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Stability in cellular neural networks with a piecewise constant argument

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ABSTRACT

In this paper, by using the concept of differential equations with piecewise constant arguments of generalized type [1–4], a model of cellular neural networks (CNNs) [5,6] is developed. The Lyapunov–Razumikhin technique is applied to find sufficient conditions for the uniform asymptotic stability of equilibria. Global exponential stability is investigated by means of Lyapunov functions. An example with numerical simulations is worked out to illustrate the results.

1. Introduction

CNNs were introduced by Chua and Yang in 1988. For a brief summary of the theory and applications of CNNs, the reader is referred to the papers [5,6]. In recent years, the dynamical behavior of delayed cellular neural networks (DCNNs) proposed in [7] has been studied and developed by many authors [8–19] as well as many applications have been found in different areas such as associative memory, image and signal processing, pattern recognition and so on. As is well known, such applications depend on the existence of an equilibrium point and its stability.

Further, CNNs have been developed by implementing impulses and delays [6–10,20–22] issuing from different reasons: In the implementation of electronic networks, the state of the networks is subject to instantaneous perturbations and experiences abrupt change at certain instants, which may be caused by the switching phenomenon, frequency change or other sudden noise. This leads to a model of cellular neural network with impulses. Due to the finite switching speed of amplifiers and transmission of signals in electronic networks or finite speed of signal propagation in biological networks, time delays exist.

It is well known that studies of differential equations with a piecewise constant argument were motivated by the fact that they represent a hybrid of continuous and discrete dynamical systems and combine the properties of both the differential and difference equations. These equations play an important role in numerous applications [23–25]. Investigation of the first order differential equations with piecewise constant arguments of delay and advanced types had been initiated in [26,27], where the method of research was based on the reduction to discrete equations. Hence, the qualitative properties of solutions which start at non-integer values cannot be achieved. Particularly, one cannot investigate the problem of stability completely, as only elements of a countable set are allowed to be discussed for initial moments. By introducing arbitrary piecewise constant functions as arguments, the concept of differential equations with a piecewise constant argument

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has been generalized in [1–4], where an integral representation formula was proposed as another approach to meet the challenges discussed above.

One of the most important novelties of the present paper is that we assume CNNs may "memorize" values of the phase variable at certain moments of time to utilize the values during middle process till the next moment. Thus, we arrive at differential equations with a piecewise constant argument. Obviously, the distances between the "memorized" moments may be very variative. Consequently, the concept of generalized type of piecewise constant argument is fruitful.

In the literature, there are many papers in which Lyapunov–Krasovskii method [28] has been successfully utilized on the stability analysis of CNNs. But, there are few results on the stability of CNNs [29–31] based on the Lyapunov–Razumikhin technique [32,33]. Moreover, it deserves to be mentioned that since differential equations with a piecewise constant argument are differential equations with a deviated argument of delay or advanced type [4,34], it is reasonable to use this technique.

The intrinsic idea of our paper is that we investigate the problem of stability for CNNs with a piecewise constant argument through two approaches based on the Lyapunov–Razumikhin method and Lyapunov functions combined with the linear matrix inequality technique [19,35,36]. In the first one, we apply proper Razumikhin technique with the peculiarity that conditions on derivative are rather vector-like but not functional. For the second one, we utilize Lyapunov functions, not functionals despite the system is a delay differential equation.

In this paper, $\mathbb{N}$ and $\mathbb{R}^+$ are the sets of natural and nonnegative real numbers, respectively, i.e., $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}^n$ denotes the n dimensional real space. The notation $X > 0$ (or $X < 0$) denotes that X is a symmetric and positive definite (or negative definite) matrix. The notations X^T and X^{-1} refer, respectively, the transpose and the inverse of a square matrix X.$\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ represent the maximal eigenvalue and minimal eigenvalue of X, respectively. The norm $\|\cdot\|$ means either one-norm: $\|x\|_1 = \sum_{i=1}^n |x_i|$, $x \in \mathbb{R}^n$ or the induced matrix 2-norm: $\|X\|_2 = \sqrt{\lambda_{\max}(X^TX)}$. $*$ refers to the element below the main diagonal of a symmetric block matrix. Let θ_i, $i \in \mathbb{N}$, denote a fixed real-valued sequence such that $0 = \theta_0 < \theta_1 < \cdots < \theta_i < \cdots$ with $\theta_i \to \infty$ as $i \to \infty$.

2. Model description and preliminaries

In this section, we will focus our attention on some preliminary results which will be used in the stability analysis of CNNs. First, let us give a general description of the mathematical model of CNNs with a piecewise constant argument:

$$x'(t) = -Ax(t) + Bf(x(t)) + Cg(x(\beta(t))) + D \tag{2.1}$$

or equivalently,

$$x_i'(t) = -a_ix_i(t) + \sum_{j=1}^n b_{ij}f_j(x_j(t)) + \sum_{j=1}^n c_{ij}g_j(x_j(\beta(t))) + d_i, \quad a_i > 0,\ i = 1, 2, \ldots, n \tag{2.2}$$

where $\beta(t) = \theta_i$ if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$, $t \in \mathbb{R}^+$, $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ is the neuron state vector, $f(x(t)) = [f_1(x_1(t)), \ldots, f_n(x_n(t))]^T \in \mathbb{R}^n$ and $g(x(\beta(t))) = [g_1(x_1(\beta(t))), \ldots, g_n(x_n(\beta(t)))]^T \in \mathbb{R}^n$ are the activation functions of neurons, $D = [d_1, \ldots, d_n]^T$ is a constant external input vector. Moreover, we have $A = \mathrm{diag}(a_1, \ldots, a_n)$, $B = (b_{ij})_{n\times n}$ and $C = (c_{ij})_{n\times n}$, where B and C denote the connection weight and the delayed connection weight matrices, respectively.

The following assumptions will be needed throughout the paper:

(H1) The activation functions $f, g \in C(\mathbb{R}^n)$ with $f(0) = 0$, $g(0) = 0$;
(H2) there exist two Lipschitz constants $L = \mathrm{diag}(L_1, \ldots, L_n)$, $\bar{L} = \mathrm{diag}(\bar{L}_1, \ldots, \bar{L}_n) > 0$ such that

$$|f_i(u) - f_i(v)| \le L_i|u - v|,$$
$$|g_i(u) - g_i(v)| \le \bar{L}_i|u - v|$$

for all $u, v \in \mathbb{R}^n$, $i = 1, 2, \ldots, n$;
(H3) there exists a positive number θ such that $\theta_{i+1} - \theta_i \le \theta$, $i \in \mathbb{N}$;
(H4) $\theta[k_3 + k_2] < 1$;
(H5) $\theta[k_2 + k_3(1 + \theta k_2)e^{\theta k_3}] < 1$,

where

$$k_1 = \sum_{i=1}^n\sum_{j=1}^n |b_{ji}|L_i, \qquad k_2 = \sum_{i=1}^n\sum_{j=1}^n |c_{ji}|\bar{L}_i \quad \text{and} \quad k_3 = \sum_{i=1}^n a_i + k_1.$$

By a solution of Eq. (2.1) on $\mathbb{R}^+$ we mean a continuous function $x(t)$ satisfying the conditions (i) the derivative $x'(t)$ exists everywhere with the possible exception of the points θ_i, $i \in \mathbb{N}$, where one-sided derivatives exist; (ii) (2.1) is satisfied on each interval $[\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$.

In the following theorem, we obtain sufficient conditions for the existence of a unique equilibrium, $x^* = (x_1^*, \ldots, x_n^*)^T$, of (2.2).

Theorem 2.1. *Suppose that the neural parameters a_i, b_{ij}, c_{ij} and Lipschitz constants $L_j, \bar{L_j}$ satisfy*

$$a_i > L_i \sum_{j=1}^{n} |b_{ji}| + \bar{L_i} \sum_{j=1}^{n} |c_{ji}|, \quad i = 1, \dots, n.$$

Then, (2.2) *has a unique equilibrium.*

The proof of the theorem is almost identical to the verification in [12] with slight changes which are caused by the piecewise constant argument.

Now we need the following lemma which provides conditions for the existence and uniqueness of solutions for arbitrary initial moment ξ.

Lemma 2.1. *Assume that conditions* (H1)–(H5) *are fulfilled. Then for all $x^0 \in \mathbb{R}^n, \theta_r \le \xi < \theta_{r+1}, r \in \mathbb{N}$, there exists a unique solution $\bar{x}(t) = x(t, \theta_r, \bar{x}^0) = (x_1(t), \dots, x_n(t))^T$ of* (2.2)*, $\theta_r \le t \le \theta_{r+1}$, such that $\bar{x}(\xi) = x^0$.*

Proof. *Existence*: Consider a solution $v(t) = x(t, \xi, x^0) = (v_1(t), \dots, v_n(t))^T$ of the equation,

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} g_j(\zeta_j) + d_i$$

on $[\theta_r, \xi]$. We need to prove that there exists a vector $\zeta = (\zeta_1, \dots, \zeta_n)^T \in \mathbb{R}^n$ such that the equation

$$v_i(t) = x_i^0 + \int_{\xi}^{t} \left[-a_i v_i(s) + \sum_{j=1}^{n} b_{ij} f_j(v_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(\zeta_j) + d_i \right] \mathrm{d}s \tag{2.3}$$

has a solution on $[\theta_r, \xi]$, and satisfies $v(\theta_r) = \zeta$. Define a norm $\|v(t)\|_0 = \max_{[\theta_r, \xi]} \|v(t)\|$ and construct the following sequences $v_i^m(t), i = 1, \dots, n, m \ge 0$.

Take $v_i^0(t) \equiv x_i^0, i = 1, \dots, n$, and sequences

$$v_i^{m+1}(t) = x_i^0 + \int_{\xi}^{t} \left[-a_i v_i^n(s) + \sum_{j=1}^{n} b_{ij} f_j(v_j^n(s)) + \sum_{j=1}^{n} c_{ij} g_j(v_j^n(\theta_r)) + d_i \right] \mathrm{d}s.$$

One can find that

$$\|v^{m+1}(t) - v^m(t)\|_0 = \max_{[\theta_r, \xi]} \|v^{m+1}(t) - v^m(t)\| \le (\theta\,(k_3 + k_2))^m \,\kappa,$$

where

$$\kappa = \theta \left((k_3 + k_2)\,\|x^0\| + \sum_{i=1}^{n} d_i \right).$$

Hence, the sequences $v_i^m(t)$ are convergent and their limits satisfy (2.3) on $[\theta_r, \xi]$ with $\zeta = v(\theta_r)$. The extension of the solution on $[\xi, \theta_{r+1}]$ is obvious. Thus, the existence is proved.

Uniqueness: It is sufficient to check that for each $t \in [\theta_r, \theta_{r+1}]$, and $x^2 = (x_1^2, \dots, x_n^2)^T, x^1 = (x_1^1, \dots, x_n^1)^T \in \mathbb{R}^n, x^2 \ne x^1$, the condition $x(t, \theta_r, x^1) \ne x(t, \theta_r, x^2)$ is valid. Let us denote solutions of (2.2) by $x^1(t) = x(t, \theta_r, x^1), x^2(t) = x(t, \theta_r, x^2)$, $x^1 \ne x^2$. Assume on the contrary that there exists $t^* \in [\theta_r, \theta_{r+1}]$ such that $x^1(t^*) = x^2(t^*)$. Then, we have

$$x_i^1 - x_i^2 = \int_{\theta_r}^{t^*} \left[-a_i \left(x_i^2(s) - x_i^1(s)\right) + \sum_{j=1}^{n} b_{ij}[f_j(x_j^2(s)) - f_j(x_j^1(s))] + \sum_{j=1}^{n} c_{ij}[g_j(x_j^2(\theta_r)) - g_j(x_j^1(\theta_r))] \right] \mathrm{d}s,$$
$$i = 1, \dots, n.$$

Taking the absolute value of both sides for each $i = 1, \dots, n$ and adding all equalities, we obtain that

$$\begin{aligned} \|x^2 - x^1\| &= \sum_{i=1}^{n} \left| \int_{\theta_r}^{t^*} \left[-a_i \left(x_i^2(s) - x_i^1(s)\right) + \sum_{j=1}^{n} b_{ij} \left[f_j(x_j^2(s)) - f_j(x_j^1(s))\right] + \sum_{j=1}^{n} c_{ij} \left[g_j(x_j^2(\theta_r)) - g_j(x_j^1(\theta_r))\right] \right] \mathrm{d}s \right| \\ &\le \sum_{i=1}^{n} \left\{ \int_{\theta_r}^{t^*} \left[a_i |x_i^2(s) - x_i^1(s)| + \sum_{j=1}^{n} L_i |b_{ji}|\, |x_i^2(s) - x_i^1(s)| + \sum_{j=1}^{n} \bar{L}_i |c_{ji}|\, |x_i^2 - x_i^1| \right] \mathrm{d}s \right\} \\ &\le \theta k_2 \|x^1 - x^2\| + \int_{\theta_r}^{t^*} k_3 \|x^1(s) - x^2(s)\| \mathrm{d}s. \end{aligned} \tag{2.4}$$

Furthermore, for $t \in [\theta_r, \theta_{r+1}]$, the following is valid:

$$\begin{aligned}\|x^1(t) - x^2(t)\| &\le \|x^1 - x^2\| + \sum_{i=1}^n \left\{ \int_{\theta_r}^t \left[a_i |x_i^2(s) - x_i^1(s)| + \sum_{j=1}^n L_i |b_{ji}|\, |x_i^2(s) - x_i^1(s)| + \sum_{j=1}^n \bar{L}_i |c_{ji}|\, |x_i^2 - x_i^1| \right] \mathrm{d}s \right\} \\ &\le (1 + \theta k_2)\, \|x^1 - x^2\| + \int_{\theta_r}^t k_3 \|x^1(s) - x^2(s)\| \mathrm{d}s.\end{aligned}$$

The Gronwall–Bellman lemma yields that

$$\|x^1(t) - x^2(t)\| \le (1 + \theta k_2)\, \mathrm{e}^{\theta k_3} \|x^1 - x^2\|. \tag{2.5}$$

Consequently, substituting (2.5) in (2.4), we obtain

$$\|x^1 - x^2\| \le \theta \left[k_2 + k_3\, (1 + \theta k_2)\, \mathrm{e}^{\theta k_3} \right] \|x^1 - x^2\|. \tag{2.6}$$

Thus, one can see that (H5) contradicts with (2.6). The lemma is proved. □

Theorem 2.2. *Suppose that conditions* (H1)–(H5) *are fulfilled. Then, for every* $(t_0, x^0) \in \mathbb{R}^+ \times \mathbb{R}^n$, *there exists a unique solution* $x(t) = x(t, t_0, x^0) = (x_1(t), \ldots, x_n(t))^T$, $t \in \mathbb{R}^+$, *of* (2.1), *such that* $x(t_0) = x^0$.

Proof. It is clear that there exists $r \in \mathbb{N}$ such that $t_0 \in [\theta_r, \theta_{r+1})$. Using Lemma 2.1 for $\xi = t_0$, there exists a unique solution $x(t) = x(t, t_0, x^0)$ of (2.2) on $[\xi, \theta_{r+1}]$. Next, applying the lemma again, one can obtain a unique solution on interval $[\theta_{r+1}, \theta_{r+2}]$. Hence, the mathematical induction completes the proof. □

Consider the equilibrium point, $x^* = (x_1^*, \ldots, x_n^*)^T$, of the system (2.1).

Definition 2.1 ([4]). The equilibrium $x = x^*$ of (2.1) is said to be uniformly stable if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|x(t_0) - x^*\| < \delta$ implies $\|x(t) - x^*\| < \varepsilon$ for all $t \ge t_0$.

Definition 2.2 ([4]). The equilibrium $x = x^*$ of (2.1) is said to be uniformly asymptotically stable if it is uniformly stable and there is a $\delta_0 > 0$ such that for every $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $T = T(\varepsilon) > 0$ such that $\|x(t) - x^*\| < \varepsilon$ for all $t > t_0 + T$ whenever $\|x(t_0) - x^*\| < \delta_0$.

Definition 2.3 ([4]). The equilibrium $x = x^*$ of (2.1) is said to be globally exponentially stable if there exist positive constants α_1 and α_2 such that the estimation $\|x(t) - x^*\| < \alpha_1 \|x(t_0) - x^*\| \mathrm{e}^{-\alpha_2 (t - t_0)}$ is valid for all $t \ge t_0$.

By means of the transformation $y(t) = x(t) - x^*$, system (2.1) can be simplified as

$$y'(t) = -Ay(t) + B\varphi(y(t)) + C\psi(y(\beta(t))), \tag{2.7}$$

where $\varphi_j(y_j(t)) = f_j(y_j(t) + x_j^*) - f_j(x_j^*)$ and $\psi_j(y_j(t)) = g_j(y_j(t) + x_j^*) - g_j(x_j^*)$ with $\varphi_j(0) = \psi_j(0) = 0$. From assumption (H2), we have $\varphi_j(\cdot)$ and $\psi_j(\cdot)$ are also Lipschitzian with $L_j, \bar{L}_j$, respectively.

It is obvious that the stability of the zero solution of (2.7) is equivalent to that of the equilibrium x^* of (2.1). Therefore, in what follows, we discuss the stability of the zero solution of (2.7).

To begin with, we introduce the following lemmas which will be used in the proof of the stability of the zero solution for CNNs with a piecewise constant argument.

Lemma 2.2 ([37]). *Given any real matrices* U, W, Z *of appropriate dimensions and a scalar* $\epsilon > 0$ *such that* $0 < W = W^T$, *then the following matrix inequality holds:*

$$U^T Z + Z^T U \le \epsilon U^T W U + \frac{1}{\epsilon} Z^T W^{-1} Z.$$

The following lemma is an important auxiliary result of the paper. It can be proved in the same way used for Theorem 2.2 in [2].

Lemma 2.3. *Let* $y(t) = (y_1(t), \ldots, y_n(t))^T$ *be a solution of* (2.7) *and* (H1)–(H5) *be satisfied. Then, the following inequality*

$$\|y(\beta(t))\| \le l \|y(t)\| \tag{2.8}$$

holds for all $t \in \mathbb{R}^+$, *where* $l = \{1 - \theta[k_2 + k_3(1 + \theta k_2)\mathrm{e}^{\theta k_3}]\}^{-1}$.

For convenience, we adopt the following notation in the sequel:

(N) Given $P > 0$, positive diagonal matrices R, S with appropriate dimensions and a real $q > 1$, denote

$$\Omega = PBR^{-1}B^TP + LRL + PCS^{-1}C^TP + qP - AP - PA,$$

or, by Schur complements, it can be rewritten as the following matrix form:

$$-\Omega = \begin{bmatrix} AP + PA - LRL - qP & PB & PC \\ * & R & 0 \\ * & * & S \end{bmatrix},$$

where $L = \mathrm{diag}(L_1, \ldots, L_n) > 0$.

We shall consider the quadratic function $V(y) = y^TPy$. The derivative of V with respect to system (2.7) is defined by

$$V'(y, z) = -y^T(AP + PA)y + 2y^TPB\varphi(y) + 2y^TPC\psi(z) \quad \text{for } y, z \in \mathbb{R}^n.$$

3. Lyapunov–Razumikhin technique

From now on, we shall need the following assumptions:

(C1) $\Omega < 0$;
(C2) $P > \bar{L}S\bar{L}$ where $\bar{L} = \mathrm{diag}(\bar{L}_1, \ldots, \bar{L}_n) > 0$.

Lemma 3.1. *Assume that conditions* (C1)–(C2) *are fulfilled, and* $y(t) : \mathbb{R}^+ \to \mathbb{R}^n$ *is a solution of* (2.7)*. Then the following conditions hold for* $V(y(t)) = y^T(t)Py(t)$*:*

(1a) $a\|y(t)\|^2 \le V(y(t)) \le b\|y(t)\|^2$, *where* $a = \lambda_{\min}(P)$ *and* $b = \lambda_{\max}(P)$*;*
(1b) $V'(y(t), y(\beta(t))) \le -c\|y(t)\|^2$ *for all* $t \ne \theta_i$ *in* $\mathbb{R}^+$ *such that* $V(y(\beta(t))) < qV(y(t))$ *with a constant* $c > 0$.

Proof. It is obvious that $a\|y(t)\|^2 \le V(y(t)) \le b\|y(t)\|^2$, where $a = \lambda_{\min}(P)$ and $b = \lambda_{\max}(P)$.

For $t \ne \theta_i$, $i \in \mathbb{N}$, the derivative of $V(y(t))$ along the trajectories of system (2.7) is given by

$$\begin{aligned} V'(y(t), y(\beta(t))) &= y'^T(t)Py(t) + y^T(t)Py'(t) \\ &= -y^T(t)(AP + PA)y(t) + 2y^T(t)PB\varphi(y(t)) + 2y^T(t)PC\psi(y(\beta(t))). \end{aligned} \tag{3.9}$$

Let $U = B^TPy(t)$, $Z = \varphi(y(t))$. By applying Lemma 2.2, we have the following inequality:

$$\begin{aligned} 2y^T(t)PB\varphi(y(t)) &= y^T(t)PB\varphi(y(t)) + \varphi^T(y(t))B^TPy(t) \\ &\le y^T(t)PBR^{-1}B^TPy(t) + \varphi^T(y(t))R\varphi(y(t)) \\ &\le y^T(t)\left(PBR^{-1}B^TP + LRL\right)y(t), \end{aligned} \tag{3.10}$$

since $\varphi^T(y(t))R\varphi(y(t)) \le y^TLRLy(t)$.

Similarly, we have

$$2y^T(t)PC\psi(y(\beta(t))) \le y^T(t)PCS^{-1}C^TPy(t) + y^T(\beta(t))\bar{L}S\bar{L}y(\beta(t)), \tag{3.11}$$

since $\psi^T(y(\beta(t)))S\psi(y(\beta(t))) \le y^T(\beta(t))\bar{L}S\bar{L}y(\beta(t))$.

Substituting (3.10) and (3.11) into (3.9) and using condition (C2), we have

$$V'(y(t), y(\beta(t))) \le y^T(t)\left(PBR^{-1}B^TP + LRL + PCS^{-1}C^TP - AP - PA\right)y(t) + y^T(\beta(t))Py(\beta(t)).$$

Then, one can conclude that

$$V'(y(t), y(\beta(t))) \le y^T(t)\Omega y(t), \quad t \ne \theta_i \tag{3.12}$$

whenever $y^T(\beta(t))Py(\beta(t)) = V(y(\beta(t))) < qV(y(t)) = y^T(t)qPy(t)$.

It follows from the condition (C1) in terms of Schur complements given in (N) and (3.12) that (1b) is valid. □

From (1a) and (1b) of the last lemma, it implies that V can be taken as a Lyapunov function for system (2.7). Now, we are ready to give sufficient conditions for uniform asymptotic stability of (2.7). To prove the following theorem we shall use the technique which was developed in paper [38].

Theorem 3.1. *Suppose that* (H1)–(H5) *and* (C1)–(C2) *hold true, then the equilibrium* x^* *of* (2.1) *is uniformly asymptotically stable.*

Proof. Fix $h_1 > 0$. Given $\varepsilon > 0$, $(\varepsilon < h_1)$, we choose $\delta_1 > 0$ such that $b\delta_1^2 \le a\varepsilon^2$. Define $\delta = \delta_1/l$ and note that $\delta < \delta_1$ as $l > 1$. We first prove uniform stability when $t_0 = \theta_j$ for some $j \in \mathbb{N}$ and then for $t_0 \ne \theta_i$ for all $i \in \mathbb{N}$, to show that this δ is the needed one in both cases.

If $t_0 = \theta_j$, where $j \in \mathbb{N}$ and $\|y(\theta_j)\| < \delta$, then $V(y(\theta_j)) < b\delta^2 < b\delta_1^2 \le a\varepsilon^2$.

We fix $k \in \mathbb{N}$ and consider the interval $[\theta_k, \theta_{k+1})$. Using (1b) in Lemma 3.1, we shall show that

$$V(y(t)) \le V(y(\theta_k)) \quad \text{for } t \in [\theta_k, \theta_{k+1}). \tag{3.13}$$

Let us set $v(t) = V(y(t))$. If (3.13) does not hold, then there must exist points η and ρ satisfying $\theta_k \le \eta < \rho < \theta_{k+1}$ and

$$v(\eta) = v(\theta_k), \qquad v(t) > v(\theta_k) \quad \text{for } t \in (\eta, \rho].$$

Based on the mean value theorem, we can find a $\zeta \in (\eta, \rho)$ satisfying the equation $\frac{v(\rho)-v(\eta)}{\rho-\eta} = v'(\zeta) > 0$.

Actually, since $v(\theta_k) < v(\zeta) < qv(\zeta)$, it follows from (1b) that $v'(\zeta) < 0$, a contradiction. Hence, (3.13) is true. As the functions V and y are continuous, one can obtain by induction that $V(y(t)) \le V(y(\theta_j))$ for all $t \ge \theta_j$. Thus, we have $a\|y(t)\|^2 \le V(y(t)) \le V(y(\theta_j)) < a\varepsilon^2$, which implies in turn that $\|y(t)\| < \varepsilon$ for all $t \ge \theta_j$. We see that evaluation of δ does not depend on the choice of $j \in \mathbb{N}$.

Now, consider the case $t_0 \in \mathbb{R}^+$ with $t_0 \ne \theta_i$ for all $i \in \mathbb{N}$. Then there exists $j \in \mathbb{N}$ such that $\theta_j < t_0 < \theta_{j+1}$. For a solution $y(t)$ satisfying $\|y(t_0)\| < \delta$, Lemma 2.3 implies that $\|y(\theta_j)\| < \delta_1$. Using a similar idea used for the case $t_0 = \theta_j$, we conclude that $\|y(t)\| < \varepsilon$ for $t \ge \theta_j$ and hence for all $t \ge t_0$, which completes the proof for the uniform stability. We note that the evaluation is independent of $j \in \mathbb{N}$ and correspondingly it is valid for all $t_0 \in \mathbb{R}^+$. Next we shall prove uniform asymptotic stability.

First, we show "uniform" asymptotic stability with respect to all elements of the sequence θ_i, $i \in \mathbb{N}$.

Fix $j \in \mathbb{N}$. For $t_0 = \theta_j$, we choose $\delta > 0$ such that $b(l\delta)^2 = ah_1^2$ holds. In view of uniform stability, one can obtain that $V(y(t)) < b\delta^2 < b(l\delta)^2$ for all $t \ge \theta_j$ and hence $\|y(t)\| < h_1$ whenever $\|y(\theta_j)\| < \delta$. In what follows, we present that this δ can be taken as δ_0 in the Definition 2.2. That is to say, given $\varepsilon > 0$, $\varepsilon < h_1$, we need to show that there exists a $T = T(\varepsilon) > 0$ such that $\|y(t)\| < \varepsilon$ for $t > \theta_j + T$ if $\|y(\theta_j)\| < \delta$.

We denote $\gamma = \frac{ac}{b}\varepsilon^2$ and $\delta_1 = l\delta$. We can find a number $\mu > 0$ such that $qs > s + \mu$ for $a\varepsilon^2 \le s \le b\delta_1^2$. Let N be the smallest positive integer such that $a\varepsilon^2 + N\mu \ge b\delta_1^2$. Choosing $t_k = k\left(\frac{b\delta_1^2}{\gamma} + \theta\right) + \theta_j$, $k = 1, 2, \ldots, N$, we aim to prove that

$$V(y(t)) \le a\varepsilon^2 + (N-k)\mu \quad \text{for } t \ge t_k,\ k = 0, 1, 2, \ldots, N. \tag{3.14}$$

It is easily seen that $V(y(t)) < b\delta_1^2 \le a\varepsilon^2 + N\mu$ for $t \ge t_0 = \theta_j$. Hence, (3.14) is true for $k = 0$. Now, assuming that (3.14) is true for some $0 \le k < N$, we will show that $V(y(t)) \le a\varepsilon^2 + (N-k-1)\mu$ for $t \ge t_{k+1}$. To prove the last inequality, we first claim that there exists a $t^* \in I_k = [\beta(t_k) + \theta, t_{k+1}]$ such that

$$V(y(t^*)) \le a\varepsilon^2 + (N-k-1)\mu. \tag{3.15}$$

Otherwise, $V(y(t)) > a\varepsilon^2 + (N-k-1)\mu$ for all $t \in I_k$. On the other side, we have $V(y(t)) \le a\varepsilon^2 + (N-k)\mu$ for $t \ge t_k$, which implies that $V(y(\beta(t))) \le a\varepsilon^2 + (N-k)\mu$ for $t \ge \beta(t_k) + \theta$. Hence, for $t \in I_k$

$$qV(y(t)) > V(y(t)) + \mu > a\varepsilon^2 + (N-k)\mu \ge V(y(\beta(t))).$$

Since $a\varepsilon^2 \le V(y(t)) \le b\|y(t)\|^2$ for $t \in I_k$, it follows from (1b) that

$$V'(y(t), y(\beta(t))) \le -c\,\|y(t)\|^2 \le -\gamma \quad \text{for all } t \ne \theta_i \text{ in } I_k.$$

Using the continuity of the function V and the solution $y(t)$, we get

$$\begin{aligned} V(y(t_{k+1})) &\le V(y(\beta(t_k) + \theta)) - \gamma(t_{k+1} - \beta(t_k) - \theta) \\ &< b\delta_1^2 - \gamma(t_{k+1} - t_k - \theta) = 0, \end{aligned}$$

which is a contradiction. Thus (3.15) holds true. Next, we show that

$$V(y(t)) \le a\varepsilon^2 + (N-k-1)\mu \quad \text{for all } t \in [t^*, \infty). \tag{3.16}$$

If (3.16) does not hold, then there exists a $\bar{t} \in (t^*, \infty)$ such that

$$V(y(\bar{t})) > a\varepsilon^2 + (N-k-1)\mu \ge V(y(t^*)).$$

Thus, we can find a $\tilde{t} \in (t^*, \bar{t})$ such that $\tilde{t} \ne \theta_i$, $i \in \mathbb{N}$, $V'(y(\tilde{t}), y(\beta(\tilde{t}))) > 0$ and $V(y(\tilde{t})) > a\varepsilon^2 + (N-k-1)\mu$. However,

$$qV(y(\tilde{t})) > V(y(\tilde{t})) + \mu > a\varepsilon^2 + (N-k)\mu \ge V(y(\beta(\tilde{t})))$$

implies that $V'(y(\tilde{t}), y(\beta(\tilde{t}))) \le -\gamma < 0$, a contradiction. Then, we conclude that $V(y(t)) \le a\varepsilon^2 + (N-k-1)\mu$ for all $t \ge t^*$ and thus for all $t \ge t_{k+1}$. This completes the induction and shows that (3.14) is valid. For $k = N$, we have

$$V(y(t)) \le a\varepsilon^2, \quad t \ge t_N = N\left(\frac{b\delta_1^2}{\gamma} + \theta\right) + t_0.$$

In the end, $\|y(t)\| < \varepsilon$ for $t > \theta_j + T$ where $T = N\left(\frac{b\delta_1^2}{\gamma} + \theta\right)$, which proves uniform asymptotic stability for $t_0 = \theta_j, j \in \mathbb{N}$.

Take $t_0 \neq \theta_i$ for all $i \in \mathbb{N}$. Then $\theta_j < t_0 < \theta_{j+1}$ for some $j \in \mathbb{N}$. $\|y(t_0)\| < \delta$ implies by Lemma 2.3 that $\|y(\theta_j)\| < \delta_1$. Hence, the argument used for the case $t_0 = \theta_j$ yields that $\|y(t)\| < \varepsilon$ for $t > \theta_j + T$ and so for all $t > t_0 + T$. □

4. Method of Lyapunov functions

In this part of our paper, Lyapunov–Krasovskii method is used for Eq. (2.7), which is a delay differential equation, but one must emphasize that Lyapunov functions, not functionals, are used.

In the following condition, the matrices A, B, C, P, R, S, L are described as in (N).

(C3) $\bar{\Omega} = PBR^{-1}B^TP + LRL + PCS^{-1}C^TP + bl^2\kappa P - AP - PA < 0$, where κ is a constant with $\kappa a \geq 1$.

Lemma 4.1. *Assume that conditions* (C2)–(C3) *are fulfilled, and $y(t)$ is a solution of* (2.7)*. Then the following conditions hold for the quadratic function $V(y(t)) = y^T(t)Py(t)$:*

(2a) *$a\|y(t)\|^2 \leq V(y(t)) \leq b\|y(t)\|^2$, where $a = \lambda_{\min}(P)$ and $b = \lambda_{\max}(P)$;*
(2b) *$V'(y(t), y(\beta(t))) \leq -c\|y(t)\|^2$ for all $t \neq \theta_i$ in $\mathbb{R}^+$ with a constant $c > 0$.*

Proof. It is easily seen that $a\|y(t)\|^2 \leq V(y(t)) \leq b\|y(t)\|^2$, where $a = \lambda_{\min}(P)$ and $b = \lambda_{\max}(P)$.

It follows from Lemma 2.3 that $V(y(\beta(t))) \leq b\|y(\beta(t))\|^2 \leq bl^2\|y(t)\|^2 \leq bl^2\kappa a\|y(t)\|^2 \leq bl^2\kappa V(y(t))$.

For $t \neq \theta_i$, $i \in \mathbb{N}$, we know from the proof of Lemma 3.1 that the derivative of $V(y(t))$ along the trajectories of system (2.7) satisfies

$$V'(y(t), y(\beta(t))) \leq y^T(t)\left(PBR^{-1}B^TP + LRL + PCS^{-1}C^TP - AP - PA\right)y(t) + y^T(\beta(t))Py(\beta(t)).$$

Hence, we get

$$V'(y(t), y(\beta(t))) \leq y^T(t)\bar{\Omega}y(t), \quad t \neq \theta_i. \tag{4.17}$$

It follows from the condition (C3) and (4.17) that (2b) is valid. □

Theorem 4.1. *Suppose that* (H1)–(H5) *and* (C2)–(C3) *hold true, then the equilibrium x^* of* (2.1) *is globally exponentially stable.*

Proof. Using Lemma 4.1, we have for $t \neq \theta_i$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{(c/b)t}V(y(t))) &= \mathrm{e}^{(c/b)t}(c/b)V(y(t)) + \mathrm{e}^{(c/b)t}V'(y(t), y(\beta(t)))\\ &\leq c\mathrm{e}^{(c/b)t}\|y(t)\|^2 - c\mathrm{e}^{(c/b)t}\|y(t)\|^2 = 0.\end{aligned}$$

Using the continuity of the function V and the solution $y(t)$, we obtain

$$\mathrm{e}^{(c/b)t}a\|y(t)\|^2 \leq \mathrm{e}^{(c/b)t}V(y(t)) \leq \mathrm{e}^{(c/b)t_0}V(y(t_0)) \leq \mathrm{e}^{(c/b)t_0}b\|y(t_0)\|^2,$$

which implies that $\|y(t)\| \leq \sqrt{\frac{b}{a}}\|y(t_0)\|\mathrm{e}^{-(c/2b)(t-t_0)}$. The theorem is proved. □

5. An illustrative example

Consider the following CNNs with a piecewise constant argument:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + \begin{pmatrix}0.5 & 0\\ 0.1 & 0.3\end{pmatrix}\begin{pmatrix}\tanh\left(\dfrac{x_1(t)}{2}\right)\\ \tanh\left(\dfrac{x_2(t)}{2}\right)\end{pmatrix} + \begin{pmatrix}0.5 & 0.1\\ 0.1 & 0.3\end{pmatrix}\begin{pmatrix}\tanh\left(\dfrac{x_1(\beta(t))}{2}\right)\\ \tanh\left(\dfrac{x_2(\beta(t))}{3}\right)\end{pmatrix} + \begin{pmatrix}1\\ 2\end{pmatrix}. \tag{5.18}$$

Clearly, we obtain

$$L = \begin{pmatrix}\dfrac{1}{2} & 0\\ 0 & \dfrac{1}{2}\end{pmatrix}, \qquad \bar{L} = \begin{pmatrix}\dfrac{1}{2} & 0\\ 0 & \dfrac{1}{3}\end{pmatrix}.$$

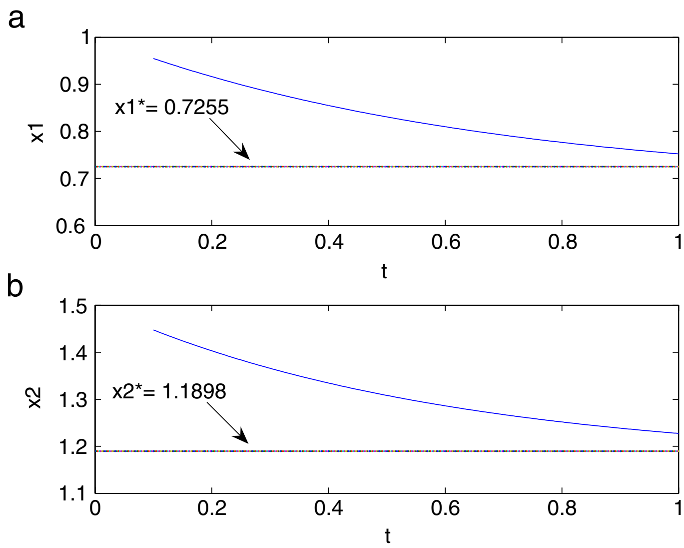

Fig. 1. Time response of state variables $x_1(t)$ and $x_2(t)$ with piecewise constant arguments in (a) and (b), respectively.

Let

$$P = \begin{pmatrix} 1.5 & 1 \\ 1 & 1.5 \end{pmatrix}, \qquad R = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \qquad S = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \qquad q = 1.2,$$

$$\beta(t) = \theta_i = \frac{i}{10}, \quad i \in \mathbb{N}.$$

By simple calculation, we can check that $k_1 = 0.45$, $k_2 = 0.4333$, $k_3 = 4.45$, $a = \lambda_{\min}(P) = 0.5$, $b = \lambda_{\max}(P) = 2.5$ and $l = 4.81$. We can choose $\kappa = 2.1$ so that $\kappa a > 1$. It follows from Theorem 2.1 that there exists a unique equilibrium such that $x^* = [0.7255, 1.1898]^T$. Then it can be easily verified that

$$\Omega = \begin{pmatrix} -2.9479 & -2.3708 \\ -2.3708 & -3.0604 \end{pmatrix} < 0, \qquad P - \bar{L}S\bar{L} = \begin{pmatrix} 0.5 & 1 \\ 1 & 1.0556 \end{pmatrix} > 0.$$

For $\theta = 1/10$, we get $\theta[k_3 + k_2] = 0.4883 < 1$ and $\theta[k_2 + k_3(1 + \theta k_2)e^{\theta k_3}] = 0.7921 < 1$. So, (H1)–(H5) and (C1)–(C2) hold. Thus, the conditions of the Theorem 3.1 for $q = 1.2$ are satisfied. Hence, (5.18) has a uniformly asymptotically stable equilibrium point. However, for the same q we have $q < bl^2\kappa$. Hence, Theorem 4.1 is not applicable. That is, using the Lyapunov–Razumikhin technique, we may take smaller q values, and that verifies it as more effective in the theoretical sense. Nevertheless, the second theorem allows us to obtain exponential evaluation of convergence to the equilibrium, which has a very important peculiarity for applications in practice.

The simulation, where the initial value is chosen as $[1, 1.5]^T$, is shown in Fig. 1 and it illustrates that all trajectories uniformly converge to the unique asymptotically stable equilibrium point x^*.

6. Conclusion

In this paper, it is the first time that CNNs with a piecewise constant argument of generalized type are investigated. There is not a restriction on the distance between switching neighbors of the argument function and the stability is discussed in the uniform version. The analysis has been available after a new approach was proposed in [1–4]. It gives new ideas not only from the modeling point of view, but also from that of theoretical opportunities to conjugate with numerical analysis, and take into account the easiness of simulations simplified by the constancy of the argument.

Moreover, comparing two main results of our paper, one can see that Theorem 3.1 allows us to analyze a larger class of equations than Theorem 4.1. At the same time, on the basis of Theorem 4.1, one can evaluate the convergence of solutions to equilibria. Application of Lyapunov functions gives an opportunity to develop further quantitative analysis such as the estimation of the domain of attraction, etc.

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