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Li–Yorke chaos in the system with impacts

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ABSTRACT

The analogue of Li–Yorke chaos [T.Y. Li, J. Yorke, Period three implies chaos, Amer. Math. Monthly 87 (1975) 985–992] for a special initial value problem of a non-autonomous impulsive differential equation is developed.

1. Introduction and preliminaries

The first mathematical definition of chaos was introduced by Li and Yorke [18], and it became one of the most discussed topics for the last several decades. Li and Yorke proved that if a map on an interval has a point of period three, then it has points of all periods. Moreover, there exists an uncountable *scrambled* subset of the interval. While the existence of periodic solutions, as it was discovered later, is a particular case of Sharkovsky's theorem [26], the scrambled set remains the feature, which distinguishes Li–Yorke chaos from other definitions [6,13]. Another fact which makes the chaos attractive for applications is that it can be developed for multidimensional systems [15,19,22,27].

The following special initial value problem for the impulsive differential equation is the main subject of the paper

$$
\begin{aligned}
&z'(t) = Az(t) + f\big(z(t)\big),\\
&\Delta|_{t=\zeta_i(t_0)} = Bz\big(\zeta_i(t_0)\big) + W\big(z\big(\zeta_i(t_0)\big)\big),\\
&z(t_0) = z_0, \quad (t_0, z_0) \in \Lambda \times \mathbb{R}^n,
\end{aligned}
\tag{1}
$$

where $z \in \mathbb{R}^n$, $t \in \mathbb{R}_+ = [0,\infty)$, $i \geqslant 0$ are integers, Cantor set $\Lambda \subset I = [0,1]$ and strictly increasing sequence of impulsive moments $\zeta_i(t_0)$, $i \leqslant \zeta_i(t_0) \leqslant i+1$, will be fully described later. We shall need the following assumptions throughout the paper:

(C1) A and B are $n \times n$ constant real valued matrices, $\det(\mathcal{I} + B) \neq 0$, where $\mathcal{I}$ is the $n \times n$ identity matrix;
(C2) the functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $W : \mathbb{R}^n \to \mathbb{R}^n$, satisfy

$$\big\|f(x_1) - f(x_2)\big\| + \big\|W(x_1) - W(x_2)\big\| \leqslant L\|x_1 - x_2\|,$$

for all $x_1, x_2 \in \mathbb{R}^n$, where $L > 0$ is a constant;
(C3) $Bx + W(x) \neq 0$, $\forall x \in \mathbb{R}^n$;

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(C4) the functions f and W are uniformly bounded so that

$$\sup_{x\in\mathbb{R}^n}\|f(x)\|+\sup_{x\in\mathbb{R}^n}\|W(x)\|=M_0<\infty;$$

(C5) the matrices A and B commute, and the real parts of all eigenvalues of the matrix $A+\ln(\mathcal{I}+B)$ are negative.

A left continuous function $z(t):[t_0,\infty)\to\mathbb{R}^n$, $t_0\in\Lambda$, is a solution of (1) if:

(i) it has discontinuities only at the points $\zeta_i(t_0)$, $i\geqslant 0$, and these discontinuities are of the first kind;
(ii) the derivative $z'(t)$ exists at each point $t\in[t_0,\infty)\backslash\{\zeta_i(t_0)\}$, and the left-sided derivative exists at points $\zeta_i(t_0)$, $i>0$;
(iii) the differential equation is satisfied by $z(t)$ on $[t_0,\infty)\backslash\{\zeta_i(t_0)\}$, and it holds for the left derivative of $z(t)$ at every point $\zeta_i(t_0)$, $i>0$;
(iv) the jump equation is satisfied by $z(t)$ for every $i\geqslant 0$.

In what follows we denote by $z(t,\xi,v)$, $\xi\in\mathbb{R}_+$, $v\in\mathbb{R}^n$, a solution of (1) with $t_0=\xi$, $z_0=v$.

Conditions (C1) and (C2) imply that for every $(t_0,z_0)\in\Lambda\times\mathbb{R}^n$ there exists a unique solution $z(t,t_0,z_0):[t_0,\infty)\to\mathbb{R}^n$, of (1) [25].

We attempt to shape the Li–Yorke chaos for system (1) by implementing a special initial value problem, where the moments of impulsive action are functionally dependent on the initial moment, and by using the results of the theory of impulsive differential equations [1–3,11,17,25].

The description of the main subject of this paper should begin with the discussion of the moments of impulses, as their generation is very important for the emergence of chaos.

Let us recall the definition of chaos for maps. Consider an infinite nonvoid compact metric space (X,ρ) with metric ρ and a continuous map $T:X\to X$. A pair $(x,x')\in X\times X$, $x\neq x'$, is called a *Li–Yorke pair* [5] if it is *proximal* and *not asymptotic*, that is, $\liminf_{i\to\infty}\rho(T^i(x),T^i(x'))=0$ and $\limsup_{i\to\infty}\rho(T^i(x),T^i(x'))>0$, respectively.

The map $T:X\to X$ is Li–Yorke chaotic, if: it has periodic points with all periods $p\in\mathbb{N}$; there exists an uncountable subset $X'\subset X$, the scrambled set, that does not contain periodic points and each pair $(x,x')\in X'\times X'$, $x\neq x'$, is a Li–Yorke pair.

One of the most effective ways to discover a chaos is to establish the topological conjugacy with the symbolic dynamics [8]. Consider the sequence space [24]

$$\Sigma_2=\{s=(s_0s_1s_2\ldots):\ s_j=0 \text{ or } 1\}$$

with the metric

$$d[s,\tilde{s}]=\sum_{i=0}^{\infty}\frac{|s_i-\tilde{s}_i|}{2^i},$$

where $\tilde{s}=(\tilde{s}_0\tilde{s}_1\ldots)\in\Sigma_2$ and the shift map $\sigma:\Sigma_2\to\Sigma_2$, with $\sigma(s)=(s_1s_2\ldots)$. The pair (Σ_2,σ) is the symbolic dynamics.

It is easy to show that the dynamics (Σ_2,σ) is Li–Yorke chaotic, and there exists a scrambled set $\Sigma_2'\subset\Sigma_2$. Moreover, if (s,s') is a Li–Yorke pair, then there exists a sequence $l_i, l_i\to\infty$ as $i\to\infty$ such that $d[\sigma^{l_i}s,\sigma^{l_i}s']\geqslant 1$.

Let $h:\Lambda\to\Lambda$, where Λ is a subset of the interval I, be a topologically conjugate map to σ. That is, there exists a homeomorphism $S:\Lambda\to\Sigma_2$ such that $S\circ h=\sigma\circ S$.

By applying the conjugacy of h and σ, one can verify that h is Li–Yorke chaotic and $\Lambda'=S^{-1}(\Sigma_2')$ is a scrambled subset of Λ.

For every $t_0\in\Lambda$, one can construct a sequence $\kappa(t_0)$ of real numbers κ_i, $i\geqslant 0$, such that $\kappa_{i+1}=h(\kappa_i)$ and $\kappa_0=t_0$. For each $p\in N$, there is $t_0\in\Lambda\backslash\Lambda'$ such that $\kappa(t_0)$ is p-periodic, i.e., $\kappa_{i+p}(t_0)=\kappa_i(t_0)$, $i\geqslant 0$.

The sequence $\zeta(t_0)=\{\zeta_i(t_0)\}$ in (1) is defined as $\zeta_i(t_0)=i+\kappa_i(t_0)$, $i\geqslant 0$.

The impulsive differential equations of type (1) play an increasingly important role in the investigation of the cardiovascular system, neural information processing in the brain, information communication and population dynamics [4,10,12,14,20,23,28,29]. It is notable that the moments of time, where the impulses are performed, are chosen dependent on the initial data. This type of problems may occur if one considers an impulsively controlled process [25]. In [21], the author considers a system of impulsive differential equations with moments of impulses generated by a sensitive map which depends on a parameter. Sensitivity is considered as a chaotic property. In this paper we try to show that solutions of (1) have Li–Yorke chaos properties. They are similar to those formulated for maps [18,24] with additional peculiarities caused by discontinuities occurring at different moments for different solutions.

2. Main results

Let us fix an interval $J\subset[0,\infty)$, a positive number ϵ, and t_0, $t_1\in I$. We introduce the distance $\|\zeta(t_0)-\zeta(t_1)\|_J=\sup_{\zeta_i(t_0),\,\zeta_i(t_1)\in J}|\zeta_i(t_0)-\zeta_i(t_1)|$, and we shall say that a pair of solutions of (1) $z(t)=z(t,t_0,z_0)$, $z_1(t)=z(t,t_1,z_1)$, $t_0,t_1\in\Lambda$, are ϵ-equivalent on J and write $z(t)(\epsilon,J)z_1(t)$, if $z(t)$ and $z_1(t)$ are defined on J, $\|\zeta(t_0)-\zeta(t_1)\|_J<\epsilon$ and $\|z(t)-z_1(t)\|<\epsilon$

for all $t \in J$ such that $t \notin \bigcup_{\zeta_i(t_0),\, \zeta_i(t_1) \in J} [\widehat{\zeta_i(t_0), \zeta_i(t_1)}]$. Here $\widehat{[a,b]}$, $a, b \in \mathbb{R}$, stands for an oriented interval, that is $\widehat{[a,b]} = [a,b]$ if $a \leqslant b$, and $\widehat{[a,b]} = [b,a]$, otherwise.

The ϵ-equivalence of two piecewise continuous functions with a small ϵ means that they have close discontinuity points, and the values of the functions are close at points that do not lie on intervals between the neighboring discontinuity points of these functions. This concept was developed in [1,2,11,16,25].

The following definitions are taken from [5,18,24] and adapted for (1).

Definition 2.1. A pair of solutions of (1) $z(t) = z(t, t_0, z_0)$, $z_1(t) = z(t, t_1, z_1)$, $t_0, t_1 \in \Lambda'$, is proximal if for each $\epsilon > 0$, $E > 0$ there exists an interval $J \subset [\max(t_0, t_1), \infty)$ with length not less than E such that $z_1(t)(\epsilon, J)z(t)$.

Definition 2.2. The solutions of (1) $z(t) = z(t, t_0, z_0)$, $z_1(t) = z(t, t_1, z_1)$, $t_0, t_1 \in \Lambda'$, are frequently (ϵ_0, ϵ_1)-separated if there exist positive numbers ϵ_0, ϵ_1 and infinitely many disjoint intervals of length not less than ϵ_1, such that $\|z_1(t) - z(t)\| > \epsilon_0$ for each t from these intervals, and none of these intervals contain a discontinuity point of $z(t)$ or $z_1(t)$.

Definition 2.3. A couple $z(t) = z(t, t_0, z_0)$, $z_1(t) = z(t, t_1, z_1)$, $t_0, t_1 \in \Lambda'$, of solutions of (1) is a Li–Yorke pair if they are proximal and (ϵ_0, ϵ_1)-separated for some positive numbers ϵ_0, ϵ_1.

Definition 2.4. Problem (1) is Li–Yorke chaotic on Λ if each couple of solutions $z(t) = z(t, t_0, z_0)$, $z_1(t) = z(t, t_1, z_1)$, with $t_0, t_1 \in \Lambda'$, $t_0 \neq t_1$, is Li–Yorke pair, and it has periodic solutions $\phi(t, t_0)$, $t_0 \in \Lambda \backslash \Lambda'$, with all periods $p \in \mathbb{N}$.

Let us fix $t_0 \in \Lambda$ and denote by $Z(t, s) = Z(t, s, t_0)$ the transition matrix [25] of the linear homogeneous system

$$\begin{aligned} &z'(t) = Az(t), \quad t \neq \zeta_i, \\ &\Delta z|_{t=\zeta_i} = Bz(\zeta_i), \end{aligned} \tag{2}$$

associated with (1).

Condition (C5) in conjunction with Theorem 34, [25], implies that there exist two positive numbers N and ω such that for all $t_0 \in I$,

$$\big\|Z(t, s, t_0)\big\| \leqslant N\mathrm{e}^{-\omega(t-s)}, \quad t \geqslant s. \tag{3}$$

It is important that constants N and ω are common for all $t_0 \in \Lambda$.

Denote $\overline{m} := \max_{|u| \leqslant 1} \|\mathrm{e}^{Au}\|$, $\underline{m} := \min_{|u| \leqslant 1} \|\mathrm{e}^{Au}\|$, and $M_1 := 1 + NM_0(\frac{1}{\omega} + \frac{\mathrm{e}^{\omega}}{1-\mathrm{e}^{-\omega}})$. Fix a number $q \geqslant 3$, such that $\frac{1}{q} < \frac{2\underline{m}}{3\overline{m}}$. Condition (C3) implies that $\eta = \min_{\|x\| \leqslant M_1}(Bx + W(x)) > 0$. The following assumptions are also needed:

(C6) $NL[\frac{2}{\omega} + \frac{\mathrm{e}^{\omega}}{1-\mathrm{e}^{-\omega}}] < 1$;
(C7) $-\omega + NL + \ln(1 + NL) < 0$;
(C8) $L < \frac{[\frac{2\underline{m}}{3\overline{m}} - \frac{1}{q}]\underline{m}\eta}{2M_1(\overline{m}+\underline{m})}$.

Lemma 2.1. *If conditions* (C1)–(C6) *are valid, and the sequence* $\kappa_i(t_0)$, $t_0 \in \Lambda$, *is periodic with a period* $p \in \mathbb{N}$, *then*:

1. *Eq.* (1) *admits a unique* p*-periodic solution* $\phi(t, t_0)$, *such that* $\|\phi(t, t_0)\| < M_1$, $t \geqslant t_0$;
2. *all periodic solutions are different.*

The verification of part 1 of this lemma replicates the proof of Theorem 58 [25]. It is easy to see that (C3) implies that there exist infinitely many different periodic solutions, since the sequences of discontinuity moments of solutions with different periods do not intersect. Thus, part 2 is also proved.

Lemma 2.2. *Assume that conditions* (C1)–(C7) *are fulfilled. Then each couple of solutions of* (1), $z(t) = z(t, t_0, z_0)$, $z_1(t) = z(t, t_1, z_1)$, *with* $t_0, t_1 \in \Lambda'$, $t_0 \neq t_1$, *is proximal.*

Proof. Fix numbers $t_0, t_1 \in \Lambda'$, $t_0 \neq t_1$, solutions $z(t) = z(t, t_0, z_0)$, $z_1(t) = z(t, t_1, z_1)$, $z_0, z_1 \in \mathbb{R}^n$, of (1), and $E, \epsilon > 0$.

Using the integral representation formula [25]

$$z(t) = Z(t, t_0)z_0 + \int_{t_0}^{t} Z(t, s) f\big(z(s)\big)\, ds + \sum_{t_0 \leqslant \zeta_i < t} Z(t, \zeta_i) W\big(z(\zeta_i)\big), \tag{4}$$

one can find that

$$\begin{aligned}\|z(t)\| &\leqslant Ne^{-\omega(t-t_0)}\|z_0\| + \int_{t_0}^{t} Ne^{-\omega(t-s)}M_0\,ds + \sum_{\zeta_i<t} Ne^{-\omega(t-\zeta_i)}M_0\\ &\leqslant Ne^{-\omega(t-t_0)}\|z_0\| + NM_0\left(\frac{1}{\omega}+\frac{\mathrm{e}^{\omega}}{1-\mathrm{e}^{-\omega}}\right).\end{aligned}$$

Denote $\bar{T}(t_0,z_0) := t_0$, if $N\|z_0\| \leqslant 1$, and $\bar{T}(t_0,z_0) := t_0 - \frac{1}{\omega}\ln\frac{1}{N\|z_0\|}$, if $N\|z_0\| > 1$.

From the last inequality it follows that $\|z(t)\| \leqslant M_1$ for $t \geqslant \bar{T}(t_0,z_0)$.

Similarly, one can find a number $\bar{T}(t_1,z_1)$ such that inequality $t \geqslant \bar{T}(t_1,z_1)$ implies $\|z_1(t)\| \leqslant M_1$. Thus, there exists a number $\bar{T}$ such that both solutions z, z_1 are in the tube with the radius M_1 if $t \geqslant \bar{T}$. By proximal property of Li–Yorke chaotic map h there exist arbitrarily large numbers $\tilde{T} > \bar{T}$, $E > 0$, such that $\|\zeta(t_1) - \zeta(t_0)\|_Q < \delta$, where $Q = (\tilde{T}, \tilde{T}+E)$. We shall find a sufficiently large E so that solutions $z(t)$, $z_1(t)$ are ϵ-equivalent on $J = (\tilde{T} + \frac{1}{2}E, \tilde{T}+E)$.

Denote $Z_1(t,s) = Z(t,s,t_0)$ and $Z_2(t,s) = Z(t,s,t_1)$, $t \geqslant s$. We have that

$$z(t) = Z_1(t,\tilde{T})z(\tilde{T}) + \int_{\tilde{T}}^{t} Z_1(t,s)f\big(z(s)\big)\,ds + \sum_{\tilde{T}\leqslant\zeta_i<t} Z_1\big(t,\zeta_i(t_0)\big)W\big(z\big(\zeta_i(t_0)\big)\big),$$

$$z_1(t) = Z_2(t,\tilde{T})z_1(\tilde{T}) + \int_{\tilde{T}}^{t} Z_2(t,s)f\big(z_1(s)\big)\,ds + \sum_{\tilde{T}\leqslant\zeta_i<t} Z_2\big(t,\zeta_i(t_1)\big)W\big(z_1\big(\zeta_i(t_1)\big)\big).$$

It is difficult to evaluate the difference between $z(t)$ and $z_1(t)$ using the last two expressions since the moments of discontinuity of $z(t)$ and $z_1(t)$ are distinct. For this reason, we assume that $\zeta_j(t_0) \leqslant \zeta_j(t_1)$ for a fixed integer j. The opposite case can be discussed similarly. We introduce the following map

$$\begin{aligned}W_j^1(z) = (\mathcal{I}+B)&\left[\big(\mathrm{e}^{A(\zeta_j(t_1)-\zeta_j(t_0))} - \mathcal{I}\big)z + \int_{\zeta_j(t_0)}^{\zeta_j(t_1)} \mathrm{e}^{A(\zeta_j(t_1)-s)} f\big(z(s)\big)\,ds\right]\\ &+ W\left((\mathcal{I}+B)\left[\mathrm{e}^{A(\zeta_j(t_1)-\zeta_j(t_0))}z + \int_{\zeta_j(t_0)}^{\zeta_j(t_1)} \mathrm{e}^{A(\zeta_j(t_1)-s)} f\big(z(s)\big)\,ds\right]\right) - \int_{\zeta_j(t_0)}^{\zeta_j(t_1)} \mathrm{e}^{A(\zeta_i(t_1)-s)} f\big(\bar{z}(s)\big)\,ds - W(z),\end{aligned}$$

where $z(t)$, $\bar{z}(t)$ are solutions of

$$z'(t) = Az(t), \tag{5}$$

such that $z(\zeta_j(t_0)) = z$ and $\bar{z}(\zeta_j(t_1)) = z(\zeta_j(t_1)+)$, where "+" indicates the right-side limit at moment $\zeta_j(t_1)$. Consider the following systems

$$\begin{aligned}&v'(t) = Av(t) + f(v),\\ &\Delta v|_{t=\zeta_i(t_0)} = Bv\big(\zeta_i(t_0)\big) + W\big(v\big(\zeta_i(t_0)\big)\big) + W_i^1\big(v\big(\zeta_i(t_0)\big)\big),\end{aligned} \tag{6}$$

and

$$\begin{aligned}&z'(t) = Az(t) + f(z),\\ &\Delta|_{t=\zeta_i(t_1)} = Bz\big(\zeta_i(t_1)\big) + W\big(z\big(\zeta_i(t_1)\big)\big).\end{aligned} \tag{7}$$

One can easily see that $M_2 = \sup_{\|z\|\leqslant M_1,\, i\in\mathbb{Z}} \|W_i^1(z)\| < \infty$, and respective solutions of two systems (6) and (7) with the same initial data coincide in the intersection of their domains only if $t \notin [\zeta_i\widehat{(t_0), \zeta_i}(t_1)]$, $i \in \mathbb{Z}$. For details [1–3] can be referred. So, if $v(t)$, $v(\tilde{T}) = z_1(\tilde{T})$, is the solution of (6), then $v(t) = z_1(t)$ for all $t \notin [\zeta_i\widehat{(t_0), \zeta_i}(t_1)]$, $i \in \mathbb{Z}$.

We have that

$$v(t) = Z_1(t,\tilde{T})v(\tilde{T}) + \int_{\tilde{T}}^{t} Z_1(t,s)f\big(v(s)\big)\,ds + \sum_{\tilde{T}\leqslant\zeta_i<t} Z_1\big(t,\zeta_i(t_0)\big)\big[W\big(v\big(\zeta_i(t_0)\big)\big) + W_1\big(v\big(\zeta_i(t_0)\big)\big)\big].$$

Consequently,

$$\begin{aligned}\|z(t)-v(t)\| &\leqslant \|z(\tilde T)-v(\tilde T)\|\|Z_1(t,\tilde T)\| + \int\limits_{\tilde T}^{t}\|Z_1(t,s)\|L\|z(s)-v(s)\|\,ds \\ &\quad + \sum_{\tilde T\leqslant \zeta_j(t_0)<t}\big\|Z_1\big(t,\zeta_j(t_0)\big)\big\|L\big\|z\big(\zeta_j(t_0)\big)-v\big(\zeta_j(t_0)\big)\big\| + \sum_{\tilde T\leqslant \zeta_j(t_0)<t}\big\|Z_1\big(t,\zeta_j(t_0)\big)\big\|\big\|W_1\big(v\big(\zeta_j(t_0)\big)\big)\big\| \\ &\leqslant 2M_1N + M_2\frac{\mathrm{e}^{\omega}}{1-\mathrm{e}^{-\omega}} + \int\limits_{\tilde T}^{t} N\mathrm{e}^{-\omega(t-s)}L\|z(s)-v(s)\|\,ds + \sum_{\tilde T\leqslant \zeta_j<t} N\mathrm{e}^{-\omega(t-\zeta_j(t_0))}L\big\|z\big(\zeta_j(t_0)\big)-v\big(\zeta_j(t_0)\big)\big\|.\end{aligned}$$

Now, applying the analogue of Gronwall–Bellman Lemma [25] for discontinuous functions, we find that

$$\begin{aligned}\|z(t)-v(t)\| &\leqslant \left(2M_1N + M_2\frac{\mathrm{e}^{\omega}}{1-\mathrm{e}^{-\omega}}\right)\mathrm{e}^{(-\omega+NL)(t-\tilde T)}\prod_{\tilde T\leqslant \zeta_j<t}(1+NL) \\ &\leqslant \left(2M_1N + M_2\frac{\mathrm{e}^{\omega}}{1-\mathrm{e}^{-\omega}}\right)\mathrm{e}^{(-\omega+NL+\ln(1+NL))(t-\tilde T)}.\end{aligned} \tag{8}$$

Last inequality implies that $\|z(t)-v(t)\|<\epsilon$ if $t>\tilde T+\frac{1}{2}E$, $t\notin[\widehat{\zeta_i(t_0),\zeta_i(t_1)}]$, $i\in\mathbb{Z}$, where

$$E > 2\frac{\ln\big(\frac{\epsilon}{2M_1N+M_2\mathrm{e}^{\omega}(1-\mathrm{e}^{-\omega})^{-1}}\big)}{-\omega+NL+\ln(1+NL)}$$

(we may assume that $\epsilon<2M_1N$, without loss of generality). That is why, if $J=(\tilde T+\frac{1}{2}E,\tilde T+E)$, then $z(t)(\epsilon,J)\phi(t)$. The lemma is proved. □

Lemma 2.3. *Assume that* (C1)–(C8) *are fulfilled. Then there exist positive numbers* ϵ_0,ϵ_1, *such that each couple of solutions* $z(t)=z(t,t_0,z_0)$, $z_1(t)=z(t,t_1,z_1)$, *with* $t_0,t_1\in\Lambda'$, $t_0\neq t_1$, *is frequently* (ϵ_0,ϵ_1)*-separated.*

Proof. Consider a pair of solutions $z(t)=z(t,t_0,z_0)$, $z_1(t)=z(t,t_1,z_1)$, with $t_0,t_1\in\Lambda'$, $t_0\neq t_1$. Assume that $\|z_0\|,\|z_1\|>M_1$. The discussion of other cases is easier.

Denote $s^0=S(t_0)=(s_0^0,s_1^0\ldots)$ and $s^1=S(t_1)=(s_0^1,s_1^1\ldots)$. It is obvious that $s^0\neq s^1$. Since the symbolic dynamics is not asymptotic, there exists a sequence of integers $m_i\to\infty$ as $i\to\infty$ such that $d[\sigma^{m_i-j}s^0,\sigma^{m_i-j}s^1]\geqslant\frac{1}{2^j}$, $0\leqslant j\leqslant m_i/2$.

Similarly to the proof of the last lemma, one can choose $m_i>2$ sufficiently large so that $\|z(t)\|<M_1$, $\|z_1(t)\|<M_1$, if $t>\frac{m_i+1}{2}$. Let us fix this m_i.

Since S is a homeomorphism and set Σ_2 is compact, for a given j, $0\leqslant j\leqslant m_i$, the set

$$P_j=\left\{(\bar s,\tilde s)\in\Sigma_2\times\Sigma_2 : d[\bar s,\tilde s]\geqslant\frac{1}{2^{m_i-j}}\right\}$$

is compact, and

$$\min_{(\bar s,\tilde s)\in P_j}\big|S^{-1}(\bar s)-S^{-1}(\tilde s)\big|=\mu_j>0,$$

$P_{j+1}\subseteq P_j$, $\mu_{j+1}\geqslant\mu_j$, $0\leqslant j<m_i-1$. Fix $i_0=m_i-2$. Then $|\kappa_i(t_0)-\kappa_i(t_1)|\geqslant\mu_{i_0}$ if $i=i_0,i_0+1$.

Similarly, we also have that there exists a positive number $\mu_0<1$ such that $|\kappa_j(t_0)-\kappa_j(t_1)|\leqslant\mu_0$ if $0\leqslant j<m_i$.

Next, we assume that $\kappa_j(t_0)<\kappa_j(t_1)$ for all j. It is easily seen that case $\kappa_j(t_0)>\kappa_j(t_1)$ can be analyzed similarly. Thus, there is a number k among i_0, i_0+1, such that $\kappa_k(t_1)-\kappa_k(t_0)>\mu_{i_0}$ and $\kappa_k(t_0)-\kappa_{k-1}(t_1)\geqslant\frac{1}{2}(1-\mu_0)$.

Clearly, (C8) implies that $\nu_1=\frac{2}{3}\frac{m\eta}{\overline{m}}-2LM_1>0$ and $\nu_2<\nu_1$, where $\nu_2=\frac{2\overline{m}LM_1}{\underline{m}}+\frac{1}{q}\eta$.

We shall show that the constants ϵ_0, ϵ_1 of Definition 2.3 can be set as $\epsilon_0=\frac{1}{q}\underline{m}\eta$, $\epsilon_1=\min\{\mu_{i_0},\frac{1}{2}(1-\mu_0)\}$.

Assume that $\|z(\zeta_k(t_0))-z_1(\zeta_k(t_0))\|<\nu_1$. Then, for $t\in[\zeta_k(t_0),\zeta_k(t_1)]$,

$$z(t)=\mathrm{e}^{A(t-\zeta_k(t_0))}(\mathcal{I}+B)z\big(\zeta_k(t_0)\big)+\int\limits_{\zeta_k(t_0)}^{t}\mathrm{e}^{A(t-s)}f\big(z(s)\big)\,ds+\mathrm{e}^{A(t-\zeta_k(t_0))}W\big(z\big(\zeta_k(t_0)\big)\big),$$

$$z_1(t)=\mathrm{e}^{A(t-\zeta_k(t_0))}z_1\big(\zeta_k(t_0)\big)+\int\limits_{\zeta_k(t_0)}^{t}\mathrm{e}^{A(t-s)}f\big(z_1(s)\big)\,ds.$$

We have that

$$\begin{aligned}\|z(t)-z_1(t)\| &= \Bigg\| e^{A(t-\zeta_k(t_0))}\big[Bz\big(\zeta_k(t_0)\big)+W\big(z\big(\zeta_k(t_0)\big)\big)\big]+e^{A(t-\zeta_k(t_0))}\big[z\big(\zeta_k(t_0)\big)-z_1\big(\zeta_k(t_0)\big)\big] \\ &\quad + \int_{\zeta_k(t_0)}^{t} e^{A(t-s)}\big(f\big(z(s)\big)-f\big(z_1(s)\big)\big)\,ds\Bigg\| \\ &\geqslant \underline{m}\eta-\overline{m}(\nu_1+2LM_1)\geqslant \epsilon_0.\end{aligned}$$

If $\|z(\zeta_k(t_0))-z_1(\zeta_k(t_0))\| > \nu_2$, then, for $t\in[\zeta_{k-1}(t_1),\zeta_k(t_0)]$,

$$z(t)=e^{A(t-\zeta_k(t_0))}z\big(\zeta_k(t_0)\big)+\int_{\zeta_k(t_0)}^{t} e^{A(t-s)}f\big(z(s)\big)\,ds,$$

$$z_1(t)=e^{A(t-\zeta_k(t_0))}z_1\big(\zeta_k(t_0)\big)+\int_{\zeta_k(t_0)}^{t} e^{A(t-s)}f\big(z_1(s)\big)\,ds,$$

and

$$\|z(t)-z_1(t)\|\geqslant \underline{m}\nu_2-\overline{m}2LM_1=\epsilon_0.$$

The lemma is proved. □

Lemmas 2.1–2.3 imply that (1) is Li–Yorke chaotic on Λ.

Remark 2.1. It seems natural to consider the chaos only for uniformly bounded on $\mathbb{R}_+$ solutions, since the domain of chaos is always assumed to be a compact set. We consider the set of all solutions, where the chaos scenario starts at the moment when a solution reaches the region $\|z(t)\|\leqslant M_1$.

Example 2.1. Consider the following initial value problem

$$\begin{aligned} x_1' &= -1/3x_2+f_1(x_1,x_2), \\ x_2' &= 1/3x_1+f_2(x_1,x_2), \quad t\neq\zeta_i(t_0), \\ \Delta x_1|_{t=\zeta_i(t_0)} &= -\frac{1}{2}x_1+W(x_1), \\ \Delta x_2|_{t=\zeta_i(t_0)} &= -\frac{1}{2}x_2, \end{aligned} \tag{9}$$

where $x_1,x_2\in\mathbb{R}, l$ is a positive constant, $f_1(s,u)=s\cos u$, $f_2(s,u)=s\sin u$, $W(s)=1+s^2$, if $|s|\leqslant l$, and $f_1(s,u)=l\cos u$, $f_2(s,u)=l\sin u$, $W(s)=1+l^2$, if $|s|>l$. One can easily see that all the functions are lipschitzian with a constant proportional to l. The matrices of coefficients are

$$A=\begin{pmatrix}0 & -1/3\\ 1/3 & 0\end{pmatrix},\qquad B=\begin{pmatrix}-1/2 & 0\\ 0 & -1/2\end{pmatrix}.$$

The matrices commute, and the eigenvalues of the matrix

$$A+\mathrm{Ln}(\mathcal{I}+B)=\begin{pmatrix}-\ln 2 & -1/3\\ 1/3 & -\ln 2\end{pmatrix}$$

have negative real parts.

Condition (C3) is obvious, since function $W(s)$ is never equal to zero. All the other conditions required by the Theorems could be easily checked with sufficiently small coefficient l. That is, the dynamics of (1) is Li–Yorke chaotic.

3. Conclusion

In this paper, we defined the features of the analogue of Li–Yorke chaos for the multidimensional discontinuous dynamics. Apparently, the question of "whether theoretical results obtained for systems with low dimensionality are still applicable for high or infinite dimensional systems" [9] has been partially answered. Taking into account Lemma 2.1 the period-doubling route to chaos can be obtained. The main modeling novelty of our paper, useful for applications, is that the moments of discontinuity are prescribed by the choice of the initial moment. The present results could be effectively used in mechanics, electronics, control theory and economics, and they could be developed further, using more delicate properties of one-dimensional maps [6,7].

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