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DYNAMICAL SYNTHESIS OF QUASI-MINIMAL SETS

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We address a nonautonomous differential equation with a pulse function, whose moments of discontinuity depend on the initial moment. Existence of a quasi-minimal set is proved. An appropriate simulation of a chaotic attractor is presented.

Keywords: Differential equations; pulse functions; Poisson stability; chaotic attractor.

1. Introduction

L. Shil'nikov emphasized in [Shil'nikov, 2002] that "...it seems quite reasonable that the role of dynamical chaos orbits should be assigned to the Poisson stable trajectories," and "...we arrive at the following problem: how can one establish the existence of the Poisson stable trajectories in the phase space of a system?"

The goal of our paper is to obtain a quasi-minimal set by inserting a generator of moments of discontinuities into a dissipative system. That is, we aim to apply *dynamical synthesis* [Brown *et al.*, 1992, 1993], which is a general technique for constructing dynamical systems with desired properties.

We believe that this approach makes a strong impact on applications, since one can investigate controllability of chaos [Fradkov, 2007; Otto *et al.*, 1990] based on similar properties of the generator-function, which are already known or can be developed, if needed. For example, it provides a tool for supporting the given degree of nonregularity, which is important for cardiac rhythm [Garfinkel *et al.*, 1992]. Another issue of relevance to the paper is the nonlinear dynamics of electrical circuits and mechanical models [Atherton, 1982; Elwakil, 2002; Tsypkin *et al.*, 1964], which convert discrete data into continuous output.

The main focus of our investigation is the following special initial value problem

$$\begin{aligned} z'(t) &= Az(t) + f(z) + v(t,t_0), \\ z(t_0) &= z_0, (t_0, z_0) \in \Lambda \times \mathbb{R}^n, \end{aligned} \tag{1}$$

where $z \in \mathbb{R}^n, t \in \mathbb{R}, i \in \mathbb{Z}$, $\mathbb{R}$ and $\mathbb{Z}$ are sets of all real numbers and integers respectively,

$$v(t,t_0) = \begin{cases} m_0 & \text{if } \zeta_{2i}(t_0) < t \le \zeta_{2i+1}(t_0), \quad i \in \mathbb{Z}, \\ 0 & \text{if } \zeta_{2i-1}(t_0) < t \le \zeta_{2i}(t_0), \quad i \in \mathbb{Z}, \end{cases}$$

where $m_0 \in \mathbb{R}^n$ is a nonzero vector. Cantor set $\Lambda \subset [0,1]$ and the strictly increasing sequence $\zeta(t_0) = \{\zeta_i(t_0)\}$, $i \in \mathbb{Z}, i \le \zeta_i(t_0) \le i+1$, will be described in the next section. The function f satisfies the Lipschitz condition with a positive constant L, A is an $n \times n$ constant real valued matrix, and there exist positive numbers N, α, such that $\|e^{At}\| \le Ne^{-\alpha t}, t \ge 0$.

It is worth mentioning that we can consider other types of equations to obtain similar results. For instance, one may assume that function f depends on t and has discontinuities of the first kind at points of $\zeta(t_0)$.

For a fixed $t_0 \in \Lambda$, system (1) is a differential equation with discontinuous right-hand side [Filippov, 1988] of a specific type where discontinuities occur on vertical planes in the (t, z)-space.

In what follows, we use the definition of solutions formulated in [Wiener, 1993] (see, also, [Akhmet, 2007]).

A function $z(t), z(t_0) = z_0$, is a solution of (1) on $\mathbb{R}$ if: (i) $z(t)$ is continuous on $\mathbb{R}$; (ii) the derivative $z'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of the points $\zeta_i(t_0)$, where left-sided derivatives exist; (iii) Equation (1) is satisfied on each interval $(\zeta_i(t_0), \zeta_{i+1}(t_0)], i \in \mathbb{Z}$.

It can be easily verified that problem (1) has a unique solution $z(t, t_0, z_0), t \in \mathbb{R}$, for each $t_0 \in \Lambda, z_0 \in \mathbb{R}^n$.

The solution $z(t) = z(t, t_0, z_0)$ satisfies the following integral equation

$$z(t) = \mathrm{e}^{A(t-t_0)} z_0 + \int_{t_0}^{t} \mathrm{e}^{A(t-s)}[f(z(s)) + v(s, t_0)]\, ds.$$

In the sequel, we assume that $\sup_{\mathbb{R}^n} |f(z)| = M_0 < \infty$, $NL < \alpha$. Fix a sequence $\zeta(t_0), t_0 \in \Lambda$. Using the standard technique, one can verify that $z(t)$ is a solution of (1), bounded on $\mathbb{R}$ if and only if it satisfies the equation

$$z(t) = \int_{-\infty}^{t} \mathrm{e}^{A(t-s)}[f(z(s)) + v(s, t_0)]\, ds,$$

and for each sequence $\zeta(t_0), t_0 \in \Lambda$, there exists a unique solution $z(t, \zeta(t_0))$ bounded on $\mathbb{R}$ and all these solutions are placed in the tube with radius $M = M_0[1 + (N/(\alpha - NL))], t \in \mathbb{R}$. Moreover, if $z(t, t_0, z_0)$ is a solution of (1), then one can obtain by Gronwall–Bellman Lemma that

$$\begin{aligned} &\|z(t, t_0, z_0) - z(t, \zeta(t_0))\| \\ &\quad \le N\|z_0 - z(t_0, \zeta(t_0))\| \mathrm{e}^{(-\alpha+NL)(t-t_0)}. \end{aligned}$$

That is, the bounded solution $z(t, \zeta(t_0))$ attracts all solutions of (1) with the same initial moment $t_0, t_0 \in \Lambda$.

Let (X, ρ) denote a metric space and $\mathbb{T}$ be either $\mathbb{R}$ or $\mathbb{Z}$. Consider a set $\mathcal{S}$ of functions defined on $\mathbb{T}$. An element $\phi(t) \in \mathcal{S}$ is called a motion if it is a solution of either a differential or a discrete equation. We say that a motion $\phi(t) \in \mathcal{S}$ is positively Poisson stable (P_+ stable) if for each $\gamma \in \mathbb{T}$ there exist two sequences $\beta_n, E_n \in \mathbb{T}$ with $\beta_n, E_n \to \infty$ such that

$$\lim_{n\to\infty} \sup_{-E_n<t<E_n} \rho[\phi(\beta_n + t), \phi(\gamma + t)] = 0.$$

A motion $\phi(t) \in \mathcal{S}$ is said to be negatively Poisson stable (P_- stable) if for each $\gamma \in \mathbb{T}$ there exist two sequences $\beta_n, E_n \in \mathbb{T}$ with $\beta_n, E_n \to \infty$ such that

$$\lim_{n\to\infty} \sup_{-E_n<t<E_n} \rho[\phi(-\beta_n + t), \phi(\gamma + t)] = 0.$$

Motion $\phi(t)$ is Poisson stable (P stable) if it is P_- and P_+ stable.

It is said that $\psi(t) \in \mathcal{S}$ is an ω-limit motion corresponding to $\phi(t) \in \mathcal{S}$, if for each $\gamma \in \mathbb{R}$ there exist two sequences $\beta_n, E_n \in \mathbb{T}$ with $\beta_n, E_n \to \infty$, and

$$\lim_{n\to\infty} \sup_{-E_n<t<E_n} \rho[\phi(\beta_n + t), \psi(\gamma + t)] = 0.$$

$\psi(t) \in \mathcal{S}$ is called an α-limit motion corresponding to $\phi(t) \in \mathcal{S}$, if for each $\gamma \in \mathbb{R}$ there exist two sequences $\beta_n, E_n \in \mathbb{T}$ with $\beta_n, E_n \to \infty$, and

$$\lim_{n\to\infty} \sup_{-E_n<t<E_n} \rho[\phi(-\beta_n + t), \psi(\gamma + t)] = 0.$$

Denote the sets, which consist of all ω-limit motions and α-limit motions by Ω_ϕ and $\mathcal{A}_\phi$, respectively.

We say that $\mathcal{S}$ is a quasi-minimal set if $\mathcal{S} = \Omega_\phi = \mathcal{A}_\phi$, where $\phi \in \mathcal{S}$.

The definition of Poisson stability can be compared with the one for solutions of nonautonomous equations in [Sell, 1971], Chapter 8.

2. Poisson Stability of a Discrete Motion

Consider the sequence space $\Sigma_2 = \{s = (s_0 s_1 s_2 \cdots) : s_j = 0 \text{ or } 1\}$ with the metric $d[s, \bar{s}] = \sum_{i=0}^{\infty} |s_i - \bar{s}_i|/2^i$, where $\bar{s} = (\bar{s}_0 \bar{s}_1 \cdots) \in \Sigma_2$, and the shift map $\sigma : \Sigma_2 \to \Sigma_2$, such that $\sigma(s) = (s_1 s_2 \cdots)$.

In addition, let us consider the space of bi-sequences

$$\begin{aligned} \Sigma^2 = \{&s = (\cdots s_{-n} \cdots s_{-1} s_0 s_1 s_2 \cdots s_n \cdots) : \\ &s_j = 0 \text{ or } 1\}. \end{aligned}$$

We introduce the following ordering in Σ^2 [Wiggins, 1990]. Given two finite sequences $s = \{s_1 \cdots s_k\}, \bar{s} = \{\bar{s}_1 \cdots \bar{s}_{k'}\}$, it is said that $s < \bar{s}$, if $k < k'$, and, if $k = k'$, then $s < \bar{s}$ if $s_i < \bar{s}_i$, where i is the first integer such that $s_i \neq \bar{s}_i$.

Thus, one can denote the sequences having length k as follows: $s_1^k < \cdots < s_{2^k}^k$, where the superscript refers to the length of the sequence and the subscript refers to a particular sequence of length k which is uniquely specified by the ordering scheme above. Denote $s^* = (\cdots s_8^3 s_6^3 s_4^3 s_2^2 s_2^1 s_1^1 s_1^2 s_3^2 s_1^3 s_3^3 s_5^3 s_7^3 \cdots)$.

Introduce the maps $B_i : \Sigma^2 \to \Sigma_2, i \in \mathbb{Z}$, such that $B_i(s) = (s_i s_{i+1} \cdots)$. From the method of construction of s^*, it follows that the following assertion is valid.

Lemma 2.1

(1) *For a fixed $j \in \mathbb{Z}$ there exist two sequences of integers k_n, l_n with $k_n, l_n \to \infty$, and*

$$\lim_{n\to\infty} \max_{-l_n \le i \le l_n} d[B_{k_n+i}(s^*), B_{j+i}(s^*)] = 0.$$

(2) *For a fixed $j \in \mathbb{Z}$ there exist two sequences of integers k_n, l_n with $k_n, l_n \to \infty$, and*

$$\lim_{n\to\infty} \max_{-l_n \le i \le l_n} d[B_{-k_n+i}(s^*), B_{j+i}(s^*)] = 0.$$

(3) *For each $s \in \Sigma^2$ and $j \in \mathbb{Z}$ one can find sequences of integers k_n, l_n with $k_n, l_n \to \infty$, and*

$$\lim_{n\to\infty} \max_{-l_n \le i \le l_n} d[B_{k_n+i}(s^*), B_j(s)] = 0.$$

(4) *For each $s \in \Sigma^2$ and $j \in \mathbb{Z}$ one can find sequences of integers k_n, l_n with $k_n, l_n \to \infty$, and*

$$\lim_{n\to\infty} \max_{-l_n \le i \le l_n} d[B_{-k_n+i}(s^*), B_j(s)] = 0.$$

Assume that there exist a homeomorphism S between Σ_2 and a set $\Lambda \subset [0,1]$, and a map $h : \Lambda \to \Lambda$ such that $S \circ h = \sigma \circ S$. That is, h and σ are topologically conjugate. It is known that Σ_2 and Λ are Cantor sets [Wiggins, 1990] and they are compact. One of the most popular examples of the map h is the logistic map $\mu x(1-x), \mu > 4$, considered on a subset of $[0,1]$ [Robinson, 1995].

For every $t_0 \in \Lambda$, one can construct a sequence $\kappa(t_0)$ of real numbers $\kappa_i, i \in \mathbb{Z}$, in the following way. If $i \ge 0$, then $\kappa_{i+1} = h(\kappa_i)$ and $\kappa_0 = t_0$. Let us show how the sequence is defined for negative i. Denote $s^0 = (s_0^0 s_1^0 \cdots) = S(t_0)$. Consider elements $\underline{s} = (0 s_0^0 s_0^1 \cdots), \bar{s} = (1 s_0^0 s_0^1 \cdots)$ of Σ_2 such that $\sigma(\underline{s}) = \sigma(\bar{s}) = s^0$ and $\underline{t} = S^{-1}(\underline{s}), \bar{t} = S^{-1}(\bar{s})$. The homeomorphism implies that $h(\bar{t}) = h(\underline{t}) = t_0$. The set $h^{-1}(t_0)$ may not consist of less than two elements $\bar{t}, \underline{t} \in \Lambda$. Each of these two values can be chosen as $\kappa_{-1}(t_0)$. Obviously, one can continue the process to $-\infty$ by choosing always one element from the set h^{-1}. We have finalized the construction of the sequence, and moreover, it is proved that $\kappa(t_0) \subset \Lambda, \kappa(t_0) = \{\kappa_i(t_0)\}, i \in \mathbb{Z}$. Thus, infinitely many sequences $\kappa(t_0)$ can be constructed for a given t_0. However, each of this type of sequence is unique for increasing i. Fix one of the sequences and define a sequence $\zeta(t_0) = \{\zeta_i\}$, $\zeta_i = i + \kappa_i, i \in \mathbb{Z}$. If we denote by Π the set of all such sequences $\{\zeta_i\}, i \in \mathbb{Z}$, then a multivalued functional $w : \Lambda \to \Pi$ is defined. In our paper, the sequence $\zeta(t_0)$ in (1) is considered to be a value of $w(t_0)$.

The above discussion shows that there exists a one-to-one correspondence between Σ^2 and Π. Denote by $\zeta(t^*)$ the sequence which corresponds to s^*. Then homeomorphism S and Lemma 2.1 imply that the following assertion is correct.

Theorem 2.1. $\Pi = \Omega_{\zeta(t^*)} = \mathcal{A}_{\zeta(t^*)}$.

3. The Quasi-Minimal Set

Denote $\mathcal{CB} = \{z(t, \zeta(t_0)) : \zeta(t_0) \in \Pi\}$. We shall show that the set $\mathcal{CB}$, which is an attractor for all solutions of (1), is a quasi-minimal set.

Theorem 3.1. *$\mathcal{CB}$ is the quasi-minimal set.*

Proof. Consider the sequence $\zeta(t^*)$, which has been defined in Theorem 2.1, and the solution $z(t, \zeta(t^*))$ of (1). We shall show that the solution is P_+ stable by considering $\gamma = 0$. The proof is very similar for any other $\gamma \in \mathbb{R}$. Fix a positive ϵ. Moreover, fix a positive ϵ_1, whose dependence on ϵ will be described below. From Theorem 2.1, it follows that there exist sufficiently large natural numbers j and m such that $|\zeta_{i+m}(t^*) - \zeta_i(t^*)| < \epsilon_1$ if $-j \le i \le j$. For the sake of simplicity, we shall write ζ_i instead of $\zeta_i(t^*)$.

We have for $t \ge -j$,

$$z(t, \zeta(t^*)) = \mathrm{e}^{A(t-\zeta_j(t^*))} z(\zeta_j, \zeta(t^*)) + \int_{\zeta_j}^{t} \mathrm{e}^{A(t-s)} [f(z(s, \zeta(t^*))) + v(s, t_0)] ds,$$

and

$$\begin{aligned} z(t + \zeta_{-j+m} - \zeta_j) &= \mathrm{e}^{A(t-\zeta_j)} z(\zeta_{-j+m}, \zeta(t^*)) \\ &\quad + \int_{\zeta_{-j+m}}^{t+\zeta_{-j+m}-\zeta_j} \mathrm{e}^{A(t+\zeta_{-j+m}-\zeta_j-s)} [f(z(s, \zeta(t^*))) \\ &\quad + v(s, t_0)] ds \\ &= \mathrm{e}^{A(t-\zeta_j)} z((\zeta_{-j+m}) \\ &\quad + \int_{\zeta_j}^{t} \mathrm{e}^{A(t-s)} [f(z(s + \zeta_{-j+m} - \zeta_j)) \\ &\quad + v(s + \zeta_{-j+m} - \zeta_j, t_0)] ds. \end{aligned}$$

Subtracting the last expression from the previous one, we obtain that

$$
\begin{aligned}
&\|z(t,\zeta(t^*)) - z(t+\zeta_{-j+m}-\zeta_j)\| \\
&\quad \le 2MNe^{-\zeta_j(t-\zeta_j)} \\
&\qquad + \int_{\zeta_j}^{t} NLe^{-\alpha(t-s)}\|z(s,\zeta(t^*)) \\
&\qquad - z(s+\zeta_{-j+m}-\zeta_j)\|ds \\
&\qquad + \int_{\zeta_j}^{t} Ne^{-\alpha(t-s)}\epsilon_1\|m_0\|ds.
\end{aligned}
$$

Next, we denote $u(t) = \|z(t,\zeta(t^*)) - z(t+\zeta_{-j+m}-\zeta_j)\|e^{\alpha t}$ and apply the following assertion.

Lemma 3.1 [Barbashin, 1970]. *Let $u(t), f(t)$ be non-negative functions integrable over the interval $t_0 \le t \le t_0+T$ and K be a positive constant. If the inequality*

$$u(t) \le f(t) + K\int_{t_0}^{t} u(s)ds, \quad t_0 \le t \le t_0+T,$$

is fulfilled then the following expression holds

$$u(t) \le f(t) + K\int_{t_0}^{t} e^{K(t-s)}f(s)ds.$$

Thus, we have

$$
\begin{aligned}
&\|z(t,\zeta(t^*)) - z(t+\zeta_{-j+m}-\zeta_j)\| \\
&\quad \le \frac{N\epsilon_1\|m_0\|}{\alpha}\left(1+\frac{1}{\alpha-NL}\right) \\
&\qquad + \frac{\alpha[2M(\alpha-NL)-\epsilon_1\|m_0\|]}{\alpha L(\alpha-NL)}e^{(-\alpha+NL)(t-\zeta_j)} \\
&\qquad + \frac{\alpha[(NL-1)(2M\alpha-\epsilon_1\|m_0\|)}{\alpha L}e^{-\alpha(t-\zeta_j)}.
\end{aligned}
$$

On the basis of the last inequality, one can see that $\|z(t,\zeta(t^*)) - z(t+\zeta_{-j+m}-\zeta_j)\| < \epsilon$ if $t \in (-E,E)$, where $E = j/2, j$ is sufficiently large, and ϵ_1 is a sufficiently small positive number. The number $\zeta_{-j+m}-\zeta_j$ is as large as m. Thus, we have proved that the solution is P_+ stable. Applying Theorem 2.1 again, one can show that $z(t,\zeta(t^*))$ is P_- stable and $\mathcal{CB} = \Omega_{z(t,\zeta(t^*))} = \mathcal{A}_{z(t,\zeta(t^*))}$, which completes the proof. ■

4. A Simulation Result

Consider the sequence $\zeta_i = i+\kappa_i, \kappa_i = 4\kappa_{i-1}(1-\kappa_{i-1}), \kappa_0 = t_0, t_0 \in [0,1], i \ge 0$, and the following system

$$
\begin{aligned}
&x'' + 2x' + 1.5x = \sin y, \\
&y' = -3y + v(t,t_0),
\end{aligned}
\tag{2}
$$

where $v(t,t_0)$ is a scalar pulse function with $m_0 = 1$. The second equation is a drive equation and the first one, the pendulum equation. Using new variables $x_1 = x, x_2 = x', x_3 = y$, one can reduce (2) to the system

$$
\begin{aligned}
&x_1' = x_2, \\
&x_2' = -1.5x_1 - 2x_2 + \sin x_3, \\
&x_3' = -3x_3 + v(t,t_0).
\end{aligned}
\tag{3}
$$

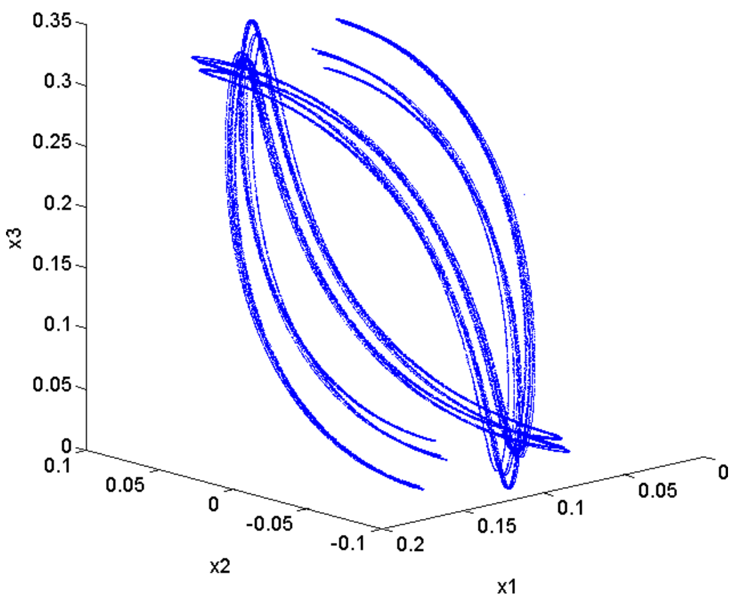

Fig. 1. The chaotic attractor by a stroboscopic sequence $(x_1(n), x_2(n), x_3(n)), 1 \le n \le 75\,000$, is observable.

One can easily verify that all eigenvalues of the matrix of coefficients have negative real parts. Fix $t_0 = 12/17$ and take a solution $(x_1(t), x_2(t), x_3(t))$ of the last system with the initial condition $x_1(t_0) = 0.02, x_2(t_0) = -0.025, x_3(t_0) = -0.02$. In Fig. 1 the chaotic attractor is shown by using points $(x_1(n), x_2(n), x_3(n)), n = 1, 2, 3, \ldots, 75\,000$, in $x_1x_2x_3$-space.

Remark 4.1. Using the technique of the present paper, we shall develop shadowing theorems [Anosov, 1967; Bowen, 1975; Hammel *et al.*, 1987] for system (1) in future investigation.

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