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Asymptotic behavior of solutions of differential equations with piecewise constant arguments

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Abstract

The main goal of the work is to obtain sufficient conditions for the asymptotic equivalence of a linear system of ordinary differential equations and a quasilinear system of differential equations with piecewise constant argument.

Keywords: Asymptotic equivalence; Quasilinear systems; Piecewise constant argument of generalized type

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1. Introduction and preliminaries

Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$ be the sets of all integers, natural and real numbers, respectively. Denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$. Fix two real valued sequences θ_i, ζ_i, $i \in \mathbb{Z}$, such that $\theta_i < \theta_{i+1}$, $\theta_i \le \zeta_i \le \theta_{i+1}$ for all $i \in \mathbb{Z}$, $|\theta_i| \to \infty$ as $|i| \to \infty$.

In the present work we shall consider the equations

$$z'(t) = Cz(t) + f(t, z(t), z(\gamma(t))), \tag{1}$$

and

$$x'(t) = Cx(t), \tag{2}$$

where $x, z \in \mathbb{R}^n$, $t \in \mathbb{R}$, C is a constant $n \times n$ real valued matrix, $f \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ is a real valued $n \times 1$ function, $\gamma(t) = \zeta_i$, if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$. The following assumptions will be needed throughout the work:

(C1) there exists a number $L > 0$ such that

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \le L(\|x_1 - x_2\| + \|y_1 - y_2\|) \tag{3}$$

for all $t \in \mathbb{R}^n$, $x_j, y_j \in \mathbb{R}^n$, $j = 1, 2$, and the condition

$$f(t, 0, 0) = 0, t \in \mathbb{R}, \tag{4}$$

is satisfied;

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(C2) there exists a number $\bar{\theta} > 0$ such that $\theta_{i+1} - \theta_i \leq \bar{\theta}$, $i \in \mathbb{Z}$;

and condition (4) implies that (1) admits the zero solution.

The theory of differential equations with piecewise constant argument (EPCA) was initiated in [4,5], and has been developed intensively in the last few decades. For a brief summary of the theory, the reader is referred to the book by Wiener [10]. System (1) is a differential equation with piecewise constant argument of generalized type (EPCAG), introduced in [1]; and it is more general than an EPCA.

One can easily see that Eq. (1) has the form of a functional differential equation

$$z'(t) = Cz(t) + f(t, z(t), z(\zeta_i)), \tag{5}$$

if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$.

Definition 1.1. A function $z(t) \in C(\mathbb{R})$ is a solution of (1) if:

(i) the derivative $z'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of the points θ_i, $i \in \mathbb{Z}$, where the one-sided derivatives exist;
(ii) the equation is satisfied for $z(t)$ on each interval (θ_i, θ_{i+1}), $i \in \mathbb{Z}$, and it holds for the right derivative of $z(t)$ at the points θ_i, $i \in \mathbb{Z}$.

Definition 1.2 (*[9]*). A homeomorphism $\mathcal{H}$ between the sets of solutions $x(t)$ and $z(t)$ is called an asymptotic equivalence if $z(t) = \mathcal{H}(x(t))$ implies that $x(t) - z(t) \to 0$ as $t \to \infty$.

Apparently, the problem of asymptotic equivalence has not been considered for EPCA (EPCAG) yet. Results closest to our investigation can be found in recent publications [6–8], and in the book [10], where the asymptotic and global stability of solutions of EPCA has been addressed.

In the following lemma a correspondence between points $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$ and the solutions of (1) in the sense of Definition 1.1 is established. Using this result we can say that the definition of the IVP for our system is similar to that for ordinary differential equations, although it is an equation with a deviating argument. The proof of the assertion is very similar to that of Lemma 3.1 [1].

Lemma 1.1. *A function $z(t) = z(t, t_0, z_0)$, $z(t_0) = z_0$, where t_0 is a fixed real number, is a solution of* (1) *in the sense of Definition* 1.1 *if and only if it is a solution of the following integral equation:*

$$z(t) = \mathrm{e}^{C(t-t_0)} z_0 + \int_{t_0}^{t} \mathrm{e}^{C(t-s)} f(s, z(s), z(\gamma(s)))\mathrm{d}s. \tag{6}$$

There exist positive numbers M and m such that $m \leq \|\mathrm{e}^{C(t-s)}\| \leq M$ if $t, s \in [\theta_i, \theta_{i+1}]$ for all $i \in \mathbb{Z}$. From now on we make the assumption:

(C3) $ML\bar{\theta}\mathrm{e}^{ML\bar{\theta}} < 1$, $2ML\bar{\theta} < 1$, $M^2 L\bar{\theta}\{\frac{ML\bar{\theta}\mathrm{e}^{ML\bar{\theta}}+1}{1-ML\bar{\theta}\mathrm{e}^{ML\bar{\theta}}} + ML\bar{\theta}\mathrm{e}^{ML\bar{\theta}}\} < m$.

Lemma 1.2. *Assume that conditions* (C1)–(C3) *are fulfilled, and fix $i \in \mathbb{Z}$. Then for every $(\xi, z_0) \in [\theta_i, \theta_{i+1}] \times \mathbb{R}^n$ there exists a unique solution $z(t) = z(t, \xi, z_0)$ of* (5) *on $[\theta_i, \theta_{i+1}]$.*

Theorem 1.1. *Assume that conditions* (C1)–(C3) *are fulfilled. Then for every $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$ there exists a unique solution $z(t) = z(t, t_0, z_0)$ of* (1) *in the sense of Definition* 1.1 *such that $z(t_0) = z_0$.*

The last two assertions can be verified in exactly the same way as Lemma 1.1 and Theorem 1.1 from [2].

2. Main results

In this section we consider the main result of the work, a theorem about the asymptotic equivalence of systems (1) and (2). The theorem is a development of V. Yakubovich's result [9,11]. Similar results for impulsive and ordinary differential equations are obtained in [2,3]. Let $\alpha = \min_j \Re\lambda_j$ and $\beta = \max_j \Re\lambda_j$, where $\Re\lambda_j$ denotes the real part of the eigenvalue λ_j of the matrix C. Let m_α and m_β be the maxima of the orders of Jordan cells corresponding to eigenvalues with real part equal to α and β, respectively. Clearly, there exist constants κ_1, κ_2 such that $\|\mathrm{e}^{Ct}\| \leq \kappa_1 t^{m_\beta - 1}\mathrm{e}^{\beta t}$ and $\|\mathrm{e}^{-Ct}\| \leq \kappa_2 t^{m_\alpha - 1}\mathrm{e}^{-\alpha t}$ for all $t \in R_+ = [0, \infty)$. The following conditions are to be assumed:

(C4) $\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq \eta(t)(\|x_1 - x_2\| + \|y_1 - y_2\|)$ for all $(t, x_1), (t, x_2)(t, y_1), (t, y_2) \in R_+ \times R^n$, and for some nonnegative function $\eta(t) \leq L$ defined on R_+, the constant L is the same as in (C1);

(C5) $l_0 := \int_0^\infty t^{m_\beta + m_\alpha - 2} \mathrm{e}^{(\beta-\alpha)t} \eta(t)\,\mathrm{d}t < \infty$.

The following lemma can be easily proved by direct substitution.

Lemma 2.1. *If $z(t)$ is a solution of (1), then there is a solution $u(t)$ of the equation*

$$u' = \mathrm{e}^{-Ct} f(t, \mathrm{e}^{Ct} u, \mathrm{e}^{C\gamma(t)} u(\gamma(t))) \tag{7}$$

such that

$$z(t) = \mathrm{e}^{Ct} u(t). \tag{8}$$

Conversely, if $u(t)$ is a solution of (7) then $y(t)$ in (8) is a solution of (2).

Lemma 2.2. *If conditions* (C1)–(C5) *are valid, then every solution of (7) is bounded on R_+ and for each solution u of (7) there exists a constant vector $c_u \in \mathbb{R}^n$ such that $u(t) \to c_u$ as $t \to \infty$.*

Proof. Let $u(t) = u(t, t_0, u_0)$ denote a solution of (7) satisfying $u(t_0) = u_0$, $t_0 \geq 0$. By Theorem 1.1 and Lemma 2.1 the solution $u(t)$ exists on $\mathbb{R}$ and is unique. Like for Lemma 1.1, we can verify that

$$u(t) = u_0 + \int_{t_0}^t \mathrm{e}^{-Cs} f(s, \mathrm{e}^{Cs} u(s), \mathrm{e}^{C\gamma(s)} u(\gamma(s)))\,\mathrm{d}s, \quad t \geq t_0.$$

By using (C4) and $f(t, 0, 0) = 0$, we see that

$$\|u(t)\| \leq \|u_0\| + \int_{t_0}^t \kappa_2 \kappa_1 s^{m_\alpha - 1} \mathrm{e}^{-\alpha s} \eta(s)[s^{m_\beta - 1} \mathrm{e}^{\beta s} \|u(s)\| + \gamma(s)^{m_\beta - 1} \mathrm{e}^{\beta\gamma(s)} \|u(\gamma(s))\|]\mathrm{d}s, \quad t \geq t_0.$$

One can find a positive number K and an integer j such that

$$\max_{|s| \leq \bar{\theta}} \frac{\zeta_i^{m_\beta - 1} \mathrm{e}^{\beta\zeta_i}}{(\zeta_i + s)^{m_\beta - 1} \mathrm{e}^{\beta(\zeta_i + s)}} < K, \quad i \geq j.$$

Conditions (C2) and (C5) imply that the integer j can be taken sufficiently large that for a positive number $l < 1$ the following inequalities hold:

$$(1 + K)\kappa_2 \kappa_1 \int_{\theta_i}^{\theta_{i+1}} t^{m_\beta + m_\alpha - 2} \mathrm{e}^{(\beta-\alpha)t} \eta(t)\,\mathrm{d}t \leq l, \quad i \geq j. \tag{9}$$

Using (9) and the expression

$$u(t) = u(\theta_i) + \int_{\theta_i}^t \mathrm{e}^{-Cs} f(s, \mathrm{e}^{Cs} u(s), \mathrm{e}^{C\gamma(s)} u(\gamma(s)))\,\mathrm{d}s$$

we can easily find that $\|u(\zeta_i)\| \leq \frac{1}{1-l} \|u(\theta_i)\|$ for all $i \geq j$. Define

$$M_1 = \|u_0\| + \int_0^{\theta_j} \kappa_2 \kappa_1 s^{m_\alpha - 1} \mathrm{e}^{-\alpha s} \eta(s)[s^{m_\beta - 1} \mathrm{e}^{\beta s} \|u(s)\| + \gamma(s)^{m_\beta - 1} \mathrm{e}^{\beta\gamma(s)} \|u(\gamma(s))\|]\,\mathrm{d}s.$$

We have that

$$\|u(t)\| \leq M_1 + \kappa_2 \kappa_1 \int_{\theta_j}^t s^{m_\beta + m_\alpha - 2} \mathrm{e}^{(\beta-\alpha)s} \eta(s) \|u(s)\| + \frac{1 + K}{1 - l} \|u(\beta(s))\|\,\mathrm{d}s, \quad t \geq \theta_j,$$

where $\beta(t) = \theta_i$, if $t \in [\theta_i, \theta_{i+1})$, $i \in \mathbb{Z}$. Denote $\|u\|_t = \sup_{\xi \in [\theta_j, t]} \|u(\xi)\|$. Let us first show that

$$\|u\|_t \leq M_1 + \kappa_2 \kappa_1 \int_{\theta_j}^t s^{m_\beta + m_\alpha - 2} \mathrm{e}^{(\beta-\alpha)s} \eta(s) \frac{2 - l + K}{1 - l} \|u(\xi)\|_s\,\mathrm{d}s, \quad t \geq \theta_j. \tag{10}$$

Since $\theta_j \leq \beta(s) \leq s$ for $s \geq \theta_j$, we have that $\|u(\beta)\|_t = \sup_{[\theta_j,t]}\|u(\beta(\xi))\| = \sup_{[\theta_j,\beta(t)]}\|u(\xi)\| \leq \sup_{[\theta_j,t]}\|u(\xi)\| = \|u(\xi)\|_t$. Hence,

$$\|u(t)\| \leq M_1 + \kappa_2\kappa_1 \int_{\theta_j}^{t} s^{m_\beta+m_\alpha-2} e^{(\beta-\alpha)s} \eta(s) \frac{2-l+K}{1-l} \|u(\xi)\|_s \, ds, \quad t \geq \theta_j.$$

If $\|u(t)\| = \|u\|_t$ for a given $t \geq \theta$, then inequality (10) is valid. Suppose that $\|u(t)\| < \|u\|_t$ for a given t. Then, by the definition of the sup-norm, there is a moment $\tilde{t} \in [\theta_j, t]$ such that $\|u(\tilde{t})\| = \|u\|_t$. Hence,

$$\begin{aligned}\|u\|_t = \|u(\tilde{t})\| &\leq M_1 + \kappa_2\kappa_1 \int_{\theta_j}^{\tilde{t}} s^{m_\beta+m_\alpha-2} e^{(\beta-\alpha)s} \eta(s) \frac{2-l+K}{1-l} \|u(\xi)\|_s \, ds \\ &\leq M_1 + \kappa_2\kappa_1 \int_{\theta_j}^{t} s^{m_\beta+m_\alpha-2} e^{(\beta-\alpha)s} \eta(s) \frac{2-l+K}{1-l} \|u(\xi)\|_s \, ds,\end{aligned}$$

as $\tilde{t} \leq t$. So, (10) is valid. Now, setting $\psi(t) = \|u\|_t$ and applying the Gronwall–Bellman lemma to

$$\psi(t) \leq M_1 + \kappa_2\kappa_1 \int_{\theta_j}^{t} s^{m_\beta+m_\alpha-2} e^{(\beta-\alpha)s} \eta(s) \frac{2-l+K}{1-l} \psi(s) \, ds, \quad t \geq \theta_j,$$

we obtain that $|u(t)| \leq M$ for all $t \in R_+$, where $M = M_1 e^{\kappa_2\kappa_1 l_0 \frac{2-l+K}{1-l}}$.

To prove the second part of the theorem, we first note that

$$\left| \int_{t_0}^{t} e^{-Cs} f(s, e^{Cs}u(s), e^{C\gamma(s)}u(\gamma(s))) \, ds \right| \leq M\kappa_2\kappa_1(1+K) \int_{0}^{\infty} t^{m_\beta+m_\alpha-2} e^{(\beta-\alpha)t} \eta(t) dt < \infty.$$

So we may define

$$c_u = u_0 + \int_{t_0}^{\infty} e^{-Cs} f(s, e^{Cs}u(s), e^{C\gamma(s)}u(\gamma(s))) \, ds. \tag{11}$$

It follows that

$$u(t) = c_u - \int_{t}^{\infty} e^{-Cs} f(s, e^{Cs}u(s), e^{C\gamma(s)}u(\gamma(s))) ds,$$

which completes the proof. □

Theorem 2.1. *If conditions* (C1)–(C5) *are valid, then every solution* $y(t)$ *of* (1) *possesses an asymptotic representation of the form* $z(t) = e^{Ct}[c + o(1)]$, *where* $c \in \mathbb{R}^n$ *is a constant vector and for a solution* $u(t)$ *of* (7),

$$o(1) = -\int_{t}^{\infty} e^{-Cs} f(s, e^{Cs}u(s), e^{C\gamma(s)}u(\gamma(s))) \, ds.$$

Proof. The proof follows from Lemmas 2.1 and 2.2.

Theorem 2.2. *Assume that conditions* (C1)–(C5) *are fulfilled, and*

(C6) $\lim_{t\to\infty} \int_t^\infty (s-t)^{m_\alpha-1} s^{m_\beta-1} e^{\alpha(t-s)} e^{\beta s} \eta(s) ds = 0.$

Then (1) *and* (2) *are asymptotically equivalent.*

Proof. In view of Lemma 2.2,

$$\begin{aligned} z(t) &= e^{Ct}\left[c_u - \int_{t}^{\infty} e^{-Cs} f(s, e^{Cs}u(s), e^{C\gamma(s)}u(\gamma(s))) ds\right] \\ &= x(t) - \int_{t}^{\infty} e^{C(t-s)} f(s, e^{Cs}u(s), e^{C\gamma(s)}u(\gamma(s))) ds, \end{aligned}$$

where $x(t) = \mathrm{e}^{Ct}c_u$ is a solution of (2) and $u(t) = u(t, t_0, u_0)$ is a solution of (7). It is clear that a given u_0 results in a homeomorphism between $x(t)$ and $y(t)$. Indeed, we have that $z_0 = \mathrm{e}^{Ct_0}u_0$ and $x_0 = \mathrm{e}^{Ct_0}c_u$. Therefore, it is sufficient to show that for some t_0, expression (11) defines a homeomorphism between $u_0 \in \mathbb{R}^n$ and $c_u \in \mathbb{R}^n$. Let us take $t_0 = \theta_i$, where i is sufficiently large to satisfy (9), and moreover to satisfy

$$\int_{\theta_i}^{\infty} t^{m_\beta+m_\alpha-2}\mathrm{e}^{(\beta-\alpha)t}\eta(t)\,\mathrm{d}t < l_1, \tag{12}$$

where the fixed positive number l_1 is such that $l_1 < 1$ and, moreover, $l_1(1+K)\kappa_2\kappa_1\mathrm{e}^{\kappa_2\kappa_1 l_1\frac{2-l+K}{1-l}} < 1$. Let us fix $u_0^1, u_0^2 \in \mathbb{R}^n$ and $u_j(t) = u(t, t_0, u_0^j)$, $j = 1, 2$. We have that

$$\begin{aligned}\|u_1(t) - u_2(t)\| \le{}& \|u_0^1 - u_0^2\| \\ &+ \int_{t_0}^{t} \mathrm{e}^{-Cs}[f(s, \mathrm{e}^{Cs}u_1(s), \mathrm{e}^{C\gamma(s)}u_1(\gamma(s))) - f(s, \mathrm{e}^{Cs}u_2(s), \mathrm{e}^{C\gamma(s)}u_2(\gamma(s)))]\mathrm{d}s.\end{aligned}$$

Applying the method used in Lemma 2.2 to the last inequality, one can obtain that $\|u_1(t) - u_2(t)\| \le \|u_0^1 - u_0^2\|\mathrm{e}^{\kappa_2\kappa_1 l_1\frac{2-l+K}{1-l}}$, $t \ge t_0$. Define

$$c_u^j = u_0^j + \int_{t_0}^{\infty} \mathrm{e}^{-Cs}f(s, \mathrm{e}^{Cs}u_j(s), \mathrm{e}^{C\gamma(s)}u_j(\gamma(s)))\,\mathrm{d}s, \quad j = 1, 2. \tag{13}$$

It is easy to obtain the following inequalities: $(1 - l_1(1+K)\kappa_2\kappa_1\mathrm{e}^{\kappa_2\kappa_1 l_1\frac{2-l+K}{1-l}})\|u_0^1 - u_0^2\| \le \|c_u^1 - c_u^2\| \le (1 + l_1(1+K)\kappa_2\kappa_1\mathrm{e}^{\kappa_2\kappa_1 l_1\frac{2-l+K}{1-l}})\|u_0^1 - u_0^2\|$. Thus, we find that there exists a bi-continuous and one-to-one correspondence between u_0 and c_u. In view of (C6), we also see that $x(t) - z(t) \to 0$ as $t \to \infty$, which completes the proof of the theorem. □

Remark 2.1. In [9], p. 199, and [11] one can find Yakubovich's theorem on the asymptotic equivalence of the quasilinear and linear systems of ordinary differential equations. The sufficient condition for asymptotic equivalence was the inequality

$$\int_{t_0}^{\infty} s^{m_\beta+p-2}\mathrm{e}^{\beta s}\eta(s)\mathrm{d}s < \infty, \tag{14}$$

where p is the maximum of the orders of Jordan cells corresponding to eigenvalues with zero real parts, provided they exist, and $p = 1$, otherwise.

One can easily see that condition (14) is stronger than (C5), (C6) if $\alpha > 0$. The next example illustrates this fact for EPCAG.

Example 2.1. Consider the second-order equation

$$y'' - 3y' + 2y + b(t)\sin^2\left(y\left(\left[t + \frac{1}{2}\right]\right)\right) = 0 \tag{15}$$

where $[\cdot]$ is the greatest integer function, $b(t)$ is a continuous function defined on R_+, and we assume that

$$|b(t)| < \frac{K_1}{1+t}\mathrm{e}^{-2t}, \qquad \text{for all } t \in R_+,$$

where K_1 is a positive number. One can see that the piecewise constant function is a γ function with sequences $\theta_i = i, i \in \mathbb{Z}, \zeta_i = i + 1/2, i \in \mathbb{Z}$.

We shall prove, using the results of the work, that Eq. (15) is asymptotically equivalent to the equation

$$x'' - 2x = 0, \tag{16}$$

which has solutions $x(t) = c_1\mathrm{e}^{t} + c_1\mathrm{e}^{2t}$.

Eq. (15) can be transformed to a system of form (1) with the matrix

$$C = \begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix}$$

and with the vector-function $f = (0, b)(t)\sin^2(z_1([t + \frac{1}{2}]))$. One can easily find that $\alpha = 1, \beta = 2, m_\alpha = p = 1, m_\beta = 1$, and $\eta(t)$ can be chosen to equal $\frac{K_1}{1+t}e^{-2t}$. Clearly, conditions (C1), (C2) and (C4) are valid. Condition (C3) is fulfilled if K_1 is sufficiently small. The constant l_0 in (C5) is equal to K_1. Let us check if (C6) holds. We have that

$$\int_t^\infty (s-t)^{m_\alpha - 1} s^{m_\beta - 1} e^{\alpha(t-s)} e^{\beta s} \eta(s) ds = \int_t^\infty e^{(t-s)} e^{2s} \frac{K_1}{1+s} e^{-2s} ds \leq \frac{K_1}{1+t}.$$

The last inequality implies that (C6) is fulfilled. Thus, the asymptotic behavior of the solutions of (15) is the same as that of the solutions of the linear equation (16). Now, let us turn to (14). One can see that the integral

$$\int_{t_0}^\infty s^{m_\beta + p - 2} e^{\beta s} \eta(s) ds = \int_{t_0}^\infty \frac{K_1}{1+s} ds$$

does not converge. That is, Yakubovich's result is not applicable.

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