eta_0) - \zeta(n_0 T + t_2(\eta_1), \eta_1)\right\| < \epsilon,$$

and hence $\eta_2 = \eta_{x(t)}(\theta_2, \eta_0) \in B_\delta$, where $\theta_2 = 2n_0 T + t_1(\eta_0) + t_2(\eta_1)$.

One can continue in the same manner to construct a sequence $\left\{t_j\right\}$, which satisfies $0 \le t_j \le 2T,\ j \ge 1$, and

$$\left\|\eta_{x(t)}(t, \eta_0) - \zeta(t - \theta_i, \eta_i)\right\| \le \frac{|\mu| M_g}{L_f}\left(e^{L_f (n_0+2)T} - 1\right) \tag{5.2}$$

for $t \in [\theta_i, \theta_i + (n_0 + 2)T]$, where $\eta_i = \eta_{x(t)}(\theta_i, \eta_0) \in B_\delta,\ i \ge 0,\ \theta_0 = 0$, and

$$\theta_i = i n_0 T + \sum_{j=1}^{i} t_j,\ i \ge 1. \tag{5.3}$$

We emphasize that for any $i \ge 1$ it is true that $\theta_i$ belongs to $[i n_0 T, i(n_0 + 2)T]$ and $\theta_i - \theta_{i-1} = n_0 T + t_i \le (n_0 + 2)T$. The procedure of the proof for $t \in [\theta_i, \theta_{i+1}]$ is illustrated in Fig. 4.

In the remaining part of the proof, we will demonstrate the boundedness of $\eta_{x(t)}(t, \eta_0) - p(t)$, which implies the boundedness of $\eta_{x(t)}(t, \eta_0)$.

For a fixed $i$, using the couple of relations $\zeta(t - \theta_i, \eta_i) = \eta_i + \int_{\theta_i}^t f(\zeta(s - \theta_i, \eta_i))\mathrm{d}s$ and $p(t) = p(\theta_i) + \int_{\theta_i}^t f(p(s))\mathrm{d}s$ one can obtain the following inequality:

$$\|\zeta(t - \theta_i, \eta_i) - p(t)\| \le (\delta + \rho)e^{2L_f T},\ \theta_i \le t \le \theta_i + 2T.$$

Hence, we have $\|\zeta(t_{i+1}(\eta_i), \eta_i)\| \le \rho + (\delta + \rho)e^{2L_f T}$.

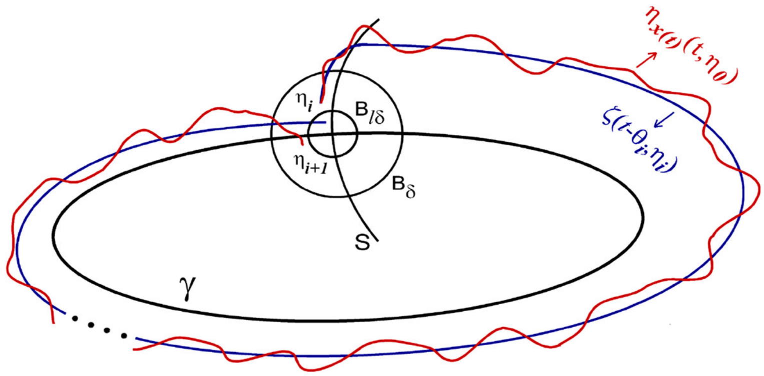



**Fig. 4** Schematic representation of proof of Theorem 5.1. The trajectory in *red* shows the function $\eta_{x(t)}(t, \eta_0)$, while the trajectory in *blue* represents $\zeta(t-\theta_i, \eta_i)$, where the sequence $\{\theta_i\}$ is defined in (5.3) and $\eta_i = \eta_{x(t)}(\theta_i, \eta_0)$. The presented illustration covers the two-dimensional case of the proof on the time interval $[\theta_i, \theta_{i+1}]$ for an arbitrary $i \geq 0$. In the figure, $B_{l\delta}$ and $B_\delta$ denote the open balls centered at the origin with radii $l\delta$ and $\delta$, respectively. At the moment $t = \theta_{i+1}$, the solution $\zeta(t-\theta_i, \eta_i)$ belongs to $B_{l\delta}$ and $\eta_{i+1}$ is inside $B_\delta$ (Color figure online)

Since the point $\zeta(t_{i+1}(\eta_i), \eta_i)$ is on the surface $S$, according to (9.8), it is true for $t \in \mathbb{R}_+$ that $\|\zeta(t + t_{i+1}(\eta_i), \eta_i) - p(t)\| \leq 4K_1 \left\|P^{-1}(0)\right\| \left[\rho + (\delta+\rho)\, e^{2L_f T}\right]$. Thus, we find

$$\left\|\eta_{x(t)}(t, \eta_0) - p(t)\right\| \leq \frac{|\mu|\, M_g}{L_f}\left(e^{L_f(n_0+2)T} - 1\right) + H_0(\delta, \rho),\ t \in \mathbb{R}_+, \tag{5.4}$$

where $H_0(\delta, \rho)$ is the maximum of the numbers $(\delta+\rho)e^{2L_f T}$ and $4K_1 \left\|P^{-1}(0)\right\| \left[\rho + (\delta+\rho)\, e^{2L_f T}\right]$. It is worth noting that $H_0(\delta, \rho) \to 0$ as $\delta \to 0$ and $\rho \to 0$, and $\left\|\eta_{x(t)}(t, \eta_0) - p(t)\right\|$ can be made arbitrarily small by suitable choices of $\mu$, $\delta$, and $\rho$.

Consequently, any solution $\eta_{x(t)}(t, \eta_0)$, where $\eta_0 \in B_\delta$, is bounded on $\mathbb{R}_+$ and remains near the limit cycle, in accordance with formula (5.4).

In compliance with the results of Theorems 3.1 and 4.1, the set $H$ exhibits sensitivity and contains infinitely many unstable periodic solutions. For each chaotic $x(t)$ the trajectories of (2.4) starting inside the ball $B_\delta$ constitute a subfamily of $H$ and behave chaotically around the limit cycle $\gamma$. Therefore, the entrainment of the limit cycle by chaos takes place in system (2.3). □

Given the presence of chaos in (2.1), we have obtained chaos for the couple $(x(t), y(t))$, so that one can talk not only of the entrainment by chaos but also of the extension of chaos to a higher-dimensional system.

## 6 Examples

We consider the system

$$\begin{aligned} u_1' &= \alpha u_1 - u_2 - u_1(u_1^2 + u_2^2), \\ u_2' &= u_1 + \alpha u_2 - u_2(u_1^2 + u_2^2), \end{aligned} \tag{6.1}$$

which is in the form of (2.2), where $\alpha$ is a positive number and

$$f(u_1, u_2) = \begin{pmatrix} \alpha u_1 - u_2 - u_1(u_1^2 + u_2^2) \\ u_1 + \alpha u_2 - u_2(u_1^2 + u_2^2) \end{pmatrix}.$$

One can verify that $p(t) = \left(\sqrt{\alpha}\cos t, \sqrt{\alpha}\sin t\right)$ is a periodic solution of (6.1). Evaluating $A(t) = \frac{\partial f(p(t))}{\partial u}$ gives us

$$A(t) = \begin{pmatrix} -2\alpha\cos^2 t & -1 - \alpha\sin(2t) \\ 1 - \alpha\sin(2t) & -2\alpha\sin^2 t \end{pmatrix}. \tag{6.2}$$

Evidently, the first multiplier of the corresponding variational system is $\rho_1 = 1$, and, according to Lemma 7.3 (Hale 1980), $\rho_2 = \exp\left(\int_0^{2\pi} tr\,A(s)\mathrm{d}s\right) = e^{-4\pi\alpha}$. Thus, the periodic solution $p(t)$ is asymptotically orbitally stable according to the Andronov–Witt theorem.

As the generator we will make use of the Duffing equation in the form

$$x'' + D_1 x' + D_2 x^3 = \lambda\cos t, \tag{6.3}$$

where $D_1$, $D_2$, and $\lambda$ are constants. Defining the variables $x_1 = x$ and $x_2 = x'$, Eq. (6.3) can be rewritten as

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -D_1 x_2 - D_2 x_1^3 + \lambda\cos t. \end{aligned} \tag{6.4}$$

*Example 1* Consider system (6.4) with $D_1 = 0.05$, $D_2 = 1$, and $\lambda = 7.5$ such that the system possesses chaotic motions seen through simulations (Thompson and Stewart 2002). Perturbing system (6.1) with solutions of (6.4) and setting $\alpha = 9$, we obtain the following system:

$$\begin{aligned} y_1' &= 9y_1 - y_2 - y_1(y_1^2 + y_2^2) + 0.5x_1(t), \\ y_2' &= y_1 + 9y_2 - y_2(y_1^2 + y_2^2) + 3.6x_2(t). \end{aligned} \tag{6.5}$$

In Fig. 5a, b, we depict the chaotic trajectories of systems (6.4) and (6.5), respectively. The initial data $x_1(0) = 3.05$, $x_2(0) = 4.153$, $y_1(0) = 2.8$, $y_2(0) = 0.5$ are used. Figure 5b shows the chaotic motion in a neighborhood of the limit cycle of (6.1). The pictures support the results of the present study predicting entrainment by chaos.

Figure 6 depicts the Poincaré sections, which are obtained by marking the trajectories of systems (6.4) and (6.5) with $x_1(0) = 2$, $x_2(0) = 3$, $y_1(0) = 3$, $y_2(0) = 0$ stroboscopically at times $t$, which are integer multiples of $2\pi$. Figure 6a represents the strange attractor of the first system, and Fig. 6b demonstrates the entrained chaotic behavior.

Now, to show through simulations the replication of sensitivity, we consider two initially close solutions of system (6.4) + (6.5), one with the initial data $x_1(0) =$

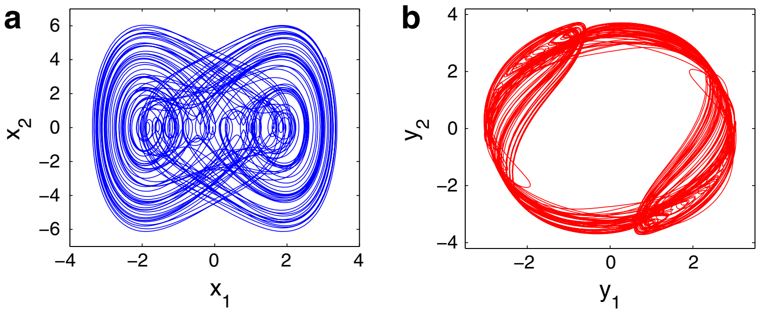



**Fig. 5** The chaotic behavior of system (6.4) is pictured in (**a**), and the chaotic motion generated around the limit cycle is shown in (**b**)

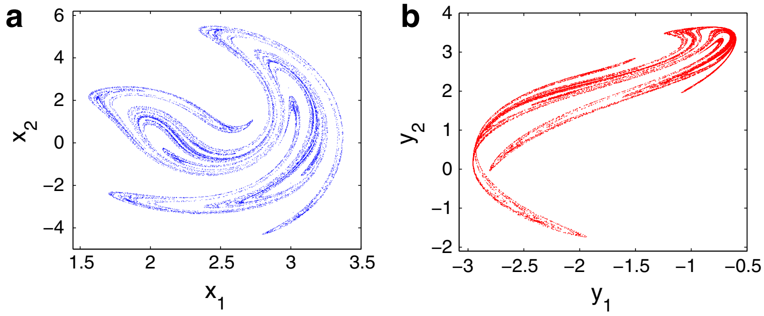



**Fig. 6** Poincaré sections of systems (6.4) and (6.5)

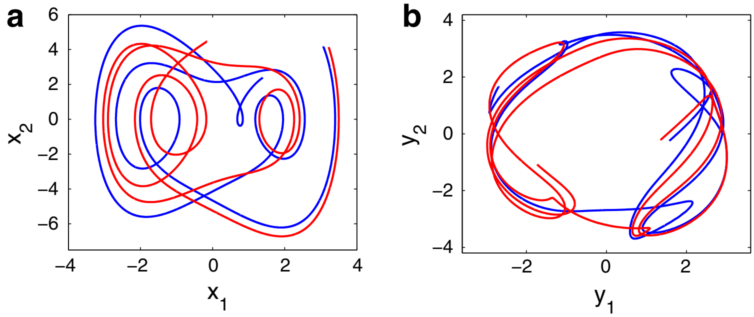



**Fig. 7** The picture in **a** shows two initially close trajectories corresponding to system (6.4) that eventually diverge. The replication of sensitivity is observed in **b**, where the blue and red trajectories are initially close to each other and are then separated

3.07, $x_2(0) = 4.18$, $y_1(0) = 1.57$, $y_2(0) = -0.25$, presented in blue, and another with the initial data $x_1(0) = 3.22$, $x_2(0) = 4.14$, $y_1(0) = 1.35$, $y_2(0) = -0.22$, pictured in red. In Fig. 7, we present these trajectories. Figure 7a shows the existence of sensitivity in system (6.4), while Fig. 7b illustrates the replication of this feature.

Formula (3.3) implies that the strength of the system (2.3)'s sensitivity is proportional to the strength of the chaotic perturbation, $\mu g(x)$, used in the system. Therefore, despite the fact that the extension of sensitivity is guaranteed by Theorem 3.1, if one considers (2.3) with weak perturbations, it may not be visible in simulation results. On the other hand, according to formula (5.2), strong perturbations may diminish the cyclical behavior of the chaotic solutions. For that reason, given the strength of the perturbation used in system (6.5), Fig. 7b displays the extension of sensitivity but does not indicate cyclical behavior.

We will continue with an example that demonstrates the extension of a period-doubling cascade.

*Example 2* In the paper Sato et al. 1983, it is mentioned that system (6.4), in which $\lambda$ is considered as a parameter, displays period-doubling bifurcations for the coefficients $D_1 = 0.3,\ D_2 = 1$, and the sequence of bifurcation parameter values accumulates at $\lambda = \lambda_\infty \equiv 40$ such that the system admits infinitely many unstable periodic orbits.

To illustrate the entrainment by chaos, system (6.4), with the specified coefficients and $\lambda = \lambda_\infty$, will be utilized as the generator. Let us use the solutions of (6.4) to perturb (6.1) and build the system

$$\begin{aligned} y_1' &= \alpha y_1 - y_2 - y_1\left(y_1^2 + y_2^2\right) + \mu x_1(t),\\ y_2' &= y_1 + \alpha y_2 - y_2\left(y_1^2 + y_2^2\right) + \mu x_2(t), \end{aligned} \tag{6.6}$$

where $\mu$ is a nonzero constant.

We will make use of the Lyapunov function $V(y_1, y_2) = y_1^2 + y_2^2$ to show the validity of condition (A5) for system (6.6). One can verify that

$$V'_{(6.6)}(y_1, y_2) = -2\sqrt{y_1^2 + y_2^2}\left[\left(y_1^2 + y_2^2 - \alpha\right)\sqrt{y_1^2 + y_2^2} - \frac{\mu}{\sqrt{y_1^2 + y_2^2}}\left(x_1 y_1 + x_2 y_2\right)\right].$$

Let us fix a positive number $r_0$ and suppose that $\sqrt{y_1^2 + y_2^2} > \sqrt{\alpha} + r_0$. Under this condition we have $\left(y_1^2 + y_2^2 - \alpha\right)\sqrt{y_1^2 + y_2^2} > r_0^3 + 3r_0^2\sqrt{\alpha} + 2r_1\alpha$. Since the chaotic attractor of system (6.4) satisfies $|x_1| < 6$ and $|x_2| < 15$, we obtain that $\left|\frac{\mu}{\sqrt{y_1^2+y_2^2}}\left(x_1 y_1 + x_2 y_2\right)\right| \le 21\left|\mu\right|$. Therefore, if $|\mu|$ is sufficiently small so that $|\mu| \le r_0^3/21$, then $V'_{(6.6)}(y_1, y_2) < 0$ for $\sqrt{y_1^2 + y_2^2} > \sqrt{\alpha} + r_1$, and condition $(A5)$ holds for system (6.6).

In conformity with the preceding discussion, one can identify a bounded region $G$ in $\mathbb{R}^2$ such that for sufficiently small $|\mu|$, Massera's theorem (Massera 1950; Yoshizawa 1975) implies the existence of a periodic solution of the planar system (6.6) inside the region $G$ for each periodic $(x_1(t), x_2(t))$. Moreover, all these periodic solutions are unstable.

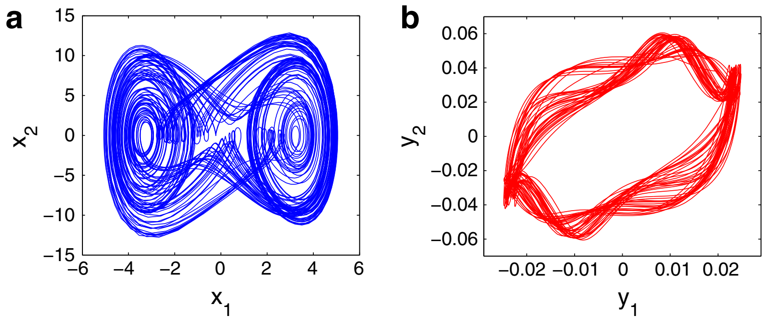



**Fig. 8** Chaotic trajectories of unidirectionally coupled systems (6.4) and (6.6). The coefficients $D_1 = 0.3,\ D_2 = 1,\ \lambda = 40,\ \alpha = 0.002$, and $\mu = 0.008$ are used in the simulation

In Fig. 8, the trajectories of systems (6.4) and (6.6), where $\alpha = 0.002$ and $\mu = 0.008$, with $x_1(0) = 3.5,\ x_2(0) = -2,\ y_1(0) = 0.02,\ y_2(0) = 0.038$, are seen. Figure 8a illustrates the chaotic behavior of system (6.4) and Fig. 8b shows the irregular motion around the limit cycle.

## 7 Discussion

This section is devoted to discussions and simulations of the entrainment of toroidal attractors by chaos and entrainment in Chua's oscillators, as well as controlling and synchronization problems. We start with the demonstration of chaos generation around tori.

### 7.1 Chaotic Tori

In previous parts of the paper, we discussed the entrainment of limit cycles by chaos. Now the question is whether a similar approach is possible around tori. In this part, we will investigate numerically the problem of capture of chaos by toroidal attractors.

Let us consider the following system (Hale and Koçak 1991; Langford 1985):

$$\begin{aligned} u_1' &= (\lambda - 3)u_1 - 0.25u_2 + u_1\left(u_3 + 0.2(1 - u_3^2)\right), \\ u_2' &= 0.25u_1 + (\lambda - 3)u_2 + u_2\left(u_3 + 0.2(1 - u_3^2)\right), \\ u_3' &= \lambda u_3 - (u_1^2 + u_2^2 + u_3^2), \end{aligned} \tag{7.1}$$

where $\lambda$ is a parameter.

For small and positive values of the parameter $\lambda$, system (7.1) admits an asymptotically stable equilibrium point with a positive $u_3$ coordinate close to the origin. At $\lambda \approx 1.68$, the equilibrium point loses its stability, and a hyperbolic, asymptotically orbitally stable limit cycle emerges. At the parameter value $\lambda = 2$, the periodic orbit is still asymptotically orbitally stable, but not hyperbolic. For $\lambda > 2$, the limit cycle is no

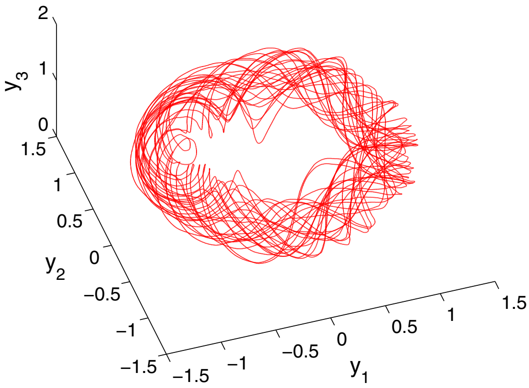



**Fig. 9** The emergence of chaotic motion around a torus demonstrates entrainment by chaos

longer stable, and an attracting invariant torus is formed near the periodic orbit. With the increasing values of λ, the invariant torus grows rapidly (Hale and Koçak 1991).

To produce chaotic motions around the torus, we use the chaotic Lorenz system (Lorenz 1963)

$$\begin{aligned} x_1' &= -10x_1 + 10x_2, \\ x_2' &= -x_1x_3 + 28x_1 - x_2, \\ x_3' &= x_1x_2 - (8/3)x_3 \end{aligned} \tag{7.2}$$

as the generator and set up the following system:

$$\begin{aligned} y_1' &= (\lambda - 3)y_1 - 0.25y_2 + y_1\left(y_3 + 0.2(1 - y_3^2)\right) + 0.003x_1(t), \\ y_2' &= 0.25y_1 + (\lambda - 3)y_2 + y_2\left(y_3 + 0.2(1 - y_3^2)\right) + 0.004x_2(t), \\ y_3' &= \lambda y_3 - (y_1^2 + y_2^2 + y_3^2) + 0.002x_3(t), \end{aligned} \tag{7.3}$$

where $\lambda = 2.003$.

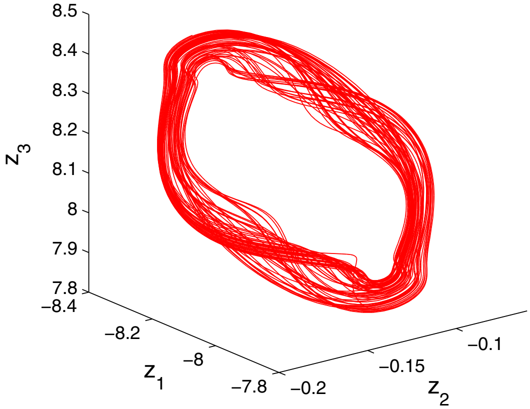
Figure 9 shows the trajectory of system (7.3) with $y_1(0) = 0.0793$, $y_2(0) = -1.1761$, $y_3(0) = 0.9449$, where $x(t)$ is a solution of (7.2) with $x_1(0) = -6.7453$, $x_2(0) = 0.3435$, $x_3(0) = 32.7629$. One can see that the motion is chaotic and surrounds the torus.

## 7.2 Entrainment in Chua's Oscillators

We continue the discussion by presenting a simulation result for the entrainment by chaos in a Chua oscillator.

Using system (6.4) with $D_1 = 0.3$, $D_2 = 1$, and $\lambda = 40$ as the generator of chaos, we demonstrated in Sect. 6 that system (6.6), with $\alpha = 0.002$ and $\mu = 0.008$, exhibits motions that behave chaotically and cyclically, so that entrainment by chaos is present.

**Fig. 10** Chaotic and cyclic motion generated by perturbed Chua system (7.4)




Now we consider a Chua oscillator that admits an asymptotically stable equilibrium in dimensionless form (Chua et al. 1993) and perturb it with the solutions of (6.6):

$$\begin{aligned} z_1' &= (21.32/5.75)[z_2 - 0.13396z_1 + 0.48993(|z_1+1| + |z_1-1|)] + 0.5y_1(t),\\ z_2' &= z_1 - z_2 + z_3 + 2y_2(t),\\ z_3' &= -7.8351z_2 - (1.38166392/12)z_3 + 3y_2(t). \end{aligned} \tag{7.4}$$

Note that system (6.6), which itself is a perturbed system, is the generator. According to Akhmet and Fen (2013a), we must observe chaotic behavior in the oscillator. We consider a trajectory of system (6.4) + (6.6) with initial data $x_1(0) = 3.5$, $x_2(0) = -2$, $y_1(0) = 0.02$, $y_2(0) = 0.038$ and plot the corresponding trajectory of (7.4) with $z_1(0) = -8.016$, $z_2(0) = -0.084$, $z_3(0) = 7.792$ in Fig. 10. It confirms that chaotic motions around a cycle emerge in the perturbed Chua system, and this is a manifestation of entrainment by chaos.

The obtained result highlights the possibility of employing existing cyclic chaos to generate a new one in systems with stable equilibria, and particularly in Chua's oscillators. Furthermore, it is seen in Fig. 10 that the resulting motion resembles the spiral Chua attractor, which occurs in the case of a period-doubling cascade (Chua et al. 1993; Lakshmanan and Rajasekar 2003).

### 7.3 Controlling Chaos

The Pyragas control method (Pyragas 1992; Schöll and Schuster 2008; Zelinka et al. 2010; Fradkov 2007) is an effective instrument for stabilizing the unstable periodic orbits of chaotic systems. It is also very useful for visually discerning periodic solutions, which are otherwise indistinguishable in the set of irregular motions.

As an example, we will describe the procedure for stabilizing unstable periodic solutions of systems of the form (2.1) + (2.3).

It is demonstrated in Gonzales-Miranda 2004 that to apply the Pyragas control method to the chaotic Duffing oscillator given by the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -0.10x_2 + 0.5x_1\left(1 - x_1^2\right) + 0.24\sin t, \end{aligned} \tag{7.5}$$

one can construct the corresponding control system

$$\begin{aligned} z_1' &= z_2, \\ z_2' &= -0.10z_2 + 0.5z_1\left(1 - z_1^2\right) + 0.24\sin(z_3) + C\left[z_2(t-\tau_0) - z_2(t)\right], \\ z_3' &= 1, \end{aligned} \tag{7.6}$$

where $q(t) = C\left[z_2(t-\tau_0) - z_2(t)\right]$ is the control law and the parameter $C$ represents the strength of the perturbation. An unstable $2\pi$-periodic solution can be stabilized by choosing the value $\tau_0 = 2\pi$.

Using system (7.5) as the generator, we set up the following system:

$$\begin{aligned} y_1' &= 7y_1 - y_2 - y_1(y_1^2 + y_2^2) + 5x_1(t), \\ y_2' &= y_1 + 7y_2 - y_2(y_1^2 + y_2^2) + 4(x_2(t) + x_2^3(t)). \end{aligned} \tag{7.7}$$

According to the theoretical discussions, system (7.5) + (7.7) is chaotic, and there is entrainment by chaos such that (7.7) exhibits chaotic motions around the limit cycle of system (6.1) with $\alpha = 7$.

Our current objective is to show numerically how to control the chaos of system (7.5) + (7.7). We suggest that if a periodic solution of the generator system (7.5) is stabilized, then the chaos of system (7.5) + (7.7) will be controlled.

To apply the Pyragas method to control the chaos of (7.5) + (7.7), we set up the system

$$\begin{aligned} v_1' &= 7v_1 - v_2 - v_1(v_1^2 + v_2^2) + 5z_1(t), \\ v_2' &= v_1 + 7v_2 - v_2(v_1^2 + v_2^2) + 4(z_2(t) + z_2^3(t)), \end{aligned} \tag{7.8}$$

where $z_1(t)$ and $z_2(t)$ refer to the first and second coordinates of the solutions of the control system (7.6).

Let us consider the solution of (7.6) + (7.8) with initial data $z_1(0) = 0.2,\ z_2(0) = 0.4,\ z_3(0) = 0,\ v_1(0) = -2.5$, and $v_2(0) = 0.8$. We allow system (7.6) + (7.8) to evolve freely by taking $C = 0$ until $t = 70$, and at that moment we switch on the control and use $C = 0.84$. When $t = 210$, the control mechanism is switched off, and thenceforth, the value $C = 0$ is utilized. Figure 11, which depicts chaos control, shows the $z_2$ and $v_2$ coordinates of the solution. It can be observed that after switching off the control mechanism, the stabilized $2\pi$-periodic solution of system (7.5) + (7.7) loses its stability, and chaos emerges again. One can obtain similar graphs for the other coordinates of (7.6) + (7.8).

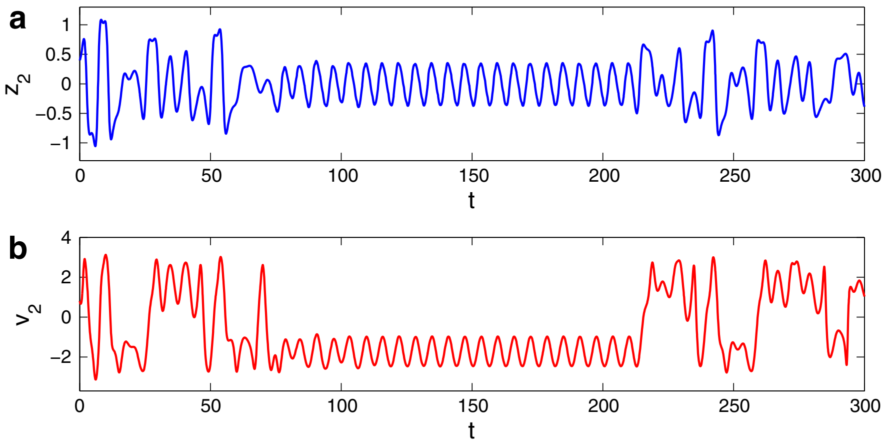



**Fig. 11** Application of Pyragas control method to system (7.5) + (7.7) by means of system (7.6) + (7.8). **a** Graph of $z_2$ coordinate. **b** Graph of $v_2$ coordinate

## 7.4 Entrainment and Synchronization

In this subsection, we will show that our results cannot be considered generalized synchronization (GS) results (Rulkov et al. 1995).

GS characterizes the dynamics of a response system that is driven by the output of a chaotic driving system (Rulkov et al. 1995; Kocarev and Parlitz 1996; Hunt et al. 1997; Abarbanel et al. 1996; Gonzales-Miranda 2004). Suppose that the dynamics of the drive and response are governed by the following systems with a skew product structure:

$$x' = D(x) \tag{7.9}$$

and

$$y' = R(y, K(x)), \tag{7.10}$$

respectively, where $x \in \mathbb{R}^p$, $y \in \mathbb{R}^q$. Synchronization (Rulkov et al. 1995) is said to occur if there exist sets $I_x$, $I_y$ of initial conditions and a transformation $\phi$, defined on the chaotic attractor of (7.9), such that for all $x(0) \in I_x$, $y(0) \in I_y$ the relation $\lim_{t\to\infty} \|y(t) - \phi(x(t))\| = 0$ holds. In this case, a motion that starts on $I_x \times I_y$ collapses onto a manifold $M \subset I_x \times I_y$ of synchronized motions. The transformation $\phi$ is not required to exist for the transient trajectories. When $\phi$ is the identity, the identical synchronization takes place (Pecora and Carroll 1990; Gonzales-Miranda 2004). The case of differentiable $\phi$ is considered in (Hunt et al. 1997).

It was formulated in the paper Kocarev and Parlitz 1996 that GS occurs if and only if for all $x_0 \in I_x$, $y_{10}, y_{20} \in I_y$ the following asymptotic stability criterion holds:

$$\lim_{t\to\infty} \|y(t, x_0, y_{10}) - y(t, x_0, y_{20})\| = 0,$$

where $y(t, x_0, y_{10}),\ y(t, x_0, y_{20})$ denote the solutions of (7.10) with the initial data $y(0, x_0, y_{10}) = y_{10},\ y(0, x_0, y_{20}) = y_{20}$ and the same $x(t),\ x(0) = x_0$.

To compare our approach with that of GS, let us apply the auxiliary system method (Abarbanel et al. 1996; Gonzales-Miranda 2004) to indicate the presence or absence of GS in the couple (2.1) + (2.3) [(2.5) + (2.3)], considered this time as drive-response systems (as is accepted in synchronization theory).

Let us start the procedure where the couple (6.4) + (6.6), where $D_1 = 0.3,\ D_2 = 1,\ \lambda = 40,\ \alpha = 0.002$, and $\mu = 0.008$ such that the entrainment by chaos takes place as demonstrated in Sect. 6. The corresponding auxiliary system is

$$z_1' = 0.002z_1 - z_2 - z_1\left(z_1^2 + z_2^2\right) + 0.008x_1(t),$$
$$z_2' = z_1 + 0.002z_2 - z_2\left(z_1^2 + z_2^2\right) + 0.008x_2(t), \qquad (7.11)$$

which is an identical copy of system (6.6).

The projection of the stroboscopic plot of the six-dimensional system (6.4) + (6.6) + (7.11) on the $y_1 - z_1$ plane is depicted in Fig. 12. The figure is obtained by marking the trajectory with the initial data $x_1(0) = 3.5,\ x_2(0) = -2,\ y_1(0) = -0.01,\ y_2(0) = -0.03,\ z_1(0) = 0.02,\ z_2(0) = 0.038$ at times $t$ that are integer multiples of $2\pi$ and by omitting the first 4,000 iterations. It is observable in Fig. 12 that the stroboscopic plot is not on the line $z_1 = y_1$, and therefore GS does not take place in system (6.4) + (6.6).

To have a more detailed comparison of the present results with GS, let us consider a Rössler–Lorenz couple. GS was observed in Abarbanel et al. 1996; Gonzales-Miranda 2004 with specific values of coefficients and perturbations. Let us take into account the couple with our particular data that issues from the present investigations.

Consider the Lorenz system

$$u_1' = -10u_1 + 10u_2,$$
$$u_2' = -u_1u_3 + 350u_1 - u_2,$$
$$u_3' = u_1u_2 - (8/3)u_3. \qquad (7.12)$$

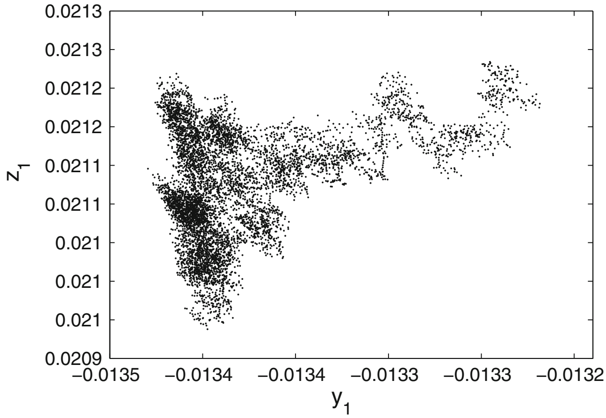



**Fig. 12** Projection of stroboscopic plot of system (6.4) + (6.6) + (7.11) on $y_1 - z_1$ plane

According to Sparrow (1982), system (7.12) possesses a globally attracting limit cycle. We perturb system (7.12) with the solutions of the chaotic Rössler system (Rössler 1976),

$$\begin{aligned} x_1' &= -(x_2 + x_3), \\ x_2' &= x_1 + 0.2x_2, \\ x_3' &= 0.2 + x_3(x_1 - 5.7), \end{aligned} \tag{7.13}$$

and set up the system

$$\begin{aligned} y_1' &= -10y_1 + 10y_2 + 2.8x_1(t), \\ y_2' &= -y_1y_3 + 350y_1 - y_2 + 7x_2(t), \\ y_3' &= y_1y_2 - (8/3)y_3 + 4.5x_3(t). \end{aligned} \tag{7.14}$$

Using the solution of system (7.13) with $x_1(0) = 2.1$, $x_2(0) = -7.7$, $x_3(0) = 0.1$, we represent the trajectory of system (7.14) corresponding to the initial data $y_1(0) = -19.2$, $y_2(0) = -63.9$, $y_3(0) = 296.1$ in Fig. 13a. The projection of the same trajectory on the $y_1 - y_2$ plane is shown in Fig. 13b. The simulation results show that chaotic motions appear near the limit cycle. Moreover, the irregular behavior of the $y_3$ coordinate is illustrated in Fig. 14.

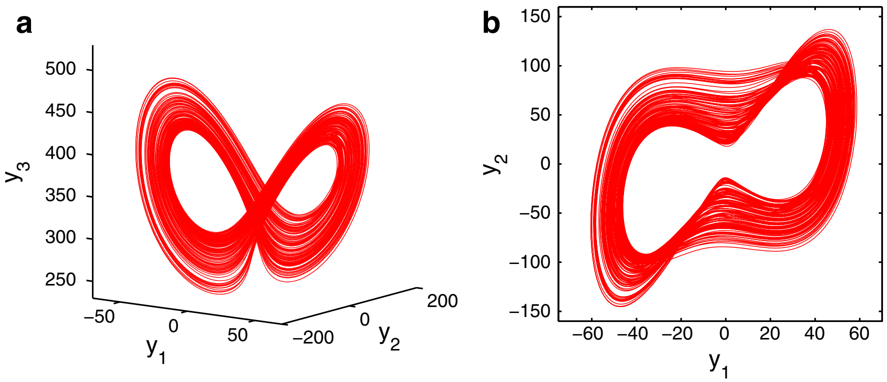



**Fig. 13** Chaotic trajectory of system (7.14) near limit cycle

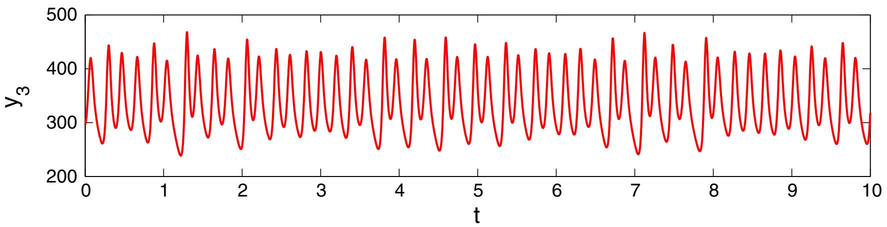



**Fig. 14** Irregular behavior of $y_3$ coordinate of system (7.14)

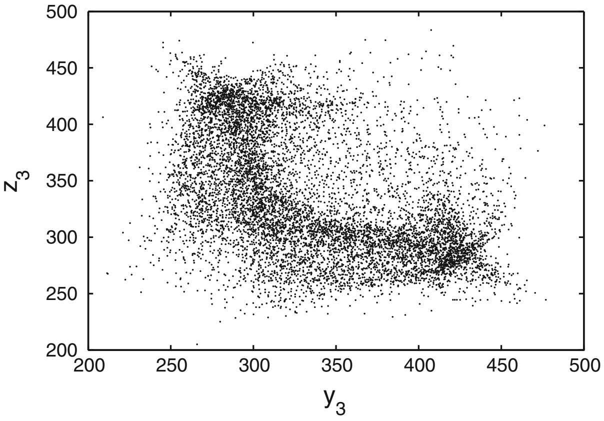



**Fig. 15** Application of the auxiliary system approach to the system (7.13) + (7.14) indicates that GS does not exist for the couple

For system (7.13) + (7.14), we can construct the corresponding auxiliary system of the form

$$\begin{aligned} z_1' &= -10z_1 + 10z_2 + 2.8x_1(t), \\ z_2' &= -z_1z_3 + 350z_1 - z_2 + 7x_2(t), \\ z_3' &= z_1z_2 - (8/3)z_3 + 4.5x_3(t). \end{aligned} \tag{7.15}$$

We show in Fig. 15 the projection of the stroboscopic plot of system (7.13) + (7.14) + (7.15) on the $y_3 - z_3$ plane. The initial data $x_1(0) = 2.1,\ x_2(0) = -7.7,\ x_3(0) = 0.1,\ y_1(0) = -19.2,\ y_2(0) = -63.9,\ y_3(0) = 296.1,\ z_1(0) = -14.9,\ z_2(0) = -75.6,\ z_3(0) = 325.4$ are used and the first 200 iterations are omitted. To have GS indicated by the auxiliary system approach, we need the stroboscopic plot to be placed on the line $z_3 = y_3$. Since this is not the case, as seen in Fig. 15, we can conclude that the entrainment by chaos is not GS.

Next, let us use the auxiliary system approach to analyze the couple (7.2) + (7.3) with $\lambda = 2.003$ such that the entrainment by chaos takes place as shown in Sect. 7.1. The auxiliary system in this case is

$$\begin{aligned} z_1' &= -0.997z_1 - 0.25z_2 + z_1\left(z_3 + 0.2(1 - z_3^2)\right) + 0.003x_1(t), \\ z_2' &= 0.25z_1 - 0.997z_2 + z_2\left(z_3 + 0.2(1 - z_3^2)\right) + 0.004x_2(t), \\ z_3' &= 2.003z_3 - (z_1^2 + z_2^2 + z_3^2) + 0.002x_3(t). \end{aligned} \tag{7.16}$$

Making use of the initial data $x_1(0) = -6.74,\ x_2(0) = 0.34,\ x_3(0) = 32.76,\ y_1(0) = 0.07,\ y_2(0) = -1.17,\ y_3(0) = 0.94,\ z_1(0) = 0.85,\ z_2(0) = -0.24,\ z_3(0) = 0.74$ and omitting the first 200 iterations, we depict in Fig. 16 the projection of the stroboscopic plot of system (7.13) + (7.14) + (7.15) on the $y_3 - z_3$ plane. One can see in Fig. 15 that the stroboscopic plot is not placed on the line $z_3 = y_3$.

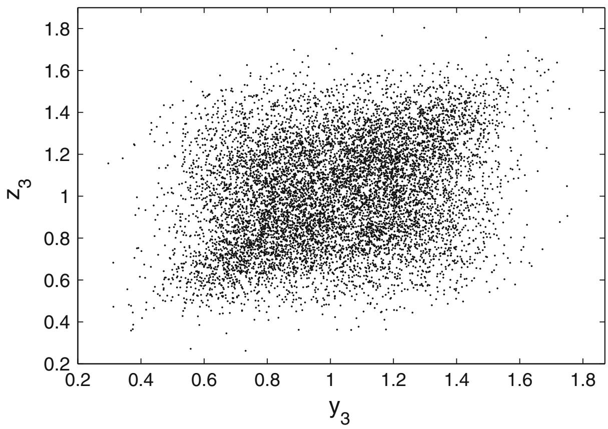



**Fig. 16** Application of the auxiliary system approach to the system (7.2) + (7.3) reveals that entrainment of toroidal attractors by chaos is not GS

Therefore, we conclude that GS is not achieved in the dynamics of the coupled system (7.13) + (7.14).

## 8 Conclusions

The concept of entrainment is extended to introduce the notion of the entrainment of limit cycles by chaos. Our theoretical results can be effectively adapted to arbitrarily high-dimensional systems that possess asymptotically orbitally stable limit cycles. Examples of such systems can be found in mechanics, electronics, economics, neural sciences, chemistry, and population dynamics (Kostova et al. 2004; Jiang et al. 2004; Wang 2008; Morton and Beran 1999; Parlitz and Lauterborn 1985; D'Humieres et al. 1982; Zhang et al. 2006). Employing the method presented, one can obtain motions that behave cyclically and chaotically at the same time.

We prove the presence of chaos through the notions of period-doubling cascade and sensitivity. It is known (Robinson 1995; Lorenz 1963) that sensitivity is the main ingredient of chaos. Verifying other ingredients of chaos, namely, the transitivity and density of periodic motions, is more difficult.

The entrainment of toroidal attractors by chaos and entrainment in Chua's oscillators are demonstrated numerically. Moreover, the existence of unstable periodic solutions is evidenced through the Pyragas method (Pyragas 1992) and simulations.

One of the important peculiarities of our approach is that entrainment by chaos cannot be embedded as a part of synchronization theory (Rulkov et al. 1995; Kocarev and Parlitz 1996; Hunt et al. 1997; Abarbanel et al. 1996; Gonzales-Miranda 2004).

Cyclical behavior in chaotic attractors has been widely observed in the literature. We note the famous Rössler attractor, Chua's spiral attractor, and even the classical Lorenz attractor, where one can observe two-center cyclical behavior, as examples. Our results for obtaining cyclical behavior are different from those presented in the literature in that exogenous perturbations are applied in our case. In fact, the mechanism proposed in this study could be the unsuspected underlying force that gives rise to some chaotic attractors discussed in the literature.

Some of our results, for example, the boundedness of solutions around the limit cycle, can be obtained if one applies the results in Hirsch et al. 1977 on the existence of invariant manifolds and their persistence under perturbation or by reduction to discrete equations with respect to both phase and time variables (Hartman 1964).

The conclusions of this paper can be replicated in cases in which cycles are attracting when time decreases to $-\infty$. Another theoretically challenging problem is to consider hyperbolic cycles, as well as critical cases (Aulbach 1981; Hale and Stokes 1960; Mitropolskij and Lykova 1973). Moreover, our results are useful for generating multidimensional chaos, especially if one requires a rigorous proof of the phenomenon (Marotto 1978). We can formally compare our results with those of Ruelle and Takens 1971 on the appearance of turbulence through three successive Hopf bifurcations. Unlike Ruelle and Takens, we observe chaos to emerge after fewer than three bifurcations and we use chaotic perturbations.

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## 9 Appendix: Motion Near the Limit Cycle

In this part, we provide the needed information from the proof of the Andronov–Witt theorem (Farkas 2010) and clarify the decay of the solutions regarding the initial value.

Without loss of generality, let us assume that $p(0) = 0$ and $p'(0) = (\bar{p}_1, 0, 0, \ldots, 0)$ for some positive number $\bar{p}_1$.

According to our assumption that system (2.7) admits the number 1 as a simple characteristic multiplier and the remaining $n-1$ characteristic multipliers are smaller than one in modulus, system (2.7) has a real fundamental matrix $\Phi(t)$ of the form $\Phi(t) = P(t)\begin{pmatrix} 1 & 0 \\ 0 & e^{B_1 t}\end{pmatrix}$, where $P(t)$ is a regular, continuously differentiable $T$-periodic matrix and $B_1$ is an $(n-1)\times(n-1)$ matrix all of whose eigenvalues have negative real parts.

We emphasize that for an arbitrary solution $u(t)$ of Eq. (2.2), the differential equation satisfied by the function $z(t) = u(t) - p(t)$ is

$$z' = A(t)z + \varphi(t, z), \tag{9.1}$$

where $A(t) = \frac{\partial f(p(t))}{\partial u}$ and $\varphi(t, z) = f(p(t) + z) - f(p(t)) - A(t)z$. It is clear that $\varphi(t + T, z) = \varphi(t, z)$ and $\varphi(t, 0) = \varphi_z(t, 0) = 0$ for all $t \in \mathbb{R}_+$.

Since $\varphi_z(t, z) = o(1)$ as $z \to 0$ uniformly in $t \in \mathbb{R}_+$, there exist numbers $L_\varphi > 0$ and $\widetilde{\delta}(L_\varphi) > 0$ such that if $\|z_1\| < \widetilde{\delta}(L_\varphi)$, $\|z_2\| < \widetilde{\delta}(L_\varphi)$, then the inequality $\|\varphi(t, z_1) - \varphi(t, z_2)\| \le L_\varphi \|z_1 - z_2\|$ holds uniformly for $t \in \mathbb{R}_+$.

Suppose that $a = (0, a_2, a_3, \ldots, a_n)$ is an $n$-dimensional vector that is orthogonal to $p'(0)$. There exist positive numbers $K_1$ and $\alpha$ such that $\|\Phi(t)a\| \le K_1 \|a\| e^{-\alpha t}$ for all $t \in \mathbb{R}_+$. Moreover, if $\|a\| < \widetilde{\delta}(L_\varphi)/(2K_1)$, then a solution $z(t, a)$ of (9.1) exists on $[0, \infty)$ and satisfies the following inequality:

$$\|z(t,a)\| \le 2K_1 \|a\| e^{-\alpha t/2}, \quad t \ge 0. \tag{9.2}$$

A solution $\zeta(t, \zeta_0)$ to (2.2) satisfies the relation $\zeta(t, \zeta_0) = z(t, a) + p(t)$, where $z(t, a)$ is a solution to (9.1) with $z(0, a) = \zeta_0$. Additionally, the equation

$$\zeta_0 = P(0)a - \widetilde{h}(a) \tag{9.3}$$

holds, where $\widetilde{h}(a) = \big(\widetilde{h}_1(a_2, \ldots, a_n), 0, \ldots, 0\big)$, for some continuously differentiable function $\widetilde{h}_1$, and $\widetilde{h}(a) = o(\|a\|)$.

Suppose that $\zeta_0 = (\zeta_1^0, \zeta_2^0, \ldots, \zeta_n^0)$ and $p_{ij}$ are the coordinates of the matrix $P(0)$, where $i, j = 1, 2, \ldots, n$. Equation (9.3) is equivalent to

$$\zeta_1^0 + \sum_{i=2}^{n} q_i \zeta_i^0 - h(\eta_2^0, \zeta_3^0, \ldots, \zeta_n^0) = 0, \tag{9.4}$$

where $q_i$, $i = 2, \ldots, n$, are constants and $h$ is a continuously differentiable function such that

$$h(\zeta_2^0, \ldots, \zeta_n^0) = o\left(\left(\sum_{i=2}^{n} (\zeta_i^0)^2\right)^{1/2}\right).$$

Denote by $S$ the $(n-1)$ dimensional, $C^1$ manifold determined by the equation

$$x_1 + \sum_{i=2}^{n} q_i x_i - h(x_2, x_3, \ldots, x_n) = 0. \tag{9.5}$$

The hypersurface $S$ crosses the orbit $\gamma$, which is defined by Eq. (2.6), transversally, so that for any solution $\zeta(t, \zeta_0)$ starting on this initial manifold we have $\|\zeta(t, \zeta_0) - p(t)\| \to 0$ exponentially as $t \to \infty$.

We will now prove that for each number $l \in (0, 1)$ there exists a natural number $n_0 = n_0(l)$ such that if $\zeta_0$ belongs to $S$, then

$$\|\zeta(n_0 T, \zeta_0)\| \le l \|\zeta_0\|. \tag{9.6}$$

Let $\overline{\epsilon} = 1/\left(2\left\|P^{-1}(0)\right\|\right)$. It is possible to find a number $\overline{\delta}(\overline{\epsilon}) > 0$ such that if

$$\|a\| < \min\left\{\widetilde{\delta}(L_\varphi)/(2K_1), \overline{\delta}(\overline{\epsilon})\right\},$$

then the inequality

$$\left\|\widetilde{h}(a)\right\| < \overline{\epsilon} \|a\| \tag{9.7}$$

is valid.

Let us fix a solution $\zeta(t,\zeta_0)$ such that $\zeta_0$ belongs to $S$. In the case $\|a\| < \min\left\{\widetilde{\delta}(L_\varphi)/(2K_1), \overline{\delta}(\epsilon)\right\}$, taking advantage of (9.3) and (9.7) one can find that $\|a\| \leq 2\left\|P^{-1}(0)\right\|\|\zeta_0\|$, and, according to (9.2), we have

$$\|\zeta(t,\zeta_0) - p(t)\| \leq 4K_1 \left\|P^{-1}(0)\right\| \|\zeta_0\| e^{-\alpha t/2},\ t \geq 0. \tag{9.8}$$

Let us fix an arbitrary number $l \in (0,1)$. There exists a natural number $n_0 = n_0(l)$ such that $4K_1\left\|P^{-1}(0)\right\| e^{-\alpha T n_0/2} < l$. Making use of (9.8) we obtain that $\|\zeta(n_0T,\zeta_0) - p(n_0T)\| < l\,\|\zeta_0\|$. Since $p(n_0T) = 0$, inequality (9.6) holds.

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