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ALMOST PERIODIC SOLUTIONS OF THE LINEAR DIFFERENTIAL EQUATION WITH PIECEWISE CONSTANT ARGUMENT

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Abstract. The paper is concerned with the existence and stability of almost periodic solutions of linear systems with piecewise constant argument

$$\frac{dx(t)}{dt} = A(t)x(t) + \sum_{j=-N}^{N} A_j(t)x([t+j]) + f(t). \tag{1}$$

where $t \in R, x \in R^n, [\cdot]$ is the greatest integer function. The Wexler inequality [1]-[4] for the Cauchy's matrix is used. The results can be easily extended for the quasilinear case. A new technique of investigation of equations with piecewise argument, based on an integral representation formula, is proposed.
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1 Introduction and Preliminaries

Let Z, R be sets of all integers and real numbers respectively, $||\cdot||$ be the euclidean norm in $R^n, n \in N$, $C_b(R)$ a set of all uniformly continuous and bounded on R functions, $\mathcal{PC}$ a set of all piecewise continuous, right continuous with discontinuities of the first type at points $t = i, i \in Z$, functions. We introduce in $C_b(R)$ and in $\mathcal{PC}$ the sup-norm $||\phi||_0 = \sup_R ||\phi(t)||$.

For $f \in C_b(R)$ and $\tau \in R$ the translate of f by τ is the function $Q_\tau f = f(t+\tau), t \in R$.

A number $\tau \in R$ is called $\epsilon-$ translation number of a function $f \in C_b(R)$ if $||Q_\tau f - f|| < \epsilon$ for every $t \in R$.

Definition 1.1 A function $f \in C_b(R)$ is called almost periodic if for every $\epsilon \in R, \epsilon > 0$, there exists a respectively dense set of $\epsilon-$ translations of $f(t)$.

Denote the set of all such functions as $\mathcal{AP}(R)$.
The aim of this paper is to investigate the problem of existence and exponential stability of an almost periodic solution of system (1).

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The theory of differential equations with piecewise constant argument (EPCA) of the type

$$\frac{dx(t)}{dt} = f(t, x([t-p_1]), x([t-p_2]), \ldots, x([t-p_m])), \tag{2}$$

where $[\cdot]$ signifies the greatest integer function, was initiated in [5, 6] and developed by many authors [7]- [15]. These systems have been under intensive investigation for the last twenty years. They describe hybrid dynamical systems and combine the properties of both differential and difference equations. Applications of these equations to the problems of biology can be found in [12, 16].

The existing method of investigation of EPCA, as it was proposed by its founders [5, 6], is based on the reduction of EPCA to discrete equations, and it has been the only method to prove assertions about EPCA until now. In our paper, we use another approach [17] to the problem. In fact, we are dealing with the construction of an equivalent integral equation. Since we do not need additional assumptions on the reduced discrete equations for investigating EPCAG, the new method requires more easily verifiable conditions, similar to those for ordinary differential equations.

Another novelty in our investigation is that we consider equations with deviated argument of mixed (advanced-delayed) type. Even in the case of advanced argument, there are certain difficulties if we try to define a solution for increasing t [18]. J. Hale remarked in [19] that "these equations (of mixed type) seem to dictate that boundary conditions should be specified in order to obtain a solution in the way as one does for elliptic partial differential equations." We regard the boundedness of the solution on R as a boundary condition in our investigation. Similar arguments were used in [20] - [22] to investigate various problems for ordinary and functional differential equations. The practical use of theory of equations of mixed (advanced-delayed) type for populations dynamics is considered in [23].

The existence of almost periodic solutions is one of the most interesting subjects of the theory of differential equations (see, for example, [24]-[27] and the references cited there). This problem has been considered in the context of EPCA in many papers, such as [9, 12, 14, 28].

To solve the problems of the present paper, we intend to apply our knowledge about the almost periodicity of discontinuous solutions of impulsive systems [3, 4],[29]-[32]. One should not be surprised by the relation between EPCA and impulsive differential equations. This possibility was mentioned in [5] for EPCA, and in [33] for differential equations with discontinuous right hand side.

We will need the following assumptions:

(C_1) $A(t), A_j(t) \in \mathcal{AP}(R)$ are $n \times n$ matrices, $f \in \mathcal{AP}(R)$ is an $n \times 1$ vector function.

Let $X(t,s), X(s,s) = I$ be the Cauchy's matrix of an associate homogeneous linear system

$$\frac{dx}{dt} = A(t)x. \tag{3}$$

One of our basic assumptions is the following:

(C_2) $\exists\{a,b\} \subset R, b \geq 1, a > 0$, such that

$$||X(t,s)|| \leq b\exp(-a(t-s)), t \geq s. \tag{4}$$

LEMMA 1.1 *[1]-[4] Let $A(t) \in \mathcal{AP}(R)$ and the condition (C_3) be valid. Then*

$$||X(t+\tau, s+\tau) - X(t,s)|| < \epsilon\exp(-\frac{a}{2}(t-s)), \tag{5}$$

if τ is $\frac{a\epsilon}{2b^2}-$ translation number of $A(t)$.

Remark 1.1 Lemma 1 is due to D. Wexler [1] and is of principal significance for the article. It states that the matrix $X(t,s)$ is "diagonal almost periodic". Information relevant to this assertion can be found in the book by W.A. Coppel (see Proposition 4 from Lecture 8 of [34]).

2 Main result

In this section we consider the problem of existence of an almost periodic solution of (1) and investigate stability of the solution.

2.1 Existence of the almost periodic solution

The following definition of a solution of EPCA which is a slightly changed form of the corresponding definition for EPCA [10, 11] can be given.

DEFINITION 2.1 *A function $x(t)$ is a solution of (1) on R if:*

(i) $x(t)$ is continuous on R;

(ii) the derivative $x'(t)$ exists at each point $t \in R$, with the possible exception of the points $i, i \in Z$, where one-sided derivatives exist;

(iii) equation (1) is satisfied on each interval $[i, i+1), i \in Z$.

It is obvious that the derivative of a solution $x(t)$ is a function from $\mathcal{PC}$,

Denote

$$F_\phi(t) = \sum_{j=-N}^{N} A_j(t)\psi([t+j]) + f(t),$$

where $\psi(t) \in C_b(R)$.

The following is one of the most important assertions for our method of investigation of EPCAG.

LEMMA 2.1 *A function $x(t) \in C_b(R)$ is a solution of (1) if and only if*

$$x(t) = \int_{-\infty}^{t} X(t,s)F_x(s)ds. \tag{6}$$

Proof.

Necessity. Assume that $x(t) \in C_b(R)$ is a solution of (1). Denote

$$\phi(t) = \int_{-\infty}^{t} X(t,s)F_x(s)ds. \tag{7}$$

By a straightforward calculation we can see that the function $\phi(t)$ is bounded and continuous on R.

Assume that $t \neq i, i \in Z$. Then

$$\phi'(t) = A(t)\phi(t) + F_x(t)$$

and

$$x'(t) = A(t)x(t) + F_x(t).$$

Hence,

$$[\phi(t) - x(t)]' = A(t)[\phi(t) - x(t)].$$

Calculating the limit values at $t = j, j \in Z$, we find that

$$\phi'(j \pm 0) = A(j \pm 0)\phi(j \pm 0) + F_x(j \pm 0)$$

$$x'(j \pm 0) = A(j \pm 0)x(j \pm 0) + F_x(j \pm 0).$$

Consequently,

$$[\phi(t) - x(t)]'|_{t=j+0} = [\phi(t) - x(t)]'|_{t=j-0}.$$

Thus, $[\phi(t)-x(t)]$ is a continuously differentiable function on R, satisfying (4). That is, $[\phi(t) - x(t)] = 0$ on R.

Sufficiency. Suppose that (6) is valid and $x(t) \in C_b(R)$. Fix $i \in Z$ and consider the interval $[i, i+1)$. If $t \in (i, i+1)$, then by differentiating one can see that $x(t)$ satisfies (1). Moreover, considering $t \to i+$, and taking into account that $[\cdot]$ is a right-continuous function, we obtain that $x(t)$ satisfies (1) on $[i, i+1)$. The lemma is proved.

Further we shall define a discontinuous almost periodic function, which is a particular case of the function considered in [1] -[4], [29] -[31].

A number $\tau \in R$ is called an $\epsilon-$ translation number of a function $\phi(t) \in \mathcal{PC}$ if $||\phi(t+\tau) - \phi(t)|| < \epsilon$ for every $t \in R, |t-i| > \epsilon, i \in Z$.

A set $S \subset R$ is said to be relatively dense if there exists a number $l > 0$, such that $[a, a+l] \cap S \neq \emptyset$ for all $a \in R$.

Definition 2.2 A function $\phi(t) \in \mathcal{PC}$ is said to be piecewise continuous almost periodic if for every $\epsilon \in R, \epsilon > 0$, there exists a relatively dense set of $\epsilon-$ translation numbers of ϕ.

Denote by $\mathcal{WAPC}(R)$ the set of all such functions .

Lemma 2.2 *[24, 25] For every* $\epsilon > 0$, *the set of all* $\epsilon-$ *translation numbers of a function* $\phi \in \mathcal{AP}(R)$, *contains a relatively dense set of numbers* $\{m_i\gamma\}, m_i \in Z, i \in Z$, *where* γ *can be taken arbitrarily small.*

Lemma 2.3 *For every function* $\phi \in \mathcal{AP}(R)$, *and a number* $\epsilon > 0$, *there exists a relatively dense set* Ω *of* $\epsilon-$ *translation numbers of* ϕ *such that if* $\omega \in \Omega$ *then* $|\omega - i| < \epsilon$ *for some* $i \in Z$.

Proof. By Lemma 2.2 there exists a relatively dense set $\Omega_1 = \{m^i\gamma\}, 0 < \gamma < \frac{\epsilon}{2}, m^i \in Z, i \in Z$, of $\frac{\epsilon}{2}-$ translation numbers of a function ϕ.

Let $L > 0$ be a real number such that $(a, a+L) \cap \Omega_1 \neq \emptyset$ for all $a \in R$.

One can easily see that there exists a relatively dense set of numbers $\Omega_2 = \{\bar{m}^i\gamma\}, \bar{m}^i \in Z, i \in Z$, such that $|\bar{m}^i\gamma - i| < \frac{\epsilon}{2}, i \in Z$.

We can take $L > 1+\frac{\epsilon}{2}$, such that every interval of length L has a nonempty intersection with Ω_1 and Ω_2. Since for every couple of numbers $m^k\gamma \in \Omega_1$ and $\bar{m}^i\gamma \in \Omega_2$, belonging to the same interval with length L it is true that $|m^k\gamma - \bar{m}^i\gamma| < L$, all possible differences $m - \bar{m}$ of such numbers can be equal only to finite values, let say $n_j, j = \overline{1,p}$. Choose for every $j = \overline{1,p}$ a couple of representatives $\{m^j\gamma, \bar{m}^j\gamma\} \in \Omega_1 \times \Omega_2$ such that $m^j - \bar{m}^j = n_j$, and fix the set of all such pairs. Denote

$$\lambda = \max_{j=\overline{1,p}} |m^j\gamma|, \; L_1 = L + 2\lambda.$$

Let $J = (a, a+L_1)$ be an interval with length L_1. Then $J_1 = (a+\lambda, a+\lambda+L) \subset J$, and in J_1 there exist $\{m^k\gamma, \bar{m}^i\gamma\} \in \Omega_1 \times \Omega_2$. It is obvious that exists a number n_j such that $m^k - \bar{m}^i = n_j$, and, hence, $m^k - \bar{m}^i = m^j - \bar{m}^j, m^k - m^j = \bar{m}^i - \bar{m}^j$. Denote $\omega = (m^k - m^j)\gamma$. Obviously $\omega \in J$, and ω is $\epsilon-$ translation number of ϕ. Now, we have that

$$|\omega - (i-j)| = |(\bar{m}^i - \bar{m}^j)\gamma - (i-j)| \leq |\bar{m}^i\gamma - i| + |\bar{m}^j\gamma - j| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

That is, set Ω is relatively dense. The Lemma is proved.

Lemma 2.4 *If* $\phi \in \mathcal{AP}(R)$, *then* $\phi([t+i]) \in \mathcal{WAPC}(R)$ *for every* $i \in Z$.

Proof. We prove the Lemma only for function $\phi([t]), (i = 0)$, as for the general case we have that $\phi([t+i]) = \phi([t]+i), i \in Z$.

Fix $\epsilon > 0$. Without loss of generality we assume that $\epsilon < \frac{1}{2}$. Since $\phi \in C_b(R)$ there exists $\delta = \delta(\frac{\epsilon}{2})$ such that $||\phi(t_1) - \phi(t_2)|| < \frac{\epsilon}{2}$ if $|t_1 - t_2| < \delta$. Denote $\epsilon_1 = \min(\frac{\epsilon}{2}, \delta)$. By using Lemma 2.3 one can show that there exists a respectively dense set Ω of ϵ_1- translation numbers of ϕ such that if $\omega \in \Omega$ then $|\omega - i| < \epsilon_1$ for some $i \in Z$. Fix $t \in R$ such that $|t-j| > \epsilon, j \in Z$. Then, there exists $k \in Z$ such that $k + \epsilon < t < k - \epsilon + 1$, and, hence, $k + \frac{\epsilon}{2} + i < t + \omega < k + 1 + i - \frac{\epsilon}{2}$. Thus,

$$||\phi([t+\omega]) - \phi([t])|| = ||\phi(k+i) - \phi(k)|| \le$$
$$||\phi(k+i) - \phi(k+\omega)|| + ||\phi(k+\omega) - \phi(k)|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Lemma is proved.

Using the method of common translations and the last Lemma one can prove that the following assertion is valid.

LEMMA 2.5 *Let $\phi, A_j \in \mathcal{AP}(R), j = \overline{-N, N}$, then $\sum_{j=-N}^{N} A_j(t)\psi([t+j]) \in \mathcal{WAPC}(R)$.*

Let T be an operator on $\mathcal{AP}(R)$ such that

$$T\phi = \int_{-\infty}^{t} X(t,s)F_\phi(s)ds. \tag{8}$$

Denote $l = \max\{\sup_R ||A_j(t)||, j = \overline{-N, N}\}$. We assume that l is sufficiently small so that

(C_3) $\frac{bl(2N+1)}{a} < 1$.

THEOREM 2.1 *Assume that $(C_1)-(C_3)$ are valid. Then there exists a unique almost periodic solution $\xi_0(t)$ of (1).*

Proof. By using Lemma 2.5 one can prove that $F_\phi(t) \in \mathcal{WAPC}(R)$, and on the basis of Lemma 1.1 and method of common translation numbers one can verify that there is a respectively dense set $\Gamma \subset R$ of $\epsilon-$ translation numbers for F_ϕ such that if $\tau \in \Gamma$ then (5) is valid for $t \ge s$ and

$$\begin{aligned}
&||T\phi(t+\tau) - T\phi(t)|| = ||\int_{-\infty}^{t} X(t+\tau,s)F_\phi(s)ds - \int_{-\infty}^{t} X(t,s)F_\phi(s)ds|| \\
&\le \int_{-\infty}^{t} ||X(t+\tau,s+\tau)||||F_\phi(s+\tau) - F_\phi(s)||ds + \int_{-\infty}^{t} ||X(t+\tau,s+\tau) \\
&-X(t,s)||||F_\phi(s)||ds \le \int_{-\infty}^{t} b\exp(-a(t-s))\epsilon ds \\
&+\sum_{i<t}\int_{i-\epsilon}^{i+\epsilon} b\exp(-a(t-s))2M_\phi ds + \int_{-\infty}^{t} \epsilon\exp(-\frac{a}{2}(t-s))M_\phi ds \\
&= [\frac{b+2M_\phi}{a} + \frac{2bM_\phi\exp(a(1+\epsilon))}{1-\exp(-a)}]\epsilon.
\end{aligned} \tag{9}$$

where $M_\phi = \sup_t ||F_\phi||$. Thus $T\phi \in \mathcal{AP}(R)$. Moreover if $\phi_1, \phi_2 \in \mathcal{AP}(R)$ then for every $t \geq t_0$

$$||T\phi_1(t) - T\phi_2(t)|| = || \int_{-\infty}^t X(t,s)F_{\phi_1}(s)ds - \int_{-\infty}^t X(t,s)F_{\phi_2}(s)ds|| \leq$$
$$\int_{-\infty}^t ||X(t,s)||||F_{\phi_1}(s) - F_{\phi_2}(s)||ds \leq$$
$$\int_{-\infty}^t b\exp(-a(t-s))lk||\phi_1 - \phi_2||_0 ds = \frac{bl(2N+1)}{a}||\phi_1 - \phi_2||_0.$$

The last inequality implies that

$$||T\phi_1(t) - T\phi_2(t)||_0 \leq \frac{bl(2N+1)}{a}||\phi_1 - \phi_2||_0, \tag{10}$$

and the condition (C_3) follows that the operator $T : \mathcal{AP}(R) \to \mathcal{AP}(R)$ is contractive. Thus there exists a unique fixed point $\xi_0 \in \mathcal{AP}(R)$ of T which is a solution of (1).
Theorem is proved.

2.2 Exponential stability of the almost periodic solution

Let $x(t)$ be a solution of (1) such that:

(D_1)

$$x(j) = \pi^j \in R, \text{ if } \quad i = -N, -N+1, \ldots, 0,$$

(D_2) $x(t)$ satisfies the equation (1) for all $t \geq 0$.

Definition 2.3 The solution $\xi_0(t)$ is called uniformly exponential stable if there exists a number $\alpha \in R, \alpha > 0$, such that for every $\epsilon > 0$ there exists a number $\delta = \delta(\epsilon)$ such that an inequality $\max_{-N \leq j \leq 0} ||\pi^j - \xi_0(j)|| < \delta$ implies that there exists a unique solution $x(t)$ which satisfies conditions $(D_1), (D_2)$ and $||x(t) - \xi_0(t)|| < \epsilon \exp(-\alpha t)$ for all $t \geq 0$.

Remark 2.1 Analyzing the definitions of Lyapunov stability for different types of equations it is possible to stress the following two generic conditions. The first one is the closeness of initial values (initial functions) for a solution whose stability is tested and a neighbour solution. The second condition is that the process after the initial moment of time should be governed only by a differential equation. These circumstances are described by conditions (D_1) and (D_2). And evidently, (D_2) generates the condition for ignoring the positive values of j to define the stability conditions.

Suppose additionally that for (1) the following conditions are valid:

(C_4) $A_j(t) = 0, t \in R, j = 2, N$;

(C_5) $l < \frac{1}{b}\min\{1, \frac{a}{N+2}, \frac{\sum_{j=-N}^{1} \mathrm{e}^{-\sigma j}}{a+\sigma}\}$.

The following system is given

$$\frac{dv}{dt} = A(t)v + \sum_{j=-N}^{1} A_j(t)v([t+j]), \tag{11}$$

with the initial condition

$$v(j) = v^j, j = \overline{-N, 0}. \tag{12}$$

where $v^j, j = \overline{-N, 0}$, is a sequence of $n-$vectors.

Consider the following auxiliary assertion.

Lemma 2.6 *Assume that $(C_1)-(C_4)$ are valid, and $bl < 1$. Then initial value problem (11),(12) has a unique solution for $t \geq 0$.*

Proof. Consider the interval $[0, 1]$. We have that

$$v(t) = X(t,0)v(0) + \int_0^t X(t,s)[A_1(s)v([s+1]) + \sum_{j=-N}^{0} A_j(t)v([s+j])]ds =$$

$$X(t,0)v(0) + \int_0^t X(t,s)A_1(s)v(1)ds + \int_0^t X(t,s)\sum_{j=-N}^{0} A_j(t)v^j ds.$$

Consequently, for existence and uniqueness of the solution on interval $[0, 1]$, the equation

$$w(t) = X(t,0)v(0) + \int_0^t X(t,s)A_1(s)\eta ds + \int_0^t X(t,s)\sum_{j=-N}^{0} A_j(t)v^j ds,$$

where $\eta \in R^n$ should be solved with respect to w and η such that $w(1) = \eta$. Consider the sequence $w_k(t), k \geq 0$, such that $w_0(t) = v(0)$ and for $k \geq 0$,

$$w_{k+1}(t) = X(t,0)v(0) + \int_0^t X(t,s)A_1(s)w_k(1)ds + \int_0^t X(t,s)\sum_{j=-N}^{0} A_j(t)v^j ds.$$

One can easily see that, if $bl < 1$, the sequence is convergent to the unique solution $v(t)$ on $[0, 1]$. Inductively we can prove the assertion for all $t \geq 0$. The lemma is proved.

The following theorem is valid.

THEOREM 2.2 *Assume that $(C_1)-(C_5)$ are valid. Then the almost periodic solution $\xi_0(t)$ of (1) is uniformly exponential stable.*

Proof. Consider a solution $x(t), x(j)=\pi^j, j=\overline{-N,0}, t\geq 0$, of (1). One can see that $v(t)=x(t)-\xi_0(t)$ is a solution of problem (11),(12) with

$$v^j=\pi^j-\xi(j), j=\overline{-N,0}. \tag{13}$$

By the last lemma solution $x(t)$ exists and unique. Thus we can reduce the problem of stability of $\xi_0(t)$ to the problem of the stability of the zero solution $v=0$ of (11).

Let us fix $\epsilon>0$ and denote $K(l,\delta)=\frac{b\delta}{1-bl\frac{1}{a+\sigma}\sum_{j=-N}^{1}\mathrm{e}^{-\sigma j}}$, where $\delta,\sigma\in R, 0<\delta, 0<\sigma<a$. Take δ so small such that $K(l,\delta)<\epsilon$. Fix the sequence v^j such that $\max_{j=\overline{-N,0}}||v^j||<\delta$ and let

$$\Psi_\pi=\{\phi\in C_b[-N,+\infty)|\phi(j)=v^j, j=\overline{-N,0}, ||\phi(t)||\leq K(l,\delta)\exp(-\sigma t),\},$$

where $C_b[-N,+\infty)$ is composed of restrictions of all functions from $C_b(R)$ on $[-N,+\infty)$. On Ψ_π define an operator Π such that if $\phi\in\Psi_\pi$ then

$$\Pi\phi=\begin{cases}\phi(t), & \text{if} \quad t\in[-N,0];\\ X(t,0)v^0+\int_0^t X(t,s)\sum_{j=-N}^{1}A_j(s)\phi([s+j])ds & \text{otherwise}.\end{cases}$$

We shall show that $\Pi:\Psi_\pi\to\Psi_\pi$. Really, for $t\geq 0$ it is true that

$$||\Pi\phi||\leq b\exp(-at)\delta+\int_0^t b\exp(-a(t-s))lK(l,\delta)\sum_{j=-N}^{1}\exp(-\sigma[s+j]ds\leq$$

$$\exp(-\sigma t)b[\delta+lK(l,\delta)\frac{\sum_{j=-N}^{1}\mathrm{e}^{-\sigma j}}{a+\sigma}]=K(l,\delta)\exp(-\sigma t).$$

Differentiating $\Pi\phi$ on $[0,\infty)\backslash Z$ it is easy to show that $[\Pi\phi]'$ is uniformly bounded on $[0,\infty)\backslash Z$ function and, hence, $\Pi\phi$ is a uniformly continuous function.

Let $\phi_1,\phi_2\in\Psi_\pi$. Then

$$||\Pi\phi_1-\Pi\phi_2||\leq\int_0^t b\exp(-a(t-s))l(N+2)||\phi_1-\phi_2||_1ds\leq\frac{bl(N+2)}{a}||\phi_1-\phi_2||_1,$$

where $||\phi||_1=\sup_{t\geq 0}||\phi(t)||$.

From $(C5)$ it follows that there is a unique fixed point v of the operator $\Pi:\ \Psi_\pi\to\Psi_\pi$ which is a solution $v(t)$ of (11), and $x(t)=v(t)+\xi_0(t)$ is a unique solution of (1) which satisfies conditions $D_1), D_2)$. Theorem is proved.

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