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# Shunting inhibitory cellular neural networks with chaotic external inputs

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Taking advantage of external inputs, it is shown that shunting inhibitory cellular neural networks behave chaotically. The analysis is based on the Li-Yorke definition of chaos. Appropriate illustrations which support the theoretical results are depicted. © *2013 AIP Publishing LLC*.
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**Cellular neural networks have been paid much attention in the past two decades. Exceptional role in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing has been played by shunting inhibitory cellular neural networks (SICNNs). Chaotic dynamics is an object of great interest in neural networks theory. This is natural since chaotic outputs have been obtained for several types of neural networks. According to the design of neural networks, solutions of some of them can be used as an input for another ones. In our paper, we realize this idea by considering SICNNs to obtain chaos through chaotic external inputs. This is the first time that a theoretically approved chaos is obtained in SICNNs.**

## I. INTRODUCTION

A class of cellular neural networks, introduced by Bouzerdoum and Pinter,[1] is the *SICNNs*, which have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing.[2–8]

The model in its most original formulation[1] is as follows. Consider a two-dimensional grid of processing cells, and let $C_{ij}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, denote the cell at the $(i, j)$ position of the lattice. Denote by $N_r(i,j)$ the $r$–neighborhood of $C_{ij}$ such that

$$N_r(i,j) = \{C_{kl} : \max\{|k-i|, |l-j|\} \le r, 1 \le k \le m, 1 \le l \le n\}.$$

In *SICNNs*, neighboring cells exert mutual inhibitory interactions of the shunting type. The dynamics of the cell $C_{ij}$ is described by the following nonlinear ordinary differential equation:

$$\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(t))x_{ij} + L_{ij}(t), \qquad (1.1)$$

where $x_{ij}$ is the activity of the cell $C_{ij}$; $L_{ij}(t)$ is the external input to $C_{ij}$; the constant $a_{ij}$ represents the passive decay rate of the cell activity; $C_{ij}^{kl} \ge 0$ is the connection or coupling strength of postsynaptic activity of the cell $C_{kl}$ transmitted to the cell $C_{ij}$; and the activation function $f(x_{kl})$ is a positive continuous function representing the output or firing rate of the cell $C_{kl}$.

The chaos phenomenon has been observed in the dynamics of neural networks,[9–20] and chaotic dynamics applying as external inputs are useful for separating image segments,[10] information processing,[16,17] and synchronization of neural networks.[21–23] Aihara *et al.*[9] proposed a model of a single neuron with chaotic dynamics by considering graded responses, relative refractoriness, and spatio-temporal summation of inputs. Chaotic solutions of both the single chaotic neuron and the chaotic neural network composed of such neurons were demonstrated numerically in Ref. 9. Focusing on the model proposed in Ref. 9, dynamical properties of a chaotic neural network in chaotic wandering state were studied concerning sensitivity to external inputs in Ref. 20. On the other hand, in Ref. 10, Aihara's chaotic neuron model is used as the fundamental model of elements in a network, and the synchronization characteristics in response to external inputs in a coupled lattice based on a Newman-Watts model are investigated. Besides, in Refs. 16 and 17, a network consisting of binary neurons which do not display chaotic behavior is considered; and by means of the reduction of synaptic connectivities, it is shown that the state of the network in which cycle memories are embedded reveals chaotic wandering among memory attractor basins. Moreover, it is mentioned that chaotic wandering among memories is considerably intermittent. Chaotic solutions to the Hodgkin-Huxley equations with periodic forcing have been discovered in Ref. 11. Ref. 12 indicates the existence of chaotic solutions in the Hodgkin-Huxley model with its original parameters. An analytical proof for the existence of chaos through period-doubling cascade in a discrete-time neural network is given in Ref. 18, and the problem of creating a robust chaotic neural network is handled in Ref. 19. Confirming one more time that the chaos phenomenon can be observed in the dynamics of neural networks, the results obtained in the present study make contribution to the development of neural networks theory.

The existence and the stability of periodic, almost periodic and anti-periodic solutions of *SICNNs* have been published in Refs. 24–33. The main novelty of the present paper is the verification of the chaotic behavior in *SICNNs*. To prove the existence of chaos, we apply the technique based on the Li-Yorke definition,[34] and make use of *chaotic external inputs* in the networks. We say that the external inputs

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are chaotic if they belong to a collection of functions which satisfy the ingredients of chaos. That is, we consider members of a chaotic set as external input terms, and, as a result, we obtain solutions which display chaotic behavior.

The first mathematically rigorous definition of chaos is introduced by Li and Yorke[34] for one dimensional difference equations. According to Ref. 34, a continuous map $F : J \to J$, where $J$ is an interval, exhibits chaos if: (i) For every natural number $p$, there exists a $p$–periodic point of $F$ in $J$; (ii) There is an uncountable set $S \subset J$, containing no periodic points, such that for every $s_1, s_2 \in S$ with $s_1 \neq s_2$, we have $\limsup_{k\to\infty} |F^k(s_1) - F^k(s_2)| > 0$ and $\liminf_{k\to\infty} |F^k(s_1) - F^k(s_2)| = 0$; (iii) For every $s \in S$ and periodic point $\sigma \in J$, we have $\limsup_{k\to\infty} |F^k(s) - F^k(\sigma)| > 0$.

Generalizations of Li-Yorke chaos to high dimensional difference equations are provided in Refs. 35–38. According to the results of Ref. 35, if a repelling fixed point of a differentiable map has an associated homoclinic orbit that is transversal in some sense, then the map must exhibit chaotic behavior. More precisely, if a multidimensional differentiable map has a snap-back repeller, then it is chaotic. Marotto's Theorem is used in Ref. 36 to prove rigorously the existence of Li-Yorke chaos in a spatiotemporal chaotic system. Furthermore, the notion of Li-Yorke sensitivity, which links the Li-Yorke chaos with the notion of sensitivity, is studied in Ref. 37, and generalizations of Li-Yorke chaos to mappings in Banach spaces and complete metric spaces are considered in Ref. 38. In the present paper, we develop the concept of Li-Yorke chaos to continuous and multidimensional dynamics of *SICNNs*.

Existence of a chaotic attractor in *SICNNs* with impulses was numerically observed in Ref. 39 without a theoretical support, as well it is the case for Ref. 40. Our results can be extended to impulsive systems,[41] but they will be very specific.

## II. PRELIMINARIES

Throughout the paper, $\mathbb{R}$ and $\mathbb{N}$ will stand for the sets of real and natural numbers, respectively, and the norm $\|u\| = \max_{(i,j)}|u_{ij}|$ will be used, where $u = \{u_{ij}\} = (u_{11}, \ldots, u_{1n}, \ldots, u_{m1} \ldots, u_{mn}) \in \mathbb{R}^{m\times n}$ and $m, n \in \mathbb{N}$.

Suppose that $\mathcal{B}$ is a collection of continuous functions $\psi(t) = \{\psi_{ij}(t)\}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, such that $\sup_{t\in\mathbb{R}} \|\psi(t)\| \le M$, where $M$ is a positive real number. We start by describing the ingredients of Li-Yorke chaos for the collection $\mathcal{B}$.

We say that a couple $(\psi(t), \tilde{\psi}(t)) \in \mathcal{B}\times\mathcal{B}$ is proximal if for arbitrary small $\epsilon > 0$ and arbitrary large $E > 0$, there exist infinitely many disjoint intervals of length not less than $E$ such that $\|\psi(t) - \tilde{\psi}(t)\| < \epsilon$, for each $t$ from these intervals. On the other hand, a couple $(\psi(t), \tilde{\psi}(t)) \in \mathcal{B}\times\mathcal{B}$ is called frequently $(\epsilon_0, \Delta)$–separated if there exist positive real numbers $\epsilon_0, \Delta$ and infinitely many disjoint intervals of length not less than $\Delta$, such that $\|\psi(t) - \tilde{\psi}(t)\| > \epsilon_0$, for each $t$ from these intervals. It is worth saying that the numbers $\epsilon_0$ and $\Delta$ depend on the functions $\psi(t)$ and $\tilde{\psi}(t)$.

A couple $(\psi(t), \tilde{\psi}(t)) \in \mathcal{B}\times\mathcal{B}$ is a Li–Yorke pair if they are proximal and frequently $(\epsilon_0, \Delta)$–separated for some positive numbers $\epsilon_0$ and $\Delta$. Moreover, an uncountable set $\mathcal{C} \subset \mathcal{B}$ is called a scrambled set if $\mathcal{C}$ does not contain any periodic functions and each couple of different functions inside $\mathcal{C}\times\mathcal{C}$ is a Li–Yorke pair.

$\mathcal{B}$ is called a Li–Yorke chaotic set if: (i) there exists a positive real number $T_0$ such that $\mathcal{B}$ possesses a periodic function of period $kT_0$, for any $k \in \mathbb{N}$; (ii) $\mathcal{B}$ possesses a scrambled set $\mathcal{C}$; (iii) for any function $\psi(t) \in \mathcal{C}$ and any periodic function $\tilde{\psi}(t) \in \mathcal{B}$, the couple $(\psi(t), \tilde{\psi}(t))$ is frequently $(\epsilon_0, \Delta)$–separated for some positive real numbers $\epsilon_0$ and $\Delta$.

One can obtain a new Li-Yorke chaotic set from a given one as follows. Suppose that $h : \mathbb{R}^{m\times n} \to \mathbb{R}^{\bar{m}\times\bar{n}}$ is a function which satisfies for all $u_1, u_2 \in \mathbb{R}^{m\times n}$ that

$$L_1\|u_1 - u_2\| \le \|h(u_1) - h(u_2)\| \le L_2\|u_1 - u_2\|, \tag{2.2}$$

where $L_1$ and $L_2$ are positive numbers. One can verify that if the collection $\mathcal{B}$ is Li-Yorke chaotic, then the collection $\mathcal{B}_h$ whose elements are of the form $h(\psi(t))$, $\psi(t) \in \mathcal{B}$ is also Li-Yorke chaotic.

The following conditions are needed in the paper:

(C1) $\gamma = \min_{(i,j)} a_{ij} > 0$;
(C2) There exist positive numbers $M_{ij}$ such that $\sup_{t\in\mathbb{R}} |L_{ij}(t)| \le M_{ij}$;
(C3) There exists a positive number $M_f$ such that $\sup_{s\in\mathbb{R}} |f(s)| \le M_f$;
(C4) There exists a positive number $L_f$ such that $|f(s_1) - f(s_2)| \le L_f|s_1 - s_2|$ for all $s_1, s_2 \in \mathbb{R}$;
(C5) $M_f \max_{(i,j)} \frac{\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}}{a_{ij}} < 1$;
(C6) $\frac{\bar{c}(L_f K_0 + M_f)}{\gamma} < 1$, where $\bar{c} = \max_{(i,j)} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}$ and $K_0 = \frac{\max_{(i,j)} \frac{M_{ij}}{a_{ij}}}{1 - M_f \max_{(i,j)} \frac{\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}}{a_{ij}}}$.

Using the theory of quasilinear equations,[42] one can verify that a bounded on $\mathbb{R}$ function $x(t) = \{x_{ij}(t)\}$ is a solution of the network (1.1) if and only if the following integral equation is satisfied

$$x_{ij}(t) = -\int_{-\infty}^{t} e^{-a_{ij}(t-s)} \left[ \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(s)) x_{ij}(s) - L_{ij}(s) \right] ds. \tag{2.3}$$

A result about the existence of bounded on $\mathbb{R}$ solutions is as follows.

*Lemma 2.1. For any $L(t) = \{L_{ij}(t)\}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, there exists a unique bounded on $\mathbb{R}$ solution $\phi_L(t) = \{\phi_L^{ij}(t)\}$ of the network (1.1) such that $\sup_{t\in\mathbb{R}} \|\phi_L(t)\| \le K_0$.*

*Proof.* Consider the set $C_0$ of continuous functions $u(t) = \{u_{ij}(t)\}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, such that $\|u\|_1 \le K_0$, where $\|u\|_1 = \sup_{t\in\mathbb{R}} \|u(t)\|$. Define on $C_0$ the operator $\Pi$ as

$$(\Pi u)_{ij}(t) \equiv -\int_{-\infty}^{t} e^{-a_{ij}(t-s)} \times \left[ \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(u_{kl}(s)) u_{ij}(s) - L_{ij}(s) \right] ds,$$

where $u(t) = \{u_{ij}(t)\}$ and $\Pi u(t) = \{(\Pi u)_{ij}(t)\}$. If $u(t)$ belongs to $C_0$, then

$$|(\Pi u)_{ij}(t)| \le \int_{-\infty}^{t} e^{-a_{ij}(t-s)} \left[ \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} |f(u_{kl})(s)||u_{ij}(s)| + |L_{ij}(s)| \right] ds \le \frac{1}{a_{ij}} \left( M_{ij} + M_f K_0 \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} \right).$$

Accordingly, we have $\|\Pi u\|_1 \le \max_{(i,j)} \frac{M_{ij}}{a_{ij}} + M_f K_0 \max_{(i,j)} \frac{\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}}{a_{ij}} = K_0$. Therefore, $\Pi(C_0) \subset C_0$.

On the other hand, for any $u, v \in C_0$,

$$\begin{aligned} |(\Pi u)_{ij}(t) - (\Pi v)_{ij}(t)| &\le \int_{-\infty}^{t} e^{-a_{ij}(t-s)} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} |f(u_{kl}(s))u_{ij}(s) - f(u_{kl}(s))v_{ij}(s)| ds \\ &\quad + \int_{-\infty}^{t} e^{-a_{ij}(t-s)} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} |f(u_{kl}(s))v_{ij}(s) - f(v_{kl}(s))v_{ij}(s)| ds \\ &\le (L_f K_0 + M_f) \max_{(i,j)} \frac{\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}}{a_{ij}} \|u - v\|_1. \end{aligned}$$

Thus, $\|\Pi u - \Pi v\|_1 \le (L_f K_0 + M_f) \max_{(i,j)} \frac{\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}}{a_{ij}} \|u - v\|_1$, and condition ($C6$) implies that the operator $\Pi$ is contractive. Consequently, for any $L(t)$, there exists a unique bounded on $\mathbb{R}$ solution $\phi_L(t)$ of the network (1.1) such that $\sup_{t\in\mathbb{R}} \|\phi_L(t)\| \le K_0$. □

For a given $L(t) = \{L_{ij}(t)\}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, let us denote by $x_L(t, x_0) = \{x_L^{ij}(t, x_0)\}$ the unique solution of the *SICNNs* (1.1) with $x_L(0, x_0) = x_0$. We note that the solution $x_L(t, x_0)$ is not necessarily bounded on $\mathbb{R}$.

Consider the collection $\mathscr{L}$ of functions with elements of the form $L(t) : \mathbb{R} \to \mathbb{R}^{m\times n}$ such that $\sup_{t\in\mathbb{R}} \|L(t)\| \le H_0$, where $H_0 = \max_{(i,j)} M_{ij}$. In the present paper, we assume that $\mathscr{L}$ is an equicontinuous family on $\mathbb{R}$. Suppose that $\mathscr{A}$ is the collection of functions consisting of the bounded on $\mathbb{R}$ solutions $\phi_L(t)$ of system (1.1), where $L(t) \in \mathscr{L}$.

The following assertion confirms the attractiveness of the set $\mathscr{A}$.

*Lemma 2.2. For any $x_0 \in \mathbb{R}^{m\times n}$ and $L(t) = \{L_{ij}(t)\}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we have $\|x_L(t, x_0) - \phi_L(t)\| \to 0$ as $t \to \infty$.*

*Proof*. Making use of the relation

$$x_L^{ij}(t, x_0) - \phi_L^{ij}(t) = e^{-a_{ij}t}\left(x_L^{ij}(0, x_0) - \phi_L^{ij}(0)\right) - \int_0^t e^{-a_{ij}(t-s)} \left[ \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(x_L^{kl}(s, x_0)) x_L^{ij}(s, x_0) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(\phi_L^{kl}(s))\phi_L^{ij}(s) \right] ds,$$

we obtain for $t \ge 0$ that

$$\begin{aligned} |x_L^{ij}(t, x_0) - \phi_L^{ij}(t)| &\le e^{-a_{ij}t} |x_L^{ij}(0, x_0) - \phi_L^{ij}(0)| + M_f \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} \int_0^t e^{-a_{ij}(t-s)} |x_L^{ij}(s, x_0) - \phi_L^{ij}(s)| ds \\ &\quad + L_f K_0 \int_0^t e^{-a_{ij}(t-s)} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} |x_L^{kl}(s, x_0) - \phi_L^{kl}(s)| ds. \end{aligned}$$

The last inequality implies the following:

$$e^{\gamma t} \|x_L(t, x_0) - \phi_L(t)\| \le \|x_0 - \phi_L(0)\| + \bar{c}(L_f K_0 + M_f) \int_0^t e^{\gamma s} \|x_L(s, x_0) - \phi_L(s)\| ds, \ t \ge 0.$$

Applying Gronwall-Bellman Lemma, one can attain that

$$\|x_L(t, x_0) - \phi_L(t)\| \le \|x_0 - \phi_L(0)\| e^{[\bar{c}(L_f K_0 + M_f) - \gamma]t}, \ t \ge 0.$$

Consequently, $\|x_L(t, x_0) - \phi_L(t)\| \to 0$ as $t \to \infty$, in accordance with condition ($C6$). □

Our purpose in the next part is to prove rigorously that if the collection $\mathscr{L}$ is chaotic in the sense of Li-Yorke, then the same is true for $\mathscr{A}$. In other words, if the external input terms $L_{ij}(t)$ behave chaotically, then the dynamics of the *SICNNs* is also chaotic.

## III. CHAOTIC DYNAMICS

The replication of the ingredients of Li-Yorke chaos from the collection $\mathscr{L}$ to the collection $\mathscr{A}$ will be affirmed in the following two lemmas, and the main conclusion will be stated in Theorem 3.1. We start with the following lemma, which indicates existence of proximality in the collection $\mathscr{A}$.

*Lemma 3.1. If a couple of functions* $(L(t), \tilde{L}(t)) \in \mathscr{L} \times \mathscr{L}$ *is proximal, then the same is true for the couple* $(\phi_L(t), \phi_{\tilde{L}}(t)) \in \mathscr{A} \times \mathscr{A}$.

*Proof.* Fix an arbitrary small positive number $\epsilon$ and an arbitrary large positive number $E$. Set $R = 2\left(M_f K_0 \max_{(i,j)} \frac{\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}}{a_{ij}} + \max_{(i,j)} \frac{M_{ij}}{a_{ij}}\right)$ and $0 < \alpha \leq \frac{\gamma - \bar{c}(L_f K_0 + M_f)}{1 + \gamma - \bar{c}(L_f K_0 + M_f)}$. Suppose that a given pair $(L(t), \tilde{L}(t)) \in \mathscr{L} \times \mathscr{L}$ is proximal. There exist a sequence of real numbers $\{E_q\}$ satisfying $E_q \geq E$ for each $q \in \mathbb{N}$ and a sequence $\{t_q\}$, $t_q \to \infty$ as $q \to \infty$, such that $\|L(t) - \tilde{L}(t)\| < \alpha\epsilon$ for each $t$ from the disjoint intervals $J_q = [t_q, t_q + E_q]$, $q \in \mathbb{N}$. Let us denote $\phi_L(t) = \{\phi_L^{ij}(t)\}$ and $\phi_{\tilde{L}}(t) = \{\phi_{\tilde{L}}^{ij}(t)\}$.

Fix $q \in \mathbb{N}$. For $t \in J_q$, using the relation (2.3), one can reach up for any $i$ and $j$ that

$$\phi_L^{ij}(t) - \phi_{\tilde{L}}^{ij}(t) = -\int_{-\infty}^{t} e^{-a_{ij}(t-s)} \left[ \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(\phi_L^{kl}(s))\phi_L^{ij}(s) - L_{ij}(s) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(\phi_{\tilde{L}}^{kl}(s))\phi_{\tilde{L}}^{ij}(s) + \tilde{L}_{ij}(s) \right] ds.$$

By means of the last equation, one can obtain that

$$|\phi_L^{ij}(t) - \phi_{\tilde{L}}^{ij}(t)| \leq 2\left( M_f K_0 \frac{\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}}{a_{ij}} + \frac{M_{ij}}{a_{ij}} \right) e^{-a_{ij}(t-t_q)} + \frac{\alpha\epsilon}{a_{ij}}(1 - e^{-a_{ij}(t-t_q)}) + \bar{c}(L_f K_0 + M_f)\int_{t_q}^{t} e^{-a_{ij}(t-s)} \|\phi_L(s) - \phi_{\tilde{L}}(s)\| ds.$$

Accordingly, we have

$$e^{\gamma t}\|\phi_L(t) - \phi_{\tilde{L}}(t)\| \leq R e^{\gamma t_q} + \frac{\alpha\epsilon}{\gamma}(e^{\gamma t} - e^{\gamma t_q}) + \bar{c}(L_f K_0 + M_f)\int_{t_q}^{t} e^{\gamma s}\|\phi_L(s) - \phi_{\tilde{L}}(s)\| ds, \ t \in J_q.$$

Application of Gronwall's Lemma to the last inequality implies for $t \in J_q$ that

$$\|\phi_L(t) - \phi_{\tilde{L}}(t)\| \leq \frac{\alpha\epsilon}{\gamma - \bar{c}(L_f K_0 + M_f)} \times \left(1 - e^{[\bar{c}(L_f K_0 + M_f) - \gamma](t - t_q)}\right) + R e^{[\bar{c}(L_f K_0 + M_f) - \gamma](t - t_q)}.$$

Suppose that the number $E$ is sufficiently large such that $E > \frac{2}{\gamma - \bar{c}(L_f K_0 + M_f)} \ln\left(\frac{R}{\alpha\epsilon}\right)$. In this case, if $t$ belongs to the interval $[t_q + E/2, t_q + E_q]$, then $R e^{[\bar{c}(L_f K_0 + M_f) - \gamma](t - t_q)} < \alpha\epsilon$.

Thus, for $t \in [t_q + E/2, t_q + E_q]$, the following inequality is valid:

$$\|\phi_L(t) - \phi_{\tilde{L}}(t)\| < \left(1 + \frac{1}{\gamma - \bar{c}(L_f K_0 + M_f)}\right)\alpha\epsilon \leq \epsilon.$$

Consequently, since the last inequality holds for each $t$ from the disjoint intervals $J_q^1 = [t_q + E/2, t_q + E_q]$, $q \in \mathbb{N}$, the couple $(\phi_L(t), \phi_{\tilde{L}}(t)) \in \mathscr{A} \times \mathscr{A}$ is proximal. □

Now, let us continue with the replication the second main ingredient of Li-Yorke chaos in the next lemma.

*Lemma 3.2. If a couple* $(L(t), \tilde{L}(t)) \in \mathscr{L} \times \mathscr{L}$ *is frequently* $(\epsilon_0, \Delta)$*–separated for some positive real numbers* $\epsilon_0$ *and* $\Delta$*, then there exist positive real numbers* $\epsilon_1$ *and* $\bar{\Delta}$ *such that the couple* $(\phi_L(t), \phi_{\tilde{L}}(t)) \in \mathscr{A} \times \mathscr{A}$ *is frequently* $(\epsilon_1, \bar{\Delta})$*–separated.*

*Proof.* Suppose that a given couple $(L(t), \tilde{L}(t)) \in \mathscr{L} \times \mathscr{L}$ is frequently $(\epsilon_0, \Delta)$ separated, for some $\epsilon_0 > 0$ and $\Delta > 0$. In this case, there exist infinitely many disjoint intervals $J_q, q \in \mathbb{N}$, each with length not less than $\Delta$, such that $\|L(t) - \tilde{L}(t)\| > \epsilon_0$, for each $t$ from these intervals. Without loss of generality, assume that these intervals are all closed subsets of $\mathbb{R}$. In that case, one can find a sequence $\{\Delta_q\}$ satisfying $\Delta_q \geq \Delta$, $q \in \mathbb{N}$, and a sequence $\{d_q\}$, $d_q \to \infty$ as $q \to \infty$, such that for each $q \in \mathbb{N}$, the inequality $\|L(t) - \tilde{L}(t)\| > \epsilon_0$ holds for $t \in J_q = [d_q, d_q + \Delta_q]$ and $J_p \cap J_q = \varnothing$ whenever $p \neq q$.

In the proof, we will verify the existence of positive numbers $\epsilon_1, \bar{\Delta}$ and infinitely many disjoint intervals $J_q^1 \subset J_q, q \in \mathbb{N}$, each with length $\bar{\Delta}$, such that the inequality $\|\phi_L(t) - \phi_{\tilde{L}}(t)\| > \epsilon_1$ holds for each $t$ from the intervals $J_q^1, q \in \mathbb{N}$.

According to the equicontinuity of $\mathscr{L}$, one can find a positive number $\tau < \Delta$, such that for any $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \tau$, the inequality

$$|(L_{ij}(t_1) - \tilde{L}_{ij}(t_1)) - (L_{ij}(t_2) - \tilde{L}_{ij}(t_2))| < \frac{\epsilon_0}{2}, \qquad (3.4)$$

holds for all $1 \le i \le m, 1 \le j \le n$.

Suppose that for each $q \in \mathbb{N}$, the number $s_q$ denotes the midpoint of the interval $J_q$. That is, $s_q = d_q + \Delta_q/2$. Let us define a sequence $\{\theta_q\}$ through the equation $\theta_q = s_q - \tau/2$.

Let us fix an arbitrary $q \in \mathbb{N}$. One can find integers $i_0, j_0$, such that

$$|L_{i_0j_0}(s_q) - \tilde{L}_{i_0j_0}(s_q)| = \|L(s_q) - \tilde{L}(s_q)\| > \epsilon_0. \qquad (3.5)$$

Making use of the inequality (3.4), for all $t \in [\theta_q, \theta_q + \tau]$, we have

$$\begin{aligned} &|L_{i_0j_0}(s_q) - \tilde{L}_{i_0j_0}(s_q)| - |L_{i_0j_0}(t) - \tilde{L}_{i_0j_0}(t)| \\ &\quad \le |(L_{i_0j_0}(t) - \tilde{L}_{i_0j_0}(t)) - (L_{i_0j_0}(s_q) - \tilde{L}_{i_0j_0}(s_q))| < \frac{\epsilon_0}{2}, \end{aligned}$$

and therefore by means of (3.5), we obtain that the inequality

$$|L_{i_0j_0}(t) - \tilde{L}_{i_0j_0}(t)| > |L_{i_0j_0}(s_q) - \tilde{L}_{i_0j_0}(s_q)| - \frac{\epsilon_0}{2} > \frac{\epsilon_0}{2}, \quad (3.6)$$

is valid for all $t \in [\theta_q, \theta_q + \tau]$.

For each $i$ and $j$, one can find numbers $\zeta_{ij}^q \in [\theta_q, \theta_q + \tau]$ such that

$$\int_{\theta_q}^{\theta_q+\tau} (L(s) - \tilde{L}(s))ds = \tau(L_{11}(\zeta_{11}^q) - \tilde{L}_{11}(\zeta_{11}^q), \ldots, L_{mn}(\zeta_{mn}^q) - \tilde{L}_{mn}(\zeta_{mn}^q)).$$

Thus, according to the inequality (3.6), we attain that

$$\left\| \int_{\theta_q}^{\theta_q+\tau} (L(s) - \tilde{L}(s))ds \right\| \ge \tau |L_{i_0j_0}(\zeta_{i_0j_0}^q) - \tilde{L}_{i_0j_0}(\zeta_{i_0j_0}^q)| > \frac{\tau\epsilon_0}{2}. \qquad (3.7)$$

For $t \in [\theta_q, \theta_q + \tau]$, using the couple of relations

$$\phi_L^{ij}(t) = \phi_L^{ij}(\theta_q) - \int_{\theta_q}^{t} \left[ a_{ij} + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(\phi_L^{kl}(s)) \right] \phi_L^{ij}(s) ds + \int_{\theta_q}^{t} L_{ij}(s) ds,$$

and

$$\phi_{\tilde{L}}^{ij}(t) = \phi_{\tilde{L}}^{ij}(\theta_q) - \int_{\theta_q}^{t} \left[ a_{ij} + \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(\phi_{\tilde{L}}^{kl}(s)) \right] \phi_{\tilde{L}}^{ij}(s) ds + \int_{\theta_q}^{t} \tilde{L}_{ij}(s) ds,$$

it can be verified that

$$\begin{aligned} \phi_L^{ij}(\theta_q + \tau) - \phi_{\tilde{L}}^{ij}(\theta_q + \tau) &= \int_{\theta_q}^{\theta_q+\tau} (L_{ij}(s) - \tilde{L}_{ij}(s))ds + (\phi_L^{ij}(\theta_q) - \phi_{\tilde{L}}^{ij}(\theta_q)) - \int_{\theta_q}^{\theta_q+\tau} a_{ij} \Big( \phi_L^{ij}(s) - \phi_{\tilde{L}}^{ij}(s) \Big) ds \\ &\quad - \int_{\theta_q}^{\theta_q+\tau} \left[ \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(\phi_L^{kl}(s)) \phi_L^{ij}(s) - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(\phi_{\tilde{L}}^{kl}(s)) \phi_{\tilde{L}}^{ij}(s) \right] ds. \end{aligned}$$

Hence, one can confirm that

$$\begin{aligned} \|\phi_L(\theta_q + \tau) - \phi_{\tilde{L}}(\theta_q + \tau)\| \ge & \left\| \int_{\theta_q}^{\theta_q+\tau} (L(s) - \tilde{L}(s))ds \right\| - \|\phi_L(\theta_q) - \phi_{\tilde{L}}(\theta_q)\| - \max_{(i,j)} \left| \int_{\theta_q}^{\theta_q+\tau} a_{ij} \Big( \phi_L^{ij}(s) - \phi_{\tilde{L}}^{ij}(s) \Big) ds \right| \\ & - \max_{(i,j)} \left| \int_{\theta_q}^{\theta_q+\tau} \left[ \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(\phi_L^{kl}(s)) \phi_L^{ij}(s) - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(\phi_{\tilde{L}}^{kl}(s)) \phi_{\tilde{L}}^{ij}(s) \right] ds \right|. \end{aligned} \qquad (3.8)$$

Let us denote $\bar{\gamma} = \max_{(i,j)} a_{ij}$. The inequalities (3.7) and (3.8) together imply that

$$\begin{aligned} \max_{t \in [\theta_q, \theta_q+\tau]} \|\phi_L(t) - \phi_{\tilde{L}}(t)\| \ge \|\phi_L(\theta_q + \tau) - \phi_{\tilde{L}}(\theta_q + \tau)\| > \frac{\tau\epsilon_0}{2} \\ - [1 + \tau\bar{\gamma} + \tau\bar{c}(L_f K_0 + M_f)] \max_{t \in [\theta_q, \theta_q+\tau]} \|\phi_L(t) - \phi_{\tilde{L}}(t)\|. \end{aligned}$$

Therefore, we have $\max_{t\in[\theta_q,\theta_q+\tau]}\|\phi_L(t)-\phi_{\tilde{L}}(t)\| > \bar{\epsilon}$, where $\bar{\epsilon} = \frac{\tau\epsilon_0}{2[2+\tau\bar{\gamma}+\tau\bar{c}(L_f K_0+M_f)]}$.

Suppose that $\max_{t\in[\theta_q,\theta_q+\tau]}\|\phi_L(t)-\phi_{\tilde{L}}(t)\| = \|\phi_L(\xi_q) -\phi_{\tilde{L}}(\xi_q)\|$ for some $\xi_q \in [\theta_q, \theta_q+\tau]$. Define

$$\bar{\Delta} = \min\left\{\frac{\tau}{2}, \frac{\bar{\epsilon}}{4(H_0+K_0\bar{\gamma}+M_f K_0\bar{c})}\right\}$$

and let

$$\theta_q^1 = \begin{cases} \xi_q, & \text{if } \xi_q \le \theta_q+\tau/2 \\ \xi_q-\bar{\Delta}, & \text{if } \xi_q > \theta_q+\tau/2. \end{cases}$$

For $t \in [\theta_q^1, \theta_q^1+\bar{\Delta}]$, by the help of the integral equation

$$\phi_L^{ij}(t)-\phi_{\tilde{L}}^{ij}(t) = (\phi_L^{ij}(\xi_q)-\phi_{\tilde{L}}^{ij}(\xi_q)) + \int_{\xi_q}^{t}\Big(L_{ij}(s)-\tilde{L}_{ij}(s)\Big)ds - \int_{\xi_q}^{t} a_{ij}\Big(\phi_L^{ij}(s)-\phi_{\tilde{L}}^{ij}(s)\Big)ds$$
$$-\int_{\xi_q}^{t}\left[\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(\phi_L^{kl}(s))\phi_L^{ij}(s) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(\phi_{\tilde{L}}^{kl}(s))\phi_{\tilde{L}}^{ij}(s)\right]ds,$$

we obtain that

$$\|\phi_L(t)-\phi_{\tilde{L}}(t)\| \ge \|\phi_L(\xi_q)-\phi_{\tilde{L}}(\xi_q)\| - \max_{(i,j)}\left|\int_{\xi_q}^{t}\Big(L_{ij}(s)-\tilde{L}_{ij}(s)\Big)ds\right| - \max_{(i,j)}\left|\int_{\xi_q}^{t} a_{ij}\Big(\phi_L^{ij}(s)-\phi_{\tilde{L}}^{ij}(s)\Big)ds\right|$$
$$-\max_{(i,j)}\left|\int_{\xi_q}^{t}\left[\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(\phi_L^{kl}(s))\phi_L^{ij}(s) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(\phi_{\tilde{L}}^{kl}(s))\phi_{\tilde{L}}^{ij}(s)\right]ds\right|$$
$$> \bar{\epsilon} - 2\bar{\Delta}(H_0+K_0\bar{\gamma}+M_f K_0\bar{c}) \ge \frac{\bar{\epsilon}}{2}.$$

Consequently, for each $t$ from the intervals $J_q^1 = [\theta_q^1, \theta_q^1+\bar{\Delta}]$, $q\in\mathbb{N}$, the inequality $\|\phi_L(t)-\phi_{\tilde{L}}(t)\| > \epsilon_1$ holds, where $\epsilon_1 = \bar{\epsilon}/2$, and the length of these intervals are $\bar{\Delta}$. □

The following theorem, which is the main result of the present article, indicates that the network (1.1) is chaotic, provided that the external inputs are chaotic.

**Theorem 3.1.** *If $\mathscr{L}$ is a Li-Yorke chaotic set, then the same is true for $\mathscr{A}$.*

*Proof.* Assume that the set $\mathscr{L}$ is Li-Yorke chaotic. Under the circumstances, there exists a positive number $T_0$ such that for any natural number $k$, $\mathscr{L}$ possesses a periodic function of period $kT_0$. One can confirm that $L(t)\in\mathscr{L}$ is $kT_0$–periodic if and only if $\phi_L(t)\in\mathscr{A}$ is $kT_0$–periodic. Therefore, the set $\mathscr{A}$ contains a $kT_0$–periodic function for any natural number $k$.

Next, suppose that $\mathscr{L}_S$ is a scrambled set inside $\mathscr{L}$ and take into account the collection $\mathscr{A}_S$ with elements of the form $\phi_L(t)$, where $L(t)\in\mathscr{L}_S$. Since $\mathscr{L}_S$ is uncountable, the set $\mathscr{A}_S$ is also uncountable. Due to the one-to-one correspondence between the periodic functions inside $\mathscr{L}$ and $\mathscr{A}$, no periodic functions exist inside $\mathscr{A}_S$.

According to Lemmas 3.1 and 3.2, $\mathscr{A}_S$ is a scrambled set. Moreover, Lemma 3.2 implies that each couple of functions inside $\mathscr{A}_S\times\mathscr{A}_P$ is frequently $(\epsilon_1,\bar{\Delta})$–separated for some positive real numbers $\epsilon_1$ and $\bar{\Delta}$, where $\mathscr{A}_P$ denotes the set of all periodic functions inside $\mathscr{A}$. Consequently, the set $\mathscr{A}$ is Li-Yorke chaotic. □

*Remark 3.1. Combining the main result presented in Theorem 3.1 with the result of Lemma 2.2, one can conclude that a chaotic attractor takes place in the dynamics of system (1.1).*

## IV. EXAMPLES

To actualize the results of the paper, one needs a source of external inputs, $L_{ij}(t)$, which are ensured to be chaotic in the Li-Yorke sense. For this reason, in the first example, we will take into account *SICNNs* whose external inputs are relay functions with chaotically changing switching moments. Then, to support our new theoretical results, we will make use of the solutions of this network as external inputs for another *SICNNs*, which is the main illustrative object for the results of the paper. To increase the flexibility of our method for applications, we will also take advantage of nonlinear functions to build chaotic inputs.

*Example 1*. Let us introduce the following *SICNNs*:

$$\frac{dz_{ij}}{dt} = -b_{ij}z_{ij} - \sum_{D_{kl}\in N_1(i,j)} D_{ij}^{kl} g(z_{kl}(t))z_{ij} + \nu_{ij}(t,t_0), \quad (4.9)$$

in which $i, j = 1, 2, 3$,

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 8 & 4 & 7 \\ 10 & 6 & 5 \\ 6 & 4 & 1 \end{pmatrix},$$

$$\begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix} = \begin{pmatrix} 0.006 & 0 & 0.001 \\ 0.009 & 0.002 & 0.003 \\ 0 & 0.005 & 0.004 \end{pmatrix}.$$

In Eq. (4.9), $D_{ij}$ denotes the cell at the $(i, j)$ position of the lattice, and for each $i, j$, the relay function $\nu_{ij}(t, t_0)$ is defined by the equation

$$\nu_{ij}(t, t_0) = \begin{cases} \alpha_{ij}, & \text{if} \quad \zeta_{2q}(t_0) < t \le \zeta_{2q+1}(t_0), \\ \beta_{ij}, & \text{if} \quad \zeta_{2q-1}(t_0) < t \le \zeta_{2q}(t_0), \end{cases}$$

where $t_0 \in [0, 1]$ and the numbers $\zeta_q(t_0)$, $q \in \mathbb{Z}$, denote the switching moments, which are the same for all $i, j$. The switching moments are defined through the formula $\zeta_q(t_0) = q + \kappa_q(t_0)$, $q \in \mathbb{Z}$, where the sequence $\{\kappa_q(t_0)\}$, $\kappa_0(t_0) = t_0$ is generated by the logistic equation $\kappa_{q+1}(t_0) = 3.9\kappa_q(t_0)(1 - \kappa_q(t_0))$, which is chaotic in the Li-Yorke sense.[34] More information about the dynamics of relay systems and replication of chaos can be found in Refs. 43–47.

In system (4.9), let $g(s) = s^2$ and $\alpha_{ij} = 1$, $\beta_{ij} = 2$ for all $i, j$. By results of Ref. 43, the family $\{\nu_{ij}(t, t_0)\}$, $t_0 \in [0, 1]$ is chaotic in the sense of Li-Yorke, and the collection $\mathscr{L}$ consisting of elements of the form $z(t) = \{z_{ij}(t)\}$, where $z(t)$ are bounded on $\mathbb{R}$ solutions of (4.9), is a Li-Yorke chaotic set.

Next, we consider the simulations of the network (4.9). Figure 1 represents the chaotic solution $z(t) = \{z_{ij}(t)\}$ of (4.9) with $z_{11}(t_0) = 0.1678, z_{12}(t_0) = 0.3956, z_{13}(t_0) = 0.1987$, $z_{21}(t_0) = 0.1261$, $z_{22}(t_0) = 0.2405$, $z_{23}(t_0) = 0.3012$, $z_{31}(t_0) = 0.2412$, $z_{32}(t_0) = 0.3942, z_{33}(t_0) = 1.6692$, where $t_0 = 0.45$.

In Example 1, to procure a Li-Yorke chaotic set, we used *SICNNs* in the form of (1.1) where the terms $L_{ij}(t)$ are replaced by relay functions $\nu_{ij}(t, t_0)$, whose switching moments change chaotically. Now, to support the results of the present paper, we will construct another *SICNNs*, but this time we will use external inputs of the form $L_{ij}(t) = h_{ij}(z(t))$, where $z(t)$ are the chaotic solutions of the network (4.9) and $h(v) = \{h_{ij}(v)\}$ is a nonlinear function, which satisfies the inequality (2.2).

*Example 2*. Consider the following *SICNNs*

$$\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} - \sum_{C_{kl} \in N_1(i,j)} C_{ij}^{kl} f(x_{kl}(t))x_{ij} + L_{ij}(t), \qquad (4.10)$$

in which $i, j = 1, 2, 3$,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 5 & 12 & 2 \\ 6 & 4 & 8 \\ 2 & 9 & 3 \end{pmatrix},$$

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} 0.02 & 0.04 & 0.06 \\ 0.04 & 0.07 & 0.09 \\ 0.03 & 0.04 & 0.08 \end{pmatrix},$$

and $f(s) = \frac{1}{2}s^3$. One can calculate that

$$\sum_{C_{kl} \in N_1(1,1)} C_{11}^{kl} = 0.17, \quad \sum_{C_{kl} \in N_1(1,2)} C_{12}^{kl} = 0.32, \quad \sum_{C_{kl} \in N_1(1,3)} C_{13}^{kl} = 0.26,$$
$$\sum_{C_{kl} \in N_1(2,1)} C_{21}^{kl} = 0.24, \quad \sum_{C_{kl} \in N_1(2,2)} C_{22}^{kl} = 0.47, \quad \sum_{C_{kl} \in N_1(2,3)} C_{23}^{kl} = 0.38,$$
$$\sum_{C_{kl} \in N_1(3,1)} C_{31}^{kl} = 0.18, \quad \sum_{C_{kl} \in N_1(3,2)} C_{32}^{kl} = 0.35, \quad \sum_{C_{kl} \in N_1(3,3)} C_{33}^{kl} = 0.28.$$

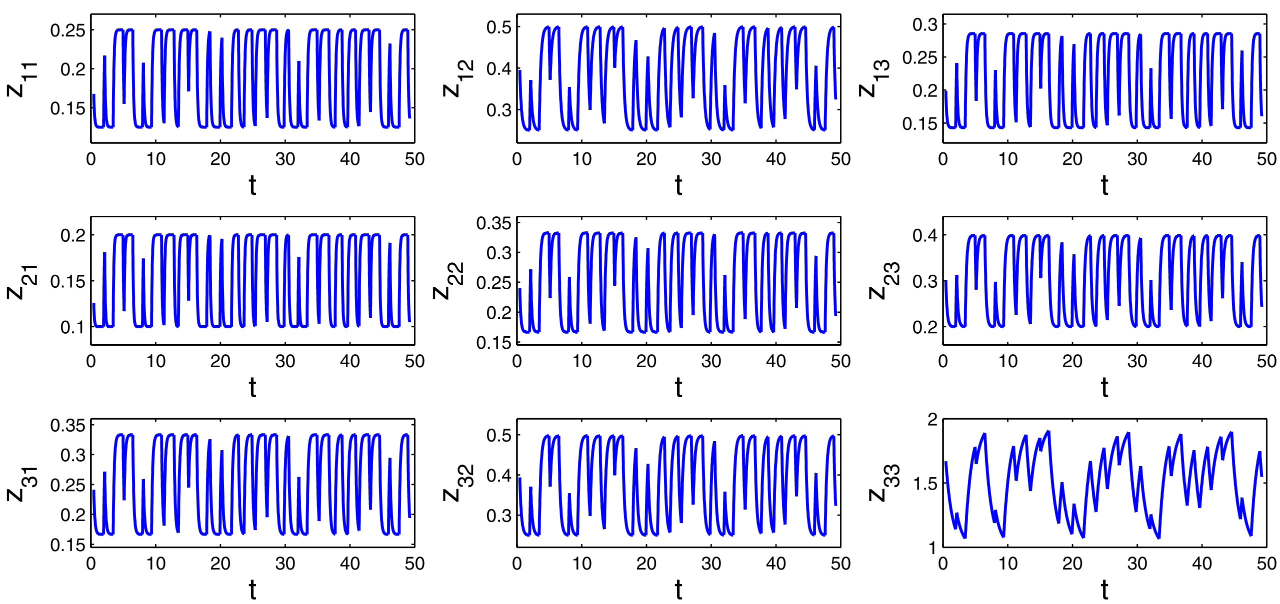


FIG. 1. The chaotic behavior of the *SICNNs* (4.9).

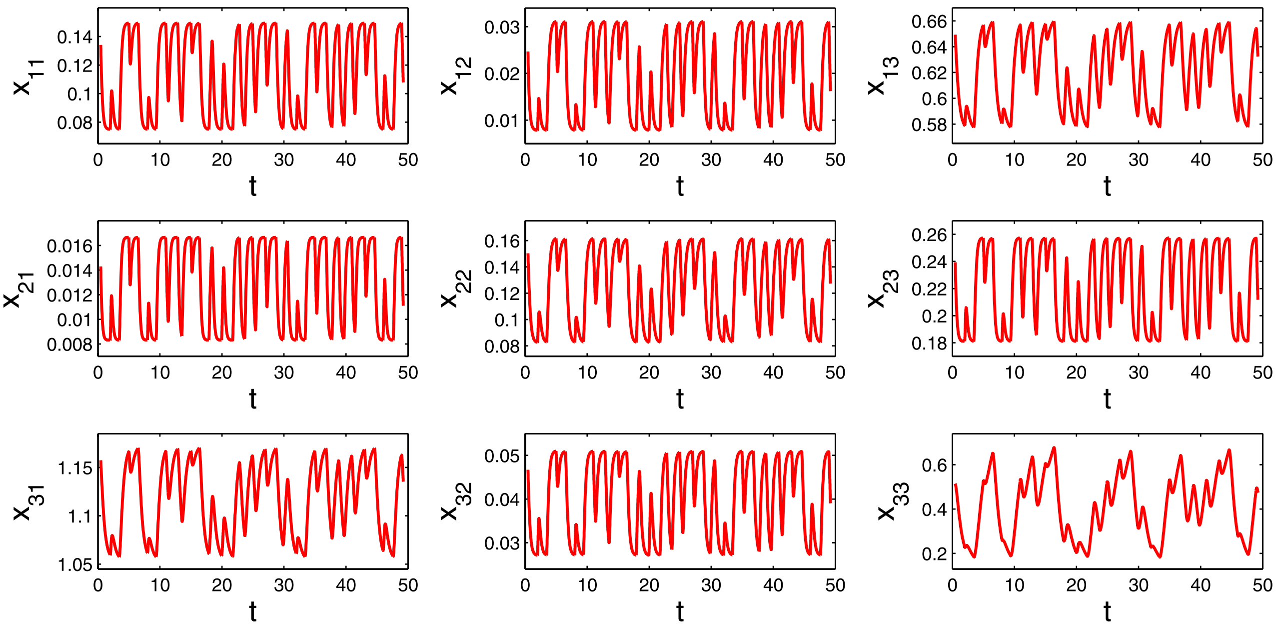


FIG. 2. The chaotic behavior of the *SICNNs* (4.10).

In the previous example, we obtained a network whose solutions behave chaotically. Now, we will make these solutions as external inputs for (4.10), with the help of a nonlinear function $h$.

Define a function $h(v) = \{h_{ij}(v)\}$, where $v = \{v_{ij}\}$, $i, j = 1, 2, 3$, through the equations $h_{11}(v) = 2v_{11} + \sin(v_{11})$, $h_{12}(v) = \frac{3}{2}v_{12}^2$, $h_{13}(v) = e^{v_{13}}$, $h_{21}(v) = \tan(\frac{v_{21}}{2})$, $h_{22}(v) = v_{22} + \arctan v_{22}$, $h_{23}(v) = \frac{v_{23}^2 - v_{23} - 1}{v_{23} - 1}$, $h_{31}(v) = \frac{2}{3}(2 + v_{31})^{3/2}$, $h_{32}(v) = \tanh(v_{32})$, $h_{33}(v) = \frac{1}{4}v_{33}^3 + \frac{1}{5}v_{33}$. We note that the inequality (2.2) can be verified by using the bounded regions where each component function $z_{ij}(t)$ lies in. Accordingly, the set $\mathscr{L}_h$ whose elements are of the form $h(z(t))$, $z(t) \in \mathscr{L}$, where $\mathscr{L}$ is the set of bounded on $\mathbb{R}$ solutions of (4.9), is Li-Yorke chaotic. Moreover, for each $z(t) \in \mathscr{L}$, we have $|h_{ij}(z(t))| \leq M_{ij}$, where $M_{11} = 0.78$, $M_{12} = 0.54$, $M_{13} = 1.35$, $M_{21} = 0.11$, $M_{22} = 0.69$, $M_{23} = 2.11$, $M_{31} = 2.41$, $M_{32} = 0.51$, and $M_{33} = 2.4$.

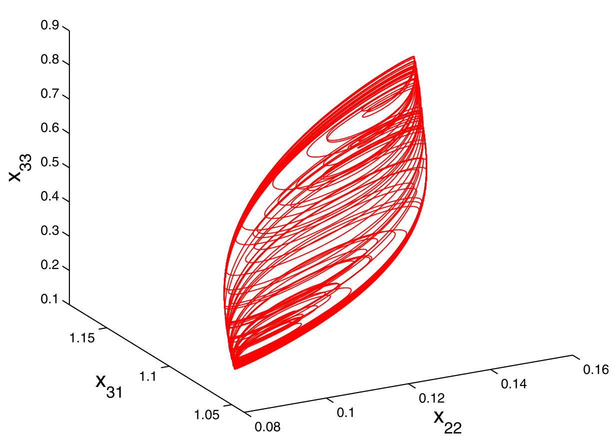


FIG. 3. The projection of the chaotic attractor of the network (4.10) on the $x_{22} - x_{31} - x_{33}$ space.

Consider the network (4.10) with $L_{ij}(t) = h_{ij}(z(t))$, where $h(z(t)) = \{h_{ij}(z(t))\} \in \mathscr{L}_h$. In this case, the condition $(C6)$ holds for (4.10) with $M_f = 0.864$, $L_f = 2.16$, $K_0 = 1.36$, $\gamma = 2$, and $\bar{c} = 0.47$. The results of Theorem 3.1 ensure us to say that the collection $\mathscr{A}$ with elements $\phi_z(t)$, $z(t) \in \mathscr{L}$ is Li-Yorke chaotic.

In *SICNNs* (4.10), we use the chaotically behaving solution $z(t) = \{z_{ij}(t)\}$ which is simulated in Example 1, and depict in Figure 2 the solution of (4.10) with $x_{11}(t_0) = 0.1341$, $x_{12}(t_0) = 0.0247$, $x_{13}(t_0) = 0.6493$, $x_{21}(t_0) = 0.0143$, $x_{22}(t_0) = 0.1503$, $x_{23}(t_0) = 0.2394$, $x_{31}(t_0) = 1.1574$, $x_{32}(t_0) = 0.0467$, and $x_{33}(t_0) = 0.5145$, where $t_0 = 0.45$. Figure 2 reveals that each cell $C_{ij}, i, j = 1, 2, 3$ behave chaotically, and this supports the result mentioned in Theorem 3.1. Moreover, Figure 3 shows the projection of the same trajectory on the $x_{22} - x_{31} - x_{33}$ space, and this figure also confirms the results of the present paper.

## V. CONCLUSION

In the paper, it is shown that *SICNNs* with chaotic external inputs admit a chaotic attractor. Considering this phenomenon with the input-output mechanism, one can say about chaos expansion among nonlinearly coupled *SICNNs*. The presented two examples considered together illustrate the possibility. Our method can be applied to other types of chaos, for example, that one analyzed through period-doubling cascade. The approach is suitable for the control of unstable periodic motions. Our results can be applied to the studies of chaotic communication, combinatorial optimization problems, and on problems that have local minima in energy (cost) functions.

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