Period-Doubling Route to Chaos in Shunting Inhibitory Cellular Neural Networks

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***Abstract*—In this study, we investigate the dynamics of shunting inhibitory cellular neural networks with external inputs in the form of relay functions. The presence of chaos through period-doubling cascade is proved theoretically. An example that confirms the theoretical results is illustrated.**

I. Introduction

In recent years, neural networks have become an important tool in bioinformatics, health care and pattern recognition problems [1]-[8]. In bioinformatics, the most common usage of neural networks is the prediction, and there are two main advantages of neural networks compared to other machine-learning methods. First, the use of neural network models to perform prediction is very efficient. In other words, computations are fast. The second one is that models based on neural networks provide high quality results for many prediction tasks such as protein secondary structure and protein solvent accessibility [1]. Neural networks have also been demonstrated to be useful in medicine and many biomedical areas. For example, neural networks can be utilized in the diagnosis of diseases, studying the pathological conditions and monitoring the progress of various treatment outcomes. Application areas of neural networks in healthcare include analysis of electrocardiography, electromyography, electroencephalography, and gait and movement of biomechanics data [3]. Moreover, neural networks are useful in cancer treatment [4], mental health [5] and many other health care areas such as medical image analysis, speech/auditory signal recognition and processing, sleep apnea detection [3]. Another phenomenon in which neural networks are useful is the pattern recognition. The popularity of neural network models to solve pattern recognition problems is due to their seemingly low dependence on domain-specific knowledge and due to the availability of efficient learning algorithms to use. The most commonly preferred neural networks for pattern recognition are feed-forward networks, which have unidirectional couplings between the layers. Although most of the neural network models are implicitly equivalent or similar to classical pattern recognition methods, there exist advantages of neural networks such as flexible procedures for finding moderately nonlinear solutions and unified approaches for feature extraction and classification [7]. Furthermore, it is mentioned in the paper [8] that chaotic neural networks can accomplish a pattern recognition task better than a standard Bayesian statistical method.

A class of cellular neural networks (CNNs) that is introduced by Bouzerdoum and Pinter [9] is shunting inhibitory cellular neural networks (SICNNs). SICNNs have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision and image processing.

The model of SICNNs in the most original formulation [9] is as follows. Consider a two-dimensional grid of processing cells, and let C_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, denote the cell at the (i, j) position of the lattice. Denote by $N_r(i, j)$ the $r-$neighborhood of C_{ij} such that

$$N_r(i,j) = \{C_{kl} : \max\{|k-i|, |l-j|\} \le r, \\ 1 \le k \le m, \ 1 \le l \le n\}.$$

In SICNNs, neighboring cells exert mutual inhibitory interactions of the shunting type. The dynamics of the cell C_{ij} is described by the nonlinear ordinary differential equation

$$\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(t))x_{ij} + L_{ij}(t), \quad (1)$$

where x_{ij} is the activity of the cell C_{ij}; $L_{ij}(t)$ is the external input to C_{ij}; the constant a_{ij} represents the passive decay rate of the cell activity; $C_{ij}^{kl} \ge 0$ is the connection or coupling strength of postsynaptic activity of the cell C_{kl} transmitted to the cell C_{ij}; and the activation function $f(x_{kl})$ is a positive continuous function representing the output or firing rate of the cell C_{kl}. The derivation and biophysical interpretation of SICNNs can be found in the paper [9].

In the present study, the external inputs will be considered in the form of a relay function with chaotically changing switching moments. More precisely, we will consider the SICNNs

$$\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(t))x_{ij} \\ +\nu_{ij}(t,\zeta,\mu), \quad (2)$$

where

$$\nu_{ij}(t,\zeta,\mu) = \begin{cases} \alpha_{ij}, & \text{if} \quad \zeta_{2q}(\mu) < t \le \zeta_{2q+1}(\mu), \\ \beta_{ij}, & \text{if} \quad \zeta_{2q-1}(\mu) < t \le \zeta_{2q}(\mu), \end{cases}$$

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$\alpha = \{\alpha_{ij}\}$ and $\beta = \{\beta_{ij}\}$ are different from each other, and the sequence $\zeta = \{\zeta_q(\mu)\}$, $q \in \mathbb{Z}$, of switching moments are the same for each i, j. The sequence ζ is defined by the formula $\zeta_q(\mu) = q + \kappa_q(\mu)$, $q \in \mathbb{Z}$, where the sequence $\{\kappa_q(\mu)\}$, $\kappa_0(\mu) \in [0,1]$, is generated through the logistic map $h(s,\mu) = \mu s(1-s)$ such that $\kappa_{q+1}(\mu) = h(\kappa_q(\mu),\mu)$.

The period-doubling cascade is the most prominent one among the discovered routes to chaos [10]. In this phenomenon, as some experimental parameter of the considered system varies, a motion with a fundamental period changes to a periodic motion with twice the period of the original oscillation. As the parameter is changed further, the same procedure occurs, and this process accumulates at a critical value of the parameter after which the chaos is observable.

The effect of relay functions to the dynamics of hyperbolic systems was investigated in papers [11]-[14]. According to the results of these papers, the usage of a relay function is convenient to produce not only chaos in the sense of Li-Yorke and Devaney but also through period-doubling cascade.

The presence of chaos in neural networks is useful for separating image segments, information processing and synchronization of neural networks. The performance of CNNs on problems that have local minima in energy (cost) functions can be improved by chaotic dynamics, since chaotic behavior of CNNs can help the network avoid local minima and reach the global optimum.

The existence and the stability of periodic, almost periodic and anti-periodic solutions of SICNNs have been investigated in papers [15]-[22]. The presence of Li-Yorke chaos in SICNNs is proved rigorously in [23]. The existence of a chaotic attractor in SICNNs with impulses was numerically demonstrated in [24] without a theoretical support. The main novelty of the present study is the verification of chaos in SICNNs through period-doubling cascade. Our results are useful for the studies of chaotic communication and combinatorial optimization problems.

II. Preliminaries

In the paper, we will make use of the notations $\delta = \max\limits_{(i,j)} \dfrac{\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}}{a_{ij}}$, $M_{ij} = \max\{|\alpha_{ij}|, |\beta_{ij}|\}$ and $M = \max\limits_{(i,j)} \dfrac{M_{ij}}{a_{ij}}$.

The following conditions are required:

(C1) $\gamma = \min\limits_{(i,j)} a_{ij} > 0$;

(C2) There exists a positive number M_f such that
$$\sup_{s\in\mathbb{R}} |f(s)| \le M_f;$$

(C3) There exists a positive number L_f such that
$$|f(s_1) - f(s_2)| \le L_f\,|s_1 - s_2|$$
for all $s_1, s_2 \in \mathbb{R}$;

(C4) $M_f\delta < 1$;

(C5) $(L_f K_0 + M_f)\delta < 1$, where $K_0 = \dfrac{M}{1 - M_f\delta}$.

In the remaining parts, the norm $\|u\| = \max\limits_{(i,j)} |u_{ij}|$ will be utilized, where $u = \{u_{ij}\}$, $i = 1,\ldots,m$, $j = 1,\ldots,n$.

We say that a function $x(t) = \{x_{ij}(t)\}$, $i = 1,\ldots,m$, $j = 1,\ldots,n$, is a solution of system (2) if: (i) $x(t)$ is continuous on $\mathbb{R}$; (ii) for each i and j, the derivatives $x'_{ij}(t)$ exist for all $t \in \mathbb{R}$ with the possible exception of the points $\zeta_q(\mu)$, $q \in \mathbb{Z}$, where one sided derivatives exist; (iii) the equations presented by (2) are satisfied on each interval $(\zeta_q(\mu), \zeta_{q+1}(\mu)]$, $q \in \mathbb{Z}$.

It can be verified that a bounded on $\mathbb{R}$ function $x(t) = \{x_{ij}(t)\}$ is a solution of system (2) if and only if the integral equation
$$x_{ij}(t) = -\int_{-\infty}^{t} \mathrm{e}^{-a_{ij}(t-s)} \Big[\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(s)) x_{ij}(s) - \nu_{ij}(s,\zeta,\mu) \Big] ds$$
is satisfied for each i and j.

As indicated in the proof of Lemma 2.1 [23], making use of the last integral equation, one can confirm the validity of the following assertion.

Lemma 2.1: For any sequence $\zeta = \{\zeta_q\}$, $\zeta_0 \in [0,1]$, there exists a unique bounded on $\mathbb{R}$ solution $\phi_\zeta(t) = \left\{\phi_\zeta^{ij}(t)\right\}$ of system (2) such that $\sup_{t\in\mathbb{R}} \|\phi_\zeta(t)\| \le K_0$.

Another result which mentions the attractiveness of the bounded on $\mathbb{R}$ solutions of system (2) is indicated in Lemma 2.2, which follows from Theorem 2 [22].

Lemma 2.2: For a fixed sequence $\zeta = \{\zeta_q\}$, all solutions of system (2) converge exponentially to the unique bounded on $\mathbb{R}$ solution $\phi_\zeta(t)$.

III. The Main Result

We will consider the sensitivity and the presence of chaos through period-doubling cascade in this section.

Let us describe the sensitivity of the logistic map on the interval $[0,1]$, which is invariant under the iterations of the map for $0 < \mu \le 4$ [25]. For any positive integer l_0, let us denote by $h^{l_0}(\kappa_0(\mu),\mu)$ the point $\kappa_{l_0}(\mu)$. The logistic map $h(s,\mu)$ is called sensitive on the interval $[0,1]$ if there exists a positive number $\overline{\epsilon} \le 1$ such that for any $\kappa_0(\mu) \in [0,1]$ and $\delta > 0$ there exists $\overline{\kappa}_0(\mu) \in [0,1]$ and a positive integer q_0 such that $|\kappa_0(\mu) - \overline{\kappa}_0(\mu)| < \delta$ and $|h^{q_0}(\kappa_0(\mu),\mu) - h^{q_0}(\overline{\kappa}_0(\mu),\mu)| > \overline{\epsilon}$ [26].

We say that system (2) is sensitive if there exist positive numbers ϵ_0 and Δ such that for every sequence $\zeta = \{\zeta_q\}$, $\zeta_0 \in [0,1]$, and arbitrary $\delta > 0$ there exist a sequence $\overline{\zeta} = \left\{\overline{\zeta}_q\right\}$, $\overline{\zeta}_0 \in [0,1]$, and an interval $J \subset (\max\left\{\zeta_0, \overline{\zeta}_0\right\}, \infty)$, with a length no less than Δ, such that $\left|\zeta_0 - \overline{\zeta}_0\right| < \delta$ and $\left\|\phi_\zeta(t) - \phi_{\overline{\zeta}}(t)\right\| > \epsilon_0$ for $t \in J$.

In the proof of the next theorem, the notation $\widehat{[a,b]}$, $a, b \in \mathbb{R}$, $a \ne b$, will stand for an oriented interval. That is $\widehat{[a,b]} = [a,b]$ if $a < b$, and $\widehat{[a,b]} = [b,a]$, otherwise.

Theorem 3.1: If the logistic map $h(s,\mu)$ is sensitive on the interval $[0,1]$, then system (2) is also sensitive.

Proof. Fix the parameter μ such that the logistic map $h(s,\mu)$ exhibits sensitivity on the interval $[0,1]$. Take an arbitrary sequence sequence $\zeta = \{\zeta_q\}$ satisfying $\zeta_0 \in [0.1]$ and an arbitrary number $\delta > 0$. In this case one can find a sequence $\overline{\zeta} = \{\overline{\zeta}_q\}$, $\overline{\zeta}_0 \in [0,1]$, such that $\left|\zeta_0 - \overline{\zeta}_0\right| < \delta$ and $\left|\zeta_{q_0} - \overline{\zeta}_{q_0}\right| > \overline{\epsilon}$ for some positive integer q_0.

For each t from the interval $[\widehat{\zeta_{q_0}, \overline{\zeta}_{q_0}}]$, one can confirm that $\max\limits_{(i,j)} \left|\nu_{ij}(t,\zeta,\mu) - \nu_{ij}(t,\overline{\zeta},\mu)\right| = \|\alpha - \beta\|$. Therefore, we have

$$\max_{(i,j)} \left| \int_{\zeta_{q_0}}^{\overline{\zeta}_{q_0}} \left(\nu_{ij}(s,\zeta,\mu) - \nu_{ij}(s,\overline{\zeta},\mu)\right) ds \right| > \overline{\epsilon}\, \|\alpha - \beta\|.$$

For $t \in [\widehat{\zeta_{q_0}, \overline{\zeta}_{q_0}}]$, using the couple of relations

$$\phi^{ij}_{\zeta}(t) = \phi^{ij}_{\zeta}(\zeta_{q_0}) + \int_{\zeta_{q_0}}^{t} \nu_{ij}(s,\zeta,\mu) ds - \int_{\zeta_{q_0}}^{t} \Big[a_{ij} + \sum_{C_{kl}\in N_r(i,j)} C^{kl}_{ij} f(\phi^{kl}_{\zeta}(s))\Big] \phi^{ij}_{\zeta}(s) ds,$$

and

$$\phi^{ij}_{\overline{\zeta}}(t) = \phi^{ij}_{\overline{\zeta}}(\zeta_{q_0}) + \int_{\zeta_{q_0}}^{t} \nu_{ij}(s,\overline{\zeta},\mu) ds - \int_{\zeta_{q_0}}^{t} \Big[a_{ij} + \sum_{C_{kl}\in N_r(i,j)} C^{kl}_{ij} f(\phi^{kl}_{\overline{\zeta}}(s))\Big] \phi^{ij}_{\overline{\zeta}}(s) ds,$$

it can be verified that

$$\begin{aligned} &\left\|\phi_{\zeta}(\overline{\zeta}_{q_0}) - \phi_{\overline{\zeta}}(\overline{\zeta}_{q_0})\right\| > \overline{\epsilon}\, \|\alpha - \beta\| \\ &- \left\|\phi_{\zeta}(\zeta_{q_0}) - \phi_{\overline{\zeta}}(\zeta_{q_0})\right\| \\ &- \max_{(i,j)} \left| \int_{\zeta_{q_0}}^{\overline{\zeta}_{q_0}} a_{ij} \left(\phi^{ij}_{\zeta}(s) - \phi^{ij}_{\overline{\zeta}}(s)\right) ds \right| \\ &- \max_{(i,j)} \left| \int_{\zeta_{q_0}}^{\overline{\zeta}_{q_0}} \Big[\sum_{C_{kl}\in N_r(i,j)} C^{kl}_{ij} f(\phi^{kl}_{\zeta}(s)) \phi^{ij}_{\zeta}(s) \right. \\ &\left. - \sum_{C_{kl}\in N_r(i,j)} C^{kl}_{ij} f(\phi^{kl}_{\overline{\zeta}}(s)) \phi^{ij}_{\overline{\zeta}}(s) \Big] ds \right|. \end{aligned} \tag{3}$$

Set $\overline{\gamma} = \max\limits_{(i,j)} a_{ij}$, $\overline{c} = \sum_{C_{kl}\in N_r(i,j)} C^{kl}_{ij}$ and $\overline{M} = \max\limits_{(i,j)} M_{ij}$. The inequality (3) implies that

$$\max_{t\in[\widehat{\zeta_{q_0}, \overline{\zeta}_{q_0}}]} \left\|\phi_{\zeta}(t) - \phi_{\overline{\zeta}}(t)\right\| > \epsilon_1,$$

where $\epsilon_1 = \dfrac{\overline{\epsilon}\, \|\alpha - \beta\|}{2 + \overline{\gamma} + \overline{c}(L_f K_0 + M_f)}$.

Suppose that

$$\max_{t\in[\widehat{\zeta_{q_0}, \overline{\zeta}_{q_0}}]} \left\|\phi_{\zeta}(t) - \phi_{\overline{\zeta}}(t)\right\| = \left\|\phi_{\zeta}(\xi) - \phi_{\overline{\zeta}}(\xi)\right\|,$$

for some $\xi \in [\widehat{\zeta_{q_0}, \overline{\zeta}_{q_0}}]$. Define

$$\Delta = \min\left\{ \frac{\overline{\epsilon}}{2}, \frac{\overline{\epsilon}}{4\left(\overline{M} + K_0\overline{\gamma} + M_f K_0 \overline{c}\right)} \right\}$$

and

$$\theta = \begin{cases} \xi, & \text{if } \xi \le (\zeta_{q_0} + \overline{\zeta}_{q_0})/2 \\ \xi - \Delta, & \text{if } \xi > (\zeta_{q_0} + \overline{\zeta}_{q_0})/2 \end{cases}.$$

One can show that for each t from the interval $J = [\theta, \theta + \Delta]$ the inequality $\left\|\phi_{\zeta}(t) - \phi_{\overline{\zeta}}(t)\right\| > \epsilon_0$ holds, where $\epsilon_0 = \epsilon_1/2$. Consequently, system (2) is sensitive. □

For a natural number p, the sequence $\zeta = \{\zeta_q(\mu)\}$ is called $p-$periodic if $\zeta_{q+p}(\mu) = p + \zeta_q(\mu)$ for all $q \in \mathbb{Z}$. That is, $\kappa_{q+p} = \kappa_q$ for all $q \in \mathbb{Z}$.

To discuss the chaos through period-doubling cascade in system (2), we need the values of the parameter μ, which are between 3.57 and 4, such that the period-doubling cascade accumulates there to provide the chaotic structure for the logistic map $h(s,\mu) = \mu s(1-s)$. In the sequel, we fix one of these values, and notate it as μ_∞.

Consider the sequence of period-doubling bifurcation values $\{\mu_r\}$, $r \in \mathbb{N}$, $\mu_r \to \mu_\infty$ as $r \to \infty$, for the logistic map [27]. We say that system (2) admits the chaos through period-doubling cascade at $\mu = \mu_\infty$, if for each periodic sequence $\zeta = \{\zeta_q(\mu)\}$, where $\zeta_0(\mu) \in [0,1]$ and μ is equal either to μ_r, $r \in \mathbb{N}$, or μ_∞, the unique bounded on $\mathbb{R}$ solution $\phi_\zeta(t)$ of system (2) is periodic. It is worth noting that the periodic solutions which correspond to different sequences ζ do not coincide, and as a result system (2) with $\mu = \mu_\infty$ possesses infinitely many periodic solutions. Moreover, the instability of these periodic solutions are ensured by Theorem 3.1.

In the proof of the next theorem, we will verify that for a $p-$periodic sequence ζ, the bounded on $\mathbb{R}$ solution $\phi_\zeta(t)$ of system (2) is $p-$periodic if p is even, and it is $2p-$periodic if p is odd. Table I summarizes this result by means of different values of the parameter μ.

TABLE I
THE RELATION BETWEEN THE PERIODS OF THE SEQUENCE ζ AND THE CORRESPONDING BOUNDED ON $\mathbb{R}$ SOLUTION $\phi_\zeta(t)$

Range of μ	Period of ζ	Period of $\phi_\zeta(t)$
$1 < \mu < 3$	1	2
$3 < \mu < 3.4494$	2	2
$3.4494 < \mu < 3.5440$	4	4
$3.5440 < \mu < 3.5644$	8	8
$3.5644 < \mu < 3.5687$	16	16
$3.5687 < \mu < 3.5696$	32	32
...	...	...
$3.6265 < \mu < 3.6304$	6	6
...	...	...
$3.7382 < \mu < 3.7411$	5	10
...	...	...
$3.8284 < \mu < 3.8415$	3	6
...	...	...

Theorem 3.2: System (2) with $\mu = \mu_\infty$ admits the chaos through period-doubling cascade.

Proof. Fix the parameter μ such that the sequence ζ is $p-$periodic for some natural number p. We shall consider the cases p is even and odd separately. Let us begin with the case

p is even. In this case, for all i and j, the function $\nu_{ij}(t, \zeta, \mu)$ is $p-$periodic.

Making use of the relation

$$\phi_\zeta^{ij}(t) = -\int_{-\infty}^{t} e^{-a_{ij}(t-s)} \Big[\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} f(\phi_\zeta^{kl}(s))\phi_\zeta^{ij}(s) - \nu_{ij}(s,\zeta,\mu)\Big] ds,$$

one can obtain that

$$\left|\phi_\zeta^{ij}(t) - \phi_\zeta^{ij}(t+p)\right| \leq \int_{-\infty}^{t} e^{-a_{ij}(t-s)} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl} \Big| f(\phi_\zeta^{kl}(s))\phi_\zeta^{ij}(s) - f(\phi_\zeta^{kl}(s+p))\phi_\zeta^{ij}(s+p)\Big| ds.$$

Therefore, for each i, j we have

$$\left|\phi_\zeta^{ij}(t) - \phi_\zeta^{ij}(t+p)\right| \leq (M_f + K_0 L_f) \frac{\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}}{a_{ij}} \sup_{t\in\mathbb{R}} \left\|\phi_\zeta(t) - \phi_\zeta(t+p)\right\|.$$

The last inequality implies that

$$[1-(L_f K_0 + M_f)]\delta \sup_{t\in\mathbb{R}} \left\|\phi_\zeta(t) - \phi_\zeta(t+p)\right\| \leq 0.$$

Consequently, in the case p is even, we have that $\phi_\zeta(t) = \phi_\zeta(t+p)$ for all $t \in \mathbb{R}$.

On the other hand, if p is odd, then the relay function $\nu_{ij}(t, \zeta, \mu)$ is $2p-$periodic. Under the circumstances, a discussion similar to the one above implies that $\phi_\zeta(t) = \phi_\zeta(t+2p)$ for all $t \in \mathbb{R}$. □

A corollary of Theorem 3.2 is that the network (2) obeys the Feigenbaum universality [28], since the same is true for the logistic map.

IV. An Example

Consider the SICNN

$$\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} - \sum_{C_{kl}\in N_1(i,j)} C_{ij}^{kl} f(x_{kl}(t))x_{ij} + \nu_{ij}(t,\zeta,\mu), \qquad (4)$$

in which $i, j = 1, 2, 3,$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 2 & 4 & 4 \\ 3 & 2 & 7 \\ 1 & 5 & 3 \end{pmatrix},$$

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} 0.004 & 0.007 & 0.001 \\ 0.002 & 0.001 & 0.005 \\ 0.006 & 0.003 & 0.008 \end{pmatrix},$$

and $f(s) = 3s^2$. Set $\alpha_{ij} = 0.4$, $\beta_{ij} = 1.8$ for all i, j. One can evaluate that $\sum_{C_{kl}\in N_1(1,1)} C_{11}^{kl} = 0.014$, $\sum_{C_{kl}\in N_1(1,2)} C_{12}^{kl} = 0.02$, $\sum_{C_{kl}\in N_1(1,3)} C_{13}^{kl} = 0.014$, $\sum_{C_{kl}\in N_1(2,1)} C_{21}^{kl} = 0.023$, $\sum_{C_{kl}\in N_1(2,2)} C_{22}^{kl} = 0.037$, $\sum_{C_{kl}\in N_1(2,3)} C_{23}^{kl} = 0.025$, $\sum_{C_{kl}\in N_1(3,1)} C_{31}^{kl} = 0.012$, $\sum_{C_{kl}\in N_1(3,2)} C_{32}^{kl} = 0.025$, $\sum_{C_{kl}\in N_1(3,3)} C_{33}^{kl} = 0.017$.

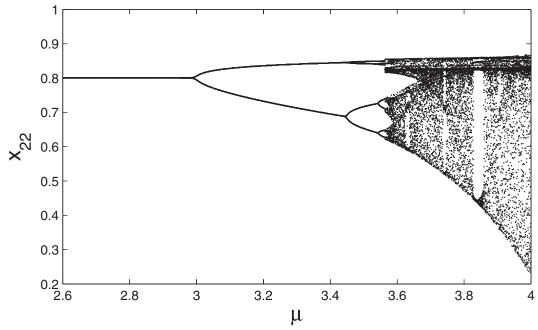

Fig. 1. The bifurcation diagram for the $x_{22}-$coordinate of the SICNN (4).

Figure 1 depicts the bifurcation diagram for the cell C_{22} of the SICNN (4). In the range of the parameter values $\mu > 3.57$, successive intervals of chaos and stable periodic solutions are observable in the diagram. In the regions where stable periodic solutions exist, for a fixed value of μ, the bifurcation diagram represents the values of the stable periodic solutions at $t = \zeta_0(\mu)$, where $\zeta_0(\mu)$ is the initial term of the sequence $\zeta = \{\zeta_q(\mu)\}$. It is worth noting that for $\mu_r < \mu < \mu_{r+1}$ there are 2^r different choices for the periodic sequence ζ with periodicity 2^r, and this is the reason for the observation of 2^r different stable periodic solutions in the diagram for these values of the parameter. Figure 1 supports our theoretical results such that period-doubling bifurcations appear in the dynamics of system (4).

Let us take $\mu = 3.8$ in system (4). For this value of the parameter, the logistic map $h(s, \mu)$ admits a positive Lyapunov exponent, and therefore, exhibits sensitivity [29]. According to the results of Theorem 3.1 and Theorem 3.2, the SICNN (4) is chaotic through the period-doubling cascade. Moreover, Lemma 2.2 implies that a chaotic attractor takes place. The bifurcation diagram presented in Figure 1 also confirms the presence of chaos for this parameter value. Making use of the compact region in which the chaotic attractor of system (4) takes place, one can confirm that the function f satisfies the conditions $(C2)$ and $(C3)$ with the constants $M_f = 8.67$ and $L_f = 10.2$. Moreover, the conditions $(C4)$, $(C5)$ hold with $M = 1.8$, $K_0 = 2.15$, and $\delta = 0.0185$.

Consider system (4) with $\zeta = \{\zeta_q(3.8)\}$ such that $\zeta_0(3.8) = 0.81$. Figure 1 depicts the behavior of the cell C_{22} of the network (4) corresponding to the initial data $x_{11}(t_0) = 0.25$, $x_{12}(t_0) = 0.11$, $x_{13}(t_0) = 0.12$, $x_{21}(t_0) = 0.14$, $x_{22}(t_0) = 0.24$, $x_{23}(t_0) = 0.05$, $x_{31}(t_0) = 0.72$, $x_{32}(t_0) = 0.08$ and $x_{33}(t_0) = 0.14$, where $t_0 = 0.81$. One can find out that the other coordinates have similar graphs, but they are not just simulated here. It is seen in Figure 2 that the represented solution behaves chaotically, and this supports our theoretical results.

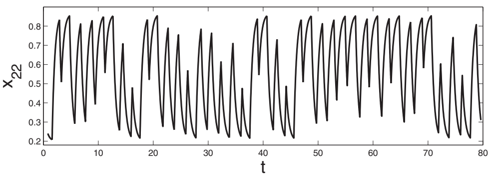

Fig. 2. Chaotic behavior in the $x_{22}-$coordinate of the SICNN (4).

V. Conclusion and Outlook

In the present study, we proved the existence of chaos through period-doubling cascade in SICNNs rigorously. Our results ensure us to say that discontinuous external inputs are appropriate to attain chaos in the dynamics of SICNNs. Moreover, the obtained chaos is controllable, for instance, by the OGY and Pyragas control methods applied to the original chaos.

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