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Stability analysis of recurrent neural networks with piecewise constant argument of generalized type

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ABSTRACT

In this paper, we apply the method of Lyapunov functions for differential equations with piecewise constant argument of generalized type to a model of recurrent neural networks (RNNs). The model involves both advanced and delayed arguments. Sufficient conditions are obtained for global exponential stability of the equilibrium point. Examples with numerical simulations are presented to illustrate the results.

1. Introduction

Recurrent neural networks (RNNs), especially Hopfield neural networks, cellular neural networks (CNNs) and delayed cellular neural networks (DCNNs), have been deeply investigated in recent years due to their applicability in solving image processing, pattern recognition, associative memory, and optimization problems (Chua, 1998; Chua & Roska, 1990, 1992; Chua & Yang, 1988a, 1988b; Civalleri, Gilli, & Pandolfi, 1993; Forti & Tesi, 1995; Hopfield, 1984; Michel, Farrell, & Porod, 1989).

It is well known that applications of RNNs depend crucially on the dynamical behavior of the networks. In these applications, stability and convergence of neural networks are prerequisites. However, in the design of neural networks one is interested not only in the uniform asymptotic stability (Akhmet & Aruğaslan, 2009; Akhmet, Aruğaslan, & Yılmaz, 2010) but also in the global exponential stability, which guarantees a neural network to converge fast enough in order to achieve fast response. In addition, in the analysis of dynamical neural networks for parallel computation and optimization, to increase the rate of convergence to the equilibrium point of the networks and reduce the neural computing time, it is necessary to ensure a desired exponential convergence rate of the networks' trajectories, starting from arbitrary initial states to the equilibrium point which corresponds to the optimal solution. Thus, from the mathematical and engineering points of view, it is required that the neural networks have a unique equilibrium point which is globally exponentially stable. Therefore, the problem of stability analysis of RNNs has received great attention and many results on this topic have been reported in the literature; see, e.g., Arik (2002), Cao (2001), Chen (2001), Chen and Amari (2001), Huang, Cao, and Wang (2002), Liao, Wu, and Yu (2002), Mohamad and Gopalsamy (2003), Park (2006), Song (2008), Xu, Chu, and Lu (2006); Xu, Lamb, Ho, and Zoua (2005), Zeng and Wang (2006a, 2006b, 2006c), Zeng, Wang, and Liao (2003), Zhang (2003), Zhang (2005), Zhang and Wang (2007), Zhang, Wang, and Liu (2008); Zhang, Wei, and Xu (2004, 2007) and Zhou and Cao (2002) and the references therein.

Lyapunov functions and functionals are among the most popular tools in studying the problem of the stability for RNNs (see Arik, 2002; Belair, Campbell, & Driessche, 1996; Cao, 2001, 1999, 2000; Chen, 2001; Chen & Amari, 2001; Driessche & Zou, 1998; Huang et al., 2002; Liao et al., 2002; Mohamad & Gopalsamy, 2003; Park, 2006; Xu et al., 2006, 2005; Zeng & Wang, 2006a, 2006c; Zhang, 2003, 2005; Zhang et al., 2004, 2007; Zhou & Cao, 2002). However, it is difficult to construct Lyapunov functions or functionals that satisfy the strong conditions required in classical stability theory. In this paper, we investigate some new stability conditions for the RNN model based on the second Lyapunov method. Although this model includes both advanced and delayed arguments, it deserves to be mentioned that new stability conditions are given in terms of inequalities, and it is

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known that for equations with deviating argument this method necessarily utilizes functionals (Cooke & Wiener, 1984; Hale, 1997; Krasovskii, 1963; Wiener, 1993).

The theory of differential equations with piecewise constant argument (EPCAs) was initiated in Cooke and Wiener (1984) and Shah and Wiener (1983). These equations have been under intensive investigation for the last twenty years. They represent a hybrid of continuous and discrete dynamical systems and combine the properties of both the differential and difference equations. In fact, the theory of EPCAs is based on the reduction of the EPCAs to discrete equations, and it has been the main and possibly a unique method of stability analysis for these equations (Cooke & Wiener, 1984; Wiener, 1993). Hence, one cannot investigate the problem of stability completely, as only elements of a countable set are allowed to be discussed as initial moments by this method. By introducing arbitrary piecewise constant functions as arguments, the concept of differential equations with piecewise constant argument has been generalized in Akhmet (2006, 2007) and Akhmet (2008a, 2008b). All of these equations are reduced to equivalent integral equations such that one can investigate many problems which have not been solved properly by using discrete equations, i.e., the existence and uniqueness of solutions, and stability. Since we do not need additional assumptions on the reduced discrete equations, the new method requires more easily verifiable conditions, similar to those for ordinary differential equations.

To the best of our knowledge, the equations with piecewise constant arguments were not considered as models of RNNs, except possibly in Akhmet et al. (2010) and Akhmet and Yılmaz (in press). In Akhmet (2006), Akhmet (2007), Akhmet (2008a), Akhmet (2008b), Akhmet and Aruğaslan (2009), Akhmet and Büyükadalı(2010), Akhmet et al. (2010), Akhmet, Aruğaslan, and Yılmaz (in press) and Akhmet and Yılmaz (in press) we discuss stability problems. Unlike in these papers, in Akhmet et al. (in press) the stability was analyzed by the second Lyapunov method. Nevertheless, this is the first time that the second method has been applied to the equations, whose arguments in the present paper are not only delayed but also advanced. Moreover, one should emphasize that there is an opportunity of application of the Lyapunov function technique to estimate domains of attraction, which has a particular interest in evaluating the performance of RNNs (Xu et al., 2006; Yang, Liao, Li, & Evans, 2006).

The crucial novelty of the paper is that the system is of mixed type; in other words, the argument can be advanced during the process. In the literature, biological reasons for the argument to be delayed have been discussed well (Hoppensteadt & Peskin, 1992; Murray, 2002). Due to the finite switching speed of amplifiers and transmission of signals in electronic networks or finite speed for signal propagation in neural networks, time delays exist (Chua & Roska, 1992, 1990; Chua & Yang, 1988b; Civalleri et al., 1993). In the present paper, we proceed from the fact that delayed as well as advanced arguments play a significant role in electromagnetic fields; see, for example, Driver (1979), where the *symmetry of the physics laws* was emphasized with respect to *time reversal*. Consequently, one can suppose that analysis of neural networks, which is based on electrodynamics, may result in the comprehension of the deviation, especially the advanced one, in the models more clearly. Therefore, in the future analysis of RNNs, the systems introduced in the present paper can be useful. Furthermore, different types of deviation of the argument may depend on the emergence of traveling waves in CNNs (Weng & Wu, 2003). Understanding the structure of such traveling waves is important due to their potential applications, including image processing (see, for example, Chua, 1998; Chua & Roska, 1992, 1990; Chua & Yang, 1988a, 1988b; Hsu, Lin, & Shen, 1999). On the other hand, the importance of anticipation for biology, which can be modeled with advanced arguments, is mentioned by some authors. For instance, in Buck and Buck (1968), it is supposed that synchronization of biological oscillators may request anticipation of counterparts' behavior.

2. Model formulation and preliminaries

Let $\mathbb{N}$ and $\mathbb{R}^+$ be the sets of natural and nonnegative real numbers, respectively; i.e., $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{R}^+ = [0, \infty)$. Denote the n-dimensional real space by $\mathbb{R}^n$, $n \in \mathbb{N}$, and the norm of a vector $x \in \mathbb{R}^n$ by $\|x\| = \sum_{i=1}^n |x_i|$. We fix two real-valued sequences θ_i, ζ_i, $i \in \mathbb{N}$, such that $\theta_i < \theta_{i+1}, \theta_i \le \zeta_i \le \theta_{i+1}$ for all $i \in \mathbb{N}, \theta_i \to \infty$ as $i \to \infty$, and shall consider the following RNN model described by differential equations with piecewise constant argument of generalized type:

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(t)) + \sum_{j=1}^n c_{ij} g_j(x_j(\gamma(t))) + I_i,$$
$$a_i > 0,\ i = 1, 2, \ldots, n, \tag{2.1}$$

where $\gamma(t) = \zeta_k$, if $t \in [\theta_k, \theta_{k+1})$, $k \in \mathbb{N}$, $t \in \mathbb{R}^+$, n corresponds to the number of units in a neural network, $x_i(t)$ stands for the state vector of the ith unit at time t, $f_j(x_j(t))$ and $g_j(x_j(\gamma(t)))$ denote, respectively, the measures of activation to its incoming potentials of the unit j at time t and $\gamma(t)$, b_{ij}, c_{ij}, I_i are real constants, b_{ij} means the strength of the jth unit on the ith unit at time t, c_{ij} infers the strength of the jth unit on the ith unit at time $\gamma(t)$, I_i signifies the external bias on the ith unit and a_i represents the rate with which the ith unit will reset its potential to the resting state in isolation when it is disconnected from the network and external inputs.

Let us clarify why the system (2.1) is of mixed type (Hale, 1997), that is, the argument can change its deviation character during the motion. The argument is deviated if it is advanced or delayed. Fix $k \in \mathbb{N}$, and consider the system on the interval $[\theta_k, \theta_{k+1})$. Then, the identification function $\gamma(t)$ is equal to ζ_k. If the argument t satisfies $\theta_k \le t < \zeta_k$, then $\gamma(t) > t$ and (2.1) is an equation with advanced argument. Similarly, if $\zeta_k < t < \theta_{k+1}$, then $\gamma(t) < t$ and (2.1) is an equation with delayed argument. Consequently, Eq. (2.1) changes the type of deviation of the argument during the process. In other words, the system is of mixed type.

The following assumptions will be needed throughout the paper:

(A1) the activation functions $f_j, g_j \in C(\mathbb{R}^n)$ satisfy $f_j(0) = 0, g_j(0) = 0$ for each $j = 1, 2, \ldots, n$;
(A2) there exist Lipschitz constants $L_i^1, L_i^2 > 0$ such that

$$|f_i(u) - f_i(v)| \le L_i^1 |u - v|,$$
$$|g_i(u) - g_i(v)| \le L_i^2 |u - v|$$

for all $u, v \in \mathbb{R}^n$, $i = 1, 2, \ldots, n$;
(A3) there exists a positive number θ such that $\theta_{i+1} - \theta_i \le \theta, i \in \mathbb{N}$;
(A4) $\theta\,[m_1 + 2m_2]\, e^{m_1\theta} < 1$;
(A5) $\theta \left[m_2 + m_1(1 + m_2\theta)e^{m_1\theta}\right] < 1$,

where

$$m_1 = \max_{1\le i\le n}\left(a_i + L_i^1 \sum_{j=1}^n |b_{ji}|\right), \qquad m_2 = \max_{1\le i\le n}\left(L_i^2 \sum_{j=1}^n |c_{ji}|\right).$$

In our paper we assume that the solutions of Eq. (2.1) are continuous functions. But the deviating argument $\gamma(t)$ is discontinuous. Thus, in general, the right-hand side of (2.1) has discontinuities at moments $\theta_i, i \in \mathbb{N}$. As a result, we consider the solutions of the equations as functions, which are continuous and continuously differentiable within intervals $[\theta_i, \theta_{i+1}), i \in \mathbb{N}$. In other words, by a solution $x(t) = (x_1(t), \ldots, x_n(t))^T$ of (2.1) we mean a continuous function on $\mathbb{R}^+$ such that the derivative $x'(t)$ exists at each point $t \in \mathbb{R}^+$, with the possible exception of the points $\theta_i, i \in \mathbb{N}$, where a one-sided derivative exists, and the differential equation (2.1) is satisfied by $x(t)$ on each interval (θ_i, θ_{i+1}) as well.

In the following theorem, we obtain sufficient conditions for the existence of a unique equilibrium, $x^* = (x_1^*, \ldots, x_n^*)^T$, of (2.1).

Theorem 2.1. *Suppose that* (A1) *and* (A2) *hold. If the neural parameters a_i, b_{ij}, c_{ij} satisfy*

$$a_i > L_i^1 \sum_{j=1}^{n} |b_{ji}| + L_i^2 \sum_{j=1}^{n} |c_{ji}|, \quad i = 1, \ldots, n,$$

then (2.1) *has a unique equilibrium $x^* = (x_1^*, \ldots, x_n^*)^T$.*

The proof of the theorem is almost identical to that of Theorem 2.1 in Mohamad and Gopalsamy (2003), and thus we omit it here.

The next theorem provides conditions for the existence and uniqueness of solutions on $t \geq t_0$. The proof of the assertion is similar to that of Theorem 1.1 in Akhmet (2007) and Theorem 2.2 in Akhmet et al. (2010). But, for convenience of the reader, we give the full proof of the assertion.

Theorem 2.2. *Assume that conditions* (A1)–(A4) *are fulfilled. Then, for every $(t_0, x^0) \in \mathbb{R}^+ \times \mathbb{R}^n$, there exists a unique solution $x(t) = x(t, t_0, x^0) = (x_1(t), \ldots, x_n(t))^T$, $t \geq t_0$, of* (2.1)*, such that $x(t_0) = x^0$.*

Proof (*Existence*)**.** Fix $k \in \mathbb{N}$. We assume without loss of generality that $\theta_k \leq \zeta_k < t_0 \leq \theta_{k+1}$. To begin with, we shall prove that, for every $(t_0, x_0) \in [\theta_k, \theta_{k+1}] \times \mathbb{R}^n$, there exists a unique solution $x(t) = x(t, t_0, x^0) = (x_1(t), \ldots, x_n(t))^T$, of (2.1) such that $x(t_0) = x^0 = (x_1^0, \ldots, x_n^0)^T$.

Let us denote for simplicity $z(t) = x(t, t_0, x^0)$, $z(t) = (z_1, \ldots, z_n)^T$, and consider the equivalent integral equation

$$z_i(t) = x_i^0 + \int_{t_0}^{t} \left[-a_i z_i(s) + \sum_{j=1}^{n} b_{ij} f_j(z_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(z_j(\zeta_k)) + I_i \right] \mathrm{d}s.$$

Define a norm $\|z(t)\|_0 = \max_{[\zeta_k, t_0]} \|z(t)\|$ and construct the following sequences $z_i^m(t),\ z_i^0(t) \equiv x_i^0,\ i = 1, \ldots, n,\ m \geq 0$ such that

$$z_i^{m+1}(t) = x_i^0 + \int_{t_0}^{t} \left[-a_i z_i^m(s) + \sum_{j=1}^{n} b_{ij} f_j(z_j^m(s)) + \sum_{j=1}^{n} c_{ij} g_j(z_j^m(\zeta_k)) + I_i \right] \mathrm{d}s.$$

One can find that

$$\|z^{m+1}(t) - z^m(t)\|_0 \leq [\theta(m_1 + m_2)]^m \, \tau,$$

where

$$\tau = \theta \left[(m_1 + m_2) \, \|x^0\| + \sum_{i=1}^{n} I_i \right].$$

Thus, there exists a unique solution $z(t) = x(t, t_0, x^0)$ of the integral equation on $[\zeta_k, t_0]$. Then, conditions (A1) and (A2) imply that $x(t)$ can be continued to θ_{k+1}, since it is a solution of the ordinary differential equations

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(\zeta_k)) + I_i, \quad a_i > 0,\ i = 1, 2, \ldots, n$$

on $[\theta_k, \theta_{k+1})$. Next, again, using same argument, we can continue $x(t)$ from $t = \theta_{k+1}$ to $t = \zeta_{k+1}$, and then to θ_{k+2}. Hence, the mathematical induction completes the proof.

Uniqueness: Denote by $x^1(t) = x(t, t_0, x^1)$, $x^2(t) = x(t, t_0, x^2)$ the solutions of (2.1), where $\theta_k \leq t_0 \leq \theta_{k+1}$. It is sufficient to check that, for every $t \in [\theta_k, \theta_{k+1}]$, $x^2 = (x_1^2, \ldots, x_n^2)^T$, $x^1 = (x_1^1, \ldots, x_n^1)^T \in \mathbb{R}^m$, $x^2 \neq x^1$ implies that $x^1(t) \neq x^2(t)$. Then, we have that

$$\begin{aligned} \|x^1(t) - x^2(t)\| &\leq \|x^1 - x^2\| + \sum_{i=1}^{n} \left\{ \int_{t_0}^{t} \left[a_i |x_i^2(s) - x_i^1(s)| + \sum_{j=1}^{n} L_i^1 |b_{ji}| \, |x_i^2(s) - x_i^1(s)| + \sum_{j=1}^{n} L_i^2 |c_{ji}| \, |x_i^2(\zeta_k) - x_i^1(\zeta_k)| \right] \mathrm{d}s \right\} \\ &\leq \left(\|x^1 - x^2\| + \theta m_2 \|x^1(\zeta_k) - x^2(\zeta_k)\| \right) + \int_{t_0}^{t} m_1 \|x^1(s) - x^2(s)\| \mathrm{d}s. \end{aligned}$$

The Gronwall–Bellman Lemma yields that

$$\|x^1(t) - x^2(t)\| \leq \left(\|x^1 - x^2\| + \theta m_2 \|x^1(\zeta_k) - x^2(\zeta_k)\| \right) \mathrm{e}^{m_1 \theta}.$$

In particular,

$$\|x^1(\zeta_k) - x^2(\zeta_k)\| \leq \left(\|x^1 - x^2\| + \theta m_2 \|x^1(\zeta_k) - x^2(\zeta_k)\| \right) \mathrm{e}^{m_1 \theta}.$$

Thus,

$$\|x^1(t) - x^2(t)\| \leq \left(\frac{\mathrm{e}^{m_1 \theta}}{1 - m_2 \theta \mathrm{e}^{m_1 \theta}} \right) \|x^1 - x^2\|. \tag{2.2}$$

On the other hand, assume on the contrary that there exists $t \in [\theta_k, \theta_{k+1}]$ such that $x^1(t) = x^2(t)$. Hence,

$$\begin{aligned} \|x^1 - x^2\| &= \sum_{i=1}^{n} \left| \int_{t_0}^{t} \left[-a_i \left(x_i^2(s) - x_i^1(s) \right) + \sum_{j=1}^{n} b_{ij} \left[f_j(x_j^2(s)) - f_j(x_j^1(s)) \right] + \sum_{j=1}^{n} c_{ij} \left[g_j(x_j^2(\zeta_k)) - g_j(x_j^1(\zeta_k)) \right] \right] \mathrm{d}s \right| \\ &\leq \sum_{i=1}^{n} \left\{ \int_{t_0}^{t} \left[a_i |x_i^2(s) - x_i^1(s)| + \sum_{j=1}^{n} L_i^1 |b_{ji}| \, |x_i^2(s) - x_i^1(s)| + \sum_{j=1}^{n} L_i^2 |c_{ji}| \, |x_i^2(\zeta_k) - x_i^1(\zeta_k)| \right] \mathrm{d}s \right\} \\ &\leq \theta m_2 \|x^1(\zeta_k) - x^2(\zeta_k)\| + \int_{t_0}^{t} m_1 \|x^1(s) - x^2(s)\| \mathrm{d}s. \end{aligned} \tag{2.3}$$

Consequently, substituting (2.2) in (2.3), we obtain

$$\|x^1 - x^2\| \leq \theta(m_1 + 2m_2) \mathrm{e}^{m_1 \theta} \|x^1 - x^2\|. \tag{2.4}$$

Thus, one can see that (A4) contradicts (2.4). The uniqueness is proved for $t \in [\theta_k, \theta_{k+1}]$. The extension of the unique solution on $\mathbb{R}^+$ is obvious. Hence, the theorem is proved. □

Definitions of Lyapunov stability for the solutions of the discussed system can be given in the same way as for ordinary differential equations. Let us give only one of them.

Definition 2.1 (*Akhmet, 2008a*)**.** The equilibrium $x = x^*$ of (2.1) is said to be globally exponentially stable if there exist positive constants α_1 and α_2 such that the estimation $\|x(t) - x^*\| < \alpha_1 \|x(t_0) - x^*\| e^{-\alpha_2(t-t_0)}$ is valid for all $t \geq t_0$.

System (2.1) can be simplified as follows. Substituting $y(t) = x(t) - x^*$ into (2.1) leads to

$$y_i'(t) = -a_i y_i(t) + \sum_{j=1}^{n} b_{ij}\varphi_j(y_j(t)) + \sum_{j=1}^{n} c_{ij}\psi_j(y_j(\gamma(t))), \tag{2.5}$$

where $\varphi_j(y_j(t)) = f_j(y_j(t) + x_j^*) - f_j(x_j^*)$ and $\psi_j(y_j(t)) = g_j(y_j(t) + x_j^*) - g_j(x_j^*)$ with $\varphi_j(0) = \psi_j(0) = 0$. From assumption (A2), $\varphi_j(\cdot)$ and $\psi_j(\cdot)$ are also Lipschitzian with L_j^1, L_j^2, respectively.

It is clear that the stability of the zero solution of (2.5) is equivalent to that of the equilibrium x^* of (2.1). Therefore, we restrict our discussion to the stability of the zero solution of (2.5).

First of all, we give the following lemma, which is one of the most important auxiliary results of the present paper.

Lemma 2.1. *Let $y(t) = (y_1(t), \ldots, y_n(t))^T$ be a solution of (2.5) and let (A1)–(A5) be satisfied. Then, the following inequality*

$$\|y(\gamma(t))\| \leq \lambda \|y(t)\| \tag{2.6}$$

holds for all $t \in \mathbb{R}^+$, where $\lambda = \{1 - \theta[m_2 + m_1(1 + m_2\theta)e^{m_1\theta}]\}^{-1}$.

Proof. Fix $k \in \mathbb{N}$. Then, for $t \in [\theta_k, \theta_{k+1})$,

$$y_i(t) = y_i(\zeta_k) + \int_{\zeta_k}^{t} \left[-a_i y_i(s) + \sum_{j=1}^{n} b_{ij}\varphi_j(y_j(s)) + \sum_{j=1}^{n} c_{ij}\psi_j(y_j(\zeta_k)) \right] ds,$$

where $\gamma(t) = \zeta_k$, if $t \in [\theta_k, \theta_{k+1}),\ t \in \mathbb{R}^+$. Taking absolute value of both sides for each $i = 1, 2, \ldots, n$ and adding all equalities, we obtain that

$$\begin{aligned} \|y(t)\| &\leq \|y(\zeta_k)\| + \sum_{i=1}^{n} \left\{ \int_{\zeta_k}^{t} \left[a_i|y_i(s)| + \sum_{j=1}^{n} L_j^1|b_{ij}|\,|y_j(s)| + \sum_{j=1}^{n} L_j^2|c_{ij}|\,|y_j(\zeta_k)| \right] ds \right\} \\ &= \|y(\zeta_k)\| + \int_{\zeta_k}^{t} \left[\sum_{i=1}^{n} \left(a_i + L_i^1 \sum_{j=1}^{n} |b_{ji}| \right) |y_i(s)| + \sum_{i=1}^{n}\sum_{j=1}^{n} L_i^2|c_{ji}|\,|y_i(\zeta_k)| \right] ds \\ &\leq (1 + m_2\theta)\|y(\zeta_k)\| + \int_{\zeta_k}^{t} m_1\|y(s)\| ds. \end{aligned}$$

The Gronwall–Bellman Lemma yields

$$\|y(t)\| \leq (1 + m_2\theta)e^{m_1\theta}\|y(\zeta_k)\|. \tag{2.7}$$

Furthermore, for $t \in [\theta_k, \theta_{k+1})$, we have

$$\begin{aligned} \|y(\zeta_k)\| &\leq \|y(t)\| + \int_{\zeta_k}^{t} \left[\sum_{i=1}^{n} \left(a_i + L_i^1 \sum_{j=1}^{n} |b_{ji}| \right) |y_i(s)| + \sum_{i=1}^{n}\sum_{j=1}^{n} L_i^2|c_{ji}|\,|y_i(\zeta_k)| \right] ds \\ &\leq \|y(t)\| + m_2\theta\|y(\zeta_k)\| + \int_{\zeta_k}^{t} m_1\|y(s)\| ds. \end{aligned}$$

The last inequality together with (2.7) implies that

$$\|y(\zeta_k)\| \leq \|y(t)\| + m_2\theta\|y(\zeta_k)\| + m_1\theta(1 + m_2\theta)e^{m_1\theta}\|y(\zeta_k)\|.$$

Thus, it follows from condition (A4) that

$$\|y(\zeta_k)\| \leq \lambda\|y(t)\|, \quad t \in [\theta_k, \theta_{k+1}).$$

Hence, (2.6) holds for all $t \in \mathbb{R}^+$. This completes the proof. □

3. Main results

In this section, we establish several criteria for global exponential stability of (2.5) based on the method of Lyapunov functions.

For convenience, we adopt the following notation in what follows:

$$m_3 = \frac{1}{n} \min_{1\leq i\leq n} \left(a_i - \frac{1}{2} \sum_{j=1}^{n} \left(L_j^1|b_{ij}| + L_j^2|c_{ij}| + L_i^1|b_{ji}| \right) \right).$$

Theorem 3.1. *Suppose that (A1)–(A5) hold true. Assume, furthermore, that the following inequality is satisfied:*

$$m_3 > \frac{m_2\lambda^2}{2}. \tag{3.8}$$

Then system (2.5) is globally exponentially stable.

Proof. We define a Lyapunov function by

$$V(y(t)) = \frac{1}{2}\sum_{i=1}^{n} y_i^2(t).$$

One can easily show that

$$\frac{1}{2n}\|y(t)\|^2 \leq V(y(t)) \leq \frac{1}{2}\|y(t)\|^2. \tag{3.9}$$

For $t \neq \theta_i,\ i \in \mathbb{N}$, the time derivative of V with respect to (2.5) is given by

$$\begin{aligned} V'_{(2.5)}(y(t)) &= \sum_{i=1}^{n} y_i(t)y_i'(t) \\ &= \sum_{i=1}^{n} y_i(t)\left[-a_i y_i(t) + \sum_{j=1}^{n} b_{ij}\varphi_j(y_j(t)) + \sum_{j=1}^{n} c_{ij}\psi_j(y_j(\gamma(t))) \right] \\ &\leq \sum_{i=1}^{n} \left[-a_i y_i^2(t) + \sum_{j=1}^{n} L_j^1|b_{ij}|\,|y_i(t)|\,|y_j(t)| + \sum_{j=1}^{n} L_j^2|c_{ij}|\,|y_i(t)|\,|y_j(\gamma(t))| \right] \\ &\leq \sum_{i=1}^{n} \left[-a_i y_i^2(t) + \frac{1}{2}\sum_{j=1}^{n} L_j^1|b_{ij}|(y_i^2(t) + y_j^2(t)) + \frac{1}{2}\sum_{j=1}^{n} L_j^2|c_{ij}|(y_i^2(t) + y_j^2(\gamma(t))) \right] \\ &\leq -\sum_{i=1}^{n} \left[\left(a_i - \frac{1}{2}\sum_{j=1}^{n} \left(L_j^1|b_{ij}| + L_j^2|c_{ij}| + L_i^1|b_{ji}| \right) \right) y_i^2(t) \right] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} L_i^2|c_{ji}|y_i^2(\gamma(t)) \\ &\leq -\min_{1\leq i\leq n} \left(a_i - \frac{1}{2}\sum_{j=1}^{n} \left(L_j^1|b_{ij}| + L_j^2|c_{ij}| + L_i^1|b_{ji}| \right) \right) \sum_{i=1}^{n} y_i^2(t) \end{aligned}$$

$$+\frac{1}{2}\max_{1\le i\le n}\left(L_i^2\sum_{j=1}^n|c_{ji}|\right)\sum_{i=1}^n y_i^2(\gamma(t))$$
$$\le -m_3\|y(t)\|^2+\frac{m_2}{2}\|y(\gamma(t))\|^2.$$

By using Lemma 2.1, we obtain

$$V'_{(2.5)}(y(t)) \le -m_3\|y(t)\|^2+\frac{m_2\lambda^2}{2}\|y(t)\|^2 = -\left(m_3-\frac{m_2\lambda^2}{2}\right)\|y(t)\|^2.$$

Now, define β for convenience as follows:

$$\beta = m_3-\frac{m_2\lambda^2}{2}>0.$$

Then, we have, for $t\ne\theta_i$,

$$\frac{d}{dt}(e^{2\beta t}V(y(t))) = e^{2\beta t}(2\beta)V(y(t))+e^{2\beta t}V'_{(2.5)}(y(t)) \le \beta e^{2\beta t}\|y(t)\|^2-\beta e^{2\beta t}\|y(t)\|^2=0.$$

From (3.9) and using the continuity of the function V and the solution $y(t)$, we obtain

$$e^{2\beta t}(1/2n)\|y(t)\|^2 \le e^{2\beta t}V(y(t))\le e^{2\beta t_0}V(y(t_0)) \le e^{2\beta t_0}(1/2)\|y(t_0)\|^2,$$

which implies that $\|y(t)\|\le\sqrt{n}\,\|y(t_0)\|\,e^{-\beta(t-t_0)}$. That is, system (2.5) is globally exponentially stable. □

In the next theorem, we utilize the same technique as that used in the previous theorem to find new stability conditions for RNNs by choosing a different Lyapunov function, defined as

$$V(y(t))=\sum_{i=1}^n\alpha_i|y_i(t)|,\quad \alpha_i>0,\ i=1,2,\ldots,n.$$

For simplicity of notation, let us denote

$$m_4=\min_{1\le i\le n}\left(a_i-L_i^1\sum_{j=1}^n|b_{ji}|\right).$$

Theorem 3.2. *Suppose that* (A1)–(A5) *hold true. Assume, furthermore, that the following inequality is satisfied:*

$$m_4>m_2\lambda. \tag{3.10}$$

Then system (2.5) is globally exponentially stable.

The proof of the assertion is similar to that of Theorem 3.1, so we omit it here.

4. Illustrative examples

In this section, we give three examples with simulations to illustrate our results. In what follows, we assume that the identification function $\gamma(t)$ is such that $\theta_k=k/9,\ \zeta_k=(2k+1)/18,\ k\in\mathbb{N}$.

Example 4.1. Consider the following RNNs with the argument function $\gamma(t)$:

$$\frac{dx(t)}{dt}=-\begin{pmatrix}2&0\\0&1.5\end{pmatrix}\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}+\begin{pmatrix}0.02&0.03\\0.01&1\end{pmatrix}\begin{pmatrix}\tanh(x_1(t))\\\tanh(x_2(t))\end{pmatrix}+\begin{pmatrix}0.08&1\\0.01&1\end{pmatrix}\begin{pmatrix}\tanh\left(\frac{x_1(\gamma(t))}{7}\right)\\\tanh\left(\frac{x_2(\gamma(t))}{6}\right)\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}. \tag{4.11}$$

It is easy to verify that (4.11) satisfies the conditions of Theorem 3.1 with $L_1^1=L_2^1=1$, $L_1^2=1/7$, $L_2^2=1/6$, $m_1=2.53$, $m_2=0.3333$, $m_3=0.6308$, $m_4=0.47$, $\lambda=1.7337$, Thus, according to this theorem, the unique equilibrium $x^*=(0.6011,1.3654)^T$ of (4.11) is globally exponentially stable. However, condition (3.10) of Theorem 3.2 is not satisfied.

Let us simulate a solution of (4.11) with initial condition $x_1^1(0)=x_1^0,\ x_2^1(0)=x_2^0$. Since Eq. (4.11) is of mixed type, the numerical analysis has a specific character and it should be described more carefully. One will see that this algorithm is in full accordance with the approximations made in the proof of Theorem 2.2.

We start with the interval $[\theta_0,\theta_1]$; that is, $[0,1/9]$. On this interval, Eq. (4.11) has the form

$$\frac{dx(t)}{dt}=-\begin{pmatrix}2&0\\0&1.5\end{pmatrix}\begin{pmatrix}x_1(0)\\x_2(0)\end{pmatrix}+\begin{pmatrix}0.02&0.03\\0.01&1\end{pmatrix}\begin{pmatrix}\tanh(x_1(0))\\\tanh(x_2(0))\end{pmatrix}+\begin{pmatrix}0.08&1\\0.01&1\end{pmatrix}\begin{pmatrix}\tanh\left(\frac{x_1(1/18)}{7}\right)\\\tanh\left(\frac{x_2(1/18)}{6}\right)\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix},$$

where $x_i(1/18),\ i=1,2$, are still unknown. For this reason, we will arrange approximations in the following way. Consider the sequence of equations

$$\frac{dx^{(m+1)}(t)}{dt}=-\begin{pmatrix}2&0\\0&1.5\end{pmatrix}\begin{pmatrix}x_1^{(m)}(0)\\x_2^{(m)}(0)\end{pmatrix}+\begin{pmatrix}0.02&0.03\\0.01&1\end{pmatrix}\begin{pmatrix}\tanh(x_1^{(m)}(0))\\\tanh(x_2^{(m)}(0))\end{pmatrix}+\begin{pmatrix}0.08&1\\0.01&1\end{pmatrix}\begin{pmatrix}\tanh\left(\frac{x_1^{(m)}(1/18)}{7}\right)\\\tanh\left(\frac{x_2^{(m)}(1/18)}{6}\right)\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix},$$

where $m=0,1,2,\ldots$, with $x_1^0(t)\equiv x_1^0,\ x_2^0(t)\equiv x_2^0$. We evaluate the solutions, $x^{(m)}(t)$, by using MATLAB 7.8, and stop the iterations at $(x_1^{(500)}(t),x_2^{500}(t))$. Then, we assign $x_1(t)=x_1^{(500)}(t),\ x_2(t)=x_2^{(500)}(t)$ on the interval $[\theta_0,\theta_1]$. Next, a similar operation is done on the interval $[\theta_1,\theta_2]$. That is, we construct the sequence $(x_1^{(m)},x_2^{(m)})$ of solutions again for the system

$$\frac{dx^{(m+1)}(t)}{dt}=-\begin{pmatrix}2&0\\0&1.5\end{pmatrix}\begin{pmatrix}x_1^{(m)}(0)\\x_2^{(m)}(0)\end{pmatrix}+\begin{pmatrix}0.02&0.03\\0.01&1\end{pmatrix}\begin{pmatrix}\tanh(x_1^{(m)}(0))\\\tanh(x_2^{(m)}(0))\end{pmatrix}+\begin{pmatrix}0.08&1\\0.01&1\end{pmatrix}\begin{pmatrix}\tanh\left(\frac{x_1^{(m)}(3/18)}{7}\right)\\\tanh\left(\frac{x_2^{(m)}(3/18)}{6}\right)\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}$$

with $x_1^0(t)\equiv x_1^{(500)}(1/9),\ x_2^0(t)\equiv x_2^{(500)}(1/9)$. Then, we reassign $x_1(t)=x_1^{(500)}(t), x_2(t)=x_2^{(500)}(t)$ on $[\theta_1,\theta_2]$. Proceeding in this way, one can obtain a simulation which demonstrates the asymptotic property.

Specifically, the simulation result with several random initial points is shown in Fig. 1. We must explain that the non-smoothness at the switching points $\theta_k,\ k\in\mathbb{N}$ is not seen by simulation. That is

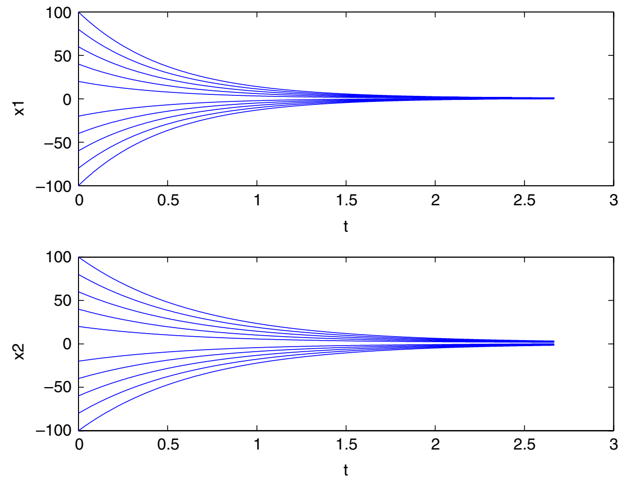

Fig. 1. Transient behavior of the RNNs in Example 4.1.

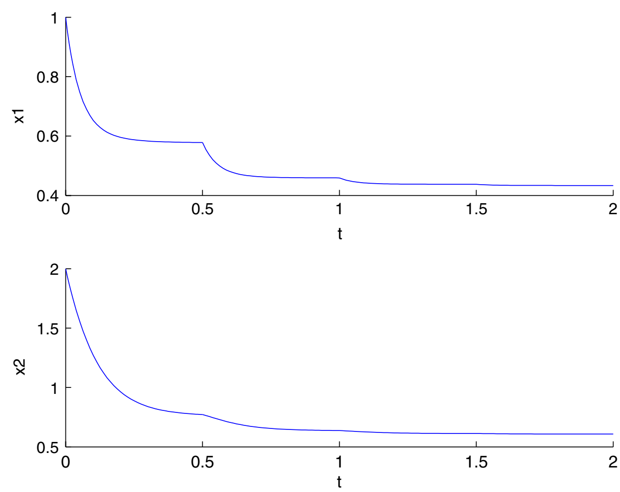

Fig. 2. The non-smoothness is seen at moments 0.5, 1, and 1.5, which are switching points of the function $\gamma(t)$.

why we have to choose the Lipschitz constants and θ small enough to satisfy the conditions of the theorems. So, the smallness "hides" the non-smoothness.

Let us now take the parameters such that the non-smoothness can be seen. Consider the following RNNs:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\begin{pmatrix} 20 & 0 \\ 0 & 10 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 8 & 0.2 \end{pmatrix}\begin{pmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{pmatrix} + \begin{pmatrix} 1 & 20 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} \tanh(x_1(\gamma(t))) \\ \tanh\left(\dfrac{x_2(\gamma(t))}{2}\right) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{4.12}$$

where $\theta_k = k/2,\ \zeta_k = (2k+1)/4,\ k \in \mathbb{N}$. One can see that θ and the Lipschitz coefficient are large this time. They do not satisfy the conditions of our theorems. It is illustrated in Fig. 2 that the non-smoothness of the solution with the initial point $[1, 2]^T$ can be seen at the switching points $\theta_k,\ k \in \mathbb{N}$. This is important for us to see that the non-smoothness of solutions expected from the equations' nature is apparent. Moreover, we can see that the solution converges to the unique equilibrium $x^* = (0.4325, 0.6065)^T$. This shows that the sufficient conditions which are found in our theorems can be elaborated further.

Example 4.2. Consider the following RNNs:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\begin{pmatrix} 2 & 0 \\ 0 & 2.5 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 1 & 0.03 \\ 0.04 & 1 \end{pmatrix}\begin{pmatrix} \tanh\left(\dfrac{x_1(t)}{4}\right) \\ \tanh(x_2(t)) \end{pmatrix} + \begin{pmatrix} 1 & 0.04 \\ 0.02 & 0.07 \end{pmatrix}\begin{pmatrix} \tanh\left(\dfrac{x_1(\gamma(t))}{4}\right) \\ \tanh\left(\dfrac{x_2(\gamma(t))}{4}\right) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.13}$$

It can be shown easily that (4.13) satisfies the conditions of Theorem 3.2 if $L_1^1 = 1/4, L_2^1 = 1, L_1^2 = 1/4, L_2^2 = 1/4, m_1 = 3.53, m_2 = 0.2550, m_3 = 0.6181, m_4 = 1.47, \lambda = 2.6693$, whereas condition (3.8) of Theorem 3.1 does not hold. Hence, it follows from Theorem 3.2 that the unique equilibrium $x^* = (0.6737, 0.6265)^T$ of (4.13) is globally exponentially stable.

Example 4.3. Consider the following system of differential equations:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0.02 & 0.03 \\ 0.04 & 0.25 \end{pmatrix}\begin{pmatrix} \tanh\left(\dfrac{x_1(t)}{4}\right) \\ \tanh\left(\dfrac{x_2(t)}{4}\right) \end{pmatrix} + \begin{pmatrix} 0.25 & 0.4 \\ 0.2 & 0.7 \end{pmatrix}\begin{pmatrix} \tanh\left(\dfrac{x_1(\gamma(t))}{4}\right) \\ \tanh\left(\dfrac{x_2(\gamma(t))}{4}\right) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.14}$$

One can see easily that the conditions of both Theorems 3.1 and 3.2 are satisfied with $L_1^1 = 1/4,\ L_2^1 = 1/4, L_1^2 = 1/4,\ L_2^2 = 1/4,\ m_1 = 3.07,\ m_2 = 0.2750,\ m_3 = 1.4081,\ m_4 = 2.93,\ \lambda = 2.1052,\ \tau = 1.1$. Thus, according to Theorems 3.1 and 3.2 the unique equilibrium $x^* = (0.4172, 0.4686)^T$ of (4.14) is globally exponentially stable.

5. Conclusion

This is the first time that the method of Lyapunov functions for differential equations with piecewise constant argument of generalized type has been applied to the model of RNNs, and this paper has provided new sufficient conditions guaranteeing the existence, uniqueness, and global exponential stability of the equilibrium point of the RNNs. In addition, our method gives new ideas not only from the modeling point of view, but also from that of theoretical opportunities since the RNN model equation involves piecewise constant arguments of both advanced and delayed types. The obtained results could be useful in the design and applications of RNNs. Furthermore, the method given in this paper may be extended to study more complex systems (Akhmet, 2009). On the basis of our results, Lyapunov functions give an opportunity to estimate domains of attraction, of particular interest in evaluating the performance of RNNs (Xu et al., 2006; Yang et al., 2006).

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