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On periodic solutions of differential equations with piecewise constant argument[☆]

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ABSTRACT

The periodic quasilinear system of differential equations with small parameter and piecewise constant argument of generalized type [M.U. Akhmet, Integral manifolds of differential equations with piecewise constant argument of generalized type, Nonlinear Anal. TMA, 66 (2007) 367–383, M.U. Akhmet, On the reduction principle for differential equations with piecewise argument of generalized type, J. Math. Anal. Appl. 336 (2007) 646–663] is addressed. We consider the critical case, when associated linear homogeneous system admits nontrivial periodic solutions. Criteria of existence of periodic solutions of such equations are obtained. One of the main auxiliary results of our paper is an analogue of Gronwall–Bellman Lemma for functions with piecewise constant and retarded–advanced type arguments. Dependence of solutions on the parameter is investigated. Appropriate examples are given to show our results.

1. Introduction

The problem of the existence of periodic solutions is one of the most interesting topics for applications. Poincaré [3] introduced the method of small parameter to investigate the problem and it has been developed by many authors (see, for example, [4,5], and the references cited therein) and this method remains as one of the most effective methods for this problem. It is important that the results obtained in this field can be extended to the bifurcation theory [6,7].

The study of differential equations with piecewise constant argument (EPCA) was initiated in [8–10]. These equations have been investigated widely using the method of reduction to discrete equations by many authors [11–21]. In [16] some limit relations between the solutions of delay differential equations with continuous arguments and the solutions of some retarded delay EPCA have been proved. The results were used to compute numerical solutions of ordinary and delay differential equations. For brief summary on theory, the reader is referred to the book by Wiener [20].

The significance of the equations for practice can be seen from the following result. The authors of [22] investigated the damped loading system subjected to a piecewise constant voltage described by the equation of charge:

$$Lq''(t) + Rq'(t) + C^{-1}q(t) = Aq\left(\frac{[Nt]}{N}\right), \tag{1}$$

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which was compared with a similar linear loading system governed by the following equation of charge

$$Lq''(t) + Rq'(t) + C^{-1}q(t) = Aq(t). \quad (2)$$

They considered, through numerical simulation, the phenomena of sensitivity on the initial data, stability and existence of oscillating solutions. On the basis of equations of type (1) we will show in Examples 5.1 and 5.2 that the results of the paper can be used for electronics, as well as for mechanical problems [13]. That is, theorems of the paper may give a new theoretical background for the scrupulous investigation of wide spectra problems of theoretical mechanics and electronics.

Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$ be the sets of all integers, natural numbers and real numbers, respectively. Denote by $\| \, . \, \|$ the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$. Fix two real-valued sequences θ_i, ζ_i, $i \in \mathbb{Z}$, such that $\theta_i < \theta_{i+1}$, $\theta_i \le \zeta_i \le \theta_{i+1}$ for all $i \in \mathbb{Z}$, $|\theta_i| \to \infty$ as $|i| \to \infty$.

In this paper we shall consider the following equation

$$z'(t) = A(t)z(t) + f(t) + \mu g(t, z(t), z(\gamma(t)), \mu), \quad (3)$$

where $z \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\mu \in J \subset \mathbb{R}$, where J is an open interval containing 0, and $\gamma(t) = \zeta_i$, if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$.

The following assumptions will be needed throughout the paper:

(C1) $A : \mathbb{R} \to \mathbb{R}^{n\times n}$, $f : \mathbb{R} \to \mathbb{R}^n$ and $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times J \to \mathbb{R}^n$ are continuous functions.

(C2) The function $g(t, x, y, \mu)$ is Lipschitzian in the second and third arguments with a positive Lipschitz constant L such that

$$\|g(t, x_1, y_1, \mu) - g(t, x_2, y_2, \mu)\| \le L(\|x_1 - x_2\| + \|y_1 - y_2\|)$$

for all $t \in \mathbb{R}$, $\mu \in J$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$.

(C3) The matrix A is uniformly bounded on $\mathbb{R}$.

(C4) There exists a number $\bar{\theta} > 0$ such that $\theta_{i+1} - \theta_i \le \bar{\theta}$, $i \in \mathbb{Z}$.

(C5) There exists a number $\theta > 0$ such that $\theta_{i+1} - \theta_i \ge \theta$, $i \in \mathbb{Z}$.

In [1,2], it was proposed to investigate differential equations of type (3), that is, the differential equations with piecewise constant argument of generalized type (EPCAG). Moreover, a new method based on the construction of an equivalent integral equation was used.

We combine that method with the method of small parameter [3,4,7] to investigate the problem of the existence of periodic solutions of Eq. (3) in the so-called critical case, when the corresponding linear homogeneous system admits nontrivial periodic solutions.

Our paper is organized in the following way. In the next section, we give known definitions and results that will be needed further. Section 3 considers continuous and differentiable dependence of solutions on the initial value and the parameter. The main result of the paper: the existence of periodic solutions of Eq. (3) is discussed in Section 4. Appropriate examples are given to illustrate the theory in the last section.

2. Preliminaries

In this section, we shall introduce some definitions and lemmas.

Definition 2.1 (*[1]*). A continuous function $z(t)$ is a solution of Eq. (3) on $\mathbb{R}$ if:

(i) The derivative $z'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of the points θ_i, $i \in \mathbb{Z}$, where the one-sided derivatives exist.

(ii) The equation is satisfied for $z(t)$ on each interval (θ_i, θ_{i+1}), $i \in \mathbb{Z}$, and it holds for the right derivative of $z(t)$ at the points θ_i, $i \in \mathbb{Z}$.

The following lemmas of this section are similar to the assertions from [23]. That is why, we provide them without proof.

Let $X(t)$ be the fundamental matrix solution of the homogeneous system, corresponding to Eq. (3),

$$x'(t) = A(t)x(t), \quad t \in \mathbb{R}, \quad (4)$$

such that $X(0) = I$, where I is an $n \times n$ identity matrix. Denote by $X(t, s) = X(t)X^{-1}(s)$, $t, s \in \mathbb{R}$ the transition matrix.

Let us now define the solutions of quasilinear system (3).

Lemma 2.1. *Suppose that (C1) is satisfied. A function $z(t) = z(t, t_0, z_0, \mu)$, where t_0 is a fixed real number, is a solution of (3) in the sense of Definition 2.1 if and only if it is a solution, on $\mathbb{R}$, of the following integral equation*

$$z(t) = X(t, t_0)z_0 + \int_{t_0}^{t} X(t, s)[f(s) + \mu g(s, z(s), z(\gamma(s)), \mu)]\mathrm{d}s. \quad (5)$$

Denote $\kappa = \sup_{t\in\mathbb{R}} \|A(t)\| < \infty$. For the transition matrix, $X(t, s)$, one can obtain the following inequatility [1,24]:

$$m \le \|X(t, s)\| \le M, \quad (6)$$

where $m = \exp(-\kappa\bar{\theta})$ and $M = \exp(\kappa\bar{\theta})$, if $t, s \in [\theta_i, \theta_{i+1}]$ for all $i \in \mathbb{Z}$.

From now on we make the following assumption:

(C6) $2|\mu|ML\bar{\theta} < 1$, $|\mu|M^2L\bar{\theta}\left\{\frac{1+|\mu|ML\bar{\theta}e^{|\mu|ML\bar{\theta}}}{1-|\mu|ML\bar{\theta}e^{|\mu|ML\bar{\theta}}} + e^{|\mu|ML\bar{\theta}}\right\} < m$.

Lemma 2.2 ([23]). *Assume that conditions*(C1)–(C6) *are fulfilled. Then for fixed $i \in \mathbb{Z}$ and every $(\xi, z_0) \in [\theta_i, \theta_{i+1}] \times \mathbb{R}^n$ there exists unique solution $z(t) = z(t, \xi, z_0, \mu)$ of Eq.* (3) *on $[\theta_i, \theta_{i+1}]$.*

From Lemma 2.2, one can obtain the following assertion.

Lemma 2.3 ([23]). *Assume that conditions*(C1)–(C6) *are fulfilled. Then for every $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$ there exists a unique solution $z(t) = z(t, t_0, z_0, \mu)$ of Eq.* (3) *in the sense of Definition 2.1 such that $z(t_0) = z_0$.*

3. Dependence of the solutions on initial value and parameter

Let us fix $t_0 \in \mathbb{R}, z_0 \in \mathbb{R}^n$ and $\mu_0 \in J$. There exists $j \in \mathbb{Z}$ such that $\theta_j \le t_0 < \theta_{j+1}$. Let us denote by $\|\ .\ \|_t$ a max-norm, $\|v\|_t = \max_{\xi\in[\theta_j,t]}\|v(\xi)\|$. Define a function $\chi(t) = \max\{t, \gamma(t)\}$. The next theorem proves continuous dependence of solutions of (3) on an initial value z_0. To prove the theorems, we use the following assertion, which is analogue of Gronwall–Bellman Lemma.

Lemma 3.1. *Let $u(t)$ be continuous, $\eta_1(t)$ and $\eta_2(t)$ nonnegative piecewise continuous scalar functions defined for $t \ge \theta_j$. Suppose that α is a nonnegative real constant and that $u(t)$ satisfies the inequality*

$$\|u(t)\| \le \alpha + \int_{\theta_j}^{t}[\eta_1(s)\,\|u(s)\| + \eta_2(s)\,\|u(\gamma(s))\|]\mathrm{d}s, \tag{7}$$

for $t \ge \theta_j$. Then the inequality

$$\|u\|_{\chi(t)} \le \alpha \exp\left(\int_{\theta_j}^{\chi(t)}[\eta_1(s) + \eta_2(s)]\mathrm{d}s\right) \tag{8}$$

is satisfied for $t \ge \theta_j$.

Proof. Let us first show that

$$\|u\|_{\chi(t)} \le \alpha + \int_{\theta_j}^{\chi(t)}[\eta_1(s) + \eta_2(s)]\|u\|_{\chi(s)}\mathrm{d}s, \quad t \ge \theta_j. \tag{9}$$

As $\chi(t) \ge \theta_j$, using (7), we have

$$\|u(\chi(t))\| \le \alpha + \int_{\theta_j}^{\chi(t)}[\eta_1(s)\,\|u(s)\| + \eta_2(s)\,\|u(\gamma(s))\|]\mathrm{d}s. \tag{10}$$

Since $\theta_j \le \gamma(s) \le \chi(s)$ for all $s \ge \theta_j$, we have that

$$\|u(\gamma)\|_{\chi(s)} = \max_{[\theta_j,\chi(s)]}\|u(\gamma(\xi))\| = \max_{[\zeta_j,\gamma(s)]}\|u(\xi)\| \le \max_{[\theta_j,\chi(s)]}\|u(\xi)\| = \|u\|_{\chi(s)}.$$

Hence, using (10), the inequality

$$\|u(\chi(t))\| \le \alpha + \int_{\theta_j}^{\chi(t)}[\eta_1(s) + \eta_2(s)]\|u\|_{\chi(s)}\mathrm{d}s$$

is satisfied.

If $\|u(\chi(t))\| = \|u\|_{\chi(t)}$ is satisfied for a given $t \ge \theta_j$, then inequality (9) follows. Suppose that $\|u(\chi(t))\| < \|u\|_{\chi(t)}$ holds. One can see that by the definition of max-norm, there is a moment $\tilde{t} \in [\theta_j, \chi(t)]$ such that $\|u\|_{\chi(t)} = \|u(\tilde{t})\|$.

Then, using (7), we have

$$\begin{aligned}\|u\|_{\chi(t)} &= \|u(\tilde{t})\| \\ &\le \alpha + \int_{\theta_j}^{\tilde{t}}[\eta_1(s)\,\|u(s)\| + \eta_2(s)\,\|u(\gamma(s))\|]\mathrm{d}s \\ &\le \alpha + \int_{\theta_j}^{\chi(\tilde{t})}[\eta_1(s) + \eta_2(s)]\|u\|_{\chi(s)}\mathrm{d}s \\ &\le \alpha + \int_{\theta_j}^{\chi(t)}[\eta_1(s) + \eta_2(s)]\|u\|_{\chi(s)}\mathrm{d}s,\end{aligned}$$

as $\chi(\tilde{t}) \le \chi(t)$. Hence, inequality (9) is valid. Now, set the function $\|u\|_{\chi(s)} = \psi(s)$, and note that $\psi(s) = \psi(\chi(s))$.

Thus we have the inequality

$$\psi(\chi(t)) \leq \alpha + \int_{\theta_j}^{\chi(t)} [\eta_1(s) + \eta_2(s)]\psi(\chi(s))\mathrm{d}s.$$

Applying Gronwall–Bellman Lemma to the last inequality, we complete the proof. □

Let us fix a number $T > 0$. Now, we set continuous dependence of solutions of (3) on an initial value z_0 by the following theorem.

Theorem 3.1. *Suppose that* (C1)–(C6) *are valid. If* $z(t) = z(t, t_0, y_0, \mu_0)$ *and* $\tilde{z}(t) = z(t, t_0, z_0 + \Delta z, \mu_0)$ *are the solutions of Eq.* (3)*, where* Δz *is an n-dimensional vector, then the inequality*

$$\|\tilde{z}(\xi) - z(\xi)\|_{\chi(t)} \leq M\|\Delta z\| \exp\left(2|\mu_0|ML(\chi(t_0 + T) - \theta_j)\right) \tag{11}$$

is satisfied for $t \in [t_0, t_0 + T]$.

The last theorem can be proved by applying Lemma 3.1. The differential dependence of a solution of Eq. (3) on an initial value is established by our next theorem, which requires the following assumption:

(C7) $g(t, x, y, \mu)$ has continuous first partial derivatives in all of its arguments $t \in \mathbb{R}, x, y \in \mathbb{R}^n, \mu \in J$.

Let us introduce the following equations

$$U'(t) = A(t)U(t) + \mu_0[A_1(t)U(t) + A_2(t)U(\gamma(t))], \tag{12}$$

$$U(t_0) = I, \tag{13}$$

where $U \in \mathbb{R}^{n\times n}$ and the functions

$$A_1(t) = \frac{\partial g}{\partial x}(t, z(t), z(\gamma(t)), \mu_0), \qquad A_2(t) = \frac{\partial g}{\partial y}(t, z(t), z(\gamma(t)), \mu_0)$$

are $n \times n$ matrices.

Theorem 3.2. *Suppose that* (C1)–(C7) *are valid. Let* $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^{\mathrm{T}}$ *be the n-tuple whose ith component is* 1 *and all others are* 0 *for* $i = 1, \ldots, n$*, and* δ *a real positive constant. If* $U(t)$ *is a solution of* (12) *and* (13) *on* $\mathbb{R}$*, and* $z(t) = z(t, t_0, z_0, \mu_0)$ *and* $\tilde{z}_i(t) = z(t, t_0, z_0+\Delta z_i, \mu_0)$ *are solutions of Eq.* (3)*, where* $\Delta z_i = \delta e_i$ *is an n-dimensional vector in the sense of Definition* 2.1*, then*

$$\tilde{z}_i(t) - z(t) - U(t)\Delta z_i = o(\Delta z_i) \tag{14}$$

is satisfied on a section $t \in [t_0, t_0 + T], T > 0$.

Proof. By Lemma 2.1, the functions $\tilde{z}_i(t)$, $z(t)$ and $U(t)$ satisfy the following integral equations:

$$\tilde{z}_i(t) = X(t, t_0)(z_0 + \Delta z_i) + \int_{t_0}^{t} X(t, s)[f(s) + \mu_0 g(s, \tilde{z}_i(s), \tilde{z}_i(\gamma(s)), \mu_0)]\mathrm{d}s,$$

$$z(t) = X(t, t_0)z_0 + \int_{t_0}^{t} X(t, s)[f(s) + \mu_0 g(s, z(s), z(\gamma(s)), \mu_0)]\mathrm{d}s,$$

$$U(t) = I + \mu_0 \int_{t_0}^{t} X(t, s)[A_1(s)U(s) + A_2(s)U(\gamma(s))]\mathrm{d}s,$$

respectively. An easy computation shows that, if $t \in [t_0, t_0 + T]$, we have

$$\begin{aligned}\tilde{z}_i(t) - z(t) - U(t)\Delta z_i = \mu_0 \int_{t_0}^{t} X(t, s)[&g(s, \tilde{z}_i(s), \tilde{z}_i(\gamma(s)), \mu_0) - g(s, z(s), z(\gamma(s)), \mu_0) \\ &- A_1(s)U(s)\Delta z_i - A_2(s)U(\gamma(s))\Delta z_i]\mathrm{d}s.\end{aligned}$$

By expanding $g(s, \tilde{z}_j(s), \tilde{z}_j(\gamma(s)), \mu_0)$ about the point $(s, z(s), z(\gamma(s)), \mu_0)$, we write

$$g(s, \tilde{z}_i(s), \tilde{z}_i(\gamma(s)), \mu_0) = g(s, z(s), z(\gamma(s)), \mu_0) + A_1(s)[\tilde{z}_i(s) - z(s)] + A_2(s)[\tilde{z}_i(\gamma(s)) - z(\gamma(s))] + \xi(s),$$

where $\xi(s) = o(\Delta z_i)$. Hence, the inequality

$$\begin{aligned}\|\tilde{z}_i(t) - z(t) - U(t)\Delta z_i\| \leq \zeta + |\mu_0|M \int_{t_0}^{t} [&\|A_1(s)\|\|\tilde{z}_i(s) - z(s) - U(s)\Delta z_i\| \\ &+ \|A_2(s)\|\|\tilde{z}_i(\gamma(s)) - z(\gamma(s)) - U(\gamma(s))\Delta z_i\|]\mathrm{d}s,\end{aligned}$$

where $\zeta = |\mu_0|M \int_{t_0}^{t_0+T} \|\xi(s)\|\mathrm{d}s$, is valid. Consequently, by applying Lemma 3.1 to the last inequality, we prove that (14) is true. □

As a result of the last theorem, we have shown that the initial value problem (12) and (13) is a variation of Eq. (3).

4. Existence of the periodic solutions

In this section, we prove the main result of this paper. Let us introduce the following assumptions:

(C8) The functions $A(t), f(t)$ and $g(t, x, y, \mu)$ are periodic in t with a fixed positive real period ω.
(C9) The sequences θ_i and ζ_i, $i \in \mathbb{Z}$, satisfy an (ω, p)-property, that is there is a positive integer p such that the equations $\theta_{i+p} = \theta_i + \omega$ and $\zeta_{i+p} = \zeta_i + \omega$ hold for all $i \in \mathbb{Z}$.

We consider the following version of Poincaré criterion.

Lemma 4.1. *Suppose that* (C1)–(C6), (C8) *and* (C9) *hold. Then, the solution $z(t) = z(t, t_0, x_0, \mu)$ of Eq.* (3)*, is ω-periodic if and only if*

$$z(\omega) = z(0). \tag{15}$$

Proof. If $z(t)$ is ω-periodic, then Eq. (15) is obviously satisfied. Suppose that Eq. (15) holds. Let $y(t) = z(t + \omega)$ on $\mathbb{R}$. Then, Eq. (15) can be written as $y(0) = z(0)$.

One can show that $\gamma(t + \omega) = \gamma(t) + \omega$ for all $t \in \mathbb{R}$. Hence,

$$\begin{aligned} y'(t) &= z'(t + \omega) \\ &= A(t + \omega)z(t + \omega) + f(t + \omega) + \mu g(t + \omega, z(t + \omega), z(\gamma(t + \omega)), \mu) \\ &= A(t)y(t) + f(t) + \mu g(t, y(t), y(\gamma(t)), \mu). \end{aligned}$$

That is, $y(t)$ is a solution of Eq. (3). By the uniqueness of the solution, we have $z(t) = y(t)$ on $\mathbb{R}$. The lemma is proved. □

In [25], we considered the noncritical case. Now, we suppose that the homogeneous equation, corresponding to Eq. (3), has a nontrivial ω-periodic solution.

Let $\phi_j, j = 1, \ldots, k, k \leq n$, be the solutions of Eq. (4), which form a maximal set of linearly independent ω-periodic solutions. Then, the corresponding adjoint system of (4),

$$x'(t) = -A^{\mathrm{T}}(t)x(t), \tag{16}$$

has a maximal set of linearly independent ω-periodic solutions, $\psi_j, j = 1, \ldots, k$.

We compose an $n \times k$ matrix $K_1(t)$, whose columns are solutions $\psi_j, j = 1, \ldots, k$.

Let us introduce the following condition:

(C10) $\int_0^\omega K_1^{\mathrm{T}}(s)f(s)\mathrm{d}s = 0$.

Theorem 4.1 ([4,5]). *Suppose that* (C1)–(C3), (C8) *and* (C10) *hold. Then, if Eq.* (4) *admits $k \leq n$ linearly independent ω-periodic solutions, then there exists a family of k linearly independent ω-periodic solutions of the equation*

$$z'(t) = A(t)z(t) + f(t), \tag{17}$$

of the form $z(t, \alpha) = \alpha_1\phi_1(t) + \cdots + \alpha_k\phi_k(t) + \tilde{z}(t)$, where $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a real constant vector and $\tilde{z}(t)$ is a particular ω-periodic solution of Eq. (17)*.*

Now let us investigate the question of existence of periodic solutions of (3). The next theorem is a generalization of a classical theorem due to Malkin [4] for EPCAG. The proof of this theorem for ordinary differential equations without piecewise argument can be found in [5, p. 179].

Theorem 4.2. *Suppose that conditions* (C1)–(C10) *hold and* (17) *admits a family of ω-periodic solutions $z(t, \alpha)$. Let α^0 be a solution of the equation $h(\alpha) = 0$, where the function h is given by*

$$h(\alpha) = \int_0^\omega K_1^{\mathrm{T}}(s)g(s, z(s, \alpha), z(\gamma(s), \alpha), 0)\mathrm{d}s, \tag{18}$$

such that

$$\det\left(\left.\frac{\partial h}{\partial \alpha}\right|_{\alpha=\alpha^0}\right) \neq 0.$$

Then for sufficiently small $|\mu|$ Eq. (3) *has an ω-periodic solution that converges to $z(t, \alpha^0)$ when $\mu \to 0$.*

Proof. Let $z(t)$ be a solution of (3) and let us complete the matrix $K_1(t)$ by columns $\psi_j, j = k+1, \ldots, n$, which are solutions of (16) to obtain a fundamental matrix of solutions $K(t)$. Performing the substitution $y(t) = K^{\mathrm{T}}(0)z(t)$ in (3), we obtain the equation

$$y'(t) = P(t)y(t) + r(t) + \mu F(t, y(t), y(\gamma(t)), \mu), \tag{19}$$

where

$$P(t) = K^{\mathrm{T}}(0)A(t)K^{\mathrm{T}}(0)^{-1}, \quad r(t) = K^{\mathrm{T}}(0)f(t),$$
$$F(t, y(t), y(\gamma(t)), \mu) = K^{\mathrm{T}}(0)g(t, K^{\mathrm{T}}(0)^{-1}z(t), K^{\mathrm{T}}(0)^{-1}z(\gamma(t)), \mu).$$

Denote $y(t,\alpha) = K^{\mathrm{T}}(0)z(t,\alpha)$, $\beta = (\beta_{k+1}, \ldots, \beta_n)$ and let $v(t) = y(t, \alpha, \beta)$ be a solution of (19) with the initial condition $v(0) = y(0, \alpha) + (0, \beta)^{\mathrm{T}}$. Further, let $L(t) = K^{-1}(0)K(t)$, $L_1(t) = K^{-1}(0)K_1(t)$, $L_2(t)$ be the matrix composed of the entries of the last $n-k$ columns and $n-k$ rows of the matrix $L(t)$, and $L_3(t)$ be the matrix composed of the last $n-k$ rows of $L^{\mathrm{T}}(t)$. Denote

$$U(\alpha, \beta, \mu) = \int_0^{\omega} L_1^{\mathrm{T}}(s)F(s, v(s), v(\gamma(s)), \mu)\mathrm{d}s,$$
$$V(\alpha, \beta, \mu) = (L_2^{\mathrm{T}}(\omega) - I)\beta - \mu \int_0^{\omega} L_3(s)F(s, v(s), v(\gamma(s)), \mu)\mathrm{d}s.$$

Then the ω-periodicity condition for the solution $v(t)$ takes on the form of the equations

$$U(\alpha, \beta, \mu) = 0, \tag{20}$$
$$V(\alpha, \beta, \mu) = 0. \tag{21}$$

If, in (21), taking $\mu = 0$, we obtain $\beta = 0$, and then Eq. (20) has the form

$$U(\alpha, 0, 0) = \int_0^{\omega} L_1^{\mathrm{T}}(s)F(s, y(s,\alpha), y(\gamma(s), \alpha), 0)\mathrm{d}s = 0. \tag{22}$$

Let $\alpha^0 = (\alpha_1^0, \ldots, \alpha_k^0)$ be a solution of (22). Since the function U has continuous partial derivatives with respect to $\alpha_j, j = 1, \ldots, k$, in a sufficiently small neighborhood of the point $(\alpha_0, 0, 0)$, it follows that under the assumption

$$\det\left(\left.\frac{\partial U}{\partial \alpha}\right|_{\alpha=\alpha^0}\right) \neq 0$$

the system of Eqs. (20) and (21) is solvable with respect to α and β so that the functions $\alpha_j(\mu)$ and $\beta_s(\mu)$, $j = 1, \ldots, k$, $s = k+1, \ldots, n$ are continuous and $\alpha_j(\mu) \to \alpha_j^0$, $\beta_s(\mu) \to 0$ as $\mu \to 0$.

Thus, we establish that for sufficiently small $|\mu|$, system (3) admits an ω-periodic solution, which converges to the solution $z(t, \alpha^0)$ of (17) as $\mu \to 0$. The theorem is proved. □

5. Illustrative examples

We will introduce appropriate examples in this section. These examples will show the feasibility of our theory. The equations of Duffing type are widely investigated in the field of nonlinear dynamics, and used to model many processes in mechanics and electronics [15,26]. We construct the examples with Duffing equations below.

Example 5.1. Let us consider the following EPCAG

$$q''(t) = -q(t) + 3\sin^2(t) + \mu\left(q(t) + q'\left(2\pi\left[\frac{t+\pi}{2\pi}\right]\right)\cos t\right). \tag{23}$$

The form of the perturbation of the last equation is chosen to be linear since the simulation of the solutions for the equation with retarded and advanced argument is difficult in nonlinear case.

We write the last equation in the system form

$$z'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ 3\sin^2 t \end{pmatrix} + \mu \begin{pmatrix} 0 \\ z_1(t) + z_2\left(2\pi\left[\frac{t+\pi}{2\pi}\right]\right)\cos t \end{pmatrix}. \tag{24}$$

Let us slightly generalize it as the following system

$$z'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ 3\sin^2 t \end{pmatrix} + \mu \begin{pmatrix} a\, z_1\left(2\pi\left[\dfrac{t+\pi}{2\pi}\right]\right)\sin t + b\, z_2(t) \\ c\, z_1(t) + d\, z_2\left(2\pi\left[\dfrac{t+\pi}{2\pi}\right]\right)\cos t \end{pmatrix}, \tag{25}$$

where a, b, c and d are real constants.

One can see that Eq. (24) is a particular case of (25) when $a = 0$, $b = 0$, $c = 1$, and $d = 1$.

If $\mu = 0$, Eq. (25) takes the form

$$z'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ 3\sin^2 t \end{pmatrix}. \tag{26}$$

It is easy to find 2π-periodic solutions $\psi_j, j = 1, 2$, as

$$\psi_1 = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

of the adjoint system of the last equation. Then, condition (C10) can be verified

$$\begin{aligned} \int_0^{2\pi} K_1^{\mathrm{T}}(s) f(s)\mathrm{d}s &= \int_0^{2\pi} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} 0 \\ 3\sin^2 s \end{pmatrix} \mathrm{d}s \\ &= 0. \end{aligned}$$

Hence, the family of 2π-periodic solutions of (26) is given by

$$z(t,\alpha) = \begin{pmatrix} \alpha_1 \cos t + \alpha_2 \sin t + \dfrac{3}{2} + \dfrac{\cos 2t}{2} \\ -\alpha_1 \sin t + \alpha_2 \cos t - \sin 2t \end{pmatrix}, \tag{27}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ are the parameters.

Next, let us show that Eq. (25) has a 2π-periodic solution.

The function $h(\alpha)$ in Theorem 4.2 can be evaluated as

$$\begin{aligned} h(\alpha) &= \int_0^{2\pi} K_1^{\mathrm{T}}(s) g(s, z(s,\alpha), z(\gamma(s),\alpha), 0)\mathrm{d}s, \\ &= \int_0^{2\pi} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} a\, z_1\left(2\pi\left[\dfrac{s+\pi}{2\pi}\right], \alpha\right)\sin s + b\, z_2(s,\alpha) \\ c\, z_1(s,\alpha) + d\, z_2\left(2\pi\left[\dfrac{s+\pi}{2\pi}\right], \alpha\right)\cos s \end{pmatrix} \mathrm{d}s \\ &= \int_0^{2\pi} \begin{pmatrix} -c\,\alpha_2 \sin^2 s + b\,\alpha_2 \cos^2 s \\ (a\,(\alpha_1 + 2) - b\,\alpha_1)\sin^2 s + (c\,\alpha_1 + d\alpha_2)\cos^2 s \end{pmatrix} \mathrm{d}s \\ &= \begin{pmatrix} \pi(b-c)\alpha_2 \\ \pi((a-b+c)\alpha_1 + d\alpha_2 + 2a) \end{pmatrix}. \end{aligned}$$

Suppose that $b \neq c$ and $a \neq b - c$. By straightforward calculation one can see that the zero of the equation $h(\alpha) = 0$ is $\alpha^0 = (\frac{-2a}{a-b+c}, 0)$, and the determinant is

$$\begin{aligned} \det\left(\left.\frac{\partial h}{\partial \alpha}\right|_{\alpha=\alpha^0} \right) &= \det \begin{pmatrix} 0 & \pi(b-c) \\ \pi(a-b+c) & d \end{pmatrix} \\ &= -\pi^2 (b-c)(a-b+c) \\ &\neq 0. \end{aligned}$$

Hence, using Theorem 4.2, we can conclude that for sufficiently small $|\mu|$ Eq. (25) has a 2π-periodic solution and this solution tends to $z(t, \alpha^0)$ as $\mu \to 0$. Since we know that the initial value of the solution is close to the initial value of the periodic solution of Eq. (26), and there is continuous dependence on parameter μ, one can make the following simulations with identical initial data, $z(0) = (2, 0)^{\mathrm{T}}$. They can be seen from Fig. 1, where the solid lines are graphs of the periodic solution of Eq. (26), and graphs of two coordinates of the periodic solution of Eq. (25) are near the dashed lines. Simulations are carried out by using MATLAB 7.3.

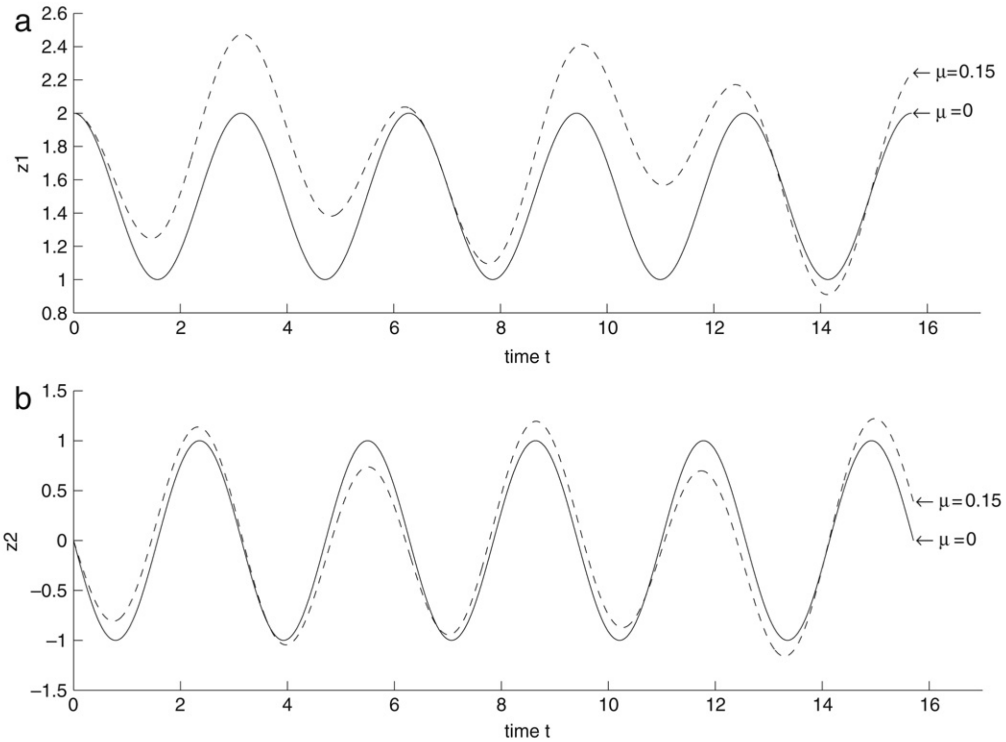

Fig. 1. Simulation of the periodic solution of (26) (solid) and the solution of (25) (dashed) which is near the periodic solution of the perturbed system if $a = 0, b = 0, c = 1, d = 1$, with identical initial data, $z(0) = (2, 0)^{\mathrm{T}}$. In (a) the first coordinates are shown, and second coordinates of the solutions are given in (b).

Example 5.2. Let us consider another example when the perturbation is nonlinear. In this case, we cannot provide a numerical simulation, but we can show the existence of periodic solutions following the result of this paper.

Consider the equation

$$z'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ 3\sin^2 t \end{pmatrix} + \mu \begin{pmatrix} z_1\left(2\pi\left[\dfrac{t+\pi}{2\pi}\right]\right)^2 \sin t + z_2(t) \\ 2\,z_1(t) + z_2\left(2\pi\left[\dfrac{t+\pi}{2\pi}\right]\right)^2 \cos t \end{pmatrix}. \tag{28}$$

Similar to the previous example, one can see that conditions (C1)–(C10) hold. The function $h(\alpha)$ can be evaluated as

$$\begin{aligned} h(\alpha) &= \int_0^{2\pi} K_1^{\mathrm{T}}(s) g(s, z(s,\alpha), z(\gamma(s),\alpha), 0)\mathrm{d}s, \\ &= \int_0^{2\pi} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} z_1\left(2\pi\left[\dfrac{s+\pi}{2\pi}\right], \alpha\right)^2 \sin s + z_2(s,\alpha) \\ 2\,z_1(s,\alpha) + z_2\left(2\pi\left[\dfrac{s+\pi}{2\pi}\right], \alpha\right)^2 \cos s \end{pmatrix} \mathrm{d}s \\ &= \int_0^{2\pi} \begin{pmatrix} -2\,\alpha_2 \sin^2 s + \alpha_2 \cos^2 s \\ ((\alpha_1+2)^2 - \alpha_1)\sin^2 s + (2\,\alpha_1 + \alpha_2^2)\cos^2 s \end{pmatrix} \mathrm{d}s \\ &= \begin{pmatrix} -\pi\,\alpha_2 \\ \pi\,((\alpha_1+2)^2 + \alpha_1 + \alpha_2^2) \end{pmatrix}. \end{aligned}$$

Then, the zeros of the equation $h(\alpha) = 0$ are $\alpha^1 = (-1, 0)$ and $\alpha^2 = (-4, 0)$. By straightforward calculation one can see that the determinant

$$\det\left(\left.\frac{\partial h}{\partial \alpha}\right|_{\alpha=\alpha^i}\right) \neq 0, \quad i = 1, 2.$$

Hence, using Theorem 4.2, we conclude that for sufficiently small $|\mu|$ Eq. (28) has two 2π-periodic solutions and these solutions tend to $z(t, \alpha^1)$ and $z(t, \alpha^2)$, respectively, as $\mu \to 0$.

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