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The complex dynamics of the cardiovascular system

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ABSTRACT

In our paper, we consider the dynamics of blood pressure, initiated in [M.U. Akhmet, G.A. Bekmukhambetova, A prototype compartmental model of blood pressure distribution, Nonlinear Anal. RWA (in press)], concentrating on the interaction between systemic arterial pressure and periphery blood pressure. A system of impulsive differential equations is applied as a model. The main result of the present paper is the existence of Devaney's chaos ingredients: sensitivity of solutions, transitivity, and existence of infinitely many periodic solutions, in the case, when the moments of discontinuity are defined as a special initial value problem. The method of creating a chaos [M.U. Akhmet, Dynamical synthesis of the quasi-minimal set, Internat. J. Bifur. Chaos (in press); M.U. Akhmet, Devaney's chaos in a relay system, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 1486–1493; M.U. Akhmet, Li–Yorke chaos in the system with impacts, J. Math. Anal. Appl. 351 (2009) 804–810] is applied. Appropriate examples are provided, including a simulation of the chaotic attractor.

1. Introduction and preliminaries

In [1] the following differential equation is considered as a model for peripheric blood pressure P_p,

$$\frac{\mathrm{d}P_p}{\mathrm{d}t} = -\frac{1}{RC}P_p + \frac{1}{C}Q(t), \tag{1}$$

where C is total arterial compliance, R—arterial resistance, $Q(t)$—continuous blood flow such that $Q(t) = \frac{P_s(t)-P_p(t)}{r}$, where P_s—is systemic arterial pressure, and r— aortic impedance.

For systemic arterial pressure the following differential equation with impulses was discussed in [2],

$$\begin{aligned} &\frac{\mathrm{d}P_s}{\mathrm{d}t} = -\frac{1}{RC}P_s, \quad t \neq \theta_i, \\ &P_s(\theta_i+) - P_s(\theta_i) = \frac{V_i}{C}, \end{aligned} \tag{2}$$

where $\theta_i = iT, i \in \mathbb{Z}$, are prescribed moments, $P_s(\theta_i+)$ is the right limit value, V_i are stroke volumes, and C is systemic arterial compliance.

In view of such phenomena as heart contraction and consecutive simultaneous change of systemic arterial pressure, the idea to investigate the cardiovascular system through the discontinuous dynamics theory is very natural.

Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$ be the sets of all integers, natural and real numbers, respectively, $\mathbb{R}_+ = [0, \infty)$. Denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$.

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The following new system is a consequence of the modeling efforts of our predecessors [2,1],

$$\frac{\mathrm{d}P_s(t)}{\mathrm{d}t} = -k_sP_s + g_s(P_s(t) - P_p(t)),$$
$$\frac{\mathrm{d}P_p(t)}{\mathrm{d}t} = -k_pP_p + g_p(P_s(t) - P_p(t)), \quad t \neq \theta_i,$$
$$\Delta P_s|_{t=\theta_i} = I_s + J_s(P_s). \tag{3}$$

The multidimensional version of this system has been introduced in [3].

We shall need the following assumptions throughout the paper:

(C1) real constants I_s, k_s, k_p are positive;
(C2) J_s, g_s, g_p are real valued continuous functions, $J_s(0) = g_s(0) = g_p(0) = 0$, $g_p(z) > 0$, $g_s(z) < 0$, if $z > 0$;
(C3) the function J_s is non-increasing;
(C4) the functions J_s, g_s, g_p satisfy the Lipschitz condition

$$|g_s(z^1) - g_s(z^2)| \leq l_s|z^1 - z^2|, \qquad |g_p(z^1) - g_p(z^2)| \leq l_p|z^1 - z^2|$$
$$|J_s(z^1) - J_s(z^1)| \leq l_J|z^1 - z^2|; \tag{4}$$

(C5) the sequence θ_i, $i \in \mathbb{Z}$, satisfies the following property: there exists a number $\omega > 0$, such that $i\omega \leq \theta_i < (i+1)\omega$, $i \in \mathbb{Z}$;
(C6) there exist positive real constants m_s, m_p such that

$$\sup_{z\geq 0} |g_s(z)| = m_s, \qquad \sup_{z\geq 0} |g_p(z)| = m_p.$$

Condition (C3) implies that $\sup_{z\geq 0} |J_0(z)| = m_J$, where m_J is a positive real number.

The theoretical background of our investigation could be found in [4–10,3,11–15,2,16,17].

The paper is organized in the following manner. In the next section we give conditions for the equation such that there exists a unique solution bounded on $\mathbb{R}$. The solution is periodic if the sequence of discontinuity moments has an appropriate property. The section also contains results on the stability and positiveness of the solutions. In Section 3, a special initial value problem is considered, using the logistic map to create the sequence of the moments jumps. We eventually introduce periodic solutions and prove the existence of these solutions for certain values of the parameter of the logistic equation. All three ingredients of the Devaney's chaos for the problem are considered in this section. Appropriate examples are provided.

2. Stability and positiveness of bounded and periodic solutions

To define the solutions of system (3), we first introduce special sets of piecewise continuous functions.

A function $P(t) : \mathbb{R} \to \mathbb{R}^2$ is from the set $\mathcal{PC}(\mathbb{R})$, if it is continuous on $\mathbb{R}$, except at the points θ_i, $i \in \mathbb{Z}$, where its first coordinate has discontinuities of the first kind, and it is left continuous at these points;

We shall also need the following set of functions. A function $P(t)$ is from the set of functions $\mathcal{PC}(\mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$, $t_0 \in \mathbb{R}$, if it is continuous on $[0, \infty)$ except at the points θ_p, $0 \leq \theta_p < \infty$, where its first coordinate has discontinuities of the first kind, and it is left continuous at these points.

One can easily show, using the theory of impulsive differential equations [16,17], that solutions of (3) are functions from $\mathcal{PC}(\mathbb{R})$ and $\mathcal{PC}(\mathbb{R}_+)$, and under the above mentioned conditions solutions of (3) $P(t, t_0, \pi_0)$ exist and are unique on $\mathbb{R}$ for all $(t_0, \pi_0) \in \mathbb{R} \times \mathbb{R}^2$, $\pi_0 = (\pi_0^s, \pi_0^p)$.

One can also find that the solution satisfies the following integral equation

$$P_s(t) = \mathrm{e}^{-k_0(t-t_0)}\pi_0^s + \int_{t_0}^{t} \mathrm{e}^{-k_s(t-u)} g_s(P_s(s) - P_p(s))\mathrm{d}u + \sum_{t_0\leq\theta_p<t} \mathrm{e}^{-k_s(t-\theta_p)}(I_s + J_s(P_s(\theta_p))),$$
$$P_p(t) = \mathrm{e}^{-k_p(t-t_0)}\pi_0^p + \int_{t_0}^{t} \mathrm{e}^{-k_p(t-s)} g_p(P_s(s) - P_p(s))\mathrm{d}s. \tag{5}$$

We shall say that the sequence θ_i has the *p-property*, $p \in \mathbb{N}$, if $\theta_{i+p} = \theta_i + p\omega$ for all integers i.

We may assume that

(C7) $L(l_s, l_p, l_J) = \sqrt{\left[\frac{l_s\sqrt{2}}{k_s} + \frac{l_J \mathrm{e}^{k_s\omega}}{1-\mathrm{e}^{-k_s\omega}}\right]^2 + \frac{(l_p\sqrt{2})^2}{k_p^2}} < 1.$

Theorem 2.1. *If conditions* (C1)–(C7) *are fulfilled, then there exists a unique solution of* (3) *bounded on* $\mathbb{R}$*. If the sequence* θ_p *has the p-property for a fixed* $p \in \mathbb{N}$*, then the bounded solution has period* $p\omega$*.*

The proof of the last assertion replicates the proof of Theorem 37 and 89 from [17]. The unique solution bounded on $\mathbb{R}$ satisfies the following integral equation

$$P_s(t) = \int_{-\infty}^{t} \mathrm{e}^{-k_s(t-s)} g_0(P_s(s) - P_p(s))\mathrm{d}s + \sum_{\theta_i<t} \mathrm{e}^{-k_s(t-\theta_p)}(I_0 + J(P_s(\theta_i))),$$

$$P_p(t) = \int_{-\infty}^{t} \mathrm{e}^{-k_p(t-s)} g_p(P_s(s) - P_p(s))\mathrm{d}s. \tag{6}$$

Let us denote the solution bounded on $\mathbb{R}$ as $\xi(t) = (\xi_s(t), \xi_p(t))$. If we use the norm $|\phi|_0 = \sup_{\mathbb{R}} |\phi(t)|$ for scalar-valued functions defined on $\mathbb{R}$, and $\|\phi\|_0 = \sup_{\mathbb{R}} \|\phi(t)\|$ for $\phi \in \mathcal{PC}(\mathbb{R})$, then every bounded solution satisfies $|\xi_s|_0 \le (\frac{m_s}{k_s} + \frac{m_J \mathrm{e}^{k_s\omega}}{1-\mathrm{e}^{-k_s\omega}}) = M_s,\ |\xi_p|_0 \le \frac{m_p}{k_p} = M_p$.

Next we obtain conditions for the positiveness of this solution. Assume additionally that

(C8) $\frac{m_s}{k_s} + \frac{m_p}{k_p} < \|I_s\| \frac{\mathrm{e}^{-k_s\omega}}{1-\mathrm{e}^{-k_s\omega}}$.

Using (6) we find that $\xi_s(t) \ge \|I_s\| \frac{\mathrm{e}^{-k_s\omega}}{1-\mathrm{e}^{-k_s\omega}} - \frac{m_s}{k_s} > 0,\ t \in \mathbb{R}$, and $|\xi_p(t)| \le \frac{m_p}{k_p},\ t \in \mathbb{R}, i = \overline{1, m}$. Hence,

$$\xi_s(t) - \xi_p(t) \ge \|I_s\| \frac{\mathrm{e}^{-k_s\omega}}{1-\mathrm{e}^{-k_s\omega}} - \frac{m_s}{k_s} - \frac{m_p}{k_p} = \delta > 0, \quad t \in \mathbb{R}, i = \overline{1, m}.$$

Using condition (C3) and (6) again we obtain that $\xi_p(t) \ge \frac{\bar{g}_p}{k_p} > 0,\ t \in \mathbb{R}$, where $\bar{g}_p$ is the minimal value of the function $g_p(z)$ for $z \in [\delta, \frac{m_s}{k_s} + \frac{m_J \mathrm{e}^{k_s\omega}}{1-\mathrm{e}^{-k_s\omega}} + \frac{m_p}{k_p}]$.

Thus, we have proved the existence of positive numbers μ_s, μ_p such that $\xi_s(t) \ge \mu_s,\ \xi_p(t) \ge \mu_p,\ t \in \mathbb{R}$.

Fix a positive number σ, $0 < \sigma < \min\{k_s, k_p\}$, denote

$$m(l_s, l_p, l_J) = 1 - \max\left\{\frac{2l_s}{k_s-\sigma} + \frac{l_J \mathrm{e}^{(k_s-\sigma)\omega}}{1-\mathrm{e}^{-(k_s-\sigma)\omega}}, \frac{2l_p}{k_p-\sigma}\right\},$$

and assume that the Lipschitz coefficients are sufficiently small so that

(C9) $m(l_s, l_p, l_J) > 0$.

Theorem 2.2. *Assume that conditions* (C1)–(C9) *are valid. Then the bounded solution* $\xi(t)$ *of* (3) *is uniformly exponentially stable.*

The proof of the last theorem for the more general case is given in [3], Theorem 3.2.

3. Blood pressure dynamics as a special initial value problem: Chaotic behavior

The phenomenal exploration of the irregular behavior of dynamical systems was done in [14,18–21]. Many papers considering the subject for specified models as well as for equations of general type have been published.

For the remainder in this paper we assume that the sequence of discontinuity points θ_i is defined by a particular function and depends on the choice of the initial moment. More precisely, consider $h(t) = h(t, \mu) \equiv \mu t^2(1-t)$, the logistic map, where $\mu > 4$ is a parameter, and $\omega = 1, I = [0, 1]$. It is known that there exists a positively invariant subset Λ of I.

For every $t_0 \in \Lambda$ one can construct a sequence $\kappa(t_0)$ of real numbers $\kappa_i, i \in \mathbb{Z}$, in the following way. If $i \ge 0$, then $\kappa_{i+1} = h(\kappa_i, \mu)$ and $\kappa_0 = t_0$. Let us show, how the sequence is defined for negative i. Denote $s^0 = S(t_0), s^0 = (s_0^0 s_1^0 \ldots)$. Consider elements $\underline{s} = (0s_0^0 s_0^1 \ldots), \bar{s} = (1s_0^0 s_0^1 \ldots)$ of Σ_2, such that $\sigma(\underline{s}) = \sigma(\bar{s}) = s^0$ and $\underline{t} = S^{-1}(\underline{s}), \bar{t} = S^{-1}(\bar{s})$. The homeomorphism implies that $h(\bar{t}, \mu) = h(\underline{t}, \mu) = t_0$. Set $h^{-1}(t_0, \mu)$ may consist of not more than two elements $\bar{t}, \underline{t} \in \Lambda$. Each of these two values can be chosen as $\kappa_{-1}(t_0, \mu)$. Obviously, one can continue the process to $-\infty$, choosing always one element from the set h^{-1}. We have finalized the construction of the sequence, and, moreover, it is proved that $\kappa(t_0, \mu) \subset \Lambda$.

Thus, infinitely many sequences $\kappa(t_0, \mu)$ can be constructed for a given t_0. However, each of this type of sequence is unique for an increasing i. Fix one of the sequences and define a sequence $\zeta(t_0) = \{\zeta_i\}, \zeta_i = i + \kappa_i, i \in \mathbb{Z}$. The sequence has a *periodicity property* if there exists $p \in \mathbb{N}$ such that $\zeta_{i+p} = \zeta_i + p, \forall i \in \mathbb{Z}$. If we denote by Π the set of all such sequences $\{\zeta_i\}, i \in \mathbb{Z}$, then a multivalued functional $w : I \to \Pi$ such that $\zeta(t_0) = w(t_0)$ is defined.

Let us introduce the following special initial value problem for the impulsive differential equation (3)

$$\begin{aligned}
&\frac{\mathrm{d}P_s(t)}{\mathrm{d}t} = -k_s P_s + g_s(P_s - P_p),\\
&\frac{\mathrm{d}P_p(t)}{\mathrm{d}t} = -k_p P_p + g_1(P_s - P_p), \quad t \ne \theta_i(t_0),\\
&\Delta P_s|_{t=\theta_i(t_0)} = I_s + J_s(P_s),\\
&P(t_0) = P_0, \quad t_0 \in I,
\end{aligned} \tag{7}$$

where π_0 is a vector from $\mathbb{R}^2$.

Our main goal for this section is to investigate various regular and irregular dynamics of the IVP solutions.

3.1. Periodic solutions revisited. Eventually periodic solutions

In what follows we assume that conditions (C1)–(C9) are valid with $\omega = 1$. Then, by previous results, for every $t_0 \in I$ such that $\kappa(t_0)$ is a p-periodic, $p \in \mathbb{N}$, there exists p-periodic uniformly exponentially stable positive solution of the IVP. Let us denote the periodic solution as $\xi(t, t_0)$. Denote by $P(t, t_0)$ a solution of (7) bounded on $\mathbb{R}$.

It is known [22] that there exists an infinite sequence of the parameter μ values, $3 < \mu_1 < \mu_2 < \cdots < \mu_k \ldots < 3.8284\ldots$, such that $\kappa(t, \mu_i)$, $i \geq 1$, has an asymptotically stable prime period—2^i point $t_i^* \in I$ with a region of attraction $(t_i^* - \delta_i, t_i^* + \delta_i)$. Beyond the value 3.8284…, there are cycles with every integer period [18].

Let $[a,\hat{\ } b]$ be an oriented interval, that is $[a,\hat{\ } b] = [a, b]$, if $a \leq b$, and $[a,\hat{\ } b] = [b, a]$, otherwise.

Fix a number j such that $h(t, \mu_j)$ has a period—2^j point t_j^* with a region of attraction $(t_j^* - \delta_j, t_j^* + \delta_j) \in I$. Denote $p = 2^j$, $t^* = t_j^*$, $\delta = \delta_j$. In what follows we shall investigate the dynamics of the IVP near the point t^*. It follows from Theorem 2.1 that there exists a p-periodic solution $\xi(t, t^*)$ of (7).

We say that a function $\phi(t)$ from $\mathcal{PC}(\mathbb{R})$ with a sequence of discontinuities θ_i is eventually $p\omega$-periodic if for every positive ϵ there exists a moment $T > 0$, such that: (1)$|\theta_{i+p} - \theta_i - p\omega| < \epsilon$, if $\theta_i > T$; (2)$\|\phi(t + p\omega) - \phi(t)\| < \epsilon$ for all $t > T$, such that $|t - \theta_i| > \epsilon$, $i \in \mathbb{R}$.

Solution $\xi(t, t^*)$ attracts all other solutions of (7) that have the same initial moment t^*. Hence, it is obvious that all solutions of IVP (7) with the same initial moment t^* are eventually p-periodic. We may assume that

(C10) $l_s + l_p + \ln(1 + l_J) < \min\{k_s, k_p\}$.

Theorem 3.1. *If conditions (C1)–(C10) are fulfilled, then solution $P(t, t_0)$, $t_0 \in (t^* - \delta, t^* + \delta)$, $t_0 \neq t^*$, is eventually p-periodic.*

Proof. Since t_0 belongs to the basin of attractiveness of t^*, $\theta(t_0)$ satisfies the condition of periodicity. Let us check if the other condition of the Definition holds for $P(t, t_0)$.

One can easily see that it is fulfilled if $\|P(t, t_0) - \xi(t, t^*)\| \to 0$, as $t \to \infty$, for all $t \notin [\theta_i(t^*),\hat{\ } \theta_i(t_0)]$.

It is difficult to evaluate the difference between $P(t, t_0)$ and $\xi(t, t^*)$ since their moments of discontinuity do not coincide. For this reason let us apply the method of B—equivalence developed in our papers [5–7].

Let us consider the following system

$$\begin{aligned}
&\frac{dQ_s(t)}{dt} = -k_s Q_s + g_s(Q_s - Q_p),\\
&\frac{dQ_p(t)}{dt} = -k_p Q_p + g_p(Q_s - Q_p), \quad t \neq \theta_i(t^*),\\
&\Delta Q_s|_{t=\theta_i(t^*)} = I_s + J_s(Q_s) + W_i(Q).\\
&Q(t_0) = P_0, \quad t_0 \in I.
\end{aligned} \tag{8}$$

The two IVPs (7) and (8) are B—equivalent [5–7], if their solutions with the same initial data coincide in their common domain only if $t \notin [\widehat{\theta_i(t_0), \theta_i}(t^*)]$, $i \in \mathbb{Z}$.

Next we shall define a function W for the last IVP such that it is B-equivalent to (7). Introduce the following system of ordinary differential equations

$$\begin{aligned}
&\frac{dR_s(t)}{dt} = -k_s R_s + g_s(R_s - R_p),\\
&\frac{dR_p(t)}{dt} = -k_p R_p + g_p(R_s - R_p),
\end{aligned} \tag{9}$$

and assume, without any loss of generality, that $\theta_i(t_0) \leq \theta_i(t^*)$, $i \in \mathbb{Z}$. Let $R(t, u, R_0)$ be a solution of (9) with the initial data u, R_0. Fix $Q = (Q_s, Q_p)$, and denote $r = (r_s, r_p)$, $r_s = I_s + J_s(R_s(\theta_i(t_0), \theta_i(t^*), Q)) + R_s(\theta_i(t_0), \theta_i(t^*), Q)$, $r_p = R_s(\theta_i(t_0), \theta_i(t^*), Q)$. If $W_i(Q) = R(\theta_i(t^*), \theta_i(t_0), r) - I_s - J_s(Q)$, then one can verify that IVPs (7) and (8) are B-equivalent [5–7]. Moreover, every $W_i(Q)$ is a continuous function, and if $\theta_i(t_0) \to \theta_i(t^*)$ as $i \to \infty$, then $W_i(Q) \to 0$ uniformly on every bounded set from $\mathbb{R}^2$.

Introduce the norm $\|a\|_1 = |a_1| + |a_2|$, if $a = (a_1, a_2)$, and denote $M = M_s + M_p$, $\kappa_1 = \max\{k_s, k_p\}$, $\kappa_2 = \min\{k_s, k_p\}$. Fix a positive $\epsilon < 2M$, and choose positive integers k, i_0 and a number $\epsilon_1 > 0$, such that

1. $\epsilon_1 e^{\kappa_2} \sum_{i=0}^{k} e^{\kappa_2 i} < \frac{\epsilon}{4}$;
2. $2M e^{-(\kappa_2 - l_s - l_p - \ln(1+l_J))k} < \frac{\epsilon}{4}$;
3. $\|W_i(Q)\| < \epsilon_1$ if $\|Q\| < M$, $i \geq i_0$.

Then for $t \geq i_0$

$$\xi_s(t) = e^{-k_s(t-i_0)}\xi_s(i_0) + \int_{i_0}^{t} e^{-k_s(t-u)} g_s(\xi_s(u) - \xi_p(u))du + \sum_{i_0 \leq \theta_i < t} e^{-k_s(t-\theta_i)}(I_s + J_s(\xi_s(\theta_i))),$$

$$\xi_p(t) = e^{-k_p(t-t_0)}\xi_p(i_0) + \int_{i_0}^{t} e^{-k_p(t-u)} g_p(\xi_s(u) - \xi_p(u))du, \tag{10}$$

and

$$Q_s(t) = e^{-k_s(t-i_0)}Q_s(i_0) + \int_{i_0}^{t} e^{-k_s(t-u)} g_s(Q_s(u) - Q_p(u))du + \sum_{i_0 \leq \theta_i < t} e^{-k_s(t-\theta_i)}(I_s + J_s(Q_s(\theta_i)) + W_i(Q(\theta_i)))$$

$$Q_p(t) = e^{-k_p(t-t_0)}Q_p(i_0) + \int_{i_0}^{t} e^{-k_p(t-u)} g_p(Q_s(u) - Q_p(u))du. \tag{11}$$

Then

$$\|\xi(t) - Q(t)\|_1 \leq e^{-\kappa_2(t-i_0)}\|\xi(i_0) - Q(i_0)\|_1 + \int_{i_0}^{t} e^{-\kappa_2(t-u)}(l_s + l_p)\|\xi(u) - Q(u)\|_1 du + \sum_{i_0 \leq \theta_i < t} e^{-\kappa_2(t-\theta_i)}(l_J\|\xi(\theta_i) - Q(\theta_i)\|_1 + \epsilon_1),$$

and for $i_0 \leq t \leq i_0 + k$,

$$\|\xi(t) - Q(t)\|_1 \leq \|\xi(i_0) - Q(i_0)\|_1 e^{-\kappa_2(t-i_0)} + \int_{i_0}^{t} e^{-\kappa_2(t-u)}(l_s + l_p)\|\xi(u) - Q(u)\|_1 du + \sum_{i_0 \leq \theta_i < t} e^{-\kappa_2(t-\theta_i)}(l_J\|\xi(\theta_i) - Q(\theta_i)\|_1 + \epsilon_1).$$

If we denote $v(t) = \|\xi(t) - Q(t)\|_1 e^{\kappa_2 t}$, then

$$v(t) \leq \left[\|\xi(i_0) - Q(i_0)\|_1 e^{\kappa_2 i_0} + \epsilon_1 e^{\kappa_2} \sum_{i=1}^{k} e^{\kappa_2 i}\right] + \int_{i_0}^{t} (l_s + l_p)v(u)du + \sum_{i_0 \leq \theta_i < t} l_J v(\theta_i).$$

Now, applying the Gronwall–Bellman Lemma for piecewise continuous functions [17] one can find that

$$v(t) \leq \left[\|\xi(i_0) - Q(i_0)\|_1 e^{\kappa_2 i_0} + \epsilon_1 e^{\kappa_2} \sum_{i=1}^{k} e^{\kappa_2 i}\right] e^{(l_s + l_p + \ln(1+l_J))(t-i_0)},$$

and

$$\|\xi(i_0 + k) - Q(i_0 + k)\|_1 \leq \|\xi(i_0) - Q(i_0)\|_1 e^{-(\kappa_2 - l_s - l_p - \ln(1+l_J))k} + e^{\kappa_2}\epsilon_1 \sum_{i=1}^{k} e^{\kappa_2 i} e^{-(\kappa_2 - l_s - l_p - \ln(1+l_J))k} < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}. \tag{12}$$

Similarly to (12), we can obtain that for $t \in [i_0 + k, i_0 + 2k]$

$$\|\xi(t) - Q(t)\|_1 \leq \frac{\epsilon}{2} e^{-(\kappa_2 - l_s - l_p - \ln(1+l_J))(t-i_0)} + e^{\kappa_2}\epsilon_1 \sum_{i=1}^{k} e^{\kappa_2 i} e^{-(\kappa_2 - l_s - l_p - \ln(1+l_J))(t-i_0)} < \frac{\epsilon}{2} + \frac{\epsilon}{4} \leq \epsilon, \tag{13}$$

and

$$\|\xi(i_0 + 2k) - Q(i_0 + 2k)\|_1 \leq \|\xi(i_0 + k) - Q(i_0 + k)\|_1 e^{-(\kappa_2 - l_s - l_p - \ln(1+l_J))k} + e^{\kappa_2}\epsilon_1 \sum_{i=1}^{k} e^{\kappa_2 i} e^{-(\kappa_2 - l_s - l_p - \ln(1+l_J))k} < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}. \tag{14}$$

Hence, by mathematical induction, for arbitrary $t \geq i_0 + k$ we have that $\|\xi(t) - Q(t)\|_1 < \epsilon$, and since ϵ can be arbitrarily small, $P(t, t_0)$, $t_0 \in (t^* - \delta, t^* - \delta)$, is eventually p-periodic. The theorem is proved. □

Since every bounded solution $P(t, t_0)$ is an attractor of all solutions of the system (9), it follows that $\xi(t, t^*)$ is an attractor of all solutions of (7) with $t_0 \in (t^* - \delta, t^* + \delta)$.

3.2. Chaos

It is known that a chaotic process takes place on a compact domain. Hence, it is natural to discuss the irregular behavior of the system only for the union of positive-valued and bounded solutions, that is positive solutions $P(t, t_0)$, which satisfy the inequalities $|P_s(t)|_0 < M_s,\ |P_p(t)|_0 < M_p$ for all $t \in \mathbb{R}$. Below, we say that a solution is bounded if it is bounded on $\mathbb{R}$. We should remark that, since each of these bounded solutions is an attractor of all solutions, that have the same initial moment. The chaotic properties are appropriate for all solutions, not only the bounded ones. So, to describe the chaos in the model we can consider only bounded solutions.

In this section we assume that $\mu > 4$. Then [23] there exists an invariant for h Cantor set $\Lambda \subset I$ such that $h(x, \mu)$ is chaotic on Λ. That is, h has sensitive dependence on initial conditions, periodic points are dense in Λ and h is topologically transitive. We may also point out that there are infinitely many orbits of h with different periods, that for each $p \in \mathbb{N}$ there exists a solution with period p, and that topological transitivity means the existence of a positive trajectory of h, dense in Λ.

Consider the sequence space [14]

$$\Sigma_2 = \{s = (s_0 s_1 s_2 \ldots) : s_j = 0 \text{ or } 1\}$$

with the metric

$$\mathrm{d}[s, t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i},$$

where $t = (t_0 t_1 \ldots) \in \Sigma_2$, and the shift map $\sigma : \Sigma_2 \to \Sigma_2$, such that $\sigma(s) = (s_1 s_2 \ldots)$. The map is continuous, $\mathrm{card}(\mathrm{Per}_n(\sigma)) = 2^n$, $\mathrm{Per}(\sigma)$ is dense in Σ_2, and there exists a dense orbit in Σ_2.

If we denote

$$I_0 = \left[0, \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\mu}}\right], \qquad A_0 = \left(\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\mu}}, \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{\mu}}\right), \qquad I_1 = \left[\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{\mu}}, 1\right].$$

then $I = I_0 \cup A_0 \cup I_1$, $\Lambda \subset I_0 \cup I_1$, $h(I_0) = h(I_1) = I$, $h(A_0) \cap I = \emptyset$.

Consider the itinerary of x, $S(x) = (s_0 s_1 \ldots)$, where $s_j = 0$, if $h^j(x) \in I_0$, and $s_j = 1$, if $h^j(x) \in I_1$. The function $S(x)$ is a homeomorphism between Λ and Σ_2, and $S \circ h = \sigma \circ S$. That is, h and σ are topologically conjugate.

Next, let us consider some useful properties of the elements of Π. We shall formulate and prove three – very important for us – consequences of the topological conjugacy of the symbolical dynamics and of the dynamics generated by the logistic map [14] in the following assertion.

Let $J \subseteq \mathbb{R}$ be an open interval. Introduce the following distance $\|\theta(t_0) - \theta(t_1)\|_J = \sup_{\theta_i(t_0), \theta_i(t_1) \in J} |\theta_i(t_0) - \theta_i(t_1)|$. One can read about the possible uses of this distance in [6].

Lemma 3.1. *If $\mu > 4$, then*

(a) *for each $\theta(t_0) \in \Pi$, arbitrarily small $\epsilon > 0$, and arbitrarily large positive number E there exists a periodical sequence $\theta(t_1) \in \Pi$ such that $|\theta(t_0) - \theta(t_1)|_J < \epsilon$, where $J = (0, E)$.*
(b) *There exists a sequence $\theta(t^*) \in \Pi$ such that for each $t_0 \in \Lambda$, and for arbitrarily small $\epsilon > 0$, and arbitrarily large positive number E there exists an integer m such that $|\theta(t_0) - \theta(t^*, m)|_J < \epsilon$, where $J = (0, E)$.*

Since the sequence θ depends on $t_0 \in I$, in (7) we shall denote the space of piecewise continuous functions as $\mathcal{PC}(t_0, \mathbb{R})$, instead of $\mathcal{PC}(\mathbb{R})$. Let us fix an interval $J \subset \mathbb{R}$, and $t_0, t_1 \in I$. We shall say that a function $\xi(t) \in \mathcal{PC}(t_0, \mathbb{R})$ is ϵ-equivalent to a function $\psi(t) \in \mathcal{PC}(t_1, \mathbb{R})$ on J and write $\xi(t)(\epsilon, J)\psi(t)$ if $\|\theta(t_s) - \theta(t_p)\|_J < \epsilon$ and $\|\xi(t) - \psi(t)\| < \epsilon$ for all t from J such that $t \notin \cup_{\theta_p(t_s), \theta_p(t_p) \in J} [\widehat{\theta_p(t_s), \theta_p}(t_p)]$.

We shall say that a bounded solution $P(t) = P(t, t_0, P_0)$, $t_0 \in \Lambda$, of (7) is sensitive with respect to the initial data if there exist positive real numbers ϵ_0, ϵ_1 such that for every $\delta > 0$ one could find a pair $(t_1, P_1) \in \Lambda \times \mathbb{R}^n$, $|t_0 - t_1| + \|P_0 - P_1\| < \delta$, and an interval J_1 in $[t_0, \infty)$ of length not less than ϵ_1 such that $\|P_1(t) - P(t)\| > \epsilon_0$, $t \in J_1$, where $P_1(t), P_1(t_1) = P_1$, is a bounded solution of (7), and there is no point of discontinuity of $P_1(t)$ and $P(t)$ in J_1.

A bounded solution $P(t) = P(t, t_0, P_0)$, $t_0 \in \Lambda$, $t \geq t_s$, of (7) is called dense in the set of all bounded solutions which start on Λ if for each bounded solution $P_1(t) = P(t, t_1, P_1)$, $t_1 \in \Lambda$, of (7) and arbitrarily large positive number E there exist an interval J of length E and a real number s such that $P(t + s)(\epsilon, J)P_1(t)$.

The set of all periodic solutions $\xi(t, t_0)$, $t_0 \in \Lambda$, of (7) is called dense in the set of all bounded solutions which start on Λ if for each bounded solution $P(t) = P(t, t_1)$, $t_1 \in \Lambda$, of (7) and each $\epsilon > 0$, $E > 0$, there exists a periodic solution $\xi(t)$ and an interval $J \subset [t_1, \infty)$ with length E such that $\phi(t)(\epsilon, J)P(t)$.

Theorem 3.2. *Assume that conditions* (C1)–(C8) *are fulfilled. Then there exists a dense solution of* (7)*.*

Proof. By Lemma 3.1(b) there exists $t^* \in \Lambda$ such that $\theta(t^*)$ is dense in Π, such that for each $t_0 \in \Lambda$, for arbitrarily small $\epsilon > 0$, and arbitrarily large positive number E there exists a positive integer m such that $|\theta(t_0) - \theta(t^*, m)|_J < \epsilon$, where

$J = (0, E)$. By Theorem 2.1 there exists a unique bounded solution $\xi^*(t) = \xi(t, t^*)$. Let us prove that $\xi^*(t)$ is the dense solution.

Consider an arbitrary solution $\xi(t) = \xi(t, t_0)$, $t_0 \in \Lambda$, of (7). Then for $t \geq 0$, we have that

$$\xi_s^*(t+m) = e^{-k_s(t+m-m)}\xi_s^*(m) + \int_m^{t+m} e^{-k_s(t+m-u)} g_s(\xi_s(u) - \xi_p(u))du + \sum_{m\leq\theta_i<t+m} e^{-k_s(t+m-\theta_i)}(I_s + J_s(\xi_s(\theta_i)))$$
$$= e^{-k_s t}\xi_s^*(m) + \int_0^{t} e^{-k_s(t-u)} g_s(\xi_s^*(u+m) - \xi_p^*(u+m))du + \sum_{m\leq\theta_i<t+m} e^{-k_s(t+m-\theta_i)}(I_s + J_s(\xi_s^*(\theta_i))). \quad (15)$$

Similarly,

$$\xi_p^*(t+m) = e^{-k_p t}\xi_p^*(m) + \int_0^{t} e^{-k_p(t-u)} g_p(\xi_s^*(u) - \xi_p^*(u))du. \quad (16)$$

Moreover,

$$\xi_s(t) = e^{-k_s t}\xi_s(0) + \int_0^{t} e^{-k_s(t-u)} g_s(\xi_s(u) - \xi_p(u))du + \sum_{0\leq\theta_i<t} e^{-k_s(t-\theta_i)}(I_s + J_s(\xi_s(\theta_i))),$$
$$\xi_p(t) = e^{-k_p t}\xi_p(0) + \int_0^{t} e^{-k_p(t-u)} g_p(\xi_s(u) - \xi_p(u))du. \quad (17)$$

Now, using the last three formulas and using the B-equivalence technique, similarly to the proof of Theorem 3.1, we can find a number E_1 sufficiently large so that there exists a subinterval J of J_1 of length E such that $\xi_*(t+m)$ and $\xi(t)$ are ϵ-equivalent on J. The theorem is proved. □

We shall need the following assumption.

(C8) $L < \frac{I_s e^{-k_s}}{16M}$.

Theorem 3.3. *Assume that conditions* (C1)–(C8) *are fulfilled. Then every bounded solution of* (7) *is sensitive with respect to the initial data.*

Proof. Let $S(t_0) = s^0 = (s_0^0, s_1^0 \ldots)$. Fix a number $t_1 \in \Lambda$ such that $S(t_1) = s^1 = (s_0^0, s_1^0 \ldots, s_{n-1}^0, s_n^1, s_{n+1}^0, s_{n+2}^0, \ldots)$, $s_n^1 \neq s_n^0$, for some $n > 0$. We have that

$$d[\sigma^i s^0, \sigma^i s^1] = \begin{cases} \dfrac{1}{2^{n-i}} & \text{if } 0 \leq i \leq n, \\ 0 & \text{if } i > n. \end{cases}$$

Take a positive number δ. Assume that $n \geq 8$, it is an even number, and that it is sufficiently large so that $|t_0 - t_1| = |S^{-1}(s^0) - S^{-1}(s^1)| < \frac{\delta}{2}$.

Since S is a homeomorphism and the set Σ_2 is compact, for a given i, $0 \leq i \leq n$, the set

$$d_i = \left\{(\bar{s}, \tilde{s}) \in \Sigma_2 \times \Sigma_2 : d[\bar{s}, \tilde{s}] \geq \frac{1}{2^{n-i}}\right\}$$

is compact, and

$$\min_{(\bar{s},\tilde{s})\in P_i} |S^{-1}(\bar{s}) - S^{-1}(\tilde{s})| = \mu_i > 0,$$

$d_{i+1} \subseteq d_i$, $\mu_{i+1} \geq \mu_i$, $0 \leq i < n$. Fix i_0, $\frac{n}{2} \leq i_0 < n-1$. Then $|\kappa_i(t_0) - \kappa_i(t_1)| > \mu_{i_0}$ if $i = i_0, i_0+1$.

We also have that $|\kappa_i(t_0) - \kappa_i(t_1)| \leq \sqrt{1 - \frac{4}{\mu}}$ if $0 \leq i < n$.

Without loss of generality, assume that $\kappa_i(t_0) < \kappa_i(t_1)$ for all i. Thus, there is a number k among i_0, i_0+1, such that $\kappa_k(t_1) - \kappa_k(t_0) > \mu_{i_0}$ and $\kappa_k(t_0) - \kappa_{k-1}(t_1) \geq \frac{1}{2}(1 - \sqrt{1 - \frac{4}{\mu}})$.

It is obvious that a solution is sensitive if its first coordinate is.

Fix a solution $P(t) = P(t, t_0)$. We shall show that the constants ϵ_0, ϵ_1 of the sensitivity definition can be taken equal to $\epsilon_0 = \frac{I_s e^{-2k_s}}{8}$, $\epsilon_1 = \min\left\{\mu_{i_0}, \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{\mu}}\right)\right\}$.

Let us fix a solution $P^1(t) = (P_s^1(t), P_p^1(t))$, $\|P_1 - P_0\| < \frac{\delta}{2}$.

Below, we consider two alternative cases. Assume, first, that $|P_s(\theta_k(t_0)) - P_s^1(\theta_k(t_0))| < \frac{I_s e^{-2k_s}}{4}$.

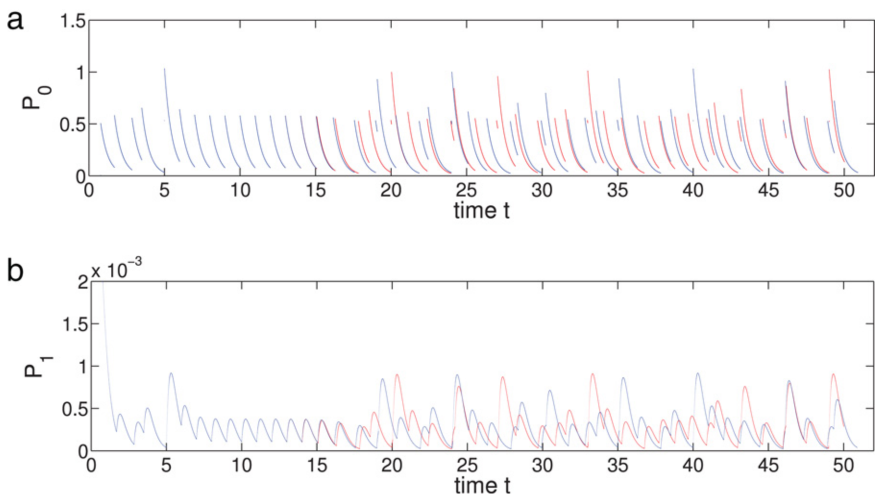

Fig. 1. Graphs of coordinates $P_0(t), P_1(t)$ are in the blue, and of coordinates $\bar{P}_0(t), \bar{P}_1(t)$ are red. The coordinates abruptly become significantly different when t is near 15, while coinciding for all t in the interval $(t_1, 15)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Then on the interval $[\theta_k(t_0), \theta_k(t_1)]$

$$P_s(t) = \mathrm{e}^{-k_s(t-\theta_k(t_0))}P_s(\theta_k(t_0)) + \int_{\theta_k(t_0)}^{t} \mathrm{e}^{-k_s(t-u)}g_s(P_s(s) - P_p(s))\mathrm{d}u + \mathrm{e}^{-k_s(t-\theta_k(t_0))}(I_s + J_s(P_s(\theta_k(t_0)))),$$

and

$$P_s^1(t) = \mathrm{e}^{-k_s(t-\theta_k(t_0))}P_s^1(\theta_k(t_0)) + \int_{\theta_k(t_0)}^{t} \mathrm{e}^{-k_s(t-u)}g_s(P_s^1(s) - P_p^1(s))\mathrm{d}u.$$

Hence,

$$|P_s(t) - P_s^1(t)| \geq |\mathrm{e}^{-k_s(t-\theta_k(t_0))}(I_s + J_s(P_s(\theta_k(t_0))))| \\ - \mathrm{e}^{-k_s(t-\theta_k(t_0))}|P_s(\theta_k(t_0)) - P_s^1(\theta_k(t_0))| - \int_{\theta_k(t_0)}^{t} \mathrm{e}^{-k_s(t-u)}2LM\mathrm{d}u \geq \epsilon_0.$$

In the case when $|P_s(\theta_k(t_0)) - P_s^1(\theta_k(t_0))| \geq \frac{I_s \mathrm{e}^{-k_s}}{4}$, we consider the interval $[\theta_{k-1}(t_1), \theta_k(t_0)]$, where

$$P_s(t) = \mathrm{e}^{-k_s(t-\theta_k(t_0))}P_s(\theta_k(t_0)) + \int_{\theta_k(t_0)}^{t} \mathrm{e}^{-k_s(t-u)}g_s(P_s(s) - P_p(s))\mathrm{d}u,$$

and

$$P_s^1(t) = \mathrm{e}^{-k_s(t-\theta_k(t_0))}P_s^1(\theta_k(t_0)) + \int_{\theta_k(t_0)}^{t} \mathrm{e}^{-k_s(t-u)}g_s(P_s^1(s) - P_p^1(s))\mathrm{d}u.$$

Consequently,

$$|P_s(t) - P_s^1(t)| \geq \mathrm{e}^{-k_s(t-\theta_k(t_0))}|P_s(\theta_k(t_0)) - P_s^1(\theta_k(t_0))| - 2LM \geq \epsilon_0.$$

The theorem is proved. □

One must say that under the additional condition $|g_p(z_1) - g_p(z_2)| \leq |z_1 - z_2|, z_1, z_2 \in \mathbb{R}$, one can easily prove that a bounded solution is sensitive in the second coordinate, too.

The next theorem can be proved similarly to the verification of Theorem 3.2 using *B*-equivalence technique and Lemma 3.1(a).

Theorem 3.4. *Assume that conditions* (C1)–(C8) *are fulfilled. Then the set of all periodic solutions* $\xi(t, t_0), t_0 \in \Lambda$, *of* (7) *is dense in the set of all bounded solutions.*

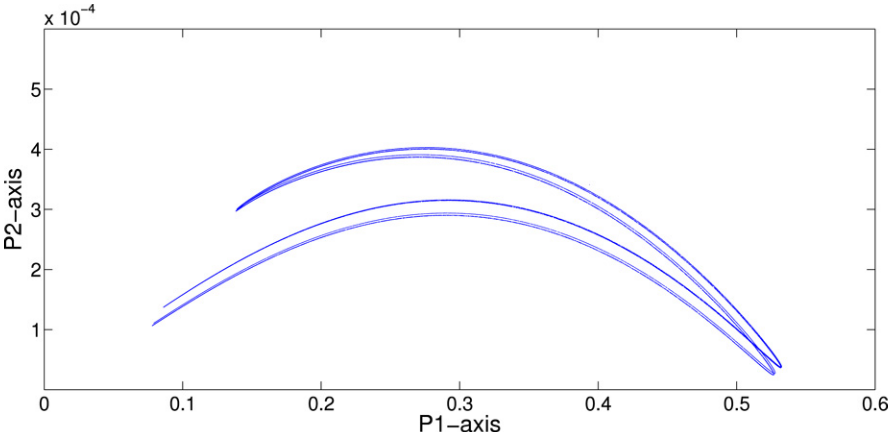

Fig. 2. The chaotic attractor by a stroboscopic sequence $P(n)$, $1 \le n \le 75\,000$, is observable.

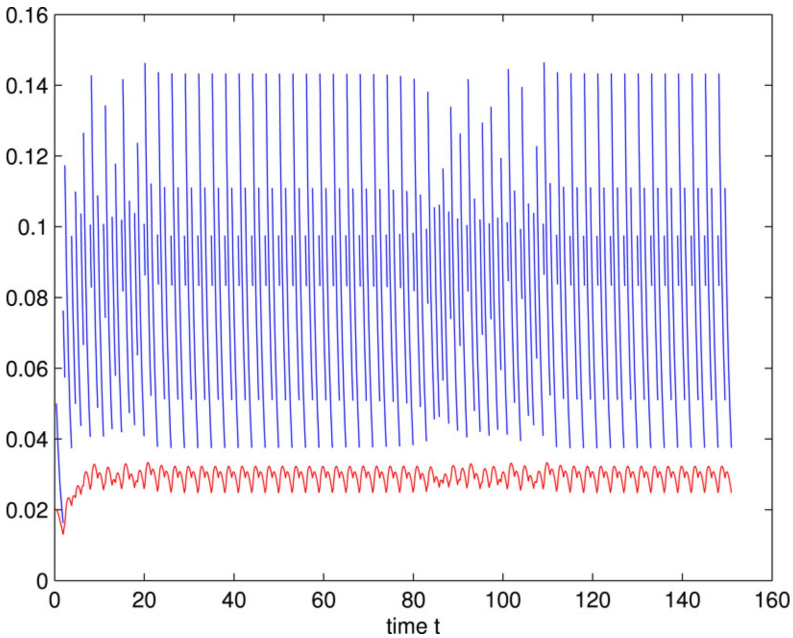

Fig. 3. Both coordinates $P_0(t)$, $P_1(t)$, which are in red and blue colors respectively, have intermittency bouts in the intervals (0, 20) and (90, 110). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

4. Example

It is not easy task to find initial moments which can illustrate the chaos of a system, if $\mu > 4$. For this reason we propose to consider values of parameter μ, which are not discussed in our paper, but which can help us demonstrate the chaotic nature of the considered initial value problem. We construct a simulation to show that the dynamics of blood pressure can exhibit sensitivity if $\mu = 4$, and intermittency for $\mu = 3.8282$. Moreover, if $\mu = 4$ we observe the chaos attractor by a stroboscopic sequence on $\mathbb{R}^2$.

Let the following system be given

$$\begin{aligned} P_0' &= -2P_0 - g_1(P_0 - P_1), \\ P_1' &= -3P_1 + g_2(P_0 - P_1), \\ \Delta P_0|_{t=\theta_i} &= 0.5, \end{aligned} \tag{18}$$

where $\theta_i = i + \xi_i$, the sequence ξ is defined recursively, $\xi_i = 4\xi_{i-1}(1 - \xi_{i-1})$, $\xi_0 = t_0$, $t_0 \in [0, 1]$, $i \ge 0$, $g_1(u) = g_2(u) = l \sin^2 u$, $W(s) = 1 + s^2$, if $|s| \le l$. One can easily see that all the functions are Lipschitzian with a constant proportional to l. In what follows we assume that $l = 10^{-4}$.

Consider two solutions $P(t) = (P_0, P_1), \bar{P}(t) = (\bar{P}_0, \bar{P}_1)$, with initial moments $t_0 = 7/9$ and $\bar{t}_0 = 7/9 + 3^{-12}$, respectively. That is, we take the initial values close to each other, and, moreover, the solutions with identical initial values, $P(t_0) = \bar{P}(\bar{t}_0) = (0.005, 0.002)$. The graphs of the coordinates of these solutions (Fig. 1) show that the solutions abruptly become different when t is between 15 and 20, despite being very close to each other for all t in the interval $(t_1, 15)$. One can conclude that the phenomenon of sensitivity is numerically observable.

Next, in Fig. 2 the chaotic attractor is shown by using points $P(n), n = 1, 2, 3, \dots, 75000$, in P_1, P_2-plane.

If one considers (18) with the sequence $\theta_i = i + \xi_i, \xi_i = 3.8282\xi_{i-1}(1 - \xi_{i-1}), \xi_0 = t_0, t_0 \in [0, 1], i \geq 0$, then the phenomenon of intermittency (i.e. irregular switching between periodic and chaotic behavior) for a solution $P(t) = (P_0, P_1)$ can be observed in Fig. 3. The coefficient's value of 3.8282 is such that the logistic map admits intermittency [14].

5. Conclusion

In this paper we continue investigation of qualitative characteristics of the system (3), which can be considered as a model of the blood pressure distribution [3]. We introduce a new initial boundary value problem for the system, and consider Eq. (7). The main modeling novelty, useful for applications, is that the moments of discontinuity are prescribed by the choice of the initial moment. Apparently, this assumption could be accepted as an appropriate deterministic condition for many real life processes. The problem is restricted for the two-dimensional model, and the investigation is confined on the chaotic behavior. We have defined the complex discontinuous dynamics proving that sensitivity and transitivity, as well as existence of a countable set of periodic solutions, are proper for the set of solutions of the problem bounded on the whole real axes. The chaotic attractor is observable.

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