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Devaney's chaos of a relay system

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ABSTRACT

We address the differential equation with a pulse function, whose moments of discontinuity depend on the initial moment. The existence of a chaotic attractor, and the complex behavior of all solutions are investigated. An appropriate simulations are presented.

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1. Introduction and preliminaries

The irregular behavior of dynamical systems [1–7] has been discovered and investigated intensively during the last decades. One of the ways to look for complex behavior on the basis of the qualitative theory of differential equations is the approach based on the topological ingredients, which were summarized in [8]. We investigate the non-autonomous differential equation with a pulse function in the right-hand side, using the topological ingredients for guidance. The moments where this function changes its value are dependent on the initial moment. Another issue of relevance to the paper is nonlinear dynamics of electric circuits, of mechanical models [9], and of control systems [10] which convert a discrete data to a continuous output. We believe that our results may be applied to models with a pulsating control, which depends on the initial data. Extremely close to our results in this sense is the investigation of relay systems. That is, linear systems which can be analyzed by means of existing linear theory, and where at certain instants the relay releases discontinuous actions in one direction or another. The discontinuities are the results of idealizations used in the representation of nonlinear characteristics. Moreover, one can see that the set of solutions of the initial value problem is not linear, either. Consequently, the system we consider concerns the nonlinear discontinuous dynamics [9–13].

In our paper, we provide the definitions of chaos and of chaotic attractors of non-autonomous differential equations, and define conditions of their existence.

We begin this section with the description of the symbolic dynamics [14,15], which is in the basis of an initial value problem with a pulse function. Consider the sequence space [8,14]

$$\Sigma_2 = \{s = (s_0 s_1 s_2 \cdots) : s_j = 0 \text{ or } 1\}$$

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with the metric

$$d[s, \bar{s}] = \sum_{i=0}^{\infty} \frac{|s_i - \bar{s}_i|}{2^i},$$

where $\bar{s} = (\bar{s}_0\bar{s}_1\cdots) \in \Sigma_2$, and the shift map $\sigma : \Sigma_2 \to \Sigma_2$, such that $\sigma(s) = (s_1s_2\cdots)$. The semidynamics (Σ_2, σ) is the symbolic dynamics [15].

The map is continuous, card $\mathrm{Per}_n(\sigma) = 2^n$, $\mathrm{Per}(\sigma)$ is dense in Σ_2, and there exists a dense orbit in Σ_2.

Dynamics of the logistic map $h(x, \mu) \equiv \mu x(1 - x),\ \mu > 0$, is another central instrument in our paper. The dynamics has a positively invariant subset $\Lambda \subseteq I = [0, 1]$, such that $\Lambda = I$, if $\mu \leqslant 4$. If $\mu > 4$, then Λ is a Cantor set, and is chaotic on Λ [14]. That is, h has sensitive dependence on the initial conditions; periodic points are dense in Λ, and there exists a solution with every natural period p; and h is topologically transitive, that is there exists a trajectory of h, dense in Λ.

If $\mu > 4$, we denote

$$I_0 = \left[0, \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\mu}}\right], \quad A_0 = \left(\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\mu}}, \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{\mu}}\right), \quad I_1 = \left[\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{\mu}}, 1\right],$$

so that $I = I_0 \cup A_0 \cup I_1,\ \Lambda \subset I_0 \cup I_1,\ h(I_0) = h(I_1) = I,\ h(A_0) \cap I = \emptyset$.

Consider the itinerary of $x,\ S(x) = (s_0s_1\cdots)$, where $s_j = 0$, if $h^j(x) \in I_0$, and $s_j = 1$, if $h^j(x) \in I_1$. The function $S(x)$ is a homeomorphism between Λ and Σ_2, and $S \circ h = \sigma \circ S$. That is, h and σ are topologically conjugate.

For every $t_0 \in \Lambda$ one can construct a sequence $\kappa(t_0)$ of real numbers $\kappa_i,\ i \in \mathbb{Z}$, in the following way. If $i \geqslant 0$, then $\kappa_{i+1} = h(\kappa_i)$ and $\kappa_0 = t_0$. Let us show, how the sequence is defined for negative i. Denote $s^0 = S(t_0),\ s^0 = (s_0^0 s_1^0\cdots)$. Consider elements $\underline{s} = (0s_0^0s_0^1\cdots),\ \bar{s} = (1s_0^0s_0^1\cdots)$ of Σ_2, such that $\sigma(\underline{s}) = \sigma(\bar{s}) = s^0$ and $\underline{t} = S^{-1}(\underline{s}),\ \bar{t} = S^{-1}(\bar{s})$. The homeomorphism implies that $h(\bar{t}) = h(\underline{t}) = t_0$. Set $h^{-1}(t_0)$ may consist of not more than two elements $\bar{t},\ \underline{t} \in \Lambda$. Each of these two values can be chosen as $\kappa_{-1}(t_0)$. Obviously, one can continue the process to $-\infty$, choosing always one element from the set h^{-1}. We have finalized the construction of the sequence, and, moreover, it is proved that $\kappa(t_0) \subset \Lambda$. Fix one of the sequences and introduce a sequence $\zeta(t_0) = \{\zeta_i\},\ \zeta_i = i + \kappa_i,\ i \in \mathbb{Z}$. The sequence $\zeta(t_0)$ has the *periodicity property* if there exists $p \in \mathbb{N}$ such that $\zeta_{i+p} = \zeta_i + p,\ \forall i \in \mathbb{Z}$. If we denote by Π the set of all such sequences $\{\zeta_i\},\ i \in \mathbb{Z}$, then a multivalued functional $w : I \to \Pi$ is defined such that each of the sequences $\zeta(t_0)$ is one of values of $w(t_0)$.

Let $J \subseteq \mathbb{R}$ be an interval. Introduce the following distance $\|\zeta(t_0) - \zeta(t_1)\|_J = \sup_{\zeta_i(t_0),\zeta_i(t_1)\in J} |\zeta_i(t_0) - \zeta_i(t_1)|$. Let us formulate two important for our discussion consequences of the topological conjugacy [14] of the symbolical dynamics and the dynamics generated by the logistic map, in the following assertion.

Lemma 1.1. *If $\mu > 4$, then*

(a) *for each $\zeta(t_0) \in \Pi$, arbitrarily small $\epsilon > 0$, and arbitrarily large positive number E, there exists a sequence $\zeta(t_1) \in \Pi$ with the periodicity property such that $|\zeta(t_0) - \zeta(t_1)|_J < \epsilon$, where $J = (0, E)$;*

(b) *there exists a sequence $\zeta(t^*) \in \Pi$ such that for each $t_0 \in \Lambda$, for arbitrarily small $\epsilon > 0$, and arbitrarily large positive number E, there exists an integer m such that $|\zeta(t_0) - \zeta(t^*, m)|_J < \epsilon$, where $J = (0, E)$.*

Let $\mathbb{Z},\ \mathbb{N}$ and $\mathbb{R}$ be the sets of all integers, natural and real numbers, respectively. Denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$.

For every $t_0 \in \Lambda$ one can construct a sequence $\kappa(t_0)$ of real numbers $\kappa_i,\ i \in \mathbb{Z}$, such that $\kappa_{i+1} = h(\kappa_i, \mu)$ and $\kappa_0 = t_0$ if $i \geqslant 0$.

Fix a nonzero vector $m_0 \in \mathbb{R}^n$. For each $\zeta(t_0),\ t_0 \in \Lambda$, we introduce a pulse function

$$f(t, t_0) = \begin{cases} m_0 & \text{if } \zeta_{2i}(t_0) < t \leqslant \zeta_{2i+1}(t_0),\ i \in \mathbb{Z}, \\ 0 & \text{if } \zeta_{2i-1}(t_0) < t \leqslant \zeta_{2i}(t_0),\ i \in \mathbb{Z}. \end{cases}$$

Is it worth mentioning that we can consider other types of pulse functions to obtain similar results, for instance, one may discuss,

$$F(t, t_0) = \begin{cases} m_0 & \text{if } \zeta_{2i}(t_0) < t \leqslant \zeta_{2i+1}(t_0),\ i \in \mathbb{Z}, \\ m_1 & \text{if } \zeta_{2i-1}(t_0) < t \leqslant \zeta_{2i}(t_0),\ i \in \mathbb{Z}, \end{cases}$$

$m_0,\ m_1 \in \mathbb{R}^n$.

The main object of our investigation is the following special initial value problem

$$\begin{aligned} z'(t) &= Az(t) + f(t, t_0), \\ z(t_0) &= z_0 \quad (t_0, z_0) \in \Lambda \times \mathbb{R}^n, \end{aligned} \tag{1}$$

where $z \in \mathbb{R}^n,\ t \in \mathbb{R}$. Following [10], we call (1) the pulse system.

The next assumption will be needed throughout the paper: A is an $n \times n$ constant real valued matrix such that $\Re\lambda_j < 0,\ j = 1, 2, \ldots, m,\ \Re\lambda_j > 0,\ j = m+1, m+2, \ldots, n$, where m is a natural number, $0 \leqslant m \leqslant n$, and $\Re\lambda_j$ denotes the real part of the eigenvalue λ_j of the matrix A. That is the matrix A is hyperbolic. Denote $\alpha = \max_j \Re\lambda_j,\ j = 1, 2, \ldots, m$, and $\beta = \min_j \Re\lambda_j,\ j = m+1, m+2, \ldots, n$.

We shall use the following definitions of solutions of (1). They coincide with the definitions for differential equations with piecewise constant arguments of generalized type [16], see also [17].

Definition 1.1. A function $z(t),\ z(t_0) = z_0$, is a solution of (1) on $\mathbb{R}$ if: (i) $z(t)$ is continuous on $\mathbb{R}$; (ii) the derivative $z'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of the points $\zeta_i(t_0),\ i \in \mathbb{Z}$, where one-sided derivatives exist; (iii) Eq. (1) is satisfied on each interval $(\zeta_i(t_0), \zeta_{i+1}(t_0)],\ i \in \mathbb{Z}$.

Definition 1.2. A solution $z(t),\ z(t_0) = z_0$, of (1) on $[t_0, \infty)$ is a continuous function such that: (i) the derivative $z'(t)$ exists at each point $t \in [t_0, \infty)$, with the possible exception of the points $\zeta_j(t_0),\ j \geqslant 0$, where left-sided derivatives exist; (ii) Eq. (1) is satisfied by $z(t)$ on each interval $(\zeta_j(t_0), \zeta_{j+1}(t_0)],\ j \geqslant 0$.

It can be easily verified that problem (1) has a unique solution in the sense of Definition 1.1, as well as Definition 1.2, for each $t_0 \in \Lambda,\ z_0 \in \mathbb{R}^n$.

In what follows we denote by $z(t, \xi, v),\ \xi \in \mathbb{R},\ v \in \mathbb{R}^n$, a solution of (1) with $t_0 = \xi,\ z_0 = v$.

There exists a constant matrix B such that $B^{-1}AB = \mathrm{diag}\{C_-, C_+\}$, where C_- and C_+ are $m \times m$ and $(n-m) \times (n-m)$ matrices, respectively, C_+ has eigenvalues with positive real part, and C_- has eigenvalues with negative real part. If we apply the linear transformation $z = Bx$ to system (1), then one can see that the pulse function f will be transformed to a pulse function. Hence, without any loss of generality, we may assume that matrix A itself has the box-diagonal form so that $A = \mathrm{diag}\{A_-, A_+\}$, where all eigenvalues of A_+ have positive real part, and all eigenvalues of A_- have negative real part. Consequently, there exist positive numbers N and ω such that:

$$\|e^{A_- t}\| \leqslant N e^{-\omega t},\ t \geqslant 0, \quad \|e^{A_+ t}\| \leqslant N e^{\omega t},\ t \leqslant 0. \tag{2}$$

Let us denote $Z(t,s) = \mathrm{diag}\{Z_-(t,s), Z_+(t,s)\},\ Z_-(t,s) = e^{A_-(t-s)},\ Z_+(t,s) = e^{A_+(t-s)},\ t,\ s \in \mathbb{R}$, where $Z(t,s)$ is the transition matrix of the linear homogeneous system of differential equations associated with (1). It can be easily checked (for a detailed explanation see [16]) that the solution $z(t) = z(t, t_0, z_0),\ t_0 \in \Lambda,\ z_0 \in \mathbb{R}^n$, of (1) has the form:

$$z(t) = Z(t, t_0)z_0 + \int_{t_0}^{t} Z(t,s) f(s, t_0) \mathrm{d}s, \tag{3}$$

and is defined on $\mathbb{R}$.

Moreover, using the standard technique one can verify that for every $t_0 \in \Lambda$ there exists a unique vector $v_0 \in \mathbb{R}^n$ such that $z(t, t_0, v_0)$ is a bounded on $\mathbb{R}$ solution of (1). Denote $z(t, t_0) = z(t, t_0, v_0)$, and $z(t, t_0) = (u(t, t_0), v(t, t_0)),\ u \in \mathbb{R}^m,\ v \in \mathbb{R}^{n-m}$. One can see that the bounded solution is equal to

$$\begin{aligned} u(t, t_0) &= \int_{-\infty}^{t} Z_-(t,s) f_-(s, t_0) \mathrm{d}s, \\ v(t, t_0) &= -\int_{t}^{\infty} Z_+(t,s) f_+(s, t_0) \mathrm{d}s, \end{aligned} \tag{4}$$

if we denote $f(t, t_0) = (f_-(t, t_0), f_+(t, t_0))$. It is easy to show that $\|z(t, t_0)\| < \frac{2N\|m_0\|}{\omega},\ t \in \mathbb{R}$. Denote $\mathscr{CB} = \{z(t, t_0) : t_0 \in \Lambda\}$.

Thus, we have that all bounded on $\mathbb{R}$ solutions of (1) are placed in the tube with the radius $\frac{2N\|m_0\|}{\omega}$. If $\kappa(t_0),\ t_0 \in \Lambda$, is a p-periodic sequence, then $z(t, t_0)$ is periodic with period p. Denote the periodic solution by $\phi(t, t_0)$.

Remark 1.1. Since sequences $\kappa(t)$ do not intersect for different $t \in \Lambda$, one can see that there exists a p-periodic solution for each $p \in \mathbb{N}$. Consequently, there are infinitely many periodic solutions of (1).

One can easily verify (see also [16]) that a solution $z(t)$ of (1) is bounded on $[0, \infty)$ if and only if $z(t) = (u(t), v(t)) = z(t, t_0, z_0),\ z_0 = (u_0, v_0)$,

$$\begin{aligned} u(t) &= Z_-(t, t_0)u_0 + \int_{t_0}^{t} Z_-(t,s) f_-(s, t_0) \mathrm{d}s, \\ v(t) &= -\int_{t}^{\infty} Z_+(t,s) f_+(s, t_0) \mathrm{d}s. \end{aligned} \tag{5}$$

We denote the solutions defined by (5) as $z(t, t_0, u_0)$. Then $\mathscr{C} = \{z(t, t_0, u_0) : t_0 \in \Lambda,\ u_0 \in \mathbb{R}^m\}$ is the set of all solutions of (1) bounded on $[0, \infty)$. One can confirm that:

$$\|z(t, t_0, u_0) - z(t, t_0)\| < N e^{-\omega(t - t_0)}(\|u_0\| + \|m_0\|/\omega), \quad t \geqslant t_0. \tag{6}$$

That is, every solution $z(t, t_0, u_0) \in \mathscr{C} \setminus \mathscr{CB}$ is attracted by a bounded solution $z(t, t_0) \in \mathscr{CB}$. These solutions have a common set of discontinuity points $\zeta(t_0)$. Thus, $\mathscr{CB}$ is an attractor with the basin $\mathscr{C}$. Obviously, $\mathscr{CB} \subset \mathscr{C}$. We intend to address the topological ingredients for $\mathscr{CB}$ and $\mathscr{C}$.

The paper is organized as follows. In Section 2, we consider the main subjects of the paper: the ingredients of the chaos, the existence of a chaotic attractor, the period-doubling cascade, and an appropriate example. The conclusion is formulated at the end of the paper.

2. The chaos

Everywhere in this section we assume that $\mu > 4$, with the exception of the part addressing the period-doubling cascade. At first we are going to describe the ingredients for solutions of the initial value problem, which do not necessarily belong to the attractor, but they are attracted by the bounded solutions from this set. Then the chaos on the attractor will be defined. Finally, we will consider the period-doubling cascade for the problem, and an illustrative example will be constructed.

Definition 2.1. We say that (1) is sensitive on Λ if there exists positive real number ϵ_0 such that for every number $t_0 \in \Lambda$ and for each $\delta > 0$ one can find a number $t_1 \in \Lambda,\ |t_0 - t_1| < \delta$, such that for each pair of solutions $z(t,t_1,u_1),\ z(t,t_0,u_0),\ u_1,\ u_0 \in \mathbb{R}^n$, there exists a moment $\xi > \max(t_0,t_1)$, which satisfies $\|z(\xi,t_1,u_1) - z(\xi,t_0,u_0)\| > \epsilon_0,\ \|z(\xi,t_1,u_1)\|,\ \|z(\xi,t_0,u_0)\| < \frac{2N\|m_0\|}{\omega} + 1$.

Definition 2.2. The set of all periodic solutions is called dense in $\mathscr{C}$ if for every solution $z(t) = z(t,t_1,u_0) \in \mathscr{C},\ t_1 \in \Lambda$, and each $\epsilon > 0,\ E > 0$, there exists a periodic solution $\phi(t,t_0),\ t_0 \in \Lambda$, and an interval $J \subset [t_1,\infty)$, with length E, such that $\|\phi(t,t_0) - z(t,t_1,z_0)\| < \epsilon,\ t \in J$.

Definition 2.3. A solution $z(t,t^*,u_0) \in \mathscr{C}$ is called dense in $\mathscr{C}$ if for every solution $z(t,t_1,z_1) \in \mathscr{C}$ and each $\epsilon > 0,\ E > 0$, there exists a number $\xi > 0$ and an interval $J \subset [\max\{t_1,t^*\},\infty)$ with length E, such that $\|z(t,t_1,z_1) - z(t+\xi,t^*,z_0)\| < \epsilon$, for all $t \in J$.

Theorem 2.1. *Problem (1) is sensitive on Λ.*

Proof. Fix $t_0 \in \Lambda,\ u_0,\ u_1 \in \mathbb{R}^n$, and solutions of (1), $z(t) = z(t,t_0,u_0),\ z_1(t) = z(t,t_1,u_1)$. Let $S(t_0) = s^0 = (s_0^0, s_1^0, \ldots)$. Take a number $t_1 \in \Lambda$ such that $S(t_1) = s^1 = (s_0^0, s_1^0, \ldots, s_{k-1}^0, s_k^1, s_{k+1}^0, s_{k+2}^0, \ldots),\ s_k^1 \neq = s_k^0$, for some $k > 0$. We have that

$$\mathrm{d}[\sigma^i s^0, \sigma^i s^1] = \begin{cases} \frac{1}{2^{k-i}} & \text{if } 0 \leqslant i \leqslant k, \\ 0 & \text{if } i > k. \end{cases}$$

Assume that k is sufficiently large so that by (6) $\|z(t,t_1,u_1)\|,\ \|z(t,t_0,u_0)\| < \frac{2N\|m_0\|}{\omega} + 1$, if $t > \min(\zeta_k(t_0),\zeta_k(t_1)) - 1$.

Since S is a homeomorphism and set Σ_2 is compact there exists a positive number $\mu_0 < 1$ so that $|\kappa_k(t_0) - \kappa_k(t_1)| > \mu_0$.

Without loss of generality, assume that $\kappa_k(t_0) < \kappa_k(t_1)$.

Denote $\bar{m} = \max_{\mu_0 \leqslant u \leqslant 1} \|e^{Au}\|,\ \underline{M} = \min_{\mu_0 \leqslant u \leqslant 1} \left\| \int_0^u e^{As} m_0\, ds \right\|$.

We shall show that the constant ϵ_0 and the moment ξ of Definition 2.1 can be taken equal to $\epsilon_0 = \frac{\underline{M}}{2(1+\bar{m})}$, and ξ and $\zeta_k(t_0)$ or $\zeta_k(t_1)$, relatively.

If $\|z(\zeta_k(t_0)) - z_1(\zeta_k(t_0))\| \leqslant \epsilon_0$, then we have that for $t \in [\zeta_k(t_0),\zeta_k(t_1)]$,

$$z(t) = e^{A(t-\zeta_k(t_0))} z(\zeta_k(t_0)) + \int_{\zeta_k(t_0)}^t e^{A(t-s)} f(s,t_0)\, ds,$$
$$z_1(t) = e^{A(t-\zeta_k(t_0))} z_1(\zeta_k(t_0)),$$

and

$$\|z(\zeta_k(t_1)) - z_1(\zeta_k(t_1))\| \geqslant \int_{\zeta_k(t_0)}^{\zeta_k(t_1)} \|e^{A(t-s)}\| \|m_0\| ds - \|e^{A(\zeta_k(t_1)-\zeta_k(t_0))}\| \|z(\zeta_k(t_0)) - z_1(\zeta_k(t_0))\| \geqslant \underline{M} - \bar{m}\epsilon_0 > \epsilon_0.$$

The theorem is proved. □

Theorem 2.2. *The set of all periodic solutions $\phi(t,t_0),\ t_0 \in \Lambda$, of (1) is dense in $\mathscr{C}$.*

Proof. Let us fix $t_1 \in \Lambda$ and $\epsilon,\ E > 0$, and denote $z(t) = (u,v) = z(t,t_1,u_0)$. Fix a positive number δ sufficiently small so that $2N\|m_0\|\delta \frac{e^{2\omega}}{1-e^{-2\omega}} < \frac{\epsilon}{2}$.

By Lemma 1.1(a) for δ and an arbitrarily large number $\widetilde{T}$, there exists a periodic sequence $\zeta(t_0) \in \Pi$ such that $\|\zeta(t_1) - \zeta(t_0)\|_Q < \delta$, where $Q = (t_1, t_1 + \widetilde{T} + E)$. We shall find numbers δ and $\widetilde{T}$ such that $\|z(t) - \phi(t,t_0)\| < \epsilon$ on $J = (t_1 + \widetilde{T}, t_1 + \widetilde{T} + E)$. Denote $M_1 = 1 + \frac{2N\|m_0\|}{\omega}$. By (6) there exists a number $\overline{T} = \overline{T}(z_0,1) > t_1$ such that $\|z(t)\| < M_1$, if $t \geqslant \overline{T}$. Denote $\phi(t) = (\phi_-,\phi_+) = \phi(t,t_0)$, the periodic solution. Assuming, without loss of generality, that $\zeta_i(t_0) \leqslant \zeta_i(t_1)\ \forall i \in \mathbb{Z}$, we have:

$$\begin{aligned}\|z(t) - \phi(t)\| &= \|u(t) - \phi_-(t)\| + \|v(t) - \phi_+(t)\| \leqslant \|u(\overline{T}) - \phi_-(\overline{T})\| \|Z_-(t,\overline{T})\| + \int_{\overline{T}}^t \|Z_-(t,s)\| \|f_-(s,t_0) - f_-(s,t_1)\| ds \\ &+ \int_{-\infty}^{\overline{T}} \|Z_-(t,s)\| \|f_-(s,t_0)\| ds \leqslant 2N e^{-\omega(t-\overline{T})} M_1 + \sum_{\overline{T} \leqslant \zeta_j(t_1) < t} \left[\int_{\zeta_{2j}(t_0)}^{\zeta_{2j}(t_1)} 2N e^{-\omega(t-s)} \|m_0\| ds \right. \\ &+ \left. \int_{\zeta_{2j+1}(t_0)}^{\zeta_{2j+1}(t_1)} 2N e^{-\omega(t-s)} \|m_0\| ds + \int_{-\infty}^{\overline{T}} e^{-\omega(t-s)} \|m_0\| ds \right] \leqslant N \left[2M_1 e^{-\omega(t-\overline{T})} + 4\delta \frac{e^{2\omega}\|m_0\|}{1-e^{-2\omega}} + \|m_0\| e^{-\omega(t-\overline{T})} 1/\omega \right].\end{aligned}$$

Now, if $\widetilde{T} \geqslant \overline{T}$ is sufficiently large so that:

$$2NM_1 \, e^{-\omega(\widetilde{T}-\overline{T})} + N\|m_0\|e^{-\omega(\widetilde{T}-\overline{T})}\frac{1}{\omega} < \frac{\epsilon}{2},$$

then $\|z(t) - \phi(t, t_0)\| < \epsilon$ for all $t \in J$. The theorem is proved. □

Theorem 2.3. *There exists a solution of (1) dense in $\mathscr{C}$.*

Proof. Fix positive $E, \ \epsilon$. By Lemma 1.1(b), there exists $t^* \in \Lambda$ such that $\zeta(t^*)$ is dense in Π. There exists a unique bounded on $\mathbb{R}$ solution $z_*(t) = (u_*, v_*) = z(t, t^*) = z(t, t^*, z_0)$. Let us prove that $z_*(t)$ is the dense solution.

Consider an arbitrary solution $z(t) = z(t, t_1, u_1), \ t_1 \in \Lambda$, of (1). There exists $\overline{T}$, such that $\|z(t)\| < M_1$, if $t > \overline{T}$. Consider an interval $J_1 = (0, \overline{T} + E_1)$, where E_1 is an arbitrarily large positive number. By Lemma 1.1(b), there exists a natural m such that:

$$\|\zeta(t_1) - \zeta(t^*, m)\|_{J_1} < \delta < \epsilon, \tag{7}$$

where δ will be defined more precisely below. We have that:

$$\begin{aligned} u_*(t+m) &= \int_{-\infty}^{t+m} Z_-(t+m, s) f_-(s, t^*) \mathrm{d}s, \\ v_*(t+m) &= -\int_{t+m}^{\infty} Z_+(t+m, s) f_+(s, t^*) \mathrm{d}s, \end{aligned} \tag{8}$$

and

$$\begin{aligned} u(t) &= Z_-(t, \overline{T}) u(\overline{T}) + \int_{\overline{T}}^{t} Z_-(t, s) f_-(s, t_1) \mathrm{d}s, \\ v(t) &= -\int_{t}^{\infty} Z_+(t, s) f_+(s, t_1) \mathrm{d}s. \end{aligned} \tag{9}$$

Now, using the last two formulas and (7), and emulating the proof of Theorem 2.2, we shall complete the proof. We have that, for $t \geqslant \overline{T}$,

$$\begin{aligned} \|u_*(t+m) - u(t)\| &= \left\| \int_{-\infty}^{t+m} Z_-(t+m, s) f_-(s, t^*) \mathrm{d}s - Z_-(t, \overline{T}) u(\overline{T}) - \int_{\overline{T}}^{t} Z_+(t, s) f_+(s, t_1) \mathrm{d}s \right\| \\ &= \|Z_-(t, \overline{T}) u(\overline{T})\| + \left\| \int_{-\infty}^{\overline{T}+m} Z_-(t+m, s) f_-(s, t^*) \mathrm{d}s \right\| + \left\| \int_{\overline{T}+m}^{t+m} Z_-(t+m, s) f_-(s, t^*) \mathrm{d}s - \int_{\overline{T}}^{t} Z_-(t, s) f_-(s, t_1) \mathrm{d}s \right\| \\ &= \|Z_-(t, \overline{T}) u(\overline{T})\| + \left\| \int_{-\infty}^{\overline{T}} Z_-(t+m, s+m) f_-(s+m, t^*) \mathrm{d}s \right\| \\ &\quad + \int_{\overline{T}}^{t} [\|Z_-(t+m, s+m) - Z_-(t, s)\| \|f_-(s+m, t^*)\| + \|Z_-(t, s)\| \|f_-(s+m, t^*) - f_-(s, t_1)\|] \mathrm{d}s \\ &\leqslant N e^{-\omega(t-\overline{T})} [M_1 + \|m_0\|/\omega] + \delta N \|m_0\|/\omega. \end{aligned}$$

Similarly,

$$\begin{aligned} \|v_*(t+m) - v(t)\| &= \left\| \int_{t+m}^{\infty} Z_+(t+m, s) f_+(s, t^*) \mathrm{d}s - \int_{t}^{\infty} Z_+(t, s) f_+(s, t_1) \mathrm{d}s \right\| \\ &\leqslant \int_{t}^{\infty} [\|Z_+(t+m, s+m) - Z_+(t, s)\| \|f_+(s, t^*)\| + \|Z_+(t, s)\| \|f_+(s, t^*) - f_+(s, t_1)\|] \mathrm{d}s \leqslant \delta N \|m_0\|/\omega. \end{aligned}$$

Fix $\widetilde{T} > \overline{T}$ and δ sufficiently large and small, respectively, so that:

$$N e^{-\omega(\widetilde{T}-\overline{T})} [M_1 + \|m_0\|/\omega] + 2\delta N \|m_0\|/\omega < \epsilon.$$

Then the last two inequalities imply that $\|z(t) - z_*(t+m)\| < \epsilon$, on the interval $J = [\widetilde{T}, \widetilde{T} + E]$, if $E_1 = \widetilde{T} + E - \overline{T}$.

The theorem is proved. □

3. The chaos on the attractor

This part of the paper is devoted to the discussion of the chaotic ingredients of bounded solutions from $\mathscr{CB}$. The first of these definitions is significantly different from its counterpart for $\mathscr{C}$, as it requires closeness of the initial values.

Definition 3.1. We say that (1) is sensitive on $\mathscr{CB}$ if there exist positive real numbers $\epsilon_0, \ \epsilon_1$ such that for each $t_0 \in \Lambda$, and for every $\delta > 0$ one can find $t_1 \in \Lambda, \ z_1 \in \mathbb{R}^n, \ \|z_1 - z_0\| + |t_0 - t_1| < \delta$, and an interval Q from $[0, \infty)$ with length no less than ϵ_1 such that $\|z(t, t_0) - z(t, t_1)\| \geqslant \epsilon_0, \ t \in Q$, and there are no points of discontinuity of $z(t, t_0), \ z(t, t_1)$ on Q.

Definition 3.2. The set of all periodic solutions is called dense in $\mathscr{CB}$ if for every solution $z(t) = z(t,t_1),\ t_1 \in \Lambda$, and each $\epsilon > 0,\ E > 0$, there exists a periodic solution $\phi(t,t_0),\ t_0 \in \Lambda$, and an interval $J \subset [t_1,\infty)$, with length E, such that $\|\phi(t,t_0) - z(t,t_1,z_0)\| < \epsilon,\ t \in J$.

Definition 3.3. A solution $z(t,t^*) \in \mathscr{CB}$ is called dense in $\mathscr{CB}$ if for every solution $z(t,t_1) \in \mathscr{CB}$, and each $\epsilon > 0,\ E > 0$, there exists a number $\xi > 0$ and an interval $J \subset [\max\{t_1,t^*\},\infty)$ with length E such that $\|z(t,t_1) - z(t+\xi,t^*)\| < \epsilon$, for all $t \in J$.

We call the attractor *chaotic* if: (i) problem (1) is sensitive in $\mathscr{CB}$; (ii) there are infinitely many periodic solutions $\phi(t,t_0),\ t_0 \in \Lambda$, and they are dense in $\mathscr{CB}$; (iii) there exists a solution $z(t,t_0),\ t_0 \in \Lambda$, which is dense in $\mathscr{CB}$.

Theorem 3.1. *The manifold $\mathscr{CB}$ is a chaotic attractor.*

Proof. Let us start with sensitivity in $\mathscr{CB}$. Fix a solution $z(t,t_0) = (u,v) = z(t,t_0,z_0),\ z_0 = (u_0,v_0)$, in $\mathscr{CB}$.

If we take into account the proof of Theorem 2.1 applied to $z(t,t_0) \in \mathscr{CB}$, we need only show that for an arbitrarily small $\delta > 0$ there exist $t_1,\ z_1$, which are considered in the proof, such that $|t_0 - t_1|,\ \|z_0 - z_1\| < \frac{\delta}{2}$, and $z_1 = (u_1,v_1) = z(t_1,t_1)$. In other words,

$$u_1 = \int_{-\infty}^{t_1} Z_-(t_1,s)f_-(s,t_1)\mathrm{d}s, \quad v_1 = -\int_{t_1}^{\infty} Z_+(t_1,s)f_+(s,t_1)\mathrm{d}s.$$

Let $(\cdots s^1_{-k}s^1_{-(k-1)}\cdots s^1_0 s^1_1\cdots)$ be a bi-infinite sequence such that $s^1_i = s^0_i,\ i < n,\ s^1_n \neq s^0_n$, where n is the number discussed in the proof of Theorem 2.1. Denote $s^1 = (s^1_0 s^1_1\cdots)$. Fix a positive $\delta_1 < \delta/2$, which will be defined more precisely below, and choose a number n sufficiently large so that $\|\zeta(t_0) - \zeta(t_1)\|_{[0,n/2]} < \delta_1$. Obviously, $|\zeta_{-k}(t_0) - \zeta_{-k}(t_1)| < \delta_1,\ k \geqslant 1$.

Now, assuming without any loss of generality that $t_0 < t_1$, we have that:

$$\begin{aligned}\|u_0 - u_1\| &= \left\|\int_{-\infty}^{t_0} Z_-(t_0,s)f_-(s,t_0)\mathrm{d}s - \int_{-\infty}^{t_1} Z_-(t_1,s)f_-(s,t_1)\mathrm{d}s\right\| \\ &\leqslant \left\|\int_{t_0}^{t_1} Z_+(t_1,s)f_+(s,t_1)\mathrm{d}s\right\| + \int_{-\infty}^{t_0}[\|Z_+(t_0,s) - Z_+(t_1,s)\|\|f_+(s,t_0)\| + \|Z_+(t_1,s)\|\|f_+(s,t_1) - f_+(s,t_0)\|]\mathrm{d}s \\ &\leqslant \|m_0\|\left\{\frac{N}{\omega}(\kappa(\delta_1) + \delta_1) + \delta_1 m^-\right\},\end{aligned}$$

where κ is a continuous function, such that $\|I - \mathrm{e}^{A_- u}\| \leqslant \kappa(\delta_1)$ if $0 \leqslant |u| < \delta_1$, $m^- = \max_{0\leqslant|u|\leqslant1}\|\mathrm{e}^{A_- u}\|$.

Similarly, we have that:

$$\begin{aligned}\|v_0 - v_1\| &= \left\|\int_{t_0}^{\infty} Z_+(t_0,s)f_+(s,t_0)\mathrm{d}s - \int_{t_1}^{\infty} Z_+(t_1,s)f_+(s,t_1)\mathrm{d}s\right\| \\ &\leqslant \left\|\int_{t_0}^{t_1} Z_+(t_1,s)f_+(s,t_0)\mathrm{d}s\right\| + \int_{t_0}^{n/2}[\|Z_+(t_0,s) - Z_+(t_1,s)\|\|f_+(s,t_0)\| + \|Z_+(t_1,s)\|\|f_+(s,t_1) - f_+(s,t_0)\|]\mathrm{d}s \\ &\quad + \int_{n/2}^{\infty}[\|Z_+(t_0,s) - Z_+(t_1,s)\|\|f_+(s,t_0)\| + \|Z_+(t_1,s)\|\|f_+(s,t_1) - f_+(s,t_0)\|]\mathrm{d}s \\ &\leqslant \|m_0\|\left\{\frac{N}{\omega}\mathrm{e}^{\omega}[\kappa_1(\delta_1) + \delta_1 + \mathrm{e}^{\omega(t_0 - \frac{n}{2})}](1-\delta_1) + \delta_1 m^+\right\},\end{aligned}$$

where κ_1 is a continuous function, such that $\|I - \mathrm{e}^{A_+ u}\| \leqslant \kappa_1(\delta_1)$ if $0 \leqslant |u| < \delta_1$, $m^+ = \max_{0\leqslant|u|\leqslant1}\|\mathrm{e}^{A_+ u}\|$.

If we suppose that n and δ_1 are sufficiently large and small, respectively, so that:

$$\|m_0\|\left\{\frac{N}{\omega}[\mathrm{e}^{\omega}(\kappa_1(\delta_1) + \delta_1 + \mathrm{e}^{\omega(t_0 - \frac{n}{2})})(1-\delta_1) + \kappa(\delta_1) + \delta_1] + \delta_1(m^+ + m^-)\right\} < \frac{\delta}{2},$$

then from the last two inequalities $\|z_1 - z_0\| < \frac{\delta}{2}$. Sensitivity is proved.

The existence of infinitely many periodic solutions is considered in the first section. The density of periodic solutions in $\mathscr{CB}$ follows immediately from Theorem 2.2. The existence of a dense solution in $\mathscr{CB}$ can be proved in exactly the same way as Theorem 2.3.

The theorem is proved. □

3.1. The period-doubling cascade and intermittency. Example

The logistic map has been used to shape the chaos in the multidimensional system. Consequently, one can expect to observe the period-doubling cascade and intermittency.

Let us consider $\mu > 0$, μ being the parameter for the logistic map $h(t,\mu) \equiv \mu t(1-t)$. It is known [4], that there exists an infinite sequence $3 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots < 3.8284\ldots$ such that $h(t,\mu_i),\ i \geqslant 1$, has an asymptotically stable prime period -2^i point t^*_i with a region of attraction $(t^*_i - \delta_i, t^*_i + \delta_i)$. And beyond the value $3.8284\ldots$, there are cycles with every integer period [3].

One can easily see that there is a 2^i-periodic solution $\phi(t, t_i^*, \mu_i)$ of (1) for each i, and for different i these periodic solutions do not coincide. The periodic solutions are in the bounded region $\|x\| < \frac{2N\|m_0\|}{\omega}$ of the space $\mathbb{R}^n$. The chaotic attractor is also placed in the region. Finally, the cascade generates infinitely many periodic solutions.

The numerical simulation of the chaos is not an easy task since even the verification of sensitivity requires two close values of the initial moment in the Cantor set Λ, which cannot be found easily. Hammel et al. [18] have given a computer-assisted proof that an approximate trajectory of the logistic map can be shadowed by a true trajectory for a long time. This result and the continuous dependence of the solutions on the sequence of discontinuity points make possible the following appropriate simulations.

To illustrate the chaotic nature of the discussed system, let us show that the chaotic attractor and intermittency can be observed in the next example.

Example 3.1. Consider the sequence $\zeta_i = i + \kappa_i,\ \kappa_i = 3.8282\kappa_{i-1}(1 - \kappa_{i-1}),\ \kappa_0 = t_0,\ t_0 \in [0, 1],\ i \geqslant 0$. The coefficient's value of 3.8282 is such that the logistic map admits intermittency [19]. Let the following pendulum equation be given:

$$x'' + 2x' + 1.5x = f_2(t, t_0), \tag{10}$$

where $f_2(t, t_0)$ is a scalar pulse function with $m_0 = 1$. Using new variables $x_1 = x,\ x_2 = x'$, one can reduce (10) to the system:

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -1.5x_1 - 2x_2 + f_2(t, t_0). \end{aligned} \tag{11}$$

One can easily verify that both eigenvalues of the matrix of coefficients have negative real parts. Fix $t_0 = 0.5$ and take a solution $(x_1(t), x_2(t))$ of the last system with the initial condition $x_1(t_0) = 0.02,\ x_2(t_0) = -0.025$. The result of simulation can be seen in Fig. 1. It demonstrates the intermittency phenomenon for the pulse mechanical model.

Next, consider Eq. (10) with $\mu = 4$, and the solution $(x_1(t), x_2(t))$ that has been chosen for the intermittency observation. In Fig. 2, the chaotic attractor is shown by using points $(x_1(n), x_2(n)),\ n = 1, 2, 3, \ldots, 75{,}000$, in $x_1,\ x_2$-plane.

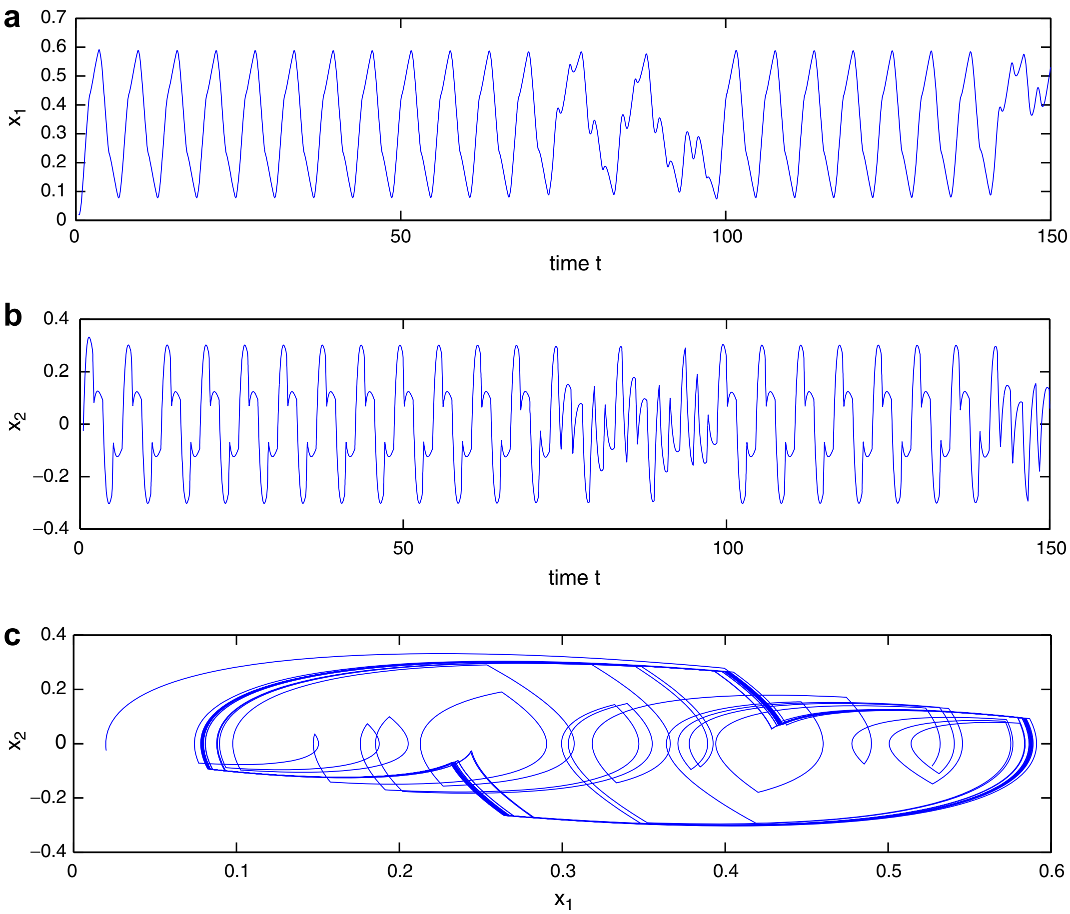

Fig. 1. Simulation results. (a) The graph of the x_1 coordinate. (b) The graph of the x_2 coordinate. (c) The trajectory of the solution $(x_1(t), x_2(t))$.

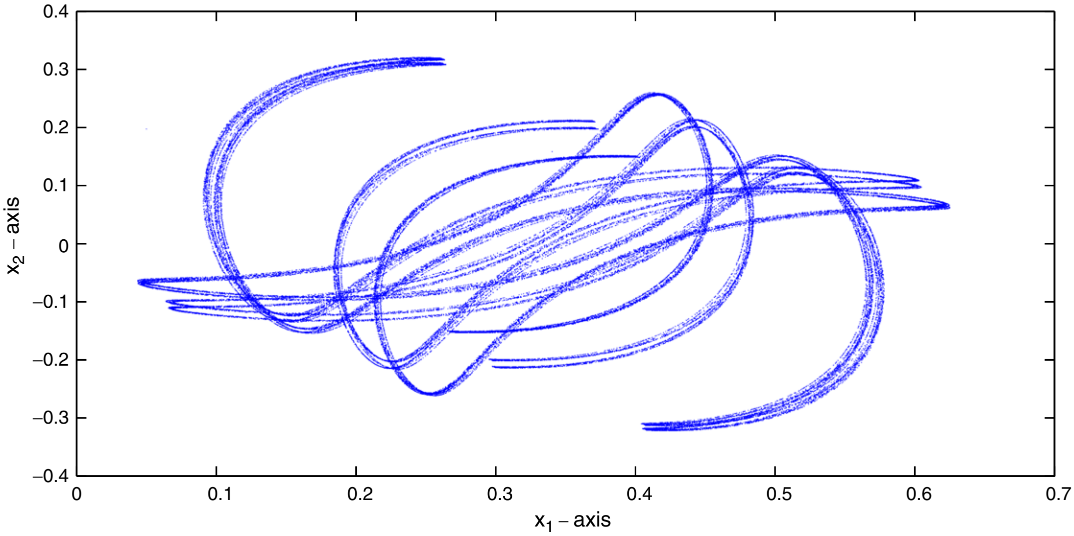

Fig. 2. The chaotic attractor by a stroboscopic sequence $(x_1(n), x_2(n))$, $1 \leqslant n \leqslant 75,000$, is observable.

4. Conclusion

We develop an approach to forming Devaney chaos in a non-autonomous system. The method can be applied to differential equations with a piecewise constant argument of generalized type [16], and to quasilinear systems with a pulse function. One can say that a new method to form the multidimensional chaos is proposed. The existence of a chaotic attractor of the initial problem is proved. The chaos is observed not only on the attractor, but in the set of all solutions.

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