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Chaos in economic models with exogenous shocks

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ABSTRACT

We investigate the generation of chaos in economic models through exogenous shocks. The perturbation is formulated as a pulse function where either values or instants of discontinuity are chaotically behaved. We provide a rigorous proof of the existence of chaos in the perturbed model. The analytical results are applied to Kaldor–Kalecki-type models of the aggregate economy subject to export and rainfall shocks, respectively. Simulations are used to demonstrate the emergence and the control of chaos. Our results shed light on a novel source of chaos in economic models and have important implications for policy-making.

1. Introduction

Irregularity is an inherent feature of economic reality. Regularity, as reflected in a constant solution of a model or a periodic and even almost periodic motion in mathematical sense, is a good assumption in engineering and natural science applications, but less so in economic models. This was pointed out in early scientific work and has been widely discussed in recent years (e.g., Baumol and Benhabib, 1989; Boldrin and Montrucchio, 1986; Day, 1983; Malthus, 1798; Marshall, 1920; Rosser, 2000). One way of introducing irregularity in economics is by allowing for stochastic processes. A different approach is generating chaos in deterministic differential equations.[1] The main property of chaos is *sensitivity*, which can be interpreted as unpredictability in real world problems. This is also known as the *butterfly effect* (Lorenz, 1963). Devaney (1989) proposed that *sensitivity* in conjunction with other properties, namely *transitivity* and *density of periodic solutions*, be considered as ingredients of chaos. Another popular way to prove theoretically the presence of chaos is by observing the *period-doubling cascade* (Gleick, 1987).

The main theoretical contribution of the present paper lies in demonstrating that *exogenous chaotic perturbations* can produce irregular motions in economic models. In mathematical terms, we augment the right-hand side of otherwise regular

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[1] As pointed out by a referee, there exists a third approach, which is somewhere in-between the two, where Iterated Function Systems generated by the optimal policy functions for a class of stochastic growth models converge to invariant distributions with support over fractal sets (Mitra and Privileggi, 2009).

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differential equations with chaotic terms and verify the intuitive idea that the resulting models admit chaotic solutions. Previous works have considered the 'endogenous' appearance of chaos in economic models, where the presence of chaos hinges on some crucial parameters (e.g. Lorenz, 1993; Zhang, 2005, and papers cited there). The principal novelty of our investigation is that we create an exogenous chaotic perturbation, plug it in a regular dynamical system, and find that similar chaos is inherited by the solutions of the new system. Such an approach has been widely used for differential equations before, but for regular disturbance functions. That is, it has been shown that an (almost) periodic perturbation function implies the existence of an (almost) periodic solution of the system. While the literature on chaos synchronization has also produced methods of generating chaos in a system by plugging in special terms that are chaotic, it relies on the asymptotic convergence between the chaotic exogenous terms and the solutions of the system (Pecora and Carroll, 1990; Gonzáles-Miranda, 2004). Instead, we provide a direct verification of the ingredients of chaos for the perturbed system. Currently, we study cases where the shocks enter the system additively, but future investigations may involve more complicated forms, where the disturbance enters as an argument of the main functions.

One can think of two types of shocks exogenous to a given economic system, say a macroeconomic model of a country. Shocks of the first type are generated by global forces that are either completely outside of human control (for example, weather phenomena) or are shaped in some worldwide marketplace (for example, commodity prices which are determined in the world markets). Zhou et al. (2002) demonstrate that the flood series in the Huaihe river basin in China over the last 500 years exhibits chaotic dynamics. Decoster et al. (1992) found evidence of chaotic motion in daily silver, copper, sugar and coffee futures prices. Wei and Leuthold (1998) show that futures prices of corn, soybeans, wheat, hogs and coffee are chaotic processes. Panas and Ninni (2000) provide strong support to the presence of chaos in daily oil product prices in the Rotterdam and Mediterranean petroleum markets. These works employ tests developed by Brock (1986) and Brock et al. (1996), among others, that aim to distinguish between random and chaotic deterministic series. While it is in general very difficult to do so, especially for high-dimensional systems and for short economic time series (Sakai and Tokumaru, 1980; Benhabib, 2008), this only implies that just as there is as yet no definite proof of the chaotic nature of economic variables, there is no definite proof of their random nature, either. Moreover, it is plausible that a hybrid of the two types of processes generates some economic data.

The second type of exogenous shocks that could affect a given economic system is shocks generated outside the system but endogenous to some other system that is linked with the former through financial, trade and information flows. In this case we can talk of the transmission of chaos from one economy to another. Multiple papers investigating the emergence of endogenous chaos in economic models have been produced. Many of them study Kaldor–Kaleckian or Keynesian models of the macroeconomy, as in Lorenz (1993), Medio and Gallo (1992), and Zhang (2005), where real output is determined along with other economic variables, such as capital stock or money supply. Suppose real output in a foreign economy affects the level of demand by this economy for the exports of the home country, and exports to the foreign economy influence the economic activity at home. Then exports to the foreign country may be viewed as an exogenous shock to the domestic economic system. The present paper points out that if real output abroad is chaotic, then the variables at home will be chaotic, as well, that is, chaos is transmitted through chaotic export shocks from the foreign to the home economy. Lorenz (1987) produces chaos in a system consisting of three similar economies linked through international trade, a six-dimensional system altogether. However, his goal is to show that multidimensional systems of the kind that generate chaos are plausible in economics, rather than to study the transmission of chaos internationally.

There is also a literature that studies the emergence of endogenous chaos in economic models with microfoundations, such as standard models of overlapping generations and models with infinitely lived representative agents (Benhabib and Day, 1980, 1982; Benhabib and Nishimura, 1979; Boldrin and Montrucchio, 1986; Deneckere and Pelikan, 1986; Grandmont, 1985). Benhabib and Day (1982) give several examples of utility functions that generate chaotic consumption trajectories in a standard, deterministic, overlapping generations model. Among others, they derive a logistic map as the optimal consumption function. Boldrin and Montrucchio (1986) and Deneckere and Pelikan (1986) show that in dynamic optimization problems satisfying the standard continuity and convexity assumptions, the optimal policy function can be chaotic. In these investigations, the discount factor plays an important role. Nishimura et al. (1994) and Nishimura and Yano (1995) show that chaotic optimal solutions can be obtained in these models even for a discount factor arbitrarily close to 1. Mitra and Sorger (1999) prove that the logistic map can be the optimal policy function of a regular dynamic optimization problem if and only if the discount factor does not exceed 1/16. We rely on the results of Benhabib and Day (1982) and Mitra and Sorger (1999), among others, to motivate our use of the logistic map in what follows.

An implication of our results is that detecting the source of chaos in an economic system is crucial for effective control of said chaos. The complex nature of economic systems implied by the presence of chaos may suggest that the evolution of economic variables is not only unpredictable, but also uncontrollable. To borrow a citation from Mendes and Mendes (2005), the common view until early 1990s was that 'a chaotic motion is generally neither predictable, nor controllable. It is unpredictable because a small disturbance will produce exponentially growing perturbation of the motion. Is it uncontrollable because small disturbances lead only to other chaotic motions and not to any stable and predictable alternative' (Dyson, 1988). As a corollary, it may seem that 'any improvement in the functioning of these economies would require a radical change to their basic structures, because the crises and booms associated with the dynamics of capitalist structures, by being chaotic manifestations, can be neither controllable nor predictable' (Mendes and Mendes, 2005).

However, developments in the study of chaos since early 1990s have provided theoretical tools to effectively control chaos (Ott et al., 1990; Pyragas, 1992; Shinbrot et al., 1990; Chen and Raton, 2000; Chen and Yu, 2003; Kapitaniak, 1996; Scholl and

Schuster, 2007). These methods rely on the sensitivity of chaotic systems to small changes, by fine-tuning the parameters of the system to nudge the dynamics toward a desired trajectory. 'In the case of chaotic systems, as these are sensitive to very small changes in the parameters, a small butterfly effect in one of them is (in most cases) all that is required to control their outcome, without changing the very nature of the controlled system in any relevant way', while 'conventional classical control techniques control the dynamics of nonlinear processes through the use of brute force, having in fact frequently to change the nature of the very system that is subject to control' (Mendes and Mendes, 2005). As a result, the cost of these control instruments is likely to be small, as well.

While the application of chaos control methods to real world economic policy-making remains an open question, numerous papers have demonstrated the potential implementation of these techniques in various economic settings. Holyst et al. (1996), Holyst and Urbanowicz (2000), Ahmed and Hassan (2000), Salarieh and Alasty (2009), and Chen and Chen (2007) control chaos in microeconomic models of firm competition, such as Cournot duopoly/oligopoly and Behrens–Feichtinger model of two competing firms (Behrens, 1992; Feichtinger, 1992). Kaas (1998) and Bala et al. (1998) implement chaos control in macroeconomic disequilibrium models, and Kopel (1997) does so in a disequilibrium model of firms with bounded rationality. Haag et al. (1997) stabilize a chaotic urban system, Mendes and Mendes (2005) control chaos in an overlapping generations model (OLG), and Wieland and Westerhoff (2005) demonstrate the possible control of chaotic exchange rate dynamics by a central bank. In all these applications control is carried out by varying the values of parameters that have a clear economic interpretation and that can be plausibly set at will by either the government or private actors, such as firms. For example, in Kaas (1998) the government varies income tax rates or government expenditures to stabilize an unstable Walrasian equilibrium, in Wieland and Westerhoff (2005) the central bank intervenes in the foreign exchange market by varying the value of the foreign exchange buy orders, and in Salarieh and Alasty (2009) chaos can be controlled either through government production tax/subsidy imposed on firms or through firms' adjustment of their production quantities.

The literature on the control of chaos originated with Ott et al. (1990). Their method (commonly known as the OGY method) relies on the observation that a chaotic set contains an infinite number of unstable periodic orbits. One can select the most desirable unstable periodic orbit, wait until the system approaches it sufficiently and apply a slight nudge to an appropriate parameter to keep the system on that orbit. Notice that the controller can choose which orbit out of infinitely many orbits to target. Particularly, the policy-makers may pick a trajectory that delivers the highest welfare, based on the preferences for levels and volatility of the variables of interest. The implementation of the method requires an observation of a slice of the chaotic attractor (called the Poincaré section). This can be done for most economic variables, data on which is collected by governments and other agencies. Finally, Ott et al. (1990) show that their approach is effective if a random noise is introduced into the system, as long as the noise variable assumes extreme values very infrequently, i.e. is sufficiently bounded. This is very convenient for the hybrid case of both deterministic chaos and random shocks present in a model.

We will focus on the OGY method due to the advantages mentioned above. We argue that correctly identifying the source of chaos in an economic system can have a significant impact on the implementation of the OGY method. The construction of the Poincaré map, an essential step, demands the knowledge of the solutions in analytical form and this is an unsolvable problem in many cases, since chaotic dynamics are non-linear. Therefore it is extremely convenient to be able to isolate and apply the OGY control directly to an exogenous shock that is driving the chaotic dynamics in a system and whose Poincaré map can be constructed (either as an analytical solution to a differential equation or through empirical analysis). This would be less resource-consuming, since it would involve modifying fewer parameters, and in some cases could be the only feasible solution.

Moreover, our findings emphasize the cost-effectiveness and importance of international cooperation in economic policy. In the case of the first type of exogenous shocks mentioned above, such as commodity prices (oil, gold, silver, etc.), whose values are determined in a global marketplace, cooperation between the major players in the market could allow to control the chaotic dynamics of these variables and would translate into control of chaos in all economies affected. For the second type of shocks, it is plausible that controlling chaos in one economy can be done most effectively with the cooperation of another country that is the source of exogenous chaotic shocks to the home economy. In the extreme case, controlling chaos in one economy can help control chaos in another economy, which in turn helps control chaos in a third country, and so on and so forth. Properly identifying the source of global chaos in one economy and controlling chaos there would then be the most effective way to control chaos worldwide and turns out to be very cheap relative to the scale of the ultimate effect.

The rest of the paper is organized as follows. In Section 2 we discuss the particular class of exogenous disturbance that we consider in this paper – i.e. exogenous perturbations that take the form of a pulse function. The mathematical investigation of the perturbed system is presented in Section 3. Simulations of a Kaldor–Kalecki model with pulsative disturbances of two types – export shocks with chaotically behaved values and rainfall shocks with chaotically behaved discontinuity instants – are shown in Section 4. We also demonstrate the application of the OGY control method to these models. Section 5 concludes.

2. Modeling the exogenous shock

Theoretically, it is clear that we need a chaotic function to model the disturbance, but in practice there is not a ready supply of such functions. For this reason, we have to either use solutions of differential equations that are known for their chaotic properties, or create functions with chaotic elements. In this paper, we employ the latter approach. In future work, we will consider how one can use actual economic time series, such as those of commodity prices, that have been tested for deterministic chaos (Barnett and Chen, 1988; Brock, 1986; Decoster et al., 1992; Frank and Stengos, 1989; Panas and Ninni, 2000; Wei and Leuthold, 1998; Zhou et al., 2002), as exogenous shocks.

We investigate exogenous perturbations that take the form of a pulse function. Consider a strictly increasing sequence of real numbers $\{\theta_i\}$ such that $|\theta_i| \to \infty$ as $|i| \to \infty$. We say that a function $p(t): \mathbb{R} \to \mathbb{R}^n$ is a *pulse function* if for each integer i there is $p_i \in \mathbb{R}^n$ such that $p(t) = p_i$ either on the interval $(\theta_i, \theta_{i+1}]$ or on the interval $[\theta_i, \theta_{i+1})$.

Consider a general form of economic models,

$$\dot{v} = H(v), \tag{2.1}$$

where $v: \mathbb{R} \to \mathbb{R}^n$ is a function of time, $v(t)$, and $H: \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in its arguments.

Perturb the model chaotically (this will be explained later) with a pulse function $\widetilde{d}_{[t/h]}$, where h is a fixed positive real number, $[s]$ denotes the largest integer that is not greater than s, so that $\widetilde{d}_{[t/h]} = \widetilde{d}_i \in \mathbb{R}^n$, if $ih \leq t < (i+1)h$, i is an integer. We obtain the following model,

$$\dot{v} = H(v) + \widetilde{d}_{[t/h]}. \tag{2.2}$$

Assume that the pulse function has only one non-zero coordinate, that is only one equation in system (2.1) is chaotically perturbed. This assumption will be justified in Section 3. This is a specific case only, and the more general case can be investigated in a similar manner. Suppose that

$$\widetilde{d}_i = (g(d_i), 0, 0, \ldots, 0) \tag{2.3}$$

for all integers i, where $g: \mathbb{R} \to \mathbb{R}$ is a continuous function.

We construct the values of the disturbance using a solution of a discrete equation:

$$d_i = F(d_{i-1}), \tag{2.4}$$

where the function $F: \mathbb{R} \to \mathbb{R}$ is continuous, and (2.4) generates chaotic exogenous shocks. Two definitions of chaos are used: Devaney chaos and chaos through period-doubling cascade.

While it is an intuitive conclusion, one has to verify rigorously whether system (2.2) admits chaos. This is the objective of our paper. One of the most convenient ways of analysis in dynamics is to consider a problem near an equilibrium. So, assume that (2.1) admits a steady state at $v = v^*$. Transform the state variables $x = v - v^*$ in (2.2). Then, near the equilibrium point the linearized model takes the form

$$\begin{aligned} \dot{x} &= Ax + f(x) + \widetilde{d}_{[t/h]}, \\ d_{[t/h]} &= F(d_{[t/h]-1}), \end{aligned} \tag{2.5}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, $f(0) = 0$. Assume that A is a matrix, all of whose eigenvalues have negative real parts, and $f: \mathbb{R}^n \to \mathbb{R}^n$ is a function continuously differentiable in its arguments.

We call the type of disturbances just described as pulsative perturbations with chaotically behaved values and prove that the perturbed system exhibits chaos. This is the first time that such disturbances are introduced in the chaos literature. There are many applications for shocks with chaotically behaved values in economics. Consider economic time series such as commodity prices, productivity indices and international trade indicators, all of which are examples of exogenous shocks in some economic models. These are usually gauged by economists at regular discrete intervals, no matter how disaggregated (year, month, day, minute, second), and it is their value that is unpredictable and irregular. Another interpretation is that some variables truly change values only at fixed times, for example, the government budget that is determined once a year, earnings of a farm that sells its produce in accordance with the seasons, a firm's capital equipment that changes with periodical investment. All of these quantities vary at regular instants of time, but their values may be irregular. Thus, pulsative perturbations with chaotically behaved values are a good approximation of reality. In this paper we provide rigorous verification of the presence of chaos in system (2.5).

To model F, one can consider the logistic map of P.-F. Verhulst (Guckenheimer and Holmes, 1997),

$$F(d_{i-1}) = \mu d_{i-1}(1 - d_{i-1}). \tag{2.6}$$

It is known that if $0 < \mu \leq 4$, then the unit interval [0, 1] is invariant under the iterations of the map, and there are values of the parameter μ such that the map is chaotic. The logistic map plays a very important role in many fields of science, and economics is not an exception. Good examples of the applications of the logistic map and its historical background are provided in Ausloos and Dirickx (2010).

Bala et al. (1998) show that for particular forms of the utility functions in a simple discrete-time model of an exchange economy with two goods under Walrasian tatonnement, the evolution of the price of the non-numeraire good is described with a logistic map. This result can be used to model commodity prices, such as prices of oil, gold, and silver, using a logistic map. Benhabib and Day (1982) obtain a logistic map as the law of motion of consumption in a simple overlapping generations model with quadratic utility function, and Mitra and Sorger (1999) verify that the logistic map can be the optimal policy function of a regular dynamic optimization problem, i.e. one satisfying some regularity assumptions, if and only if the discount factor does not exceed 1/16. We use these results to motivate our use of the logistic map to model the export shock in the Kaldor–Kalecki model of the aggregate economy in Section 4, where exports to a foreign country are a function of consumption levels there. Since consumption in the foreign country can be thought of as the solution to a regular dynamic optimization problem of foreign consumers, we describe it with a logistic map. Of course, the logistic map is only an illustrative example of a wide range of chaotic dynamics that exogenous shocks can follow. Modeling the shocks in any other way in our simulations would not alter the main message of our paper.

An alternative way to generate a pulsative chaotic perturbation is considering a pulse function with chaotically behaved discontinuity instants, the θ_i in the definition of the pulse function above (as opposed to chaotically behaved values):

$$\nu(t,d)=\begin{cases} m_0, & \text{if} \quad \theta_{2i}(d)<t\leq\theta_{2i+1}(d)\\ m_1, & \text{if} \quad \theta_{2i-1}(d)<t\leq\theta_{2i}(d),\end{cases} \tag{2.7}$$

where i is an integer and m_0 and m_1 are real numbers such that $m_0\neq m_1$. The sequence $\{\theta_i(d)\}$, which defines the discontinuity instants of the function $\nu(t, d)$, is introduced through the equation $\theta_i(d)=i+d_i$, where $d=\{d_i\}$ is a solution of Eq. (2.4). Examples of this type of shocks are natural disasters and extreme events in general, such as market crashes. They take a finite number of values (an earthquake either happens or not), but their timing is irregular.

The original system (in its linearized form) then becomes

$$\dot{x}=Ax+f(x)+\nu(t,d). \tag{2.8}$$

The theory of the systems of the form (2.8) is described in Akhmet (2009) and Akhmet and Fen (2012). In this work, we will present simulations of chaos in a Kaldor–Kalecki model subject to such shocks. We will also demonstrate the application of the OGY control method for both types of shocks described.

3. Mathematical investigation of system (2.5)

In this section, we study differential equations perturbed by a pulse function with chaotically behaved values. We first give a complete description of the perturbation, and then consider the space of all bounded solutions of the system.

We shall make use of the uniform norm $\|\Gamma\|=\sup_{\|v\|=1}\|\Gamma v\|$ for any matrix Γ.

Since all eigenvalues of the constant $n\times n$ real-valued matrix A have negative real parts, one can verify the existence of positive real numbers N and ω such that the inequality $\|e^{At}\|\leq Ne^{-\omega t}$ is valid for all $t\geq 0$.

The following four assumptions are needed throughout this section and the paper:

(C1) There exist positive real numbers M_f and M_g such that $\sup_{x\in\mathbb{R}^n}\|f(x)\|=M_f$, $\sup_{s\in\mathbb{R}}|g(s)|=M_g$;
(C2) There exists a positive real number L_f such that the inequality $\|f(x_1)-f(x_2)\|\leq L_f\|x_1-x_2\|$ holds for all $x_1, x_2\in\mathbb{R}^n$;
(C3) There exist positive real numbers L_1 and L_2 such that the inequality $L_1|s_1-s_2|\leq|g(s_1)-g(s_2)|\leq L_2|s_1-s_2|$ holds for all $s_1, s_2\in\mathbb{R}$;
(C4) $NL_f-\omega<0$.

Condition (C4) ensures that system (2.2) is weakly nonlinear. We assume that Eq. (2.4) admits a set of bounded solutions, defined for all integers. More precisely, assume that there exists a bounded set Λ of real numbers such that the values of bounded solutions are in this set. Notice that in the case of the logistic map (2.6), with $0<\mu\leq 4$, the set Λ can be taken as the unit interval [0, 1]. We shall denote by $\mathcal{D}$ the set of all bounded solutions.

To solve system (2.5), one has to solve the discrete equation (2.4), given initial value d_0, obtain a sequence $\{d_i\}$ as a solution, build a function $d_{[t/h]}=d_i$, if $t\in[ih, (i+1)h)$, and substitute this function in (2.5). The resulting system is

$$\dot{x}=Ax+f(x)+\widetilde{d}_{[t/h]}, \tag{3.1}$$

where $\widetilde{d}_{[t/h]}=(g(d_{[t/h]}), 0, 0, \ldots, 0)\in\mathbb{R}^n$. If the Lipschitz constant L_f is sufficiently small so that condition (C4) is satisfied, then for a given $d\in\mathcal{D}$ this system admits a unique, bounded on the entire real axis, solution, denoted by $\phi_d(t)$ (Hale, 1980). Let us denote by X the set of such solutions for all possible $d\in\mathcal{D}$. One can show that $\phi_d(t)$ satisfies the relation (Akhmet, 2011)

$$\phi_d(t)=\int_{-\infty}^{t}e^{A(t-s)}(f(\phi_d(s))+\widetilde{d}_{[s/h]})\,ds. \tag{3.2}$$

Let us denote $M=M_f+M_g$, where the numbers M_f and M_g are discussed in condition (C1). For any $x(t)\in X$, we have $\sup_{t\in\mathbb{R}}\|x(t)\|\leq H_0$, where $H_0=NM/\omega$. That is, all bounded solutions of system (3.1) lie in a tube with radius H_0.

In what follows, for fixed $d\in\mathcal{D}$, the function $x_d(t, x_0)$, $x_0\in\mathbb{R}^n$, will stand for the unique solution of system (3.1) with the initial condition $x_d(0, x_0)=x_0$. Notice that this solution is not necessarily bounded.

We say that a sequence $\{d_i\} \in \mathcal{D}$ is p-periodic if there exists a natural number p such that $d_{i+p} = d_i$ for each integer i. Suppose that system (2.4) admits infinitely many periodic solutions, and let us denote the set of all such solutions by $\mathcal{P}$, which is a subset of $\mathcal{D}$.

By applying the standard technique (Hale, 1980), common for quasilinear ordinary differential equations, one can prove the following two assertions. We omit their verification.

Lemma 3.1. *For every $d \in \mathcal{D}$ and $x_0 \in \mathbb{R}^n$, the inequality $\|x_d(t, x_0) - \phi_d(t)\| \leq N \| x_0 - \phi_d(0) \| e^{(NL_f - \omega)t}$ holds for all $t \geq 0$.*

Using the last lemma together with condition (C3), one can show that for every $d \in \mathcal{D}$ and any $x_0 \in \mathbb{R}^n$, $\|x_d(t, x_0) - \phi_d(t)\| \to 0$ as $t \to \infty$, and consequently $x_d(t, x_0)$ eventually enters the tube with radius H_0.

The proof of the next lemma uses representation (3.2).

Lemma 3.2. *Suppose that p is a natural number. If $d \in \mathcal{P}$ is a p-periodic sequence, then the solution $\phi_d(t)$ of system (3.1) is ph-periodic, and vice versa.*

Now, we demonstrate the chaotic properties of Eq. (2.4). We use two exact mathematical descriptions of chaos: Devaney chaos and chaos through period-doubling cascade. The former is the most theoretical known type of chaos, and the latter is convenient for simulations.

The following are the ingredients of chaos (Devaney, 1989), adapted for our needs. They hold for any map which is topologically conjugate to symbolic dynamics (Akhmet, 2011).

(i) The set $\mathcal{D}$ is called sensitive if there exists a positive real number $\bar{\epsilon}$ such that, for each sequence $\{d_i\} \in \mathcal{D}$ and an arbitrary positive real number δ, there exist a sequence $\{c_i\} \in \mathcal{D}$ and a natural number j such that $|c_i - d_i| < \delta$ for all $i \leq 0$ and $|c_j - d_j| > \bar{\epsilon}$.

(ii) The set $\mathcal{D}$ is called transitive if there exists a sequence $\{d_i^*\} \in \mathcal{D}$ such that for each $\{d_i\} \in \mathcal{D}$, an arbitrarily small positive number ϵ and an arbitrarily large natural number E, there exist a natural number m and an integer n such that $|d_i - d_{i+m}^*| < \epsilon$ for each integer i between n and $n + E$.

(iii) The set of all periodic solutions $\mathcal{P}$ of Eq. (2.4) is called dense in $\mathcal{D}$ if for each sequence $\{d_i\} \in \mathcal{D}$, an arbitrarily small positive number ϵ and an arbitrarily large natural number E, there exist a periodic sequence $\{c_i\} \in \mathcal{P}$ and an integer n such that $|c_i - d_i| < \epsilon$, for each integer i between n and $n + E$.

In our discussions of chaos, we will suppose that the set $\mathcal{D}$ is sensitive, transitive and admits a dense set of periodic solutions.

We will make use of the number $\tau = \min\{(h/2), (L_1\bar{\epsilon}h/(4(H_0\|A\| + M)[2 + h(L_f + \|A\|)]))\}$ in the next lemma, where $\bar{\epsilon}$ is that from the definition (i) of sensitivity of the set $\mathcal{D}$.

Lemma 3.3. *Suppose that the set $\mathcal{D}$ is sensitive. In this case, there exists a positive number ϵ_0 such that for each sequence $d \in \mathcal{D}$ and an arbitrary positive real number δ, there exist $c \in \mathcal{D}$ and an interval $J \subset [0, \infty)$ of length τ such that $\|\phi_c(0) - \phi_d(0)\| < \delta$ and $\|\phi_c(t) - \phi_d(t)\| > \epsilon_0$, for all $t \in J$. That is, X is sensitive.*

Proof. Fix an arbitrary sequence $d \in \mathcal{D}$ and an arbitrary positive number δ. Let us take a sufficiently small positive real number δ_0 which satisfies the inequality $(1 + (NL_2/\omega - NL_f))\delta_0 < \delta$ and a negative real number R such that $(2MN/\omega)e^{(\omega - NL_f)R} < \delta_0$.

Since the set $\mathcal{D}$ is sensitive, there exists a positive number $\bar{\epsilon}$ such that both of the inequalities $|c_i - d_i| < \delta_0$, $i \leq 0$, and $|c_j - d_j| > \bar{\epsilon}$ hold for some sequence $c \in \mathcal{D}$ and a natural number j.

First of all, we will show that $\|\phi_c(0) - \phi_d(0)\| < \delta$. According to the relation (3.2), the functions $\phi_c(t)$ and $\phi_d(t)$ satisfy the following couple of integral equations

$$\phi_c(t) = \int_{-\infty}^{t} e^{A(t-s)}(f(\phi_c(s))) + \widetilde{c}_{[s/h]})\,ds, \qquad \phi_d(t) = \int_{-\infty}^{t} e^{A(t-s)}(f(\phi_d(s))) + \widetilde{d}_{[s/h]})\,ds,$$

where $\widetilde{c}_{[s/h]} = (g(c_{[s/h]}), 0, 0, \ldots, 0) \in \mathbb{R}^n$ and $\widetilde{d}_{[s/h]} = (g(d_{[s/h]}), 0, 0, \ldots, 0) \in \mathbb{R}^n$.

Using these equations one can obtain for $R \leq t \leq 0$ that

$$e^{\omega t}\|\phi_c(t) - \phi_d(t)\| \leq \frac{2MN}{\omega}e^{\omega R} + \frac{NL_2\delta_0}{\omega}(1 - e^{-\omega(t-R)})e^{\omega t} + NL_f\int_R^t e^{\omega s}\|\phi_c(s) - \phi_d(s)\|ds.$$

Applying Gronwall's Lemma (Hale, 1980) to the last inequality, one can find that

$$\|\phi_c(0) - \phi_d(0)\| \leq \frac{NL_2\delta_0}{\omega - NL_f} + \frac{2MN}{\omega}e^{(\omega - NL_f)R} < \delta.$$

In the remaining part of the proof, we shall determine an interval $J \subset [0, \infty)$ of length τ such that the inequality $\|\phi_d(t) - \phi_c(t)\| > \epsilon_0$ is valid for all $t \in J$.

For $t \in [jh, (j+1)h]$, the functions $\phi_c(t)$ and $\phi_d(t)$ satisfy the equation

$$\phi_c(t) - \phi_d(t) = (\phi_c(jh) - \phi_d(jh)) + \int_{jh}^{t} A(\phi_c(s) - \phi_d(s))\,ds + \int_{jh}^{t} [f(\phi_c(s)) - f(\phi_d(s))]\,ds + \int_{jh}^{t} (\widetilde{c}_{[s/h]} - \widetilde{d}_{[s/h]})\,ds,$$

and evaluating at $t = (j+1)h$, one can produce the inequality

$$\|\phi_c((j+1)h) - \phi_d((j+1)h)\| \geq |g(c_j) - g(d_j)|h - \|\phi_c(jh) - \phi_d(jh)\| - \int_{jh}^{(j+1)h} (L_f + \|A\|)\|\phi_c(s) - \phi_d(s)\|\,ds.$$

By means of the last inequality, we have

$$\max_{t\in[jh,(j+1)h]} \|\phi_c(t) - \phi_d(t)\| \geq \|\phi_c((j+1)h) - \phi_d((j+1)h)\| > L_1\overline{\epsilon}h - [1 + h(L_f + \|A\|)] \max_{t\in[jh,(j+1)h]} \|\phi_c(t) - \phi_d(t)\|.$$

Therefore, $\max_{t\in[jh,(j+1)h]} \|\phi_c(t) - \phi_d(t)\| > (L_1\overline{\epsilon}h/([2 + h(L_f + \|A\|)]))$.

Suppose that on the interval $[jh, (j+1)h]$, the real valued function $\|\phi_c(t) - \phi_d(t)\|$ takes its maximum value at the point η. Let us define the number $\xi = \begin{cases} \eta, & \text{if} \quad \eta \leq jh + (h/2) \\ \eta - \tau, & \text{if} \quad \eta > jh + (h/2) \end{cases}$, and let $J = [\xi, \xi + \tau]$, which is an interval of length τ. We note that the interval J is a subset of the interval $[jh, (j+1)h]$ and depends on the sequences c and d, but its length remains the same for different sequences.

By virtue of the inequality

$$\|\phi_c(t) - \phi_d(t)\| \geq \|\phi_c(\eta) - \phi_d(\eta)\| - \left| \int_{\eta}^{t} \|A\|\,\|\phi_c(s) - \phi_d(s)\|ds \right| - \left| \int_{\eta}^{t} \|f(\phi_c(s)) - f(\phi_d(s)) + \widetilde{c}_{[s/h]} - \widetilde{d}_{[s/h]}\|ds \right|,$$

for $t \in J$, one has $\|\phi_c(t) - \phi_d(t)\| > \epsilon_0$, where $\epsilon_0 = (L_1\overline{\epsilon}h/(2[2 + h(L_f + \|A\|)]))$.

The proof is finalized. □

We shall proceed to the next ingredient of Devaney chaos. In the case when Eq. (2.4) possesses a dense sequence $d^* \in \mathcal{D}$, the following assertion is valid.

Lemma 3.4. *Suppose the set $\mathcal{D}$ is transitive. Then there exists a solution $\phi_{d^*}(t) \in X, d^* \in \mathcal{D}$ such that for each solution $\phi_d(t) \in X$, $d \in \mathcal{D}$, an arbitrarily small positive real number ϵ and an arbitrarily large natural number E, there exist a positive real number ζ and an interval $J \subset \mathbb{R}$ of length Eh, such that $\|\phi_d(t) - \phi_{d^*}(t+\zeta)\| < \epsilon$, for all $t \in J$.*

Proof. Fix an arbitrarily small positive number ϵ and an arbitrarily large natural number E. Let $d \in \mathcal{D}$ be a given solution of Eq. (2.4) and suppose that $\gamma = ((\omega(\omega - NL_f))/(2MN(\omega - NL_f) + NL_2\omega))$. Since $\mathcal{D}$ is transitive, there exist a natural number m and an integer n such that $\|d_i - d^*_{i+m}\| < \gamma\epsilon$, for all integers i satisfying $n \leq i \leq n + 2E$.

Let $\zeta = mh$. Eq. (3.2) implies that $\phi_{d^*}(t+\zeta) = \int_{-\infty}^{t} e^{A(t-s)}(f(\phi_{d^*}(s+\zeta)) + \widetilde{d^*}_{[s/h]+m})\,ds$. Therefore, for $t \in [nh, (n+2E)h]$, the equation

$$\phi_d(t) - \phi_{d^*}(t+\zeta) = \int_{-\infty}^{nh} e^{A(t-s)}(f(\phi_d(s)) - f(\phi_{d^*}(s+\zeta)) + \widetilde{d}_{[s/h]} - \widetilde{d^*}_{[s/h]+m})\,ds + \int_{nh}^{t} e^{A(t-s)}(f(\phi_d(s)) - f(\phi_{d^*}(s+\zeta)))ds$$
$$+ \int_{nh}^{t} e^{A(t-s)}(\widetilde{d}_{[s/h]} - \widetilde{d^*}_{[s/h]+m})\,ds$$

holds, where $\widetilde{d}_{[s/h]} = (g(d_{[s/h]}), 0, 0, \ldots, 0)$ and $\widetilde{d^*}_{[s/h]} = (g(d^*_{[s/h]}), 0, 0, \ldots, 0)$ are n-dimensional vectors in $\mathbb{R}^n$.

Making use of the last equation we obtain that

$$e^{\omega t}\|\phi_d(t) - \phi_{d^*}(t+\zeta)\| \leq \frac{2MN}{\omega}e^{\omega nh} + \frac{NL_2\gamma\epsilon}{\omega}(e^{\omega t} - e^{\omega nh}) + \int_{nh}^{t} NL_f e^{\omega s}\|\phi_d(s) - \phi_{d^*}(s+\zeta)\|ds.$$

Applying Gronwall's Lemma (Hale, 1980), one can arrive at the following inequality

$$\|\phi_d(t) - \phi_{d^*}(t+\zeta)\| \leq \frac{2MN}{\omega}e^{(NL_f-\omega)(t-nh)} + \frac{NL_2\gamma\epsilon}{\omega - NL_f}(1 - e^{(NL_f-\omega)(t-nh)}).$$

Suppose that the natural number E is large enough so that $Eh > (1/(\omega - NL_f))\ln(1/\gamma\epsilon)$ and let the interval J be defined as $J = [(n+E)h, (n+2E)h]$. We note that the length of the interval J is Eh. For $t \geq (n+E)h$, it is the case that $e^{(NL_f-\omega)(t-nh)} \leq e^{(NL_f-\omega)Eh} < \gamma\epsilon$. Consequently, for $t \in J$ we have $\|\phi_d(t) - \phi_{d^*}(t+\zeta)\| < ((2MN/\omega) + (NL_2/(\omega - NL_f)))\gamma\epsilon = \epsilon$.

The proof of the lemma is completed. □

For the case when the set of all periodic solutions $\mathcal{P}$ of Eq. (2.4) is dense in $\mathcal{D}$, we shall formulate an important ingredient of chaos for the set X, which states that an arbitrary function chosen from this set can be approximated by periodic functions from the set X_P of all periodic solutions of system (3.1) on intervals of arbitrary lengths. In other words, the density property of the set $\mathcal{P}$ is inherited by the set X_P. We call this property the density of the set X_P in X. The next assertion can be proved similarly to the previous lemma.

Lemma 3.5. *Suppose that the set of all periodic solutions $\mathcal{P}$ of Eq. (2.4) is dense in $\mathcal{D}$. Then for every solution $\phi_d(t) \in X$, $d \in \mathcal{D}$, an arbitrarily small positive ϵ and an arbitrarily large positive E, one can find a periodic solution $\phi_c(t) \in X_P$, $c \in \mathcal{P}$, and an interval $J \subset \mathbb{R}$ of length Eh, such that $\|\phi_c(t) - \phi_d(t)\| < \epsilon$ for $t \in J$.*

We call the combined properties expressed in Lemmas 3.3–3.5 as chaos in the sense of Devaney for the set X, and we can formulate the following theorem.

Theorem 3.1. *The set X is chaotic in the sense of Devaney provided that the set $\mathcal{D}$ is sensitive, transitive and possesses a dense set of periodic sequences.*

Next, let us describe chaos for Eq. (2.4) as obtained through period-doubling cascade.
Let us consider the equation

$$d_i = G(d_{i-1}, \mu), \tag{3.3}$$

where i is an integer and the function $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies for all $x \in \mathbb{R}$ the property that $F(x) = G(x, \mu_\infty)$, for some finite value μ_∞ of the parameter μ, which will be explained below.

Suppose that there exist a natural number k_0 and a sequence of period-doubling bifurcation values $\{\mu_m\}$ of the parameter μ, such that for each natural number m, as the parameter μ increases or decreases through μ_m, system (3.3) undergoes a period-doubling bifurcation and the previously existing stable $k_0 2^{m-1}$-periodic sequence becomes unstable and is replaced by a stable periodic sequence of period $k_0 2^m$. Moreover, the sequence $\{\mu_m\}$ of parameter values converges to a finite value μ_∞ as $m \to \infty$ and as a result, at $\mu = \mu_\infty$, there exist infinitely many unstable periodic solutions of Eq. (3.3), and consequently of Eq. (2.4), all lying in a bounded region. In this case, we say that Eq. (2.4) admits chaos through period-doubling cascade (Feigenbaum, 1983).

Since chaos through period-doubling cascade is based on the existence of infinitely many periodic solutions, if Eq. (2.4) admits chaos through period-doubling cascade, then by Lemma 3.2 the same is true for system (3.1), as stated in the following theorem. The instability of these periodic solutions can be proved using the same technique as in the proof of Lemma 3.3.

Theorem 3.2. *If Eq. (2.4) is chaotic through period-doubling cascade, then the same is true for (3.1).*

From the above discussion, Eq. (3.1), like Eq. (3.3), undergoes period-doubling bifurcations as the parameter μ increases or decreases through the values μ_m, $m \in \mathbb{N}$. In other words, the sequence $\{\mu_m\}$ of bifurcation parameters is exactly the same for both equations. It is worth pointing out that if Eq. (3.3) obeys the universality of Feigenbaum (1983), one can conclude that the same holds for Eq. (3.1). That is, when $\lim_{m\to\infty}((\mu_m - \mu_{m+1})/(\mu_{m+1} - \mu_{m+2}))$ is evaluated, the universal constant known as the Feigenbaum number 4.6692016... is achieved, and this universal number is the same for both equations, and consequently for system (2.5).

4. Chaos in a Kaldor–Kalecki model with exogenous shocks

Consider a model of the aggregate economy of a given country:

$$\begin{aligned} \dot{Y} &= \alpha[I(Y, K) - S(Y, K)], \\ \dot{K} &= I(Y, K) - \delta K, \end{aligned} \tag{4.1}$$

where Y is income, K is capital stock, I is gross investment, and S is savings. Income changes proportionally to the excess demand in the goods market, and the second equation is a standard capital accumulation equation. The constant depreciation rate δ and the adjustment coefficient α are positive. This model was studied in detail in Lorenz (1993) and Zhang (2005). It admits a stable equilibrium under certain conditions on the functions involved. We will show how perturbing it with a chaotic disturbance affects the resulting dynamics. For this purpose, let us consider the following specification of system (4.1) with $I(Y, K) = Y - aY^3 + bK$, $S(Y, K) = sY$,

$$\begin{aligned} \dot{Y} &= \alpha[(1-s)Y - aY^3 + bK], \\ \dot{K} &= Y - aY^3 + (b-\delta)K, \end{aligned} \tag{4.2}$$

where the constant parameters satisfy $\alpha > 0$, $a > 0$, $b < 0$, $0 < s < 1$ and $0 < \delta < 1$. We present the following modified systems:

$$\begin{aligned} \dot{Y} &= \alpha[(1-s)Y - aY^3 + bK] + g(d_{[t]}), \\ \dot{K} &= Y - aY^3 + (b-\delta)K, \\ d_{[t]} &= \mu d_{[t]-1}(1 - d_{[t]-1}). \end{aligned} \tag{4.3}$$

and

$$\dot{Y} = \alpha[(1-s)Y - aY^3 + bK] + \nu(t, d),$$
$$\dot{K} = Y - aY^3 + (b-\delta)K, \tag{4.4}$$

where the function $\nu(t, d)$ is defined in (2.7), and d is a solution of (2.4) and (2.6).

We introduce the perturbation only in the equation for income Y, since the equation for capital stock K can be viewed as a mechanical relation between investment and capital stock, where there is little room for exogenous influences. Income of a given country, on the other hand, is subject to many possible exogenous disturbances, such as productivity shocks and global economic fluctuations. This explains why we investigated the case of a perturbation with only one non-zero coordinate in the theoretical part. Of course, as we emphasized above, a more general case can be considered in a similar manner.

We model these perturbations as pulse functions, with chaotically behaved values in (4.3) and with chaotically behaved discontinuity instants in (4.4). Both types of disturbances are plausible: the first one is relevant if income shocks 'pulsate' at regular time intervals (that is, they change values monthly, daily, etc.). Many economic time series that are good examples of exogenous disturbances to output, such as productivity indices, international trade indicators and commodity prices, can be modeled in this way. The second case is applicable if the disturbances admit a finite number of values, but the timing of these is chaotic. For example, output shocks due to natural disasters and weather fluctuations could be described with a finite set of values (e.g. the 'Atlas of the Flood/Dryness in China for the last 500-year period' distinguishes between flood, wetness, normal level, dryness and aridity; Zhou et al., 2002), but their timing is irregular. Of course, there may be a third case, where both the values of the shocks and the instants of discontinuity evolve irregularly. We can provide simulations for this scenario, as well, and the resulting chaotic behavior would be similar to the other cases.

One can see that a steady state of (4.2) with positive coordinates

$$Y^* = \sqrt{\frac{\delta(1-s)+bs}{a\delta}}, \qquad K^* = \frac{s}{\delta}\sqrt{\frac{\delta(1-s)+bs}{a\delta}},$$

exists only if

$$\delta s < \delta + bs. \tag{4.5}$$

In the remaining part of the section, set $\alpha = 1$, $s = \delta = 1/2$, $b = -7/16$ to obtain $Y^* = 1/4\sqrt{a}$, $K^* = 1/4\sqrt{a}$. Now, the transformations $Y = y + Y^*$, $K = k + K^*$, applied to (4.2), give us the system

$$\dot{y} = (5/16)y - (7/16)k - ay^3 - (3/4)\sqrt{a}y^2,$$
$$\dot{k} = (13/16)y - (15/16)k - ay^3 - (3/4)\sqrt{a}y^2. \tag{4.6}$$

The eigenvalues for the associated linear system are $\lambda_1 = -1/2$, $\lambda_2 = -1/8$. In this case, the equilibrium is asymptotically stable, if the number a is chosen to be sufficiently small.

4.1. Perturbation with chaotically behaved values

Suppose that the home economy exports goods to a foreign country. The export flows are a function of consumption levels in the foreign country (normalized so that they lie in the interval [0, 1] and denoted by d), which evolve according to a logistic map: $d_{[t]} = \mu d_{[t]-1}(1 - d_{[t]-1})$. That is, foreign consumption is a pulse function, where the unit of time can be chosen as fine as desired (year, month, day, minute, etc.). We motivate our choice of the logistic map with the results of Benhabib and Day (1982) and Mitra and Sorger (1999), who show that in some standard optimization problems the optimal policy function is a logistic map, under certain conditions. In general, any other way of modeling chaotic shocks could be implemented. Assume that the export flows to the foreign country are determined by a cubic function, $ex_{[t]} \equiv 0.0005(d_{[t]} + d^3_{[t]})$, and $\mu = 3.8$. We multiply the export flows by the same multiplier as excess demand in the domestic goods market in the output equation, α, and since in our case $\alpha = 1$, we obtain the following system of type (4.3) introduced above:

$$\dot{Y} = (1-s)Y - aY^3 + bK + 0.0005(d_{[t]} + d^3_{[t]}),$$
$$\dot{K} = Y - aY^3 + (b-\delta)K, \tag{4.7}$$
$$d_{[t]} = 3.8d_{[t]-1}(1 - d_{[t]-1}),$$

with $g(d_{[t]}) \equiv 0.0005(d_{[t]} + d^3_{[t]})$.

Let us take $a = 0.02$ in system (4.7). Transforming the variables as $Y = y + Y^*$, $K = k + K^*$, where $Y^* = 5/2\sqrt{2}$, $K^* = 5/2\sqrt{2}$, one can obtain

$$\dot{y} = (5/16)y - (7/16)k - ay^3 - (3/4)\sqrt{a}y^2 + 0.0005(d_{[t]} + d^3_{[t]}),$$
$$\dot{k} = (13/16)y - (15/16)k - ay^3 - (3/4)\sqrt{a}y^2, \tag{4.8}$$
$$d_{[t]} = 3.8d_{[t]-1}(1 - d_{[t]-1}).$$

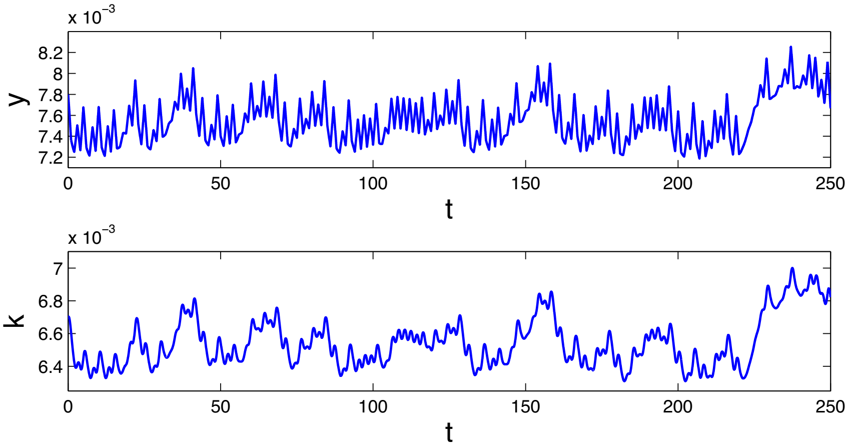

Fig. 1. The graphs of the y and k coordinates of the chaotic solution of system (4.8).

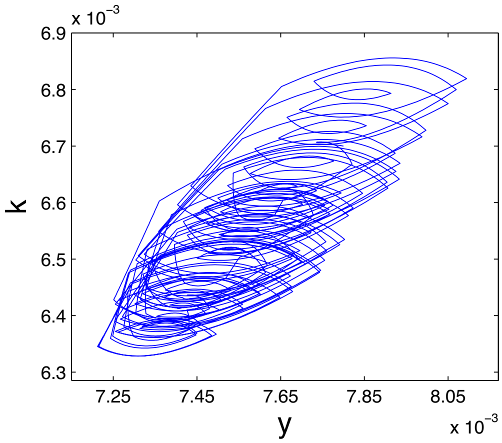

Fig. 2. The chaotic trajectory of system (4.8).

The sequence $\{d_i\}$ with $d_0 = 0.219$ is chaotic (Scholl and Schuster, 2007), and according to Theorem 3.2, the solution with $y(0) = 0.0078$, $k(0) = 0.0067$, is chaotic. In Figs. 1 and 2 the chaotic behavior of the solution is observable. Notice that both coordinates are positive.

We proceed by briefly explaining the OGY control method. Suppose that the parameter μ in the logistic map (2.6) is allowed to vary in the range $[3.8 - \varepsilon, 3.8 + \varepsilon]$, where ε is a given small number. That is, it is not possible (say, it is prohibitively costly or practically infeasible) to simply shift the value of μ to a level that generates non-chaotic dynamics. Let us consider an arbitrary solution $\{d_i\}$, $d_0 \in [0, 1]$, of the map and denote by $d^{(j)}$, $j = 1, 2, \ldots, p$, the target unstable p-periodic orbit to be stabilized. In the OGY control method (Scholl and Schuster, 2007), at each iteration step i after the control mechanism is switched on, we consider the logistic map with the parameter value $\mu = \overline{\mu}_i$, where

$$\overline{\mu}_i = 3.8\left(1 + \frac{[2d^{(j)} - 1][d_i - d^{(j)}]}{d^{(j)}[1 - d^{(j)}]}\right), \tag{4.9}$$

provided that the number on the right-hand side of the formula (4.9) belongs to the interval $[3.8 - \varepsilon, 3.8 + \varepsilon]$. In other words, we apply a perturbation in the amount of $((3.8[2d^{(j)} - 1][d_i - d^{(j)}])/(d^{(j)}[1 - d^{(j)}]))$ to the parameter $\mu = 3.8$ of the logistic map, if the trajectory $\{d_i\}$ is sufficiently close to the target periodic orbit. This perturbation makes the map behave regularly so that at each iteration step the orbit d_i is forced to be located in a small neighborhood of a previously chosen periodic orbit $d^{(j)}$. Unless the parameter perturbation is applied, the orbit d_i moves away from $d^{(j)}$ due to the instability. If $|((3.8[2d^{(j)} - 1][d_i - d^{(j)}])/(d^{(j)}[1 - d^{(j)}]|)) > \varepsilon$, we set $\overline{\mu}_i = 3.8$, so that the system evolves at its original parameter value, and wait until the trajectory $\{d_i\}$ enters a sufficiently small neighborhood of the periodic orbit $d^{(j)}$, $j = 1, 2, \ldots, p$, such that the inequality $-\varepsilon \le \frac{3.8[2d^{(j)}-1][d_i-d^{(j)}]}{d^{(j)}[1-d^{(j)}]} \le \varepsilon$ holds. If this is the case, the control of chaos is not achieved immediately after switching on the control mechanism. Instead, there is a transition time before the desired periodic orbit is stabilized. The transition time increases if the number ε decreases (Gonzáles-Miranda, 2004).

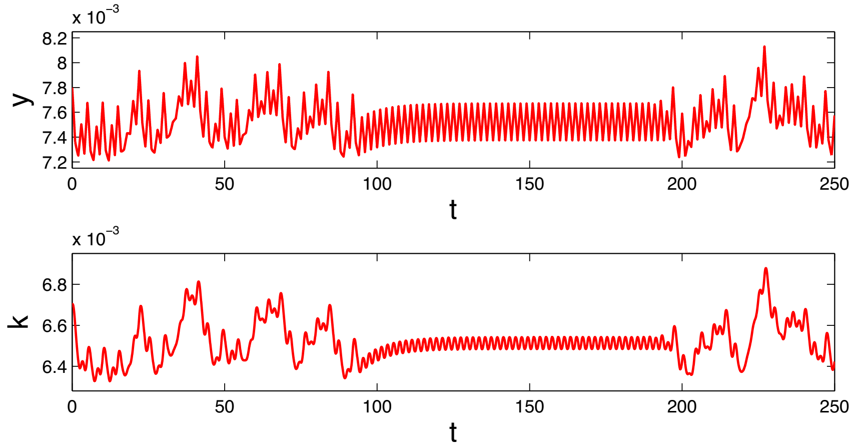

Fig. 3. OGY control method applied to system (4.8). It is seen in both panels that the 2-periodic solution of system (4.8) is stabilized.

An unstable p-periodic solution of system (4.8) can be stabilized by controlling the corresponding p-periodic solution of the third equation, that is, the p-periodic solution of the logistic map. In the next example, we stabilize a 2-periodic solution of system (4.8) by applying the OGY control around the 2-periodic orbit $d^{(1)} \approx 0.3737$, $d^{(2)} \approx 0.8894$ of the logistic map. Notice that there exist two different 2-periodic solutions of system (4.8). One of them corresponds to the 2-periodic solution $\{c_i\}$ of the logistic map with $c_0 = d^{(1)}$, and the other corresponds to the 2-periodic solution $\{\bar{c}_i\}$ with $\bar{c}_0 = d^{(2)}$.

We consider the solution of system (4.8) with the initial data $y(0) = 0.0078$, $k(0) = 0.0067$ and $d_0 = 0.219$ again and apply the OGY control method around the 2-periodic solution $\{\bar{c}_i\}$ of the logistic map, with $c_0 = d^{(2)} \approx 0.8894$. Fig. 3 shows the simulation results for $\varepsilon = 0.04$. The control mechanism is switched on at $t = 50$ and switched off at $t = 130$. The control becomes dominant approximately at $t = 100$ and its effect lasts approximately until $t = 190$, after which the instability becomes dominant and irregular behavior develops again.

The OGY control has to be applied to the logistic map, i.e. to the consumption function of the foreign economy. This highlights the importance of international economic cooperation between countries. Since foreign consumption is out of direct control of the home policy-makers, its adjustment can be only done by the foreign country in response to international negotiations. Additionally, the application of the OGY control to differential equations is feasible if a Poincaré map can be constructed, but this requires the knowledge of analytical solutions, which is a difficult task in general. Properly recognizing the unidimensional export shock as the source of the chaotic motion in the home economy will lead to locating the most effective and least costly way of stabilizing this dynamics, and in more general (and realistic) cases of higher-dimensional models of home economy may prove to be the only way of controlling chaos.

4.2. *Perturbation with chaotically behaved discontinuity instants*

Now suppose that the home economy modeled in (4.2) is perturbed with an exogenous rainfall shock, $\nu(t, d)$, that affects agricultural output and therefore the entire economy. We model rainfall as taking one of two values, where the higher value is normal rainfall, and the lower value is drought, which leads to lower agricultural production and slower output growth:

$$\nu(t,d) = \begin{cases} 0.024, & \text{if} \quad \theta_{2i}(d) < t \le \theta_{2i+1}(d) \\ 0.007, & \text{if} \quad \theta_{2i-1}(d) < t \le \theta_{2i}(d), \end{cases} \tag{4.10}$$

where d is a solution of (2.6):

$$d_i = F(d_{i-1}) = \mu d_{i-1}(1 - d_{i-1}).$$

Consider system (4.4) with $a = 10^{-5}$. Transforming the variables as described for system (4.8), one can reduce system (4.4) to the following:

$$\begin{aligned} \dot{y} &= (5/16)y - (7/16)k - ay^3 - (3/4)\sqrt{a}y^2 + \nu(t,d), \\ \dot{k} &= (13/16)y - (15/16)k - ay^3 - (3/4)\sqrt{a}y^2. \end{aligned} \tag{4.11}$$

The bifurcation diagram of system (4.11), for $2.6 \le \mu \le 4$, is depicted in Fig. 4, where successive intervals of chaos and stable periodic solutions can be observed. In the regions of stability, for fixed μ, the bifurcation diagram represents the values of the stable periodic solutions of (4.11) at $t = \theta_0(d)$, where d is a periodic sequence. Therefore, in such regions, the number of intersection points of the graph with a vertical line through a given value of μ gives the number of stable periodic solutions for that μ. For example, for $\mu = 3.2$, a vertical line intersects the diagram at two points, which means that for this value of the

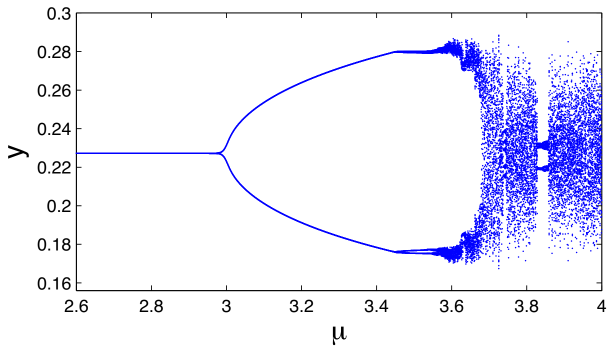

Fig. 4. Bifurcation diagram of system (4.11) with $a = 10^{-5}$.

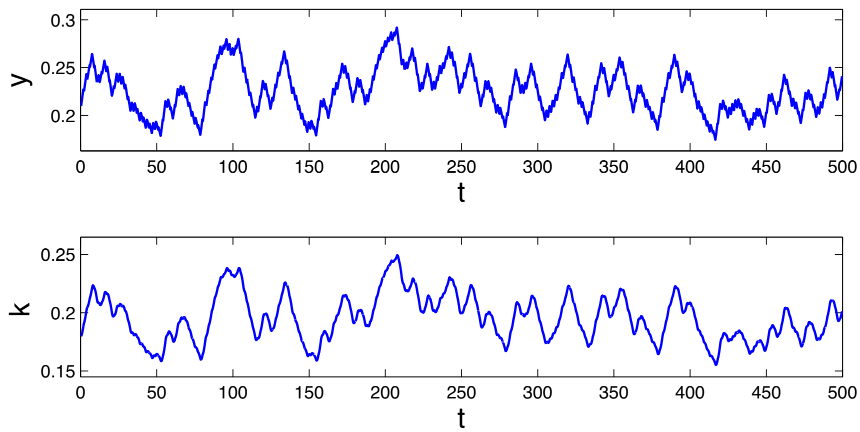

Fig. 5. The graphs of the y and k coordinates of system (4.11).

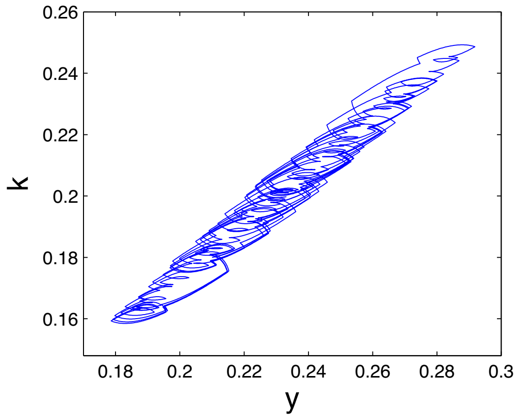

Fig. 6. The chaotic trajectory of system (4.11).

parameter there exist two stable periodic solutions. Even though the diagram indicates regions where stable solutions exist, it does not inform us about the periods of these solutions. A theoretical discussion for the periods can be found in Akhmet and Fen (2012). System (4.11) undergoes period-doubling bifurcations at the same parameter values as the logistic map and obeys the Feigenbaum universality (Feigenbaum, 1983). One can observe from the diagram that system (4.11) is chaotic for $\mu = 3.8$.

Let us investigate system (4.11) with the parameter value $\mu = 3.8$. The time series of the y and k coordinates of system (4.11) corresponding to the initial data $y(0.39) = 0.21$, $k(0.39) = 0.18$ and $d_0 = 0.39$, are graphed in Fig. 5. The trajectory of the same solution is shown in Fig. 6. In both figures, the chaotic behavior of the solution can be observed. Notice that both coordinates of the solution, y and k, are positive.

Similarly to the previous case, the unstable periodic solutions of system (4.11) can be stabilized by controlling the chaos of the logistic map, particularly, using the OGY control method (Akhmet and Fen, 2012). Of course, the policy-makers cannot control rainfall directly. To implement this method, they would need to regulate the *timing of the impact* of the rainfall on the economy, by providing assistance to farmers in order to stimulate their demand for goods and employ workers that are unemployed/underemployed due to the shortfall in agricultural production in the times of drought, and conversely tax farmers and/or agricultural workers in other times. Again, applying OGY control directly to the exogenous shock would be less costly than applying it to the entire system, and in high-dimensional models of the economy would be the only feasible approach.

5. Conclusion

This paper highlights a novel source of chaos in economic models. Unlike previous literature that studies endogenous chaos emergence, we allow chaotic exogenous shocks perturbing a system with a stable equilibrium to generate chaos there. We focus on exogenous disturbances that take the form of a pulse function. The pulsative shocks may have chaotically behaved values or chaotically behaved instants of discontinuity. Both types of shocks are plausible in economics, as is the hybrid of the two. We rigorously verify that the system perturbed with a pulsative disturbance with chaotically behaved values admits chaos. The results are applied to a model of the aggregate economy of a country subject to export shocks, which are determined by the chaotic consumption levels of a foreign economy. We show simulations of the chaotic motion, as well as the stabilized periodic solutions, obtained by implementing the OGY (Ott et al., 1990) control method. We also demonstrate chaos in a model of the aggregate economy perturbed by rainfall shocks that behave as a pulse function with chaotically behaved instants of discontinuity. The theory of this type of perturbations was developed in Akhmet (2009) and Akhmet and Fen (2012).

According to Baumol and Benhabib (1989), 'apparently random behavior may not be random at all', but a product of deterministic chaos. We argue that what we used to interpret as endogenous chaotic behavior may not be endogenous at all, but a product of exogenous chaotic shocks. For the purposes of economic policy-making, the control of chaos needs to be carried out in a very different way once its source in an exogenous chaotic perturbation is recognized. The OGY control can be applied directly to the exogenous disturbance, rather than the entire system. This will significantly reduce the costs of the policy, and in most instances will be the only feasible approach.

Our results also illustrate the *transmission of unpredictability* from one economic system to another, so that even economies that do not admit irregularity in isolation can eventually be contaminated with chaos. Thus, we provide support to the idea that unpredictability is a *global phenomenon* in economics and demonstrate a mechanism for this contagion. Considering the current extensive globalization process, this is a good depiction of reality.

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