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LYAPUNOV-RAZUMIKHIN METHOD FOR DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT

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Abstract. At the first time, Razumikhin technique is applied for differential equations with piecewise constant argument of generalized type [1, 2]. Sufficient conditions are established for stability, uniform stability and uniform asymptotic stability of the trivial solution of such equations. We also provide appropriate examples to illustrate our results.

1. **Introduction and preliminaries.** In [1]-[4], the concept of differential equations with piecewise constant argument [7, 9, 11, 22] has been generalized by considering arbitrary piecewise constant functions as arguments.

In our paper, using stability definitions from [3], we develop the Lyapunov's second method for stability of differential equations with piecewise constant argument of generalized type by employing the Razumikhin technique [13, 17].

To the best of our knowledge, there have been no results on stability obtained by the Lyapunov-Razumikhin method for differential equations with piecewise constant argument, despite they are delay differential equations.

Differential equations with piecewise constant argument play an important role in applications [5, 6], [8]-[10], [14]-[16], [21, 24] as well as they can be applied successfully to approximate solutions of delay differential equations [8, 12]. There are many interesting results of the theory of differential equations with piecewise constant argument [18, 20, 23], which includes complex behavior of solutions [9]. A great part of the theory has been summarized in [22]. The theoretical depth of investigation of these equations was determined by papers [6, 7, 19], where the reduction to discrete equations had been chosen as the main instrument of study. Consequently, analysis of solutions, starting at moments which are not integers has been unattainable. Particularly, one can not investigate the problem of stability completely, as only integers or their multiples are allowed to be discussed for initial moments.

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The approach developed in [1]-[3] has a goal to meet the challenges described above. Detailed comparison of values of a solution at a point and at the neighbour moment, where the argument function has discontinuity, helps to extend the discussion. It embraces the existence and uniqueness of solutions, dependence on initial data, and, exceptionally important is those of stability results, which we intend to consider in the present paper. To give more sense to the last words, in Example 3 at the end of our paper, we will do additional stability analysis of the results obtained by Gopalsamy and Liu [10] for the logistic equation

$$N'(t) = rN(t)(1 - aN(t) - bN([t])), \;\; t > 0, \tag{1}$$

where $[t]$ denotes the maximal integer not greater than t.

Let $\mathbb{N}$ and $\mathbb{R}^+$ be the set of natural numbers and nonnegative real numbers, respectively, i.e., $\mathbb{N} = \{0, 1, 2, 3, ...\}$, $\mathbb{R}^+ = [0, \infty)$. Denote the n-dimensional real space by $\mathbb{R}^n$, $n \in \mathbb{N}$, and the Euclidean norm in $\mathbb{R}^n$ by $\| \, . \, \|$. Fix a real-valued sequence θ_i such that $0 = \theta_0 < \theta_1 < ... < \theta_i < ...$ with $\theta_i \to \infty$ as $i \to \infty$.

In the present paper, we shall consider the following equation [1]

$$x'(t) = f(t, x(t), x(\beta(t))), \tag{2}$$

where $x \in S(\rho)$, $S(\rho) = \{x \in \mathbb{R}^n : \|x\| < \rho\}$, $t \in \mathbb{R}^+$, $\beta(t) = \theta_i$ if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{N}$.

The following assumptions will be needed throughout the paper:

(C1) $f(t, y, z) \in C(\mathbb{R}^+ \times S(\rho) \times S(\rho))$ is an $n \times 1$ real valued function;
(C2) $f(t, 0, 0) = 0$ for all $t \geq 0$;
(C3) $f(t, y, z)$ satisfies the condition

$$\|f(t, y_1, z_1) - f(t, y_2, z_2)\| \leq \ell(\|y_1 - y_2\| + \|z_1 - z_2\|) \tag{3}$$

for all $t \in \mathbb{R}^+$ and y_1, y_2, z_1, $z_2 \in S(\rho)$, where $\ell > 0$ is a Lipschitz constant;
(C4) there exists a positive number θ such that $\theta_{i+1} - \theta_i \leq \theta$, $i \in \mathbb{N}$;
(C5) $\ell\theta[1 + (1 + \ell\theta)e^{\ell\theta}] < 1$;
(C6) $3\ell\theta e^{\ell\theta} < 1$.

Let us use the following sets of functions:

$\mathcal{K} = \{\, a \in C(\mathbb{R}^+, \mathbb{R}^+) : a$ is strictly increasing and $a(0) = 0\}$,
$\Omega = \{\, b \in C(\mathbb{R}^+, \mathbb{R}^+) \, : b(0) = 0, \; b(s) > 0$ for $s > 0\}$.

The following definition is from [1].

Definition 1.1. A function $x(t)$ is a solution of (2) on $\mathbb{R}^+$ if:

(i) $x(t)$ is continuous on $\mathbb{R}^+$;
(ii) the derivative $x'(t)$ exists for $t \in \mathbb{R}^+$ with the possible exception of the points θ_i, $i \in \mathbb{N}$, where one-sided derivatives exist;
(iii) equation (2) is satisfied by $x(t)$ on each interval (θ_i, θ_{i+1}), $i \in \mathbb{N}$, and it holds for the right derivative of $x(t)$ at the points θ_i, $i \in \mathbb{N}$.

Notation 1. $K(\ell) = \dfrac{1}{1 - \ell\theta[1 + (1 + \ell\theta)e^{\ell\theta}]}$.

The following lemma is an important auxiliary result of the paper.

Lemma 1.2. *Let (C1)-(C5) be fulfilled. Then the following inequality*

$$\|x(\beta(t))\| \leq K(\ell)\,\|x(t)\| \tag{4}$$

holds for all $t \geq 0$.

Proof. Let us fix $t \in \mathbb{R}^+$. Then there exists $k \in \mathbb{N}$ such that $t \in [\theta_k, \theta_{k+1})$. We have

$$x(t) = x(\theta_k) + \int_{\theta_k}^{t} f(s, x(s), x(\theta_k))ds, \quad t \in [\theta_k, \theta_{k+1}).$$

Hence,

$$\begin{aligned} \|x(t)\| &\leq \|x(\theta_k)\| + \ell \int_{\theta_k}^{t} \left(\|x(s)\| + \|x(\theta_k)\|\right) ds \\ &\leq (1+\ell\theta)\,\|x(\theta_k)\| + \ell \int_{\theta_k}^{t} \|x(s)\|\, ds \end{aligned}$$

The Gronwall-Bellman Lemma yields that $\|x(t)\| \leq (1+\ell\theta)e^{\ell\theta}\,\|x(\theta_k)\|$. Moreover, for $t \in [\theta_k, \theta_{k+1})$ we have

$$x(\theta_k) = x(t) - \int_{\theta_k}^{t} f(s, x(s), x(\theta_k))ds.$$

Thus,

$$\begin{aligned} \|x(\theta_k)\| &\leq \|x(t)\| + \ell \int_{\theta_k}^{t} \left(\|x(s)\| + \|x(\theta_k)\|\right) ds \\ &\leq \|x(t)\| + \ell \int_{\theta_k}^{t} \left[(1+\ell\theta)e^{\ell\theta} + 1\right] \|x(\theta_k)\|\, ds \\ &\leq \|x(t)\| + \ell\theta \left[(1+\ell\theta)e^{\ell\theta} + 1\right] \|x(\theta_k)\|\,. \end{aligned}$$

It follows from condition (C5) that $\|x(\theta_k)\| \leq K(\ell)\,\|x(t)\|$ for $t \in [\theta_k, \theta_{k+1})$. Hence, (4) holds for all $t \geq 0$. □

We give the following assertion which establishes the existence and uniqueness of solutions of (2). It can be proved by using Lemma 1.1 and Theorem 1.1 in [2] with slight changes.

Theorem 1.3. *Assume that conditions (C1) and (C3)-(C6) are satisfied. Then for every* $(t_0, x_0) \in \mathbb{R}^+ \times S(\rho)$ *there exists a unique solution* $x(t) = x(t, t_0, x_0)$ *of (2) on* $\mathbb{R}^+$ *in the sense of Definition 1.1 such that* $x(t_0) = x_0$.

Definition 1.4. Let $V : \mathbb{R}^+ \times S(\rho) \to \mathbb{R}^+$. Then, V is said to belong to the class ϑ if

(i) V is continuous on $\mathbb{R}^+ \times S(\rho)$ and $V(t, 0) \equiv 0$ for all $t \in \mathbb{R}^+$;
(ii) $V(t, x)$ is continuously differentiable on $(\theta_i, \theta_{i+1}) \times S(\rho)$ and for each $x \in S(\rho)$, right derivative exists at $t = \theta_i$, $i \in \mathbb{N}$.

Definition 1.5. Given a function $V \in \vartheta$, the derivative of V with respect to system (2) is defined by

$$V'(t, x, y) = \frac{\partial V(t, x)}{\partial t} + grad_x^T V(t, x) f(t, x, y) \tag{5}$$

for all $t \neq \theta_i$ in $\mathbb{R}^+$ and $x\,, y \in S(\rho)$.

2. **Main results.** In this section, we assume that conditions (C1)-(C6) are satisfied and we will obtain the stability of the zero solution of (2) based on the Lyapunov-Razumikhin method. We can formulate the definitions of Lyapunov stability in the same way as for ordinary differential equations.

Definition 2.1. [3] The zero solution of (2) is said to be

(i) stable if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t \geq t_0$;
(ii) uniformly stable if δ is independent of t_0.

Definition 2.2. [3] The zero solution of (2) is said to be uniformly asymptotically stable if it is uniformly stable and there is a $\delta_0 > 0$ such that for every $\varepsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $T = T(\varepsilon) > 0$ such that $\|x(t, t_0, x_0)\| < \varepsilon$ for all $t > t_0 + T$ whenever $\|x_0\| < \delta_0$.

Theorem 2.3. *Assume that there exists a function* $V \in \vartheta$ *such that*

(i) $u(\|x\|) \leq V(t, x)$ *on* $\mathbb{R}^+ \times S(\rho)$, *where* $u \in \mathcal{K}$;
(ii) $V'(t, x, y) \leq 0$ *for all* $t \neq \theta_i$ *in* $\mathbb{R}^+$ *and* x, $y \in S(\rho)$ *such that* $V(\beta(t), y) \leq V(t, x)$.

Then the zero solution of (2) *is stable.*

Proof. At first, we show stability for $t_0 = \theta_j$ for some $j \in \mathbb{N}$. Then it will allow us to prove stability for an arbitrary $t_0 \in \mathbb{R}^+$ due to Lemma 1.2.

Let $\rho_1 \in (0, \rho)$. Given $\varepsilon \in (0, \rho_1)$ and $t_0 = \theta_j$, choose $\delta_1 > 0$ sufficiently small that $V(\theta_j, x(\theta_j)) < u(\varepsilon)$ if $\|x(\theta_j)\| < \delta_1$. Define $\delta = \delta_1/K(\ell)$. We note $\delta < \delta_1$ as $K(\ell) > 1$ and show that this δ is the needed one.

Let us fix $k \in \mathbb{N}$ and consider the interval $[\theta_k, \theta_{k+1})$. Using the condition (ii), we shall show that

$$V(t, x(t)) \leq V(\theta_k, x(\theta_k)) \ \text{ for } \ t \in [\theta_k, \theta_{k+1}). \tag{6}$$

Set $V(t) = V(t, x(t))$. If (6) is not true, then there exist points κ and τ, $\theta_k \leq \kappa < \tau < \theta_{k+1}$, such that

$$V(\kappa) = V(\theta_k) \ \text{ and } \ V(t) > V(\theta_k) \ \text{ for } \ t \in (\kappa, \tau].$$

By applying the Mean-Value Theorem to the function V, we get

$$\frac{V(\tau) - V(\kappa)}{\tau - \kappa} = V'(\zeta) > 0 \tag{7}$$

for some $\zeta \in (\kappa, \tau)$. Indeed, being $V(\zeta) > V(\theta_k)$, it follows from the condition (ii) that $V'(\zeta) \leq 0$, which contradicts (7). Hence, (6) is true. Using the continuity of V and $x(t)$, we can obtain by induction that

$$V(t, x(t)) \leq V(\theta_j, x(\theta_j)) \ \text{ for all } \ t \geq \theta_j. \tag{8}$$

If $\|x(\theta_j)\| < \delta$, we have $V(\theta_j, x(\theta_j)) < u(\varepsilon)$ since $\delta < \delta_1$. This together with (8) lead us to the inequality $V(t, x(t)) < u(\varepsilon)$ which implies immediately that $\|x(t)\| < \varepsilon$ for all $t \geq \theta_j$. Hence, stability for the case $t_0 = \theta_j$, $i \in \mathbb{N}$ is proved.

Now let us consider the case $t_0 \in \mathbb{R}^+$, $t_0 \neq \theta_i$ for all $i \in \mathbb{N}$. Then there is $j \in \mathbb{N}$ such that $\theta_j < t_0 < \theta_{j+1}$. Given $\varepsilon > 0$ ($\varepsilon < \rho_1$), we choose $\delta_1 > 0$ such that $V(\theta_j, x(\theta_j)) < u(\varepsilon)$ if $\|x(\theta_j)\| < \delta_1$. Take a solution $x(t)$ of (2) such that $\|x(t_0)\| < \delta$, where $\delta = \delta_1/K(\ell)$. By Lemma 1.2, $\|x(t_0)\| < \delta$ results in $\|x(\theta_j)\| < \delta_1$. Then by the discussion used for $t_0 = \theta_j$, we obtain that $\|x(t)\| < \varepsilon$ for all $t \geq \theta_j$ and hence for all $t \geq t_0$, proving the stability of the zero solution. □

Theorem 2.4. *Assume that there exists a function* $V \in \vartheta$ *such that*

(i) $u(\|x\|) \le V(t,x) \le v(\|x\|)$ *on* $\mathbb{R}^+ \times S(\rho)$, *where* $u\,, v \in \mathcal{K}$;

(ii) $V'(t,x,y) \le 0$ *for all* $t \neq \theta_i$ *in* $\mathbb{R}^+$ *and* x, $y \in S(\rho)$ *such that* $V(\beta(t),y) \le V(t,x)$.

Then the zero solution of (2) *is uniformly stable.*

Proof. Let $\rho_1 \in (0,\rho)$. Fix $\varepsilon > 0$ in the range $0 < \varepsilon < \rho_1$ and choose $\delta_1 > 0$ such that $v(\delta_1) \le u(\varepsilon)$. Define $\delta = \delta_1/K(\ell)$. Similar to the previous discussion, we consider two cases when $t_0 = \theta_j$ for some $j \in \mathbb{N}$ and another one when $t_0 \neq \theta_i$ for all $i \in \mathbb{N}$, to show that this δ is the needed one. If $t_0 = \theta_j$, where j is a fixed nonnegative integer and $\|x(\theta_j)\| < \delta$, then as a consequence of the condition (i) we have $V(\theta_j, x(\theta_j)) < v(\delta) < v(\delta_1) \le u(\varepsilon)$. Using the same argument used in the proof of Theorem 2.3, we get the inequality $V(t,x(t)) \le V(\theta_j, x(\theta_j))$ for all $t \ge \theta_j$ and see that $V(t,x(t)) < u(\varepsilon)$ for all $t \ge \theta_j$. Hence $\|x(t)\| < \varepsilon$ for all $t \ge \theta_j$. We note that evaluation of δ does not depend on the choice of $j \in \mathbb{N}$.

Now, take $t_0 \in \mathbb{R}^+$ with $t_0 \neq \theta_i$ for all $i \in \mathbb{N}$. Then there exists $j \in \mathbb{N}$ such that $\theta_j < t_0 < \theta_{j+1}$. Take a solution $x(t)$ of (2) such that $\|x(t_0)\| < \delta$. It follows by Lemma 1.2 that $\|x(\theta_j)\| < \delta_1$. From a similar idea used for the case $t_0 = \theta_j$, we conclude that $\|x(t)\| < \varepsilon$ for $t \ge \theta_j$ and indeed for all $t \ge t_0$. Finally, one can see that the evaluation is independent of $j \in \mathbb{N}$ and correspondingly for all $t_0 \in \mathbb{R}^+$. □

Theorem 2.5. *Assume that all of the conditions in Theorem* 2.4 *are valid and there exist a continuous nondecreasing function* ψ *such that* $\psi(s) > s$ *for* $s > 0$ *and a function* $w \in \Omega$. *If condition (ii) is replaced by*

(iii) $V'(t,x,y) \le -w(\|x\|)$ *for all* $t \neq \theta_i$ *in* $\mathbb{R}^+$ *and* x, $y \in S(\rho)$ *such that* $V(\beta(t),y) < \psi(V(t,x))$,

then the zero solution of (2) *is uniformly asymptotically stable.*

Proof. When $V(\beta(t),y) \le V(t,x)$, we have $V(\beta(t),y) < \psi(V(t,x))$. Then by the condition (iii), we have $V'(t,x,y) \le 0$. From Theorem 2.4, it follows that the zero solution of (2) is uniformly stable.

First, we show "uniform" asymptotic stability with respect to all elements of the sequence θ_i, $i \in \mathbb{N}$.

Fix $j \in \mathbb{N}$ and $\rho_1 \in (0,\rho)$. If $t_0 = \theta_j$ and $\delta > 0$ is such that $v(K(\ell)\delta) = u(\rho_1)$, $K(\ell) > 1$, arguments of Theorem 2.4 shows that $V(t,x(t)) < v(\delta) < v(K(\ell)\delta)$ for all $t \ge \theta_j$ and hence $\|x(t)\| < \rho_1$ if $\|x(\theta_j)\| < \delta$. In what follows, we shall present that this δ can be taken as δ_0 in the Definition 2.2 of uniform asymptotic stability. That is, for arbitrary ε, $0 < \varepsilon < \rho_1$, we need to show that there exists a $T = T(\varepsilon) > 0$ such that $\|x(t)\| < \varepsilon$ for $t > \theta_j + T$ if $\|x(\theta_j)\| < \delta$.

Set $\gamma = \inf\{w(s) \,:\, v^{-1}(u(\varepsilon)) \le s \le \rho_1\}$. We note that this set is not empty since $\varepsilon < \rho_1$ and u, $v \in \mathcal{K}$ implies that $u(\varepsilon) < v(\rho_1)$, which, in turn, leads us to the inequality $v^{-1}(u(\varepsilon)) < \rho_1$.

Denote $\delta_1 = K(\ell)\delta$. From the properties of the function $\psi(s)$, there is a number $a > 0$ such that $\psi(s) - s > a$ for $u(\varepsilon) \le s \le v(\delta_1)$.

Let N be the smallest positive integer such that $u(\varepsilon) + Na \ge v(\delta_1)$.

Choose $t_k = k(\dfrac{v(\delta_1)}{\gamma} + \theta) + \theta_j$, $k = 1,2,...,N$. We will prove that

$$V(t,x(t)) \le u(\varepsilon) + (N-k)a \;\text{ for }\; t \ge t_k, \;\; k = 0,1,2,...,N. \tag{9}$$

We have $V(t,x(t)) < v(\delta_1) \leq u(\varepsilon) + Na$ for $t \geq t_0 = \theta_j$. Hence, (9) holds for $k = 0$. Now, we suppose that (9) holds true for some $0 \leq k < N$. Let us show that

$$V(t,x(t)) \leq u(\varepsilon) + (N-k-1)a \ \text{ for } \ t \geq t_{k+1}. \tag{10}$$

Let $I_k = [\beta(t_k)+\theta, t_{k+1}]$. To prove (10), we first claim that there exists a $t^* \in I_k$ such that

$$V(t^*, x(t^*)) \leq u(\varepsilon) + (N-k-1)a. \tag{11}$$

Otherwise, $V(t,x(t)) > u(\varepsilon) + (N-k-1)a$ for all $t \in I_k$.

On the other side, we have

$$V(t,x(t)) \leq u(\varepsilon) + (N-k)a \ \text{ for } \ t \geq t_k, \tag{12}$$

which implies that $V(\beta(t), x(\beta(t))) \leq u(\varepsilon) + (N-k)a$ for $t \geq \beta(t_k) + \theta$.

Hence, for $t \in I_k$

$$\psi(V(t,x(t))) > V(t,x(t)) + a > u(\varepsilon) + (N-k)a \geq V(\beta(t), x(\beta(t))).$$

Since $v^{-1}(u(\epsilon)) \leq \|x(t)\| \leq \rho_1$ for $t \in I_k$, it follows from the hypothesis (iii) that

$$V'(t, x(t), x(\beta(t))) \leq -w(\|x(t)\|) \leq -\gamma \text{ for all } t \neq \theta_m \text{ in } I_k.$$

Using the continuity of the function V and the solution $x(t)$, we get

$$\begin{aligned} V(t_{k+1}, x(t_{k+1})) &\leq V(\beta(t_k)+\theta, x(\beta(t_k)+\theta)) - \gamma(t_{k+1} - \beta(t_k) - \theta) \\ &< v(\delta_1) - \gamma(t_{k+1} - t_k - \theta) = 0, \end{aligned}$$

which is a contradiction. Thus (11) holds, i.e. there is a $t^* \in I_k$ such that $V(t^*, x(t^*)) \leq u(\varepsilon) + (N-k-1)a$.

Next, we show that

$$V(t,x(t)) \leq u(\varepsilon) + (N-k-1)a \ \text{ for all } \ t \in [t^*, \infty). \tag{13}$$

If (13) does not hold, then there exists a $\bar{t} \in (t^*, \infty)$ such that

$$V(\bar{t}, x(\bar{t})) > u(\varepsilon) + (N-k-1)a \geq V(t^*, x(t^*)).$$

Thus, we can find a $\tilde{t} \in (t^*, \bar{t})$ such that $\tilde{t} \neq \theta_m$, $V'(\tilde{t}, x(\tilde{t}), x(\beta(\tilde{t}))) > 0$ and $V(\tilde{t}, x(\tilde{t})) > u(\varepsilon) + (N-k-1)a$. If there is no such $\tilde{t}$, then for all $t \in (t^*, \bar{t})$, $t \neq \theta_m$, we have $V'(t, x(t), x(\beta(t))) \leq 0$ or $V(t,x(t)) \leq u(\varepsilon) + (N-k-1)a$. But, $V'(t, x(t), x(\beta(t))) \leq 0$ leads to $V(\bar{t}, x(\bar{t})) \leq V(t^*, x(t^*))$, a contradiction. If $V(t,x(t)) \leq u(\varepsilon) + (N-k-1)a$, then $V(t,x(t)) < V(\bar{t}, x(\bar{t}))$ for $t \in (t^*, \bar{t})$, $t \neq \theta_m$, is also a contradiction. Hence, $\tilde{t}$ exists.

However,

$$\psi(V(\tilde{t}, x(\tilde{t}))) > V(\tilde{t}, x(\tilde{t})) + a > u(\varepsilon) + (N-k)a \geq V(\beta(\tilde{t}), x(\beta(\tilde{t})))$$

implies that $V'(\tilde{t}, x(\tilde{t}), x(\beta(\tilde{t}))) \leq -\gamma < 0$, a contradiction. Then, we conclude that $V(t,x(t)) \leq u(\varepsilon) + (N-k-1)a$ for all $t \geq t^*$ and thus for all $t \geq t_{k+1}$. This completes the induction and shows that (9) is valid. For $k = N$, we have

$$V(t,x(t)) \leq u(\varepsilon)\,, \ t \geq t_N = N(\frac{v(\delta_1)}{\gamma} + \theta) + t_0.$$

Hence, $\|x(t)\| < \varepsilon$ for $t > \theta_j + T$ where $T = N(\dfrac{v(\delta_1)}{\gamma} + \theta)$, proving uniform asymptotic stability for $t_0 = \theta_j$, $j \in \mathbb{N}$.

Consider the case $t_0 \neq \theta_i$ for all $i \in \mathbb{N}$. Then $\theta_j < t_0 < \theta_{j+1}$ for some $j \in \mathbb{N}$. $\|x(t_0)\| < \delta$ implies by Lemma 1.2 that $\|x(\theta_j)\| < \delta_1$. Hence, the argument used above for the case $t_0 = \theta_j$ yields that $\|x(t)\| < \varepsilon$ for $t > \theta_j + T$ and in turn for all $t > t_0 + T$. $\square$

In the following examples, we assume that the sequence θ_i, which is in the basis of the definition of the function $\beta(t)$, satisfies the condition (C4).

Example 1. *Consider the following linear equation*

$$x'(t) = -a(t)x(t) - b(t)x(\beta(t)) \tag{14}$$

where a and b are bounded continuous functions on $\mathbb{R}^+$ such that $|b(t)| \le a(t)$ for all $t \ge 0$. Clearly, one can check that conditions (C1)-(C2) and (C3) with a Lipschitz constant $\ell = \sup_{t\in\mathbb{R}^+} a(t)$ are fulfilled. Moreover, we assume that the sequence θ_i and ℓ satisfy (C5) and (C6). Let $V(x) = \dfrac{x^2}{2}$, then for $t \ne \theta_i$, $i \in \mathbb{N}$,

$$\begin{array}{rcl} V'(x(t)) & = & -a(t)x^2(t) - b(t)x(t)x(\beta(t)) \\ & \le & -a(t)x^2(t) + |b(t)|\,|x(t)|\,|x(\beta(t))| \\ & \le & -[a(t) - |b(t)|]x^2(t) \le 0 \end{array}$$

whenever $|x(\beta(t))| \le |x(t)|$. Since $V = x^2/2$, $V(x(\beta(t))) \le V(x(t))$ implies that $V'(x(t)) \le 0$. Thus by Theorem 2.4, the trivial solution of (14) is uniformly stable.

Next, let us investigate uniform asymptotic stability. If there are constants $\lambda > 0$, $\omega \in [0, 1)$ and $q > 1$ with $\lambda \le a(t)$, $|b(t)| \le \omega\lambda$ and $1 - q\omega > 0$, then for $\psi(s) = q^2 s$, $w(s) = (1 - q\omega)\lambda s^2$ and $V(x) = \dfrac{x^2}{2}$ as above, we obtain that

$$V'(x(t)) \le -w(|x(t)|),\ t \ne \theta_i,$$

whenever $V(x(\beta(t))) < \psi(V(x(t)))$. Theorem 2.5 implies that $x = 0$ is uniformly asymptotically stable.

The following illustration is a development of an example from [17].

Example 2. *Let us now consider a nonlinear scalar equation*

$$x'(t) = f(x(t), \mu x(\beta(t))) \tag{15}$$

where $f(x, y)$ is a continuous function with $f(0, 0) = 0$, $\dfrac{f(x, 0)}{x} = -\sigma$ for some $\sigma > 0$ satisfying $\sigma \ge \ell|\mu|$ and $|f(x_1, y_1) - f(x_2, y_2)| \le \ell(|x_1 - x_2| + |y_1 - y_2|)$. Then conditions (C1)-(C3) are valid. We take a sequence θ_i so that (C5)-(C6) hold true together with the Lipschitz constant ℓ.

Choosing $V(x) = x^2$, we get for $t \ne \theta_i$

$$\begin{array}{rcl} V'(x(t)) & = & 2x(t)f(x(t), \mu x(\beta(t))) \\ & = & 2\left[\dfrac{f(x(t), \mu x(\beta(t))) - f(x(t), 0)}{x(t)} + \dfrac{f(x(t), 0)}{x(t)}\right] x^2(t) \\ & \le & 2\left[\dfrac{\ell|\mu||x(\beta(t))|}{|x(t)|} - \sigma\right] x^2(t) \le 2(\ell|\mu| - \sigma)x^2(t) \le 0 \end{array}$$

whenever $V(x(\beta(t))) \le V(x(t))$. It follows from Theorem 2.4 that the solution $x = 0$ of (15) is uniformly stable.

Example 3. *(a logistic equation with harvesting)*

In [10], stability of the positive equilibrium $N^* = \dfrac{1}{a+b}$ *of equation (1) has been studied. Equation (1) models the dynamics of a logistically growing population subjected to a density-dependent harvesting. There,* $N(t)$ *denotes the population density of a single species and the model parameters* r*,* a *and* b *are assumed to be positive.*

Gopalsamy and Liu showed that N^* *is globally asymptotically stable if* $\alpha \geq 1$ *where* $\alpha = a/b$*. Particularly, it was shown that the equilibrium is stable for integer initial moments. The restriction is caused by the method of investigation: the reduction to difference equation. Our results are for all initial moments from* $\mathbb{R}^+$*, not only integers. Moreover, we consider uniform stability for the general case* $\beta(t)$*. Consequently, we may say that the approach of the paper allows to study stability of the class of equations in complete form.*

Let us discuss the following equation

$$N'(t) = rN(t)(1 - aN(t) - bN(\beta(t))), \;\; t > 0, \tag{16}$$

which is a generalization of (1). One can see that (1) is of type (16) when $\beta(t) = [t]$*.*

For our needs, we translate the equilibrium point N^* *to origin by the transformation* $x = b(N - N^*)$*, which takes (16) into the following form*

$$x'(t) = -r[x(t) + \frac{1}{1+\alpha}][\alpha x(t) + x(\beta(t))] \tag{17}$$

Note that $f(x,y) := -r(x + \frac{1}{1+\alpha})(\alpha x + y)$ *is a continuous function and has continuous partial derivatives for* $x, y \in S(\rho)$*. If we evaluate the first partial derivatives of the function* $f(x,y)$*, we see that*

$$\begin{aligned} |\partial f/\partial x| &\leq r(2\alpha\rho + \rho + \frac{\alpha}{1+\alpha}), \\ |\partial f/\partial y| &\leq r(\rho + \frac{1}{1+\alpha}), \end{aligned}$$

for $x, y \in S(\rho)$*.*

If we choose $\ell = r(2\alpha\rho + 2\rho + 1)$ *as a Lipschitz constant, one can see that the conditions (C1)-(C3) are fulfilled for sufficiently small* r*. In addition, we assume that* ℓ *is sufficiently small so that the conditions (C5) and (C6) are satisfied.*

Suppose that $\alpha \geq 1$ *and* $\rho < 1/(1+\alpha)$*. Then for* $V(x) = x^2$*,* $x \in S(\rho)$ *and* $t \neq \theta_i$*, we have*

$$\begin{aligned} V'(x(t), x(\beta(t))) &= -2rx(t)(x(t) + \frac{1}{1+\alpha})(\alpha x(t) + x(\beta(t))) \\ &\leq -2r(x(t) + \frac{1}{1+\alpha})(\alpha x^2(t) - |x(t)||x(\beta(t))|) \\ &\leq -2r(x(t) + \frac{1}{1+\alpha})(\alpha - 1)x^2(t) \leq 0 \end{aligned}$$

whenever $V(x(\beta(t))) \leq V(x(t))$*. Theorem 2.4 implies that the zero solution of (17) is uniformly stable. This in turn leads to uniform stability of the positive equilibrium* N^* *of (16).*

To prove uniform asymptotic stability, we need to satisfy the condition (iii) in Theorem 2.5. In view of uniform stability, given $\rho_1 \in (0, \rho)$ *we know that there exists a* $\delta > 0$ *such that* $x(t) \in S(\rho_1)$ *for all* $t \geq t_0$ *whenever* $|x(t_0)| < \delta$*. Let us* *take a constant* q *such that* $1 < q < \alpha$*, then for* $\psi(s) = q^2 s$*,* $w(s) = 2r(\alpha - q)\eta s^2$*,* $\eta = 1/(1+\alpha) - \rho_1$ *and* $V(x) = x^2$ *as before, we have*

$$V'(x(t), x(\beta(t))) \leq \quad -2r(x(t) + \frac{1}{1+\alpha})(\alpha - q)x^2(t) \leq -w(|x(t)|), \; t \neq \theta_i,$$

whenever $V(x(\beta(t))) < \psi(V(x(t)))$*. Hence the solution* $x = 0$ $(N = N^*)$ *of* (17) ((16)) *is uniformly asymptotically stable by Theorem 2.5.*

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