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Method of Lyapunov functions for differential equations with piecewise constant delay

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ABSTRACT

We address differential equations with piecewise constant argument of generalized type [5–8] and investigate their stability with the second Lyapunov method. Despite the fact that these equations include delay, stability conditions are merely given in terms of Lyapunov functions; that is, no functionals are used. Several examples, one of which considers the logistic equation, are discussed to illustrate the development of the theory. Some of the results were announced at the 14th International Congress on Computational and Applied Mathematics (ICCAM2009), Antalya, Turkey, in 2009.

1. Introduction

Cooke, Wiener and their co-authors [1–4] introduced differential equations with piecewise constant argument, which play an important role in applications [5–13,1,2,14–19,3,4,20]. By introducing arbitrary piecewise constant functions as arguments, the concept of differential equations with piecewise constant argument has been generalized in [5–7].

We should mention the following novelties of the present paper. The main and possibly a unique way of stability analysis for differential equations with piecewise constant argument has been the reduction to discrete equations [16,17,21–24,19,4]. Particularly, the problem of exploring stability with Lyapunov functions of continuous time has remained open. Moreover, the results of our paper have been developed through the concept of "total stability" [25,26], which is stability under persistent perturbations of the right-hand side of a differential equation, and they originate from a special theorem in [27]. Then, one can accept our approach as comparison of stability of equations with piecewise constant argument and ordinary differential equations. Finally, it deserves to emphasize that the direct method for differential equations with deviating argument necessarily utilizes functionals [28–30], but we use only Lyapunov functions to determine criteria of the stability, and this can be an advantage in applications.

2. The subject and method of analysis

Let $\mathbb{N}$ and $\mathbb{R}^+$ be the set of natural numbers and nonnegative real numbers, respectively, i.e., $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{R}^+ = [0, \infty)$. Denote the n-dimensional real space by $\mathbb{R}^n$, $n \in \mathbb{N}$, and the Euclidean norm in $\mathbb{R}^n$ by $\|.\|$.

Let us introduce a special notation:

$$\mathcal{K} = \{\psi : \psi \in C(\mathbb{R}^+, \mathbb{R}^+) \text{ is a strictly increasing function and } \psi(0) = 0\}.$$

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We fix a real-valued sequence $\theta_i, i \in \mathbb{N}$, such that $0 = \theta_0 < \theta_1 < \cdots < \theta_i < \cdots$ with $\theta_i \to \infty$ as $i \to \infty$, and shall consider the following equation

$$x'(t) = f(t, x(t), x(\beta(t))), \tag{2.1}$$

where $x \in B(h), B(h) = \{x \in \mathbb{R}^n : \|x\| < h\}, t \in \mathbb{R}^+$ and $\beta(t) = \theta_i$ if $t \in [\theta_i, \theta_{i+1}),\ i \in \mathbb{N}$, is an identification function.

We say that a continuous function $x(t)$ is a solution of Eq. (2.1) on $\mathbb{R}^+$ if it satisfies (2.1) on the intervals $[\theta_i, \theta_{i+1}), i \in \mathbb{N}$ and the derivative $x'(t)$ exists everywhere with the possible exception of the points $\theta_i, i \in \mathbb{N}$, where one-sided derivatives exist.

In the rest of our paper, we assume that the following conditions hold:

(C1) $f(t, u, v) \in C(\mathbb{R}^+ \times B(h) \times B(h))$ is an $n \times 1$ real valued function;
(C2) $f(t, 0, 0) = 0$ for all $t \geq 0$;
(C3) f satisfies a Lipschitz condition with constants ℓ_1, ℓ_2:

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq \ell_1 \|u_1 - u_2\| + \ell_2 \|v_1 - v_2\| \tag{2.2}$$

for all $t \in \mathbb{R}^+$ and $u_1, u_2, v_1, v_2 \in B(h)$;
(C4) there exists a constant $\theta > 0$ such that $\theta_{i+1} - \theta_i \leq \theta, i \in \mathbb{N}$;
(C5) $\theta[\ell_2 + \ell_1(1 + \ell_2\theta)e^{\ell_1\theta}] < 1$;
(C6) $\theta(\ell_1 + 2\ell_2)e^{\ell_1\theta} < 1$.

We give now some definitions and preliminary results which enable us to investigate stability of the trivial solution of (2.1).

Definition 2.1 (*[7]*)**.** The zero solution of (2.1) is said to be

(i) stable if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geq t_0$;
(ii) uniformly stable if δ is independent of t_0.

Definition 2.2 (*[7]*)**.** The zero solution of (2.1) is said to be uniformly asymptotically stable if it is uniformly stable and there is a $\delta_0 > 0$ such that for every $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $T = T(\varepsilon) > 0$ such that $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t > t_0 + T$ whenever $\|x_0\| < \delta_0$.

Next, we shall describe the method, which is in the base of our investigation. Let us rewrite the system (2.1) in the form

$$x'(t) = f(t, x(t), x(t)) + h(t, x(t), x(\beta(t))),$$

where $h(t, x(t), x(\beta(t))) = f(t, x(t), x(\beta(t))) - f(t, x(t), x(t))$. If the constant θ mentioned in (C4) is small, then we can consider $h(t, x(t), x(\beta(t)))$ as a small perturbation. That is to say, system (2.1) is a perturbed system for the following ordinary differential equation,

$$y'(t) = g(t, y(t)), \tag{2.3}$$

where $g(t, y(t)) = f(t, y(t), y(t))$.

Our intention is to consider systems (2.1) and (2.3) involved in the perturbation relation, and then extend these systems to the problem of stability based on the approach of Malkin [27].

Before applying the method, it is useful to consider a simple example. Let the following linear scalar equation with piecewise constant argument be given:

$$x'(t) = ax(t) + bx(\beta(t)) \tag{2.4}$$

where $\theta_i = ih,\ i \in \mathbb{N}$. The solution of (2.4) if $t \in [ih, (i+1)h)$ is given by [31,4]

$$x(t) = \left\{e^{a(t-ih)}\left(1 + \frac{b}{a}\right) - \frac{b}{a}\right\}\left\{e^{ah}\left(1 + \frac{b}{a}\right) - \frac{b}{a}\right\}^i x_0.$$

Then, one can easily see that the zero solution of (2.4) is asymptotically stable if and only if

$$-\frac{a(e^{ah} + 1)}{e^{ah} - 1} < b < -a. \tag{2.5}$$

On the other side, consider the following ordinary differential equation, which is associated with (2.4), and plays the role of (2.3),

$$y'(t) = ay(t) + by(t) = (a + b)y(t). \tag{2.6}$$

It is seen that the trivial solution of (2.6) is asymptotically stable if and only if

$$b < -a. \tag{2.7}$$

When the insertion of the greatest integer function is regarded as a "perturbation" of the linear equation (2.6), it is seen for (2.4) that the stability condition (2.5) is necessarily stricter than the one given by (2.7) for the corresponding "nonperturbed" equation (2.6). Moreover, it is seen that the condition (2.5) transforms to (2.7) as $h \to 0$.

If we discuss stability of Eq. (2.1) on the basis of (2.3), we expect that a comparison, similar to the relation of the conditions of (2.5) and (2.7), can be generalized. Furthermore, stability conditions for the ordinary differential equation (2.3) may not be enough for the issue system (2.1). By means of the following theorems, we demonstrate that stability of (2.1) depends on that of the corresponding ordinary differential equation (2.3).

3. Main results

The following lemma plays a crucial role in the proofs of stability theorems.

Lemma 3.1. *If the conditions* (C1)–(C6) *are fulfilled, then we have the estimation*

$$\|x(\beta(t))\| \le m\,\|x(t)\| \tag{3.8}$$

for all $t \in \mathbb{R}^+$, *where* $m = \{1 - \theta[\ell_2 + \ell_1(1+\ell_2\theta)e^{\ell_1\theta}]\}^{-1}$.

Proof. Fix $t \in \mathbb{R}^+$, then one can find $k \in \mathbb{N}$ such that $t \in I_k = [\theta_k, \theta_{k+1})$. For $t \in I_k$, we have $x(t) = x(\theta_k) + \int_{\theta_k}^t f(s, x(s), x(\theta_k))ds$, which yields to

$$\|x(t)\| \le (1+\ell_2\theta)\,\|x(\theta_k)\| + \ell_1 \int_{\theta_k}^t \|x(s)\|\,ds.$$

By the Gronwall–Bellman Lemma, we obtain $\|x(t)\| \le (1+\ell_2\theta)e^{\ell_1\theta}\|x(\theta_k)\|$. Moreover, $x(\theta_k) = x(t) - \int_{\theta_k}^t f(s, x(s), x(\theta_k))ds$, $t \in I_k$. Thus,

$$\begin{aligned}\|x(\theta_k)\| &\le \|x(t)\| + \int_{\theta_k}^t (\ell_1\,\|x(s)\| + \ell_2\,\|x(\theta_k)\|)\,ds \\ &\le \|x(t)\| + \int_{\theta_k}^t \ell_1\left[(1+\ell_2\theta)e^{\ell_1\theta} + \ell_2\right]\|x(\theta_k)\|\,ds \\ &\le \|x(t)\| + \theta\left[\ell_1(1+\ell_2\theta)e^{\ell_1\theta} + \ell_2\right]\|x(\theta_k)\|,\end{aligned}$$

proves that $\|x(\theta_k)\| \le m\|x(t)\|$ for $t \in I_k$. As the function $x(t)$ is continuous on $\mathbb{R}^+$, (3.8) holds for all $t \ge 0$. □

Next, we need the following theorem which provides conditions for the existence and uniqueness of solutions on $\mathbb{R}^+$. Since the proof of the assertion is almost identical to the one given in [5], we omit it here.

Theorem 3.1. *Suppose that conditions* (C1) *and* (C3)–(C6) *are fulfilled. Then for every* $(t_0, x_0) \in \mathbb{R}^+ \times B(h)$ *there exists a unique solution* $x(t) = x(t, t_0, x_0)$ *of* (2.1) *on* $\mathbb{R}^+$ *with* $x(t_0) = x_0$.

Let the derivative of V with respect to system (2.3) be defined by

$$V'_{(2.3)}(t,y) = \frac{\partial V(t,y)}{\partial t} + \frac{\partial V(t,y)}{\partial y} g(t,y)$$

for all t in $\mathbb{R}^+$ and $y \in B(h)$.

Theorem 3.2. *Suppose that* (C1)–(C6) *hold true and there exist a continuously differentiable function* $V : \mathbb{R}^+ \times B(h) \to \mathbb{R}^+$, $V(t,0) = 0$ *for all* $t \in \mathbb{R}^+$, *and a positive constant* α *such that*

(i) $u(\|y\|) \le V(t,y)$ *on* $\mathbb{R}^+ \times B(h)$, *where* $u \in \mathcal{K}$;
(ii) $V'_{(2.3)}(t,y) \le -\alpha\ell_2(1+m)\|y\|^2$ *for all* $(t,y) \in \mathbb{R}^+ \times B(h)$, *where m is the constant defined in Lemma* 3.1;
(iii) $\|\frac{\partial V(t,y)}{\partial y}\| \le \alpha\|y\|$.

Then the zero solution of (2.1) *is stable.*

Proof. Let $h_1 \in (0,h)$. Given $\varepsilon \in (0,h_1)$ and $t_0 \in \mathbb{R}^+$, choose $\delta > 0$ sufficiently small that $V(t_0, x(t_0)) < u(\varepsilon)$ if $\|x(t_0)\| < \delta$. If we evaluate the time derivative of V with respect to (2.1), we get for $t \ne \theta_i$

$$\begin{aligned}V'_{(2.1)}(t, x(t), x(\beta(t))) &= \frac{\partial V(t,x(t))}{\partial t} + \left\langle \frac{\partial V(t,x(t))}{\partial x}, f(t, x(t), x(\beta(t)))\right\rangle \\ &= V'_{(2.3)}(t, x(t)) + \left\langle \frac{\partial V(t,x(t))}{\partial x}, h(t, x(t), x(\beta(t)))\right\rangle.\end{aligned}$$

Hence, we have

$$\begin{aligned}V'_{(2.1)}(t, x(t), x(\beta(t))) &\le -\alpha\ell_2(1+m)\|x(t)\|^2 + \left\|\frac{\partial V(t,x(t))}{\partial x}\right\| \|h(t, x(t), x(\beta(t)))\| \\ &\le -\alpha\ell_2(1+m)\,\|x(t)\|^2 + \alpha\ell_2(1+m)\,\|x(t)\|^2 = 0,\end{aligned}$$

which implies that $V(t, x(t)) \le V(t_0, x(t_0)) < u(\varepsilon)$ for all $t \ge t_0$, proving that $\|x(t)\| < \varepsilon$. □

Theorem 3.3. *Suppose that* (C1)–(C6) *hold true and there exist a continuously differentiable function* $V : \mathbb{R}^+ \times B(h) \to \mathbb{R}^+$ *and a constant* $\alpha > 0$ *such that*

(i) $u(\|y\|) \le V(t, y) \le v(\|y\|)$ *on* $\mathbb{R}^+ \times B(h)$*, where* $u, v \in \mathcal{K}$*;*
(ii) $V'_{(2.3)}(t, y) \le -\alpha\ell_2(1+m)\|y\|^2$ *for all* $t \in \mathbb{R}^+$ *and* $y \in B(h)$*;*
(iii) $\|\frac{\partial V(t,y)}{\partial y}\| \le \alpha\|y\|$.

Then the zero solution of (2.1) *is uniformly stable.*

Proof. Let $h_1 \in (0, h)$. Fix $\varepsilon > 0$ in the range $0 < \varepsilon < h_1$ and choose $\delta > 0$ such that $v(\delta) < u(\varepsilon)$. If $t_0 \ge 0$ and $\|x(t_0)\| < \delta$, then as a consequence of the condition (i) we have $V(t_0, x(t_0)) < v(\delta) < u(\varepsilon)$. Using the same argument used in the proof of Theorem 3.2, one can obtain that $V(t, x(t)) \le V(t_0, x(t_0)) < u(\varepsilon)$ for all $t \ge t_0$. Hence $\|x(t)\| < \varepsilon$ for all $t \ge t_0$. □

Theorem 3.4. *Suppose that* (C1)–(C6) *hold true and there exist a continuously differentiable function* $V : \mathbb{R}^+ \times B(h) \to \mathbb{R}^+$*, constants* $\alpha > 0$ *and* $\tau > 1$ *such that*

(i) $u(\|y\|) \le V(t, y) \le v(\|y\|)$ *on* $\mathbb{R}^+ \times B(h)$*, where* $u, v \in \mathcal{K}$*;*
(ii) $V'_{(2.3)}(t, y) \le -\tau\alpha\ell_2(1+m)\|y\|^2$ *for all* $t \in \mathbb{R}^+$ *and* $y \in B(h)$*;*
(iii) $\|\frac{\partial V(t,y)}{\partial y}\| \le \alpha\|y\|$.

Then the zero solution of (2.1) *is uniformly asymptotically stable.*

Proof. In view of Theorem 3.2, the equilibrium $x = 0$ of (2.1) is uniformly stable. We need to show that it is asymptotically stable as well. For $t \ne \theta_i$,

$$\begin{aligned} V'_{(2.1)}(t, x(t), x(\beta(t))) &\le -\tau\alpha\ell_2(1+m)\,\|x(t)\|^2 + \alpha\ell_2(1+m)\,\|x(t)\|^2 \\ &= -(\tau-1)\alpha\ell_2(1+m)\,\|x(t)\|^2 . \end{aligned}$$

Denote $w(\|x\|) = (\tau-1)\alpha\ell_2(1+m)\|x\|^2$. Let $h_1 \in (0, h)$. Choose $\delta > 0$ such that $v(\delta) < u(h_1)$. We fix $\varepsilon > 0$ in the range $(0, h_1)$ and pick $\eta \in (0, \delta)$ such that $v(\eta) < u(\varepsilon)$. Let $t_0 \in \mathbb{R}^+$ and $\|x(t_0)\| < \delta$. We define $T = \frac{u(h_1)}{w(\eta)}$. We shall show that $\|x(\bar{t})\| < \eta$ for some $\bar{t} \in [t_0, t_0 + T]$. If this were not true, then we would have $\|x(t)\| \ge \eta$ for all $t \in [t_0, t_0 + T]$.

For $t \in [t_0, t_0 + T]$, $t \ne \theta_i$, we have

$$V'_{(2.1)}(t, x(t), x(\beta(t))) \le -w(\|x(t)\|) \le -w(\eta).$$

Since the function $V(t, x(t))$ and the solution $x(t)$ are continuous, we obtain that

$$V(t_0 + T, x(t_0 + T)) \le V(t_0, x(t_0)) - w(\eta)T < v(\delta) - w(\eta)\frac{u(h_1)}{w(\eta)} < 0,$$

which is a contradiction. Hence, $\bar{t}$ exists. Now for $t \ge \bar{t}$ we have

$$V(t, x(t)) \le V(\bar{t}, x(\bar{t})) < v(\eta) < u(\varepsilon).$$

In the end, it follows from the hypothesis (i) that $\|x(t)\| < \varepsilon$ for all $t \ge \bar{t}$ and in turn for all $t \ge t_0 + T$. □

Remark 3.1. Theorems 3.2–3.4 provide criteria for stability, which are entirely constructed on the basis of Lyapunov functions. As for the functionals, they appear only in the proofs of theorems. Although the equations include deviating arguments, and functionals are ordinarily used in the stability criteria [17,28], we see that the conditions of our paper, which guarantee stability, are definitely formulated without functionals.

Next, we want to compare our present results, which are obtained by the method of Lyapunov functions with the ones proved in [9] by employing the Lyapunov–Razumikhin technique. To this end, let us discuss the following linear equation with piecewise constant argument of generalized type taken from [9],

$$x'(t) = -a_0(t)x(t) - a_1(t)x(\beta(t)), \tag{3.9}$$

where a_0 and a_1 are bounded continuous functions on $\mathbb{R}^+$. We suppose that the sequence θ_i, $i \in \mathbb{N}$, with $\ell_1 = \sup_{t\in\mathbb{R}^+} |a_0(t)|$, $\ell_2 = \sup_{t\in\mathbb{R}^+} |a_1(t)|$, satisfies the conditions (C4)–(C6). One can check easily that conditions (C1)–(C3) are also valid. Under the assumption

$$0 \le a_0(t) + a_1(t) \le 2a_0(t), \quad t \ge 0, \tag{3.10}$$

it was obtained via the Lyapunov–Razumikhin method in [9] that the trivial solution of (3.9) is uniformly stable. Let us consider this equation using the results obtained in the present paper. We set

$$(1+m)\sup_{t\in\mathbb{R}^+} |a_1(t)| \le a_0(t) + a_1(t), \quad t \ge 0, \tag{3.11}$$

In order to apply our results, we need the following equation besides (3.9);

$$y'(t) = -(a_0(t) + a_1(t))y(t). \tag{3.12}$$

Let us define a Lyapunov function $V(y) = \frac{\alpha}{2}y^2,\ y \in B(h), \alpha > 0$. It follows from (3.11) that the derivative of $V(y)$ with respect to Eq. (3.12) is given by

$$\begin{aligned} V'_{(3.12)}(y(t)) &= -\alpha(a_0(t) + a_1(t))y^2(t) \\ &\le -\alpha\ell_2(1+m)y^2(t). \end{aligned}$$

Then, by Theorem 3.3, the zero solution of (3.9) is uniformly stable.

In addition, taking $(a_0(t) + a_1(t)) \ge \tau\ell_2(1+m), \tau > 1$, one can show that the trivial solution of (3.9) is uniformly asymptotically stable by Theorem 3.4.

We can see that theorems obtained by Lyapunov–Razumikhin method provide larger class of equations with respect to (3.9). However, from the perspective of the constructive analysis, the present method may be more preferable, since, for example, from the proof of Theorem 3.3, we have $V'_{(3.9)}(t, x(t), x(\beta(t))) \le 0$, which implies $|x(t)| \le |x(t_0)|,\ t \ge t_0$, for our specific Lyapunov function. Thus, by using the present results, it is possible to evaluate the number δ needed for (uniform) stability in Definition 2.1 as $\delta = \varepsilon$.

Besides Theorems 3.2–3.4, the following assertions may be useful for analysis of the stability of differential equations with piecewise constant argument. These theorems are important and have their own distinctive values with the newly required properties of the Lyapunov function and can be proved similarly.

Theorem 3.5. *Suppose that* (C1)–(C6) *hold true and there exist a continuously differentiable function* $V : \mathbb{R}^+ \times B(h) \to \mathbb{R}^+$ *and a positive constant* M *such that*

(i) $u(\|y\|) \le V(t, y)$ *on* $\mathbb{R}^+ \times B(h)$, *where* $u \in \mathcal{K}$;
(ii) $V'_{(2.3)}(t, y) \le -M\ell_2(1+m)\|y\|$ *for all* $t \in \mathbb{R}^+$ *and* $y \in B(h)$;
(iii) $\|\frac{\partial V(t,y)}{\partial y}\| \le M$.

Then the zero solution of (2.1) *is stable.*

Theorem 3.6. *Suppose that* (C1)–(C6) *hold true and there exist a continuously differentiable function* $V : \mathbb{R}^+ \times B(h) \to \mathbb{R}^+$ *and a positive constant* M *such that*

(i) $u(\|y\|) \le V(t, y) \le v(\|y\|)$ *on* $\mathbb{R}^+ \times B(h)$, *where* $u, v \in \mathcal{K}$;
(ii) $V'_{(2.3)}(t, y) \le -M\ell_2(1+m)\|y\|$ *for all* $t \in \mathbb{R}^+$ *and* $y \in B(h)$;
(iii) $\|\frac{\partial V(t,y)}{\partial y}\| \le M$.

Then the zero solution of (2.1) *is uniformly stable.*

Theorem 3.7. *Suppose that* (C1)–(C6) *hold true and there exist a continuously differentiable function* $V : \mathbb{R}^+ \times B(h) \to \mathbb{R}^+$, *constants* $M > 0$ *and* $\tau > 1$ *such that*

(i) $u(\|y\|) \le V(t, y) \le v(\|y\|)$ *on* $\mathbb{R}^+ \times B(h)$, *where* $u, v \in \mathcal{K}$;
(ii) $V'_{(2.3)}(t, y) \le -\tau M\ell_2(1+m)\|y\|$ *for all* $t \in \mathbb{R}^+$ *and* $y \in B(h)$;
(iii) $\|\frac{\partial V(t,y)}{\partial y}\| \le M$.

Then the zero solution of (2.1) *is uniformly asymptotically stable.*

4. Applications to the logistic equation

In this section, we are interested in the stability of the positive equilibrium $N^* = \frac{1}{a+b}$ of the following logistically growing population subjected to a density-dependent harvesting;

$$N'(t) = rN(t)[1 - aN(t) - bN(\beta(t))], \quad t > 0, \tag{4.13}$$

where $N(t)$ denotes the biomass of a single species, and r, a, b are positive parameters. There exists an extensive literature dealing with sufficient conditions for global asymptotic stability of equilibria for the logistic equation with piecewise constant argument (see [16,22,18,24,19] and the references therein). For example, Gopalsamy and Liu [16] showed that N^* is globally asymptotically stable if $a/b \ge 1$. In these papers, the initial moments are taken as integers owing to the method of investigation: reduction to difference equations. Since our approach makes it possible to take not only integers, but also all values from $\mathbb{R}^+$ as initial moments, we can consider the stability in uniform sense.

Let us also discuss the biological sense of the insertion of the piecewise constant delay [16,17,22,18,24,19]. The delay means that the rate of the population depends both on the present size and the memorized values of the population. To illustrate the dependence, one may think populations, which meet at the beginning of a season, e.g., in springtime, with their instinctive evaluations of the population state and environment and implicitly decide which living conditions to prefer and where to go [13] in line with group hierarchy, communications and dynamics and then adapt to those conditions.

By means of the transformation $x = b(N - N^*)$, Eq. (4.13) can be simplified as

$$x'(t) = -r\left[x(t) + \frac{1}{1+\gamma}\right][\gamma x(t) + x(\beta(t))], \tag{4.14}$$

where $\gamma = a/b$. Let us specify for (4.14) general conditions of Theorems 3.2–3.4. We observe that $f(x, y) := -r[x + \frac{1}{1+\gamma}][\gamma x + y]$ is a continuous function and has continuous partial derivatives for $x, y \in B(h)$. It can be found easily that

$$\ell_1 = r\left(2\gamma h + h + \frac{\gamma}{1+\gamma}\right), \qquad \ell_2 = r\left(h + \frac{1}{1+\gamma}\right).$$

One can see that (C1)–(C3) hold if r is sufficiently small. Moreover, we assume that (C4)–(C6) are satisfied.

Consider the following equation associated with (4.14);

$$y'(t) = -r(1+\gamma)y(t)\left[y(t) + \frac{1}{1+\gamma}\right]. \tag{4.15}$$

Suppose h is smaller than $\frac{1}{1+\gamma}$ and consider a Lyapunov function defined by $V(y) = \frac{\alpha}{2}y^2,\ y \in B(h), \alpha > 0$. Then,

$$\begin{aligned} V'_{(4.15)}(y(t)) &= -\alpha r(1+\gamma)y^2(t)\left[y(t) + \frac{1}{1+\gamma}\right] \\ &\le -\alpha r[1 - h(1+\gamma)]y^2(t). \end{aligned}$$

For sufficiently small h, we assume that

$$\varphi(h, m) \le \gamma \tag{4.16}$$

where

$$\varphi(h, m) = \frac{1 - h(3+m) - \sqrt{(h(1+m))^2 - 6h(1+m) + 1}}{2h}.$$

It follows from (4.16) that

$$\left(h + \frac{1}{1+\gamma}\right)(1+m) \le 1 - h(1+\gamma),$$

which implies in turn

$$V'_{(4.15)}(y(t)) \le -\alpha\ell_2(1+m)y^2(t).$$

By Theorem 3.3, the zero solution of (4.14) is uniformly stable.

Next, we consider uniform asymptotic stability. Assuming for $\tau > 1$;

$$\psi(h, m, \tau) \le \gamma \tag{4.17}$$

where

$$\psi(h, m, \tau) = \frac{1 - h\tau(3+m) - \sqrt{(h\tau(1+m))^2 - 6h\tau(1+m) + 1}}{2h},$$

we obtain that

$$\tau\left(h + \frac{1}{1+\gamma}\right)(1+m) \le 1 - h(1+\gamma).$$

One can show easily that $\psi(h, m, \tau) \ge 1$ for small h. Then for $V(y) = \frac{\alpha}{2}y^2$, we have

$$V'_{(4.15)}(y(t)) \le -\tau\alpha\ell_2(1+m)y^2(t).$$

That is, condition (iii) of Theorem 3.4 is satisfied. Thus, the trivial solution $x = 0$ of (4.14) is uniformly asymptotically stable.

In the light of the above reduction, we see that the obtained conditions are valid for the stability of the equilibrium $N = N^*$ of (4.13).

Finally, we see that the condition (4.17) is stronger than the one $\gamma \ge 1$ taken from [16]. However, our results are for all values from $\mathbb{R}^+$ as initial moments, whereas [16] considers only integers. Moreover, the piecewise constant argument is of generalized type.

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