DYNAMICAL SYNTHESIS OF QUASI-MINIMAL SETS

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We address a nonautonomous differential equation with a pulse function, whose moments of discontinuity depend on the initial moment. Existence of a quasi-minimal set is proved. An appropriate simulation of a chaotic attractor is presented.

Keywords: Differential equations; pulse functions; Poisson stability; chaotic attractor.

1. Introduction

L. Shil’nikov emphasized in [Shil’nikov, 2002] that “...it seems quite reasonable that the role of dynamical chaos orbits should be assigned to the Poisson stable trajectories,” and “...we arrive at the following problem: how can one establish the existence of the Poisson stable trajectories in the phase space of a system?”

The goal of our paper is to obtain a quasi-minimal set by inserting a generator of moments of discontinuities into a dissipative system. That is, we aim to apply dynamical synthesis [Brown et al., 1992, 1993], which is a general technique for constructing dynamical systems with desired properties.

We believe that this approach makes a strong impact on applications, since one can investigate controllability of chaos [Fradkov, 2007; Otto et al., 1990] based on similar properties of the generator-function, which are already known or can be developed, if needed. For example, it provides a tool for supporting the given degree of nonregularity, which is important for cardiac rhythm [Garfinkel et al., 1992]. Another issue of relevance to the paper is the nonlinear dynamics of electrical circuits and mechanical models [Atherton, 1982; Elwakil, 2002; Tsypkin et al., 1964], which convert discrete data into continuous output.

The main focus of our investigation is the following special initial value problem

\[ z'(t) = Az(t) + f(z) + v(t, t_0), \]

\[ z(t_0) = z_0, \]

where \( z \in \mathbb{R}^n, t \in \mathbb{R}, i \in \mathbb{Z}, \) \( R \) and \( Z \) are sets of all real numbers and integers respectively,

\[ v(t, t_0) = \begin{cases} m_0 & \text{if } \zeta_i(t_0) < t \leq \zeta_{i+1}(t_0), \quad i \in \mathbb{Z}, \\ 0 & \text{if } \zeta_{i-1}(t_0) < t \leq \zeta_i(t_0), \quad i \in \mathbb{Z}, \end{cases} \]

where \( m_0 \in \mathbb{R}^n \) is a nonzero vector. Cantor set \( \Lambda \subset [0,1] \) and the strictly increasing sequence \( \zeta(t_0) = \{\zeta_i(t_0)\}, i \in \mathbb{Z}, \) \( i \leq \zeta_i(t_0) \leq i+1, \) will be described in the next section. The function \( f \) satisfies the Lipschitz condition with a positive constant \( L, \) \( A \) is an \( n \times n \) constant real valued matrix, and there exist positive numbers \( N, \alpha, \) such that \( \|e^{At}\| \leq Ne^{-\alpha t}, t \geq 0. \)

It is worth mentioning that we can consider other types of equations to obtain similar results. For instance, one may assume that function \( f \) depends on \( t \) and has discontinuities of the first kind at points of \( \zeta(t_0). \)
For a fixed $t_0 \in \Lambda$, system (1) is a differential equation with discontinuous right-hand side [Filippov, 1988] of a specific type where discontinuities occur on vertical planes in the $(t, z)$-space.

In what follows, we use the definition of solutions formulated in [Wiener, 1993] (see, also, [Akhmet, 2007]).

A function $z(t), z(t_0) = z_0$, is a solution of (1) on $\mathbb{R}$ if: (i) $z(t)$ is continuous on $\mathbb{R}$; (ii) the derivative $z'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of the points $\zeta(t_0)$, where left-sided derivatives exist; (iii) Equation (1) is satisfied on each interval $[\zeta(t), \zeta(t+1)], t \in \mathbb{Z}$.

It can be easily verified that problem (1) has a unique solution $z(t, t_0, z_0), t \in \mathbb{R}$, for each $t_0 \in \Lambda, z_0 \in \mathbb{R}^n$.

The solution $z(t) = z(t, t_0, z_0)$ satisfies the following integral equation

$$z(t) = e^{A(t-t_0)}z_0 + \int_{t_0}^{t} e^{A(t-s)}[f(z(s)) + v(s, t_0)] \, ds,$$

In the sequel, we assume that $\sup_{\mathbb{R}} |f(z)| = M_0 < \infty, NL < \alpha$. Fix a sequence $\zeta(t_0), t_0 \in \Lambda$. Using the standard technique, one can verify that $z(t)$ is a solution of (1), bounded on $\mathbb{R}$ if and only if it satisfies the equation

$$z(t) = \int_{-\infty}^{t} e^{A(t-s)}[f(z(s)) + v(s, t_0)] \, ds,$$

and for each sequence $\zeta(t_0), t_0 \in \Lambda$, there exists a unique solution $z(t, \zeta(t_0))$ bounded on $\mathbb{R}$ and all these solutions are placed in the tube with radius $M = M_0[1 + (N/\alpha - N)L]$, $t \in \mathbb{R}$. Moreover, if $z(t, t_0, z_0)$ is a solution of (1), then one can obtain by Gronwall-Bellman Lemma that

$$\|z(t, t_0, z_0) - z(t, \zeta(t_0))\| \leq \sup_{\mathbb{R}} |z_0 - z(t, \zeta(t_0))| e^{\alpha |t-t_0|}.$$

That is, the bounded solution $z(t, \zeta(t_0))$ attracts all solutions of (1) with the same initial moment $t_0, t_0 \in \Lambda$.

Let $(X, \rho)$ denote a metric space and $T$ be either $\mathbb{R}$ or $\mathbb{Z}$. Consider a set $\mathcal{S}$ of functions defined on $T$. An element $\phi(t) \in \mathcal{S}$ is called a motion if it is a solution of either a differential or a discrete equation. We say that a motion $\phi(t) \in \mathcal{S}$ is positively Poisson stable ($P_s$ stable) if for each $\gamma \in T$ there exist two sequences $\beta_\gamma, E_\gamma \in T$ with $\beta_\gamma, E_\gamma \to \infty$ such that

$$\lim_{n \to \infty} \sup_{-E_\gamma < t < E_\gamma} \rho(\phi(\beta_\gamma + t), \phi(\gamma + t)) = 0.$$

A motion $\phi(t) \in \mathcal{S}$ is said to be negatively Poisson stable ($P_n$ stable) if for each $\gamma \in T$ there exist two sequences $\beta_n, E_n \in T$ with $\beta_n, E_n \to \infty$ such that

$$\lim_{n \to \infty} \sup_{-E_n < t < E_n} \rho(\phi(-\beta_n - t), \phi(\gamma + t)) = 0.$$
Introduce the maps $B_j : \Sigma^2 \to \Sigma_2$, $i \in \mathbb{Z}$, such that $B_0(s) = (s_0, s_1, \ldots)$. From the method of construction of $s^*$, it follows that the following assertion is valid.

Lemma 2.1

(1) For a fixed $j \in \mathbb{Z}$ there exist two sequences of integers $k_n, l_n$ with $k_n, l_n \to \infty$, and
\[
\lim_{n \to \infty} \max_{s \in \Sigma_2} \{B_{k_n + i}(s^*), B_{j + i}(s^*)\} = 0.
\]

(2) For a fixed $j \in \mathbb{Z}$ there exist two sequences of integers $k_n, l_n$ with $k_n, l_n \to \infty$, and
\[
\lim_{n \to \infty} \max_{s \in \Sigma_2} \{B_{k_n + i}(s^*), B_{j + i}(s^*)\} = 0.
\]

(3) For each $s \in \Sigma_2$ and $j \in \mathbb{Z}$ one can find sequences of integers $k_n, l_n$ with $k_n, l_n \to \infty$, and
\[
\lim_{n \to \infty} \max_{s \in \Sigma_2} \{B_{k_n + i}(s^*), B_{j + i}(s^*)\} = 0.
\]

(4) For each $s \in \Sigma_2$ and $j \in \mathbb{Z}$ one can find sequences of integers $k_n, l_n$ with $k_n, l_n \to \infty$, and
\[
\lim_{n \to \infty} \max_{s \in \Sigma_2} \{B_{k_n + i}(s^*), B_{j + i}(s^*)\} = 0.
\]

Assume that there exist a homeomorphism $S$ between $\Sigma_2$ and a set $\Lambda \subset [0, 1]$, and a map $h : \Lambda \to \Lambda$ such that $S h = \sigma \circ S$. That is, $h$ and $\sigma$ are topologically conjugate. It is known that $\Sigma_2$ and $\Lambda$ are Cantor sets [Wiggins, 1990] and they are compact. One of the most popular examples of the map $h$ is the logistic map $\mu(1 - x), \mu > 4$, considered on a subset of $[0, 1]$ [Robinson, 1995].

For every $t_0 \in \Lambda$, one can construct a sequence $\kappa(j)_{0}$ of real numbers $\kappa_i, i \in \mathbb{Z}$, in the following way. If $i \geq 0$, then $\kappa_{i+1} = h(\kappa_i)$ and $\kappa_0 = t_0$. Let us show how the sequence is defined for negative $i$. Denote $s^0 = (s_0, s_1, \ldots) = S(t_0)$. Consider elements $\lambda = (0, s_0, s_1, \ldots), \pi = (1, s_0, s_1, \ldots) \in \Sigma_2$ such that $\sigma(\lambda) = \sigma(\pi) = s^0$ and $\lambda = S^{-1}(\lambda), \pi = S^{-1}(\pi)$. The homeomorphism implies that $h(\lambda) = h(\pi) = t_0$. The set $h^{-1}(t_0)$ may not consist of less than two elements $t_{-n}, t_n \in \Lambda$. Each of these two values can be chosen as $\kappa_{-n}(t_0)$. Obviously, one can continue the process to $-\infty$ by choosing always one element from the set $h^{-1}$. We have finally the construction of the sequence, and moreover, it is proved that $\kappa(t_0) \subset \Lambda, \kappa(t_0) = \{\kappa_i(t_0), i \in \mathbb{Z}\}$. Thus, infinitely many sequences $\kappa(t_0)$ can be constructed for a given $t_0$. However, each of this type of sequence is unique for increasing $i$. Fix one of the sequences and define a sequence $\xi(t_0) = \{\xi_i\}, \xi_i = i + \kappa_i, i \in \mathbb{Z}$.

If we denote by $H$ the set of all such sequences $\{\xi_i\}, i \in \mathbb{Z}$, then a multivalued functional $w : \Lambda \to H$ is defined. In our paper, the sequence $\xi(t_0)$ in (1) is considered to be a value of $w(t_0)$.

The above discussion shows that there exists a one-to-one correspondence between $\Sigma_2$ and $H$. Denote by $\xi(t^*)$ the sequence which corresponds to $s^*$. Then homeomorphism $S$ and Lemma 2.1 imply that the following assertion is correct.

Theorem 2.1. $H = \Omega_{\xi(t^*)} = A_{\xi(t^*)}$.}

3. The Quasi-Minimal Set

Denote $\mathcal{CB} = \{z(t, \xi(t^*)) : \xi(t_0) \in \Pi\}$. We shall show that the set $\mathcal{CB}$ is an attractor for all solutions of (1), is a quasi-minimal set.

Theorem 3.1. $\mathcal{CB}$ is the quasi-minimal set.

Proof. Consider the sequence $\xi(t^*)$, which has been defined in Theorem 2.1, and the solution $z(t, \xi(t^*))$ of (1). We shall show that the solution is $P_{\xi}$ stable by considering $\gamma = 0$. The proof is very similar for any other $\gamma \in \mathbb{R}$. Fix a positive $\epsilon$. Moreover, fix a positive $\epsilon_1$, whose dependence on $\epsilon$ will be described below. From Theorem 2.1, it follows that there exist sufficiently large natural numbers $j$ and $m$ such that $|z(j + m(t^*) - \xi(t^*))| < \epsilon_1$ if $-j < i < j$. For the sake of simplicity, we shall write $\xi_i$ instead of $\xi(t^*)$.

We have for $t \geq -j$:
\[
z(t, \xi(t^*)) = e^{A(t)(\xi(t^*))}z(\xi_i, \xi_i(t^*)) + \int_{\xi_i} f(e^{A(t)(\xi_i(t^*))}) + v(s, \xi_i)ds,
\]
and
\[
z(t + \xi_{j+m} - \xi_i) = e^{A(t)(\xi_i(t^*))}z(\xi_{j+m} - \xi_i(t^*))
+ \int_{\xi_i} f(e^{A(t)(\xi_{j+m} - \xi_i) - \xi_i(t^*))} + v(s, \xi_i)ds
+ e^{A(t)(\xi_i(t^*))}z(\xi_{j+m} - \xi_i(t^*))
+ \int_{\xi_i} f(z(s + \xi_{j+m} - \xi_i(t^*)) + v(s, \xi_i)ds.
\]
Subtracting the last expression from the previous one, we obtain that
\[
\|z(t, \zeta(t^*)) - z(t + \zeta_{j+m} - \zeta_j)\| \\
\leq 2MNe^{-\alpha(t-\zeta_j)} \\
+ \int_{\zeta_j}^{t} NLe^{-\alpha(t-s)}\|z(s, \zeta(t^*)) - z(s + \zeta_{j+m} - \zeta_j)\|ds \\
+ \int_{\zeta_j}^{t} Ne^{-\alpha(t-s)}e_1\|m_0\|ds.
\]

Next, we denote \(u(t) = \|z(t, \zeta(t^*)) - z(t + \zeta_{j+m} - \zeta_j)\|\) and apply the following assertion.

**Lemma 3.1** [Barbashin, 1970]. Let \(u(t), f(t)\) be non-negative functions integrable over the interval \(t_0 \leq t \leq t_0 + T\) and \(K\) be a positive constant. If the inequality
\[
u(t) \leq f(t) + K \int_{t_0}^{t} v(s)ds, \quad t_0 \leq t \leq t_0 + T,
\]
is fulfilled then the following expression holds
\[
u(t) \leq f(t) + K \int_{t_0}^{t} v(t-s) f(s)ds.
\]

Thus, we have
\[
\|z(t, \zeta(t^*)) - z(t + \zeta_{j+m} - \zeta_j)\| \\
\leq N\epsilon_1\|m_0\|\left(1 + \frac{1}{\alpha - NL}\right)
\]
\[
+ \frac{\alpha[(NL-1)(2M\alpha - \epsilon_1\|m_0\|)]}{\alpha L}\|z(t, \zeta(t^*)) - z(t + \zeta_{j+m} - \zeta_j)\|.
\]

On the basis of the last inequality, one can see that \(\|z(t, \zeta(t^*)) - z(t + \zeta_{j+m} - \zeta_j)\| < \epsilon\) if \(t \in (-E, E)\), where \(E = j/2, j\) is sufficiently large, and \(\epsilon_1\) is a sufficiently small positive number. The number \(\zeta_{j+m} - \zeta_j\) is as large as \(m\). Thus, we have proved that the solution is \(P\) stable. Applying Theorem 2.1 again, one can show that \(z(t, \zeta(t^*))\) is \(P\) stable and \(CB = \Omega_{\alpha}(\epsilon, \|\mu\|) = A_{\epsilon}(\mu, \|\mu\|),\) which completes the proof.

## 4. A Simulation Result

Consider the sequence \(\zeta_i = i + \kappa_i, \kappa_i = 4\kappa_{i-1}(1 - \kappa_{i-1}), \kappa_0 = t_0, \kappa_i \in [0, 1], i \geq 0\), and the following system
\[
x'' + 2x' + 1.5x = \sin y, \\
y' = -3y + v(t, t_0),
\]
where \(v(t, t_0)\) is a scalar pulse function with \(m_0 = 1\). The second equation is a drive equation and the first one, the pendulum equation. Using new variables \(x_1 = x, x_2 = x', x_3 = y\), one can reduce (2) to the system
\[
x'_1 = x_2, \\
x'_2 = -1.5x_1 - 2x_2 + \sin x_3, \\
x'_3 = -3x_2 + v(t, t_0).
\]

![Fig. 1. The chaotic attractor by a stroboscopic sequence \((x_1(n), x_2(n), x_3(n)), 1 \leq n \leq 75,000\), is observable.](image-url)
One can easily verify that all eigenvalues of the matrix of coefficients have negative real parts. Fix $t_0 = 12/17$ and take a solution $(x_1(t), x_2(t), x_3(t))$ of the last system with the initial condition $x_1(t_0) = 0.02, x_2(t_0) = -0.025, x_3(t_0) = -0.02$. In Fig. 1 the chaotic attractor is shown by using points $(x_1(n), x_2(n), x_3(n))$, $n = 1, 2, 3, \ldots, 75,000$, in $x_1-x_2-x_3$-space.

Remark 4.1. Using the technique of the present paper, we shall develop shadowing theorems [Anosov, 1967; Bowen, 1975; Hammel et al., 1987] for system (1) in future investigation.

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References