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QUASILINEAR RETARDED DIFFERENTIAL EQUATIONS WITH FUNCTIONAL DEPENDENCE ON PIECEWISE CONSTANT ARGUMENT

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Abstract. We introduce a new class of differential equations, retarded differential equations with functional dependence on piecewise constant argument, $RFDEPCA$, and focus on quasilinear systems. Formulation of the initial value problem, bounded solutions, periodic and almost periodic solutions, their stability are under investigation. Illustrating examples are provided.

1. **Introduction and preliminaries.** Differential equations with piecewise constant argument of generalized type (EPCAG) were introduced in [2]-[4], and then investigated in [5]-[13]. Extended information about these systems can be found in book [5]. They contains as a subclass, differential equations with piecewise constant argument (EPCA), [35]. In paper [2], we generalized the argument-functions as well as proposed to reduce investigation of EPCAG to integral equations. Due to this innovation, it is possible, now, to analyze essentially non-linear systems. That is, systems non-linear with respect to values of solutions at the discrete moments of time, where the argument changes its constancy. While the main and unique method for EPCA is reduction to discrete equations and, hence, only those equations are considered, where values of solutions at discrete moments appear linearly [35]. Thus, we have deepen the analysis insight significantly. Our proposals were used in papers [5]-[9], [12, 13, 24] to develop the theory, and in [10, 11, 15] for applications.

Recently, in papers [27],[29]-[33],[36, 37], delay differential equations with piecewise constant argument have been investigated and interesting problems mainly related to existence of periodic and almost periodic solutions were considered. These investigations continue traditions of founders, [16, 18, 26], of theory of differential equations with piecewise constant argument (EPCA), papers [1, 19, 28, 35] and many others. This means that the constant argument is assumed as a multiple of the greatest integer function, and analysis is made on the basis of reduction to discrete equations. In fact, the simple type of the constancy and the method of investigation strongly relate to each other, since one can reduce a linear system with the greatest integer argument-function to a linear discrete equation easily. In

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the present paper, we have maximally generalized the problem considering an arbitrary piecewise constant argument-function and, it is the first time in literature that functional dependence on piecewise constant argument is considered.

Let us start description of the system considered in the paper. We begin with the argument's constancy. Denote by $\mathbb{Z}, \mathbb{R}$ the sets of all integers and real numbers respectively and by $\theta = \{\theta_i\}, \zeta = \{\zeta_i\}, i \in \mathbb{Z}$, sequences of real numbers such that the first one is strictly ordered, $|\theta_i| \to \infty$ as $|i| \to \infty$, and another one satisfies $\theta_i \leq \zeta_i \leq \theta_{i+1}, i \in \mathbb{Z}$. That is the second sequence, ζ, is not necessary strictly ordered.

We say that a function is of the $\beta-$type, and denote it $\beta(t)$, if $\beta(t) = \theta_i$ for $\theta_i \leq t < \theta_{i+1}, i \in \mathbb{Z}$. One can see that the greatest integer function $[t]$, which is equal to the maximal among all integers less than t, is a $\beta-$type function with $\theta_i = i, i \in \mathbb{Z}$. Similarly, $\beta(t) = 2[t/2]$ if $\theta_i = 2i, i \in \mathbb{Z}$. A sketch of the graph of a $\beta-$type function is seen in Figure 1.

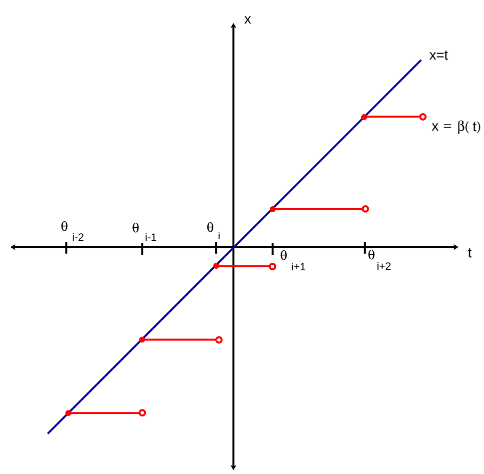

FIGURE 1. The graph of the $\beta(t)$ function.

We say that a function is of the $\gamma-$type, and denote it $\gamma(t)$, if $\gamma(t) = \zeta_i$ for $\theta_i \leq t < \theta_{i+1}, i \in \mathbb{Z}$. One can easily find, for example, that $2[\frac{t+1}{2}]$ is a $\gamma-$type function with $\theta_i = 2i - 1$, $\zeta_i = 2i$. In Figure 2 the typical graph of a $\gamma(t)$ function is seen.

Finally, we say that a function is of $\chi-$type, and denote it $\chi(t)$, if $\chi(t) = \theta_{i+1}$ for $\theta_i \leq t < \theta_{i+1}, \in \mathbb{Z}$. One can see that the function $[t + 1]$, is a $\chi-$type function with $\theta_i = i, i \in \mathbb{Z}$.

It is obvious that the most general among piecewise constant functions considered above is the $\gamma-$type function.

Let us introduce the following functional differential equations,

$$x'(t) = A_0(t)x(t) + A_1(t)x(\gamma(t)) + f(t, x_t, x_{\gamma(t)}), \tag{1}$$

where $t \in \mathbb{R}, x \in \mathbb{R}^n$. In equation (1), terms $x_t, x_{\gamma(t)}$, must be understood in the way used for FDE, [17, 21, 23]. That is, $x_t(s) = x(t+s), x_{\gamma(t)}(s) = x(\gamma(t)+s), s \in [-\tau, 0]$. Let us clarify that the argument function $\gamma(t)$ is of the alternate type. Fix

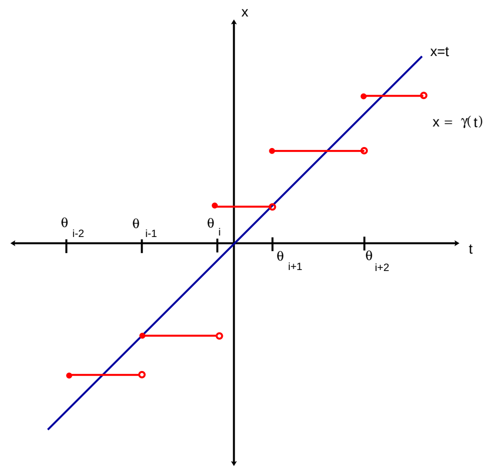

FIGURE 2. The graph of the $\gamma(t)$ function.

$k \in \mathbb{N}$ and consider the function on the interval $[\theta_k, \theta_{k+1})$. Then, the function $\gamma(t)$ is equal to ζ_k. If the argument t satisfies $\theta_k \le t < \zeta_k$, then $\gamma(t) > t$ and it is of advanced type. Similarly, if $\zeta_k < t < \theta_{k+1}$, then $\gamma(t) < t$ and, hence, it is of the delayed type. Consequently, it is worth pointing out that the equation (1) is with *alternate constancy* of argument. If the argument-function is of $\beta-$type or $\chi-$type we shall say about *retarded constancy* and *advanced constancy* of argument, respectively. It is clear that the alternate constancy is the most general among those three. One can easily see that (1) is more general than those considered in [27],[29]-[33],[36, 37]. Indeed, we have introduced equations, where the rate depends on $x_{\gamma(t)}$, instead of $x(\gamma(t))$. That is, (1) is an equation with functional dependence on piecewise constant argument. Examples of functionals with piecewise constant argument are given in Section 6, Example 1.

In this paper we extend both, differential equations with piecewise constant argument and functional differential equations to a new type differential equations. In most general sense, they are functional differential equations, since the right-hand-side is a functional. Despite that newly introduced systems are functional differential equations, they can not be analyzed only trough existing theoretical results for functional differential equations, and their peculiarity requests a new approach which combines features of continuous and discrete dynamics.

Fix a non-negative number $\tau \in \mathbb{R}$, and denote by $\mathcal{C} = C([-\tau, 0], \mathbb{R}^n), n-$ a fixed natural number, the set of all continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$, with the uniform norm $\|\phi\|_0 = \max_{[-\tau,0]} \|\phi\|$, where $||\cdot||$ is the Euclidean norm in $\mathbb{R}^n$.

Consider a subset $\mathcal{D}$ of the product $\mathbb{R} \times \mathcal{C} \times \mathcal{C}$, and introduce a continuous functional $f : \mathcal{D} \to \mathbb{R}^n$. To be concrete, we assume in this paper that $\mathcal{D} = \mathbb{R} \times \mathcal{C} \times \mathcal{C}$. Let $s \in \mathbb{R}$ be a positive number. We denote $\mathcal{C}_s = \{\phi \in \mathcal{C} | \|\phi\|_0 \le s\}$. Let $C_0(\mathbb{R})$ (respectively $C_0(\mathbb{R} \times \mathcal{C}_H \times \mathcal{C}_H)$ for a given $H \in \mathbb{R}$, $H > 0$) be the set of all bounded and continuous functions on $\mathbb{R}$ (respectively on $\mathbb{R} \times \mathcal{C}_H \times \mathcal{C}_H$).

The following assumptions will be needed throughout this article:

(C1) A_0, A_1 are $n \times n$ matrices and their elements are from $C_0(\mathbb{R})$;

(C2) $f \in C_0(\mathbb{R} \times \mathcal{C}_H \times \mathcal{C}_H)$ for each positive $H \in \mathbb{R}$;
(C3) f satisfies the Lipschitz condition in the second and third arguments:

$$\|f(t, \phi_1, \psi_1) - f(t, \phi_2, \psi_2)\| \le L(\|\phi_1 - \phi_2\|_0 + \|\psi_1 - \psi_2\|_0),$$

where (t, ϕ_1, ψ_1) and (t, ϕ_2, ψ_2) are from $\mathcal{D}$, for some positive constant L;
(C4) $\inf_{\mathbb{R}} \|A_1(t)\| > 0$;
(C5) there exist positive numbers $\bar{\theta}, \bar{\zeta} > 0$ such that $\theta_{i+1} - \theta_i \le \bar{\theta}, \zeta_{i+1} - \zeta_i \le \bar{\zeta}, i \in \mathbb{Z}$.

One can find easily that $\bar{\zeta} \le 2\bar{\theta}$.

The system (1) has the form of a functional differential equation,

$$z'(t) = A_0(t)z(t) + A_1(t)z(\zeta_i) + f(t, z_t(t), z_{\zeta_i}), \tag{2}$$

if $t \in [\theta_i, \theta_{i+1}), i \in \mathbb{Z}$. That is, this system has the structure of a functional differential equation with continuous time within intervals $[\theta_i, \theta_{i+1}), i \in \mathbb{Z}$.

2. Existence and uniqueness.

Let us introduce the initial condition for system (1). We will consider the increasing time. Fix a number $\sigma \in \mathbb{R}$, and two functions $\phi, \psi \in \mathcal{C}$. If $\gamma(\sigma) < \sigma$, we shall say that a solution $x(t)$ of equation (1) satisfies the initial condition and write $x(t) = x(t, \sigma, \phi, \psi), t \ge \sigma$, if $x_\sigma(s) = \phi(s), x_{\gamma(\sigma)}(s) = \psi(s), s \in [-\tau, 0]$. In what follows we assume that if the set $[\gamma(\sigma) - \tau, \gamma(\sigma)] \cup [\sigma - \tau, \sigma]$ is connected, then equality $\phi(s) = \psi(s + \sigma - \gamma(\sigma))$ is true for all $s \in [-\tau, \gamma(\sigma) - \sigma]$. If $\gamma(\sigma) \ge \sigma$, then we look for a solution $x(t) = x(t, \sigma, \phi), t \ge \sigma$, such that $x_\sigma(s) = \phi(s), s \in [-\tau, 0]$. Thus, if $\theta_i \le \sigma < \theta_{i+1}$, for some $i \in \mathbb{Z}$, then there are two cases of the initial condition:

- $(IC_1)]$ $x_\sigma(s) = \phi(s), \phi \in \mathcal{C}, s \in [-\tau, 0]$ if $\theta_i \le \sigma \le \zeta_i < \theta_{i+1}$;
- (IC_2) $x_\sigma(s) = \phi(s), x_{\gamma(\sigma)}(s) = \psi(s), \phi, \psi \in \mathcal{C}, s \in [-\tau, 0]$, if $\theta_i \le \zeta_i < \sigma < \theta_{i+1}$.

Considering equation (1) with these conditions we shall say about the initial value problem, IVP, for system (1). To be short, we shall say only about solutions of IVP in the form $x(t, \sigma, \phi, \psi)$. Specifying $x(t, \sigma, \phi)$ for (IC_1), if needed. Thus, we can provide the following definition, now.

Definition 2.1. A function $x(t)$ is a solution of (1) with (IC_1) or (IC_2) on an interval $[\sigma, \sigma + a)$ if:

(i) it satisfies the initial condition;
(ii) $x(t)$ is continuous on $[\sigma, \sigma + a)$;
(iii) the derivative $x'(t)$ exists for $t \ge \sigma$ with the possible exception of the points θ_i, where one-sided derivatives exist;
(iv) equation (2) is satisfied by $x(t)$ for all $t > \sigma$, except possibly points of θ and it holds for the right derivative of $x(t)$ at points θ_i.

Consider the following linear system,

$$z'(t) = A_0(t)z(t) + A_1(t)z(\gamma(t)), \tag{3}$$

which corresponds to equation (1). Systems of type (3) has been investigated in [5, 6]. In what follows we will give a short information from the book.

Let $\mathcal{I}$ be an $n \times n$ identity matrix. Denote by $X(t, s)$, $X(s, s) = \mathcal{I}$, $t, s \in \mathbb{R}$, the fundamental matrix of solutions of the system

$$x'(t) = A_0(t)x(t) \tag{4}$$

which is associated with systems (1) and (3). We introduce the following matrix-function

$$M_i(t) = X(t,\zeta_i) + \int_{\zeta_i}^{t} X(t,s)A_1(s)ds, i \in Z.$$

This matrix is very useful in what follows.

From now on we make the assumption:

(C6) For every fixed $i \in \mathbb{Z}$, $\det[M_i(t)] \neq 0$, $\forall t \in [\theta_i, \theta_{i+1}]$.

We shall call $(C6)$ *the regularity condition.*

Remark 1. It is easily seen that the last condition is equivalent to the following one:

$$\det[\mathcal{I} + \int_{\zeta_i}^{t} X(\zeta_i,s)A_1(s)ds] \neq 0,$$

for all $t \in [\theta_i, \theta_{i+1}], i \in \mathbb{Z}$.

Definition 2.2. A function $x(t)$ is a solution of (1)((3)) on $\mathbb{R}$ if:

(i) $x(t)$ is continuous;
(ii) the derivative $x'(t)$ exists for all $t \in \mathbb{R}$ with the possible exception of the points $\theta_i, i \in \mathbb{Z}$, where one-sided derivatives exist;
(iv) equation (1)((3)) is satisfied by $x(t)$ for all $t \in \mathbb{R}$, except points of θ and it holds for the right derivative of $x(t)$ at the points $\theta_i, i \in \mathbb{Z}$.

Theorem 2.3 ([5, 6])**.** *If condition* $(C1)$ *is fulfilled, then for every* $(t_0, z_0) \in \mathbb{R}\times\mathbb{R}^n$ *there exists a unique solution* $z(t) = z(t,t_0,z_0), z(t_0) = z_0$, *of (3) in the sense of Definition 2.2 if and only if condition* $(C6)$ *is valid.*

Using the last theorem one can easily prove [5, 6] that the set of the solutions of (3) is an $n-$dimensional linear space. Hence, for a fixed $t_0 \in \mathbb{R}$ there exists a fundamental matrix of solutions of (1), $Z(t) = Z(t,t_0), Z(t_0,t_0) = \mathcal{I}$, such that

$$\frac{dZ}{dt} = A_0(t)Z(t) + A_1(t)Z(\gamma(t)).$$

Let us show how to construct the fundamental matrix. Fix $i \in \mathbb{Z}$, such that $\theta_i \leq t_0 < \theta_{i+1}$ and define the matrix only for increasing t, as the construction is similar for decreasing t.

We have

$$Z(t) = M_l(t)\Big[\prod_{k=l}^{i+1} M_k^{-1}(\theta_k)M_{k-1}(\theta_k)\Big]M_i^{-1}(t_0), \tag{5}$$

if $t \in [\theta_l, \theta_{l+1}]$, for arbitrary $l > i$.

Similarly, if $\theta_j \leq t \leq \theta_{j+1} < \ldots < \theta_i \leq t_0 \leq \theta_{i+1}$, then

$$Z(t) = M_j(t)\Big[\prod_{k=j}^{i-1} M_k^{-1}(\theta_{k+1})M_{k+1}(\theta_{k+1})\Big]M_i^{-1}(t_0). \tag{6}$$

It is obtained that $Z(t,s) = Z(t)Z^{-1}(s), t,s \in \mathbb{R}$, and a solution $z(t), z(t_0) = z_0, (t_0,z_0) \in \mathbb{R}\times\mathbb{R}^n$, of (1) is equal to $z(t) = Z(t,t_0)z_0, t \in \mathbb{R}$.

One can easily see that $(C4)-(C7)$ imply the existence of positive constants m, M and $\bar{M}$ such that $m \leq \|Z(t,s)\| \leq M$, $\|X(t,s)\| \leq \bar{M}$ for $t,s \in [\theta_i,\theta_{i+1}]$, $i \in \mathbb{Z}$.

From now on we make the assumptions

(C7) $\bar{M}L(1+M)\bar{\theta} < 1.$

Lemma 2.4. *Suppose that conditions* $(C1)-(C7)$, *hold, and fix* $i \in \mathbb{Z}$. *Then, for every* $(\sigma, \phi, \psi) \in [\theta_i, \theta_{i+1}] \times \mathcal{C} \times \mathcal{C}$, *there exists a unique solution* $x(t) = x(t, \sigma, \phi, \psi)$ *of (1) on* $[\theta_i, \theta_{i+1}]$.

Proof. We will consider only (IC_1), since the proof with (IC_2) coincides with that for functional differential equations [21].

Existence. Fix $i \in \mathbb{Z}$. We assume that $\theta_i \leq \sigma\theta_{i+1}$, and consider solution $x(t, \sigma, \phi)$. Take $x_0(t) = Z(t, \sigma)\phi(\sigma)$ and define a sequence $\{x^k(t)\}, \quad k \geq 0$, by

$$\begin{aligned}
&x^k(t) = \phi(t), t \leq \sigma, \\
&x^{k+1}(t) = Z(t, \xi_i)\Big[\phi(\sigma) + \int_\sigma^{\zeta_i} X(\zeta_i, s) f(s, x^k_s, x^k_{\zeta_i}) ds\Big] \\
&\qquad\qquad + \int_{\zeta_i}^t X(t, s) f(s, x^k_s, x^k_{\zeta_i}) ds, t \in [\theta_i, \theta_{i+1}].
\end{aligned}$$

The last expression implies that

$$\max_{[\sigma, \theta_{i+1}]} \|x^{k+1}(t) - x^k(t)\| \leq [\bar{M}L(1+M)\bar{\theta}]^{k+1} M \|\phi(\sigma)\|.$$

Thus, there exists a unique solution $x(t) = x(t, \sigma, \phi)$ of the equation

$$x(t) = Z(t, \xi_i)\Big[\phi(\sigma) + \int_\sigma^{\zeta_i} X(\zeta_i, s) f(s, x_s, x_{\zeta_i}) ds\Big] \tag{7}$$

$$\qquad + \int_{\zeta_i}^t X(t, s) f(s, x_s, x_{\zeta_i}) ds, t \in [\theta_i, \theta_{i+1}]. \tag{8}$$

which is a solution of (1) on $[\theta_i, \theta_{i+1}]$ as well. This proves the existence.

Uniqueness. Denote by $x^j(t) = x(t, \sigma, \phi), j = 1, 2$, the solutions of (1), where $\theta_i \leq \sigma < \theta_{i+1}$.

We have that

$$x^1(t) - x^2(t) = Z(t, \xi_i)\{\int_\sigma^{\zeta_i} X(\zeta_i, s)[f(s, x^1_s, x^1_{\zeta_i}) - f(s, x^2_s, x^2_{\zeta_i})] ds\}$$

$$+ \int_{\zeta_i}^t X(t, s)[f(s, x^1_s, x^1_{\zeta_i}) - f(s, x^2_s, x^2_{\zeta_i})] ds.$$

Hence,

$$\|x^1(t) - x^2(t)\| \leq \bar{M}L\bar{\theta}(1+M) \max_{[\sigma, \theta_{i+1}]} \|x^1(t) - x^2(t)\|, t \in [\sigma, \theta_{i+1}].$$

Since of $(C7)$ it is possible if only $\max_{[\theta_i, \theta_{i+1}]} \|x^1(t) - x^2(t)\| = 0$. The lemma is proved. □

The next Lemma can be proved exactly in the way that used to verify Lemma 2.2 from [5], see also [6], if we use Lemma 2.4.

Lemma 2.5. *Suppose that conditions* $(C1)-(C7)$, *hold. Then, for every* $(\sigma, \phi, \psi) \in [\theta_i, \theta_{i+1}] \times \mathcal{C} \times \mathcal{C}$, *there exists a unique solution* $x(t) = x(t, \sigma, \phi, \psi), t \geq \sigma$, *of (1),* *and it satisfies the integral equation*

$$\begin{aligned} x(t) = Z(t,\sigma)[\phi(\sigma) + \int_{\sigma}^{\zeta_i} X(\sigma,s) f(s,x_s,x_{\gamma(s)})\,ds] \\ + \sum_{k=i}^{j-1} Z(t,\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s) f(s,x_s,x_{\gamma(s)})\,ds \\ + \int_{\zeta_j}^{t} X(t,s) f(s,x_s,x_{\gamma(s)})\,ds, \end{aligned} \tag{9}$$

where $\theta_i \le \sigma \le \theta_{i+1}$ *and* $\theta_j \le t \le \theta_{j+1}$, $i < j$.

3. **Bounded solutions.** We shall need the following assumptions.

(C8) $\|Z(t,s)\| \le K\mathrm{e}^{-\alpha(t-s)}$, $s \le t$, where K, α are positive numbers;
(C9) there exist positive numbers $\underline{\theta}, \underline{\zeta} > 0$ such that $\theta_{i+1} - \theta_i \ge \underline{\theta}, \zeta_{i+1} - \zeta_i \ge \underline{\zeta}, i \in \mathbb{Z}$;
(C10) $2\bar{M}L[\bar{\theta} + \bar{\zeta}\frac{K\mathrm{e}^{\alpha\bar{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}}] < 1$;
(C11) $L\frac{2K\bar{M}(1+\mathrm{e}^{\alpha\bar{\theta}})\mathrm{e}^{\alpha\bar{\theta}}\mathrm{e}^{\alpha\tau}}{\alpha} < 1$;
(C12) $LK\bar{M}\frac{\mathrm{e}^{\bar{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}} < 1$;
(C13) $2M\bar{M}L\bar{\theta} < 1$.

Lemma 3.1. *Suppose that conditions* $(C1) - (C9)$, *hold. Then, a bounded on* $\mathbb{R}$ *function* $x(t)$ *is a solution of (1) if and only if it satisfies the following integral equation*

$$\begin{aligned} x(t) = \int_{\zeta_j}^{t} X(t,s) f(s,x_s,x_{\gamma(s)})\,ds \\ + \sum_{k=-\infty}^{j-1} Z(t,\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s) f(s,x_s,x_{\gamma(s)})\,ds, \end{aligned} \tag{10}$$

where $\theta_j \le t \le \theta_{j+1}$.

Proof. We consider only sufficiency. The necessity can be proved by using (9) and $(C8)$, in very similar way to the ordinary differential equations case. Since the solution is bounded, there is a positive constant H such that $\|x(t)\| \le H$. By assumption, $f \in C_0(\mathbb{R} \times \mathcal{C}_H \times \mathcal{C}_H)$. That is, $\sup_{\mathbb{R}} \|f(s,x_s,x_{\gamma(s)})\| = M_H < \infty$, for a positive number M_H. Then

$$\|\int_{\zeta_j}^{t} X(t,s) f(s,x_s,x_{\gamma(s)})\,ds + \sum_{k=-\infty}^{j-1} Z(t,\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s) f(s,x_s,x_{\gamma(s)}) ds\|$$

$$\le |\int_{\zeta_j}^{t} \bar{M}M_H ds| + \sum_{k=-\infty}^{j-1} K\mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} \bar{M}M_H\,ds \le \bar{M}M_H[\bar{\theta} + \bar{\zeta}\frac{K\mathrm{e}^{\alpha\bar{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}}].$$

That is, the series and integral in (10) are convergent. Let us differentiate (10). We have that

$$x'(t) = \int_{\zeta_j}^{t} A_0(t)X(t,s)f(s,x_s,x_{\gamma(s)})\,ds + f(s,x_t,x_{\gamma(t)})$$
$$+ \sum_{k=-\infty}^{j-1} [A_0(t)Z(t,\theta_{k+1}) + A_1(t)Z(\zeta_j,\theta_{k+1})] \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s)f(s,x_s,x_{\gamma(s)})ds$$
$$= A_0(t)x(t) + A_1(t)x(\gamma(t)) + f(s,x_t,x_{\gamma(t)}).$$

The Lemma is proved. □

Now, we apply the result of the last Lemma to prove existence of a unique bounded on $\mathbb{R}$ solution of (1). Then we find conditions of its stability. So, we will prove that the following theorem is valid.

Theorem 3.2. *Suppose that conditions* $(C1)-(C10)$, *hold. Then, (1) admits a unique bounded of* $\mathbb{R}$ *solution. If, additionally, conditions* $(C11)-(C13)$ *are valid then the solution is exponentially stable.*

Proof. Consider the complete metric space $C_0(\mathbb{R})$ with the sup-norm $\|\phi\|_\infty = \sup_{\mathbb{R}} \|\phi(t)\|$. Define on $C_0(\mathbb{R})$ the operator Π such that

$$\Pi y(t) \equiv \int_{\zeta_j}^{t} X(t,s)f(s,y_s,y_{\gamma(s)})\,ds$$
$$+ \sum_{k=-\infty}^{j-1} Z(t,\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s)f(s,y_s,y_{\gamma(s)})\,ds, t \in [\theta_j,\theta_{j+1}].$$

One can show that $\Pi : C_0(\mathbb{R}) \to C_0(\mathbb{R})$. Let us verify that this operator is contractive. Indeed, if $u,v \in C_0(\mathbb{R})$, then

$$\|\Pi u(t) - \Pi v(t)\| \leq | \int_{\zeta_j}^{t} 2\bar{M}L\|u-v\|_1 ds + \sum_{k=-\infty}^{j-1} K\mathrm{e}^{-\alpha(t-\theta_{k+1})} \int_{\zeta_k}^{\zeta_{k+1}} 2\bar{M}L\|u-v\|_1\,ds$$
$$\leq 2\bar{M}L[\bar{\theta} + \bar{\zeta}\frac{K\mathrm{e}^{\alpha\bar{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}}]\|u-v\|_\infty.$$

Since of $(C10)$, the operator is contractive. That is, equation (1) admits a unique solution $u(t)$ from $C_0(\mathbb{R})$.

Let us investigate its stability. We apply representation (9) for this. We have that if u,v, are solutions of the equation with initial data $(\sigma,\phi,\psi),(\sigma,\eta,\pi)$, then

$$u(t) - v(t) = Z(t,\sigma)[(\phi(\sigma) - \eta(\sigma)) + \int_{\sigma}^{\zeta_i} X(\sigma,s)(f(s,u_s,u_{\gamma(s)}) - f(s,v_s,v_{\gamma(s)}))\,ds]$$
$$+ \sum_{k=i}^{j-1} Z(t,\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s)(f(s,u_s,u_{\gamma(s)}) - f(s,v_s,v_{\gamma(s)}))\,ds$$
$$+ \int_{\zeta_j}^{t} X(t,s)(f(s,u_s,u_{\gamma(s)}) - f(s,v_s,v_{\gamma(s)}))\,ds.$$

Denote by $w(t)$ the difference $u(t) - v(t)$. Then w satisfies the following integral equation

$$\begin{aligned} w(t) =& Z(t,\sigma)[(\phi(\sigma) - \eta(\sigma)) + \int_{\sigma}^{\zeta_i} X(\sigma,s)(f(s,u_s,u_{\gamma(s)}) \\ &- f(s,u_s+w_s,u_{\gamma(s)}+w_{\gamma(s)}))\,ds] \\ &+ \sum_{k=i}^{j-1} Z(t,\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s)(f(s,u_s,u_{\gamma(s)}) \\ &- f(s,u_s+w_s,u_{\gamma(s)}+w_{\gamma(s)}))\,ds + \int_{\zeta_j}^{t} X(t,s)(f(s,u_s,u_{\gamma(s)}) \\ &- f(s,u_s+w_s,,u_{\gamma(s)}+w_{\gamma(s)}))\,ds. \end{aligned} \tag{11}$$

We will solve this equation for $\sigma = 0$, initial function $w_\sigma = \phi(s) - \eta(s), \|\phi(s) - \eta(s)\| < \delta, s \in [-\tau, 0]$, where $\delta > 0$, will be precised later. assuming that $\gamma(0) \le 0$.

Fix $\epsilon > 0$ and denote $L(l,\delta) = \frac{K e^{\alpha\tau}\delta}{1 - L\frac{2K\bar{M}(1+e^{\alpha\bar{\theta}})e^{\alpha\bar{\theta}}e^{\alpha\tau}}{\alpha}}$. Take δ so small that $L(l,\delta) < \epsilon$.

Let Ψ_δ be the set of all continuous functions which are defined on $[-\tau,\infty)$ such that:

1. $\pi_\sigma(s) = \phi(s) - \eta(s), s \in [-\tau, 0]$;
2. $||\pi(t)|| \le L(l,\delta)\exp(-\frac{\alpha}{2}t)$ if $t \ge 0$,

for all $\pi \in \Psi_\delta$. If the norm is $\|\phi\|_1 = \sup_{[0,\infty)} \|\phi(t)\|$, then the set is a complete metric space. Define on Ψ_δ an operator $\tilde{\Pi}$ such that

$$\tilde{\Pi}\pi(t) = \begin{cases} \phi(t) - \eta(t), t \in [-\tau, 0] \\ Z(t,0)[(\phi(0) - \eta(0) + \int_0^{\zeta_i} X(0,s)(f(s,u_s,u_{\gamma(s)}) \\ -f(s,u_s+\pi_s,u_{\gamma(s)}+\pi_{\gamma(s)}))\,ds] \\ +\sum_{k=i}^{j-1} Z(t,\theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s)(f(s,u_s,u_{\gamma(s)}) \\ -f(s,u_s+\pi_s,u_{\gamma(s)}+\pi_{\gamma(s)}))\,ds + \int_{\zeta_j}^{t} X(t,s)(f(s,u_s,u_{\gamma(s)}) \\ -f(s,u_s+\pi_s,,u_{\gamma(s)}+\pi_{\gamma(s)}))\,ds, t \in [\theta_i,\theta_{i+1}]. \end{cases}$$

We shall show that $\tilde{\Pi} : \Psi_\delta \to \Psi_\delta$. Indeed, for $t \ge 0$ it is true that

$$\begin{aligned} ||\tilde{\Pi}\pi(t)|| \le& K\exp(-\alpha t)[\delta + \int_0^{\zeta_i} \bar{M}L(\|\pi_s\|_0 + \|\pi_{\gamma(s)}\|\|)ds] \\ &+ \sum_{k=i}^{j-1} K\exp(-\alpha(t-\theta_{k+1}) \int_{\zeta_k}^{\zeta k+1} \bar{M}L(\|\pi_s\|_0 + \|\pi_{\gamma(s)}\|_0\|)ds \\ &+ \int_{\zeta_j}^{t} \bar{M}L(\|\pi_s\|_0 + \|\pi_{\gamma(s)}\|_0\|)ds \le L(l,\delta)\exp(-\frac{\alpha}{2}t). \end{aligned}$$

Let $\pi_1, \pi_2 \in \Psi_\delta$. Then

$$||\tilde{\Pi}\pi_1 - \tilde{\Pi}\pi_2||_1 \le LK\bar{M}\frac{e^{\bar{\theta}}}{1 - e^{-\alpha\underline{\theta}}}||\pi_1 - \pi_2||_1.$$

Using a contraction mapping theorem, one can conclude that there exists a unique fixed point $v(t,\delta)$ of the operator $\tilde{\Pi} :\ \Psi_\delta \to \Psi_\delta$ which is a solution of (11).

To complete the proof we should show that there is no other solutions of the initial value problem. Consider first the interval $[\theta_0, \theta_1]$. Assume that on this interval,

(11) has two different solutions v_1, v_2 of the problem. Denote $w = v_1 - v_2, \bar{m} = \max_{[\theta_0,\theta_1]} ||w(t)||$, and assume, on contrary, that $\bar{m} > 0$. We have that on the interval

$$||w(t)|| \leq || \int_0^{\zeta_i} M\bar{M}L2\bar{m}ds + \int_{\zeta_i}^t \bar{M}L2\bar{m}ds \leq 2M\bar{M}L\bar{\theta}\bar{m}.$$

The last inequality contradicts condition $(C13)$. Now, using induction, one can easily prove the uniqueness for all $t \geq 0$. The theorem is proved. □

4. **Periodic solutions.** Assume that there are two numbers, $\omega \in \mathbb{R}, p \in \mathbb{Z}$, such that $\theta_{k+p} = \theta_k + \omega, \zeta_{k+p} = \zeta_k + \omega, k \in \mathbb{Z}$. Then denote by Q the product $\prod_{k=1}^p G_k$, where matrices G_k are equal to $M_k^{-1}(\theta_k)M_{k-1}(\theta_k), k \in \mathbb{Z}$. The matrix Q, is the monodromy matrix, and eigenvalues of the matrix, $\rho_j, j = 1, 2, \ldots, n$, are multipliers. It is clear that system (1) admits a periodic solution, if there exists a unit multiplier. Generally, all the results known for linear homogeneous ordinary differential equations based on the unit multipliers can be identically repeated for the present systems. Our main goal in this section is to study the non-critical systems, and find formulas for solutions. We assume that the system is $\omega-$ periodic. That is, in addition to the above conditions, $A_j(t+\omega) = A_j(t), j = 0, 1, f(t+\omega, \phi, \psi) = f(t, \phi, \psi), t \in \mathbb{R}$. In what follows, we assume without loss of generality that $\zeta_0 = 0$, and consider $\sigma = \zeta_0$.

Consider the solution $z(t) = z(t, 0, z_0)$. We have that

$$\begin{aligned} z(t) =& Z(t)z_0 + \sum_{k=0}^{j-1} Z(t, \theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1}, s)f(s, z_s, z_{\gamma(s)})\, ds \\ &+ \int_{\zeta_j}^t X(t, s)f(s, z_s, z_{\gamma(s)})\, ds, t \in [\theta_j, \theta_{j+1}], \end{aligned} \tag{12}$$

where $Z(t) \equiv Z(t, 0), t \in \mathbb{R}$, and

$$z(\omega) = Z(\omega)z_0 + \sum_{k=0}^{p-1} Z(\omega, \theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1}, s)f(s, z_s, z_{\gamma(s)})\, ds.$$

In papers [12, 13] we proved the Poincaré criterion for EPCAG. According to this, $z(t)$ is a periodic solution if and only if z_0 satisfies

$$[I - Z(\omega)]z_0 = \sum_{k=0}^{p-1} Z(\omega, \theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1}, s)f(s, z_s, z_{\gamma(s)})\, ds.$$

By conditions of non-criticality,

$$\det[I - Z(\omega)] \neq 0, \tag{13}$$

and the last equation admits a unique solution,

$$z^* = [I - Z(\omega)]^{-1} \sum_{k=0}^{p-1} Z(\omega, \theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1}, s)f(s, z_s, z_{\gamma(s)})\, ds.$$

Thus, we have obtained that

$$\begin{aligned} z(t) =&Z(t)[I-Z(\omega)]^{-1}\sum_{k=0}^{p-1} Z(\omega,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s)f(s,z_s,z_{\gamma(s)})\ ds \\ &+\sum_{k=0}^{j-1} Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s)f(s)\ ds+\int_{\zeta_j}^{t} X(t,s)f(s,z_s,z_{\gamma(s)})\ ds. \end{aligned} \tag{14}$$

Use formula (14) to obtain

$$\begin{aligned} z(t) =&\sum_{k=0}^{j-1} Z(t)[I-Z(\omega)]^{-1}Z^{-1}(\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s)f(s,z_s,z_{\gamma(s)})\ ds \\ &+\sum_{k=j}^{p-1} Z(t)[I-Z(\omega)]^{-1}Z(\omega)Z^{-1}(\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s)f(s,z_s,z_{\gamma(s)})\ ds \\ &+\int_{\zeta_j}^{t} X(t,s)f(s,z_s,z_{\gamma(s)})\ ds. \end{aligned} \tag{15}$$

One can easily verify by substitution that (15) is a solution , and it is a continuous function. One can construct the following *Green function* for the *periodic solution*, $G_P(t,s), t,s\in[0,\omega]$.

If $t\in[\theta_j,\theta_{j+1}), j=0,2,\ldots,p-1$, then

$$G_P(t,s)=\begin{cases} Z(t)[I-Z(\omega)]^{-1}Z^{-1}(\theta_{k+1})X(\theta_{k+1},s), s\in[\zeta_k,\zeta_{k+1}), k<j, \\ Z(t)[I-Z(\omega)]^{-1}Z(\omega)Z^{-1}(\theta_{k+1})X(\theta_{k+1},s), \\ s\in[\zeta_k,\zeta_{k+1})\backslash[\hat{\zeta_j},t], k\geq j, \\ Z(t)[I-Z(\omega)]^{-1}Z(\omega)Z^{-1}(\theta_{k+1})X(\theta_{k+1},s)+X(t,s), s\in[\hat{\zeta_j},t] \end{cases}$$

Now, apply the last formula in (15) to see that the periodic solution satisfies

$$z(t)=\int_0^\omega G_P(t,s)f(s,z_s,z_{\gamma(s)})ds.$$

Denote by $\tilde{M}=\max_{t,s\in[0,\omega]}\|G_P(t,s)\|<\infty$. By applying the last integral equation one can easily verify that the following theorem is valid.

Theorem 4.1. *Suppose that conditions $(C1)-(C7)$ are valid, inequalities (13) and $\tilde{M}L\omega<1$ hold. Then, (1) admits a unique $\omega-$ periodic solution.*

5. **Almost periodic solutions.** In this section we will continue study system (1), assuming that all notations of the last section are valid. Concerning conditions $(C1)-(C13)$, we have to say that some of them are consequences of the almost periodicity.

For $f\in C_0(\mathbb{R})$ (respectively $C_0(\mathbb{R}\times\mathcal{C}\times\mathcal{C})$ and $\tau\in\mathbb{R}$, a translation of f by τ is a function $Q_\tau f=f(t+\tau)$, $t\in\mathbb{R}$ (respectively $Q_\tau f(t,\phi,\psi)=f(t+\tau,\phi,\psi)$, $(t,\phi,\psi)\in\mathbb{R}\times\mathcal{C}\times\mathcal{C}$). A number $\tau\in\mathbb{R}$ is called an $\epsilon-$ translation number of a functional $f\in C_0(\mathbb{R})$ ($C_0(\mathbb{R}\times\mathcal{C}\times\mathcal{C})$) if $||Q_\tau f-f||<\epsilon$ for every $t\in\mathbb{R}$ ($(t,\phi,\psi)\in\mathbb{R}\times\mathcal{C}\times\mathcal{C}$). A set $S\subset\mathbb{R}$ is said to be relatively dense if there exists a number $l>0$ such that $[a,a+l]\cap S\neq\emptyset$ for all $a\in\mathbb{R}$.

Definition 5.1 ([22])**.** A function (functional) $f\in C_0(\mathbb{R})(C_0(\mathbb{R}\times\mathcal{C}\times\mathcal{C}))$ is said to be almost periodic (almost periodic in t uniformly with respect to $\phi,\psi\in\mathcal{C}_H\times\mathcal{C}_H, H>$

0,) if for every $\epsilon \in \mathbb{R}, \epsilon > 0$, there exists a relatively dense set of $\epsilon-$ translation numbers of f.

Denote by $\mathcal{AP}(\mathbb{R})$ ($\mathcal{AP}(\mathbb{R} \times \mathcal{C}_H \times \mathcal{C}_H)$) the set of all such functions.

Definition 5.2. A sequence $a_i, i \in \mathbb{Z}$, is almost periodic, if for any $\epsilon > 0$ there exists a relatively dense set of its $\epsilon-$almost periods.

Let $\zeta_i^j = \zeta_{i+j} - \zeta_i, \theta_i^j = \theta_{i+j} - \theta_i$ for all i and j. We call the family of sequences $\{\zeta_i^j\}_i, j \in \mathbb{Z}$, equipotentially almost periodic [5, 20, 25] if for an arbitrary positive ϵ there exists a relatively dense set of $\epsilon-$almost periods, common for all sequences $\{\zeta_i^j\}_i, j \in \mathbb{Z}$.

We assume that the following conditions are valid throughout this section:

(A1) $A_0, A_1 \in \mathcal{AP}(\mathbb{R})$;
(A2) $f \in \mathcal{AP}(\mathbb{R} \times \mathcal{C}_H \times \mathcal{C}_H)$ for each $H > 0$;
(A3) sequences $\zeta_i^j, j \in \mathbb{Z}$, as well sequences $\theta_i^j, j \in \mathbb{Z}$, are equipotentially almost periodic.

One can easily see that $(A1)$ implies $(C1)$. From condition $(A3)$ it follows [5, 20, 25], that there exist positive numbers $\bar{\theta}$ and $\bar{\zeta}$ for $(C5)$, and $|\theta_i|, |\zeta_i| \to \infty$, as $|i| \to \infty$. Let us prove an auxiliary assertion.

Lemma 5.3. *Let $\omega \in \mathbb{R}$ be a common $\eta-$ almost period of matrices $A_0(t), A_1(t)$, then there exists a function $R(\eta) = \frac{K\bar{M}\mathrm{e}^{\alpha\bar{\theta}}}{\alpha}\eta$ such that*

$$\|Z(t+\omega, s+\omega) - Z(t,s)\| < R(\eta)\mathrm{e}^{-\frac{\alpha}{2}(t-s)}, s \le t. \tag{16}$$

Proof. Set $W(t,s) = Z(t+\omega, s+\omega) - Z(t,s)$. Then

$$\begin{aligned}\frac{\partial W}{\partial t} =& A_0(t)W(t,s) + A_1(t)W(\gamma(t),s) + [A_0(t+\omega) - A(t)]W(t+\omega, s+\omega) \\ &+ [A_1(t+\omega) - A(t)]W(\gamma(t)+\omega, s+\omega).\end{aligned}$$

Since $W(s,s) = 0$, from the last equation it follows that

$$\begin{aligned}W(t,s) =& Z(t,s)\int_s^{\zeta_i} X(s,u)[A_0(u+\omega) - A(u)]W(u+\omega, s+\omega) \\ &+ A_1(u+\omega) - A(u)]W(\gamma(u)+\omega, s+\omega)]\, du \\ &+ \sum_{k=i}^{j-1} Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},u)[A_0(u+\omega) - A(u)]W(u+\omega, s+\omega) \\ &+ [A_1(u+\omega) - A(u)]W(\gamma(u)+\omega, s+\omega)]\, du \\ &+ \int_{\zeta_j}^t X(t,u)[A_0(u+\omega) - A(u)]W(u+\omega, s+\omega) \\ &+ [A_1(u+\omega) - A(u)]W(\gamma(u)+\omega, s+\omega)]\, du.\end{aligned}$$

Then, we have that

$$\begin{aligned}\|W(t,s)\| \leq &\int_s^{\zeta_i} K\bar{M}\eta e^{-\alpha(t-s-\bar{\theta})}du + \sum_{k=i}^{j-1}\int_{\zeta_k}^{\zeta_{k+1}} K\bar{M}\eta e^{-\alpha(t-s-\bar{\theta})}du \\ &+ \int_{\zeta_j}^{t} K\bar{M}\eta e^{-\alpha(t-s-\bar{\theta})}du \\ \leq &\int_s^t K\bar{M}\eta e^{-\alpha(t-s-\bar{\theta})}du = K\bar{M}\eta e^{-\alpha(t-s-\bar{\theta})}(t-s) \\ \leq &\frac{K\bar{M}e^{\alpha\bar{\theta}}}{\alpha}\eta e^{-\frac{\alpha}{2}(t-s)}.\end{aligned}$$

The lemma is proved. □

The following assertion can be proved by the method of common almost periods developed in [34] (see, also, [20, 25]).

Lemma 5.4 ([20])**.** *Assume that $f(t,\phi,\psi)$, and $\xi_j(t), j = 1,2,\ldots,k$, are Bohr almost periodic in t. Conditions $(C11),(A3)$ are valid. Then, for arbitrary $H,\eta > 0, 0 < \nu < \eta$, there exist a respectively dense set of real numbers Ω and integers Q, such that for $\omega \in \Omega, q \in Q$, it is true that*

1. $\|f(t+\omega,\phi,\psi) - f(t,\phi,\psi)\| < \eta$, *for all* $\phi,\psi \in \mathcal{C}_H$;
2. $\|\xi_j(t+\omega) - \xi_j(t)\| < \eta, j = 1,2,\ldots,k, t \in \mathbb{R}$;
3. $|\zeta_i^q - \omega| < \nu, i \in \mathbb{Z}$;
4. $|\theta_i^q - \omega| < \nu, i \in \mathbb{Z}$.

Let us formulate the following assertion.

Theorem 5.5. *Assume that conditions $(A1)-(A3),(C3),(C4),(C6)-(C9)$, are valid. Then, (1) admits a unique almost periodic solution. If, additionally, conditions $(C10)-(C13)$ are valid then the solution is exponentially stable.*

Proof. It follows from Theorem 3.2 that (1) admits a unique bounded on $\mathbb{R}$ solution $u(t)$, which is exponentially stable. We shall show that it is an almost periodic function. Consider the operator

$$\begin{aligned}\Pi y(t) \equiv &\int_{\zeta_j}^{t} X(t,s)f(s,y_s,y_{\gamma(s)})\,ds \\ &+ \sum_{k=-\infty}^{j-1} Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1},s)f(s,y_s,y_{\gamma(s)})\,ds,\ t\in[\theta_j,\theta_{j+1}],\end{aligned}$$

again.

It is sufficient to verify that $\Pi y(t)$ is almost periodic, if $y(t)$ is. Fix positive ϵ. Suppose that ω and q satisfy conditions of Lemma 5.4, such that ω is an $\eta-$translation number for y. We assume that $\theta_j + \eta < t < \theta_{j+1} - \eta$. Then, by Lemma

5.4 one can obtain that $t+\omega \in (\theta_i, \theta_{i+1})$ and $i = j+q$. We have that

$$\begin{aligned}
&\Pi y(t+\omega) - \Pi y(t) \\
&= \int_{\zeta_i}^{t+\omega} X(t+\omega, s) f(s, y_s, y_{\gamma(s)})\, ds \\
&\quad + \sum_{k=-\infty}^{i-1} Z(t+\omega, \theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1}, s) f(s, y_s, y_{\gamma(s)})\, ds \\
&\quad - \int_{\zeta_j}^{t} X(t,s) f(s, y_s, y_{\gamma(s)})\, ds - \sum_{k=-\infty}^{j-1} Z(t, \theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1}, s) f(s, y_s, y_{\gamma(s)})\, ds,
\end{aligned}$$

Transform the last expression to

$$\begin{aligned}
&\Pi y(t+\omega) - \Pi y(t) \qquad (17)\\
&= \int_{\zeta_i - \omega}^{t} X(t+\omega, s+\omega) f(s+\omega, y_{s+\omega}, y_{\gamma(s+\omega)})\, ds \\
&\quad + \sum_{k=-\infty}^{j-1} Z(t+\omega, \theta_{k+1+q}) \int_{\zeta_{k+q}}^{\zeta_{k+1+q}} X(\theta_{k+1+q}, s) f(s, y_s, y_{\gamma(s)})\, ds \\
&\quad - \int_{\zeta_j}^{t} X(t,s) f(s, y_s, y_{\gamma(s)})\, ds - \sum_{k=-\infty}^{j-1} Z(t, \theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1}, s) f(s, y_s, y_{\gamma(s)})\, ds \\
&= \int_{\zeta_{j+q} - \omega}^{\zeta_j} X(t+\omega, s+\omega) f(s+\omega, y_{s+\omega}, y_{\gamma(s+\omega)})\, ds \\
&\quad + \int_{\zeta_j}^{t} [X(t+\omega, s+\omega) f(s+\omega, y_{s+\omega}, y_{\gamma(s+\omega)}) - X(t,s) f(s, y_s, y_{\gamma(s)})]\, ds \\
&\quad + \sum_{k=-\infty}^{j-1} \{ Z(t+\omega, \theta_{k+1+q}) \int_{\zeta_{k+q}}^{\zeta_{k+1+q}} X(\theta_{k+1+q}, s) f(s, y_s, y_{\gamma(s)})\, ds \\
&\quad - Z(t, \theta_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1}, s) f(s, y_s, y_{\gamma(s)})\, ds \}, \qquad (18)
\end{aligned}$$

where we assume, without lost of generality, that $\zeta_i - \omega \le \zeta_j$. Let us begin with estimation of the first integral in the last expression. We have that $|\zeta_i - \omega - \zeta_j| = |\zeta_{j+q} - \zeta_j - \omega| = |\zeta_j^q - \omega| < \nu < \eta$. Now, because of the boundedness of the integrand, one can find that

$$\| \int_{\zeta_{j+q} - \omega}^{\zeta_j} X(t+\omega, s+\omega) f(s+\omega, y_{s+\omega}, y_{\gamma(s+\omega)})\, ds \| \le R_0(\eta),$$

where $R_0(\eta) \to 0$ as $\eta \to 0$. For the second integral we have that,

$$\begin{aligned}
&\|X(t+\omega, s+\omega) f(s+\omega, y_{s+\omega}, y_{\gamma(s+\omega)}) - X(t,s) f(s, y_s, y_{\gamma(s)})\| \\
\le &\|X(t+\omega, s+\omega) - X(t,s)\| \|f(s+\omega, y_{s+\omega}, y_{\gamma(s+\omega)})\| \\
&+ \|X(t,s)\| \|f(s+\omega, y_{s+\omega}, y_{\gamma(s+\omega)}) - f(s, y_s, y_{\gamma(s)})\|.
\end{aligned}$$

At first, let us observe that by “diagonal almost periodicity” of the fundamental matrix of solutions, , Lemma 34 [25], it is true that $\|X(t+\omega, s+\omega) - X(t,s)\| < R_1(\eta)$, where $R_1(\eta) \to 0$ as $\eta \to 0$.

Next, we have that

$$\begin{aligned}&\|f(s+\omega,y_{s+\omega},y_{\gamma(s+\omega)})-f(s,y_s,y_{\gamma(s)})\|\\ \leq&\|f(s+\omega,y_{s+\omega},y_{\gamma(s+\omega)})-f(s,y_{s+\omega},y_{\gamma(s+\omega)})\|\\ &+\|f(s,y_{s+\omega},y_{\gamma(s+\omega)})-f(s,y_s,y_{\gamma(s+\omega)})\|\\ &+\|f(s,y_s,y_{\gamma(s+\omega)})-f(s,y_s,y_{\gamma(s)})\|\end{aligned}$$

Let us estimate the last three norms. The first one is less than η, since of the almost periodicity of f in t. The second one is less than $L\eta$, since of the almost periodicity of y and the Lipschitz condition. To evaluate the last one, we use again Lemma 5.4, and consider $\nu>0$ so small such that $|y(t')-y(t'')|<\eta$, if $|t'-t''|<\nu$. Then we obtain that $\|y_{\gamma(t+\omega)}-y_{\gamma(t)}\|=\|y(\zeta_{i+q}+s)-y(\zeta_i+s)\|\leq\|y(\zeta_{i+q}+s)-y(\zeta_i+\omega+s)\|+\|y(\zeta_i+\omega+s)-y(\zeta_i+s)\|<2\eta$, for all $t\in\mathbb{R},s\in[0,\omega]$, since $|\zeta_{i+q}-\zeta_i-\omega|<\nu$. Thus, the third norm is less than $2L\eta$. Now, from boundedness of $\|f(s+\omega,y_{s+\omega},y_{\gamma(s+\omega)})\|$ and $\|X(t,s)\|$ it implies that

$$\|X(t+\omega,s+\omega)f(s+\omega,y_{s+\omega},y_{\gamma(s+\omega)})-X(t,s)f(s,y_s,y_{\gamma(s)})\|\leq R_2(\eta),$$

and consequently the second integral in the norm is less than $R_2(\eta)\bar{\theta}$, where $R_2(\eta)\to 0$, as $\eta\to 0$.

Let us now estimate the sum in (17),

$$\sum_{k=-\infty}^{k=j-1}\{Z(t+\omega,\theta_{k+1+q})\int_{\zeta_{k+q}}^{\zeta_{k+1+q}}X(\theta_{k+1+q},s)f(s,y_s,y_{\gamma(s)})\,ds$$

$$-Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}}X(\theta_{k+1},s)f(s,y_s,y_{\gamma(s)})\,ds\}.$$

We fix k in the sum, and consider

$$\begin{aligned}&\|Z(t+\omega,\theta_{k+1+q})\int_{\zeta_{k+q}}^{\zeta_{k+1+q}}X(\theta_{k+1+q},s)f(s,y_s,y_{\gamma(s)})\,ds\\ &-Z(t,\theta_{k+1})\int_{\zeta_k}^{\zeta_{k+1}}X(\theta_{k+1},s)f(s,y_s,y_{\gamma(s)})\,ds\|\\ \leq&\|Z(t+\omega,\theta_{k+1+q})-Z(t,\theta_{k+1})\|\|\int_{\zeta_{k+q}}^{\zeta_{k+1+q}}X(\theta_{k+1+q},s)f(s,y_s,y_{\gamma(s)})\,ds\|\\ &+\|Z(t,\theta_{k+1})\|\|\int_{\zeta_{k+q}}^{\zeta_{k+1+q}}X(\theta_{k+1+q},s)f(s,y_s,y_{\gamma(s)})\,ds\\ &-\int_{\zeta_k}^{\zeta_{k+1}}X(\theta_{k+1},s)f(s,y_s,y_{\gamma(s)})\,ds\|.\end{aligned}$$

We have that

$$\begin{aligned}&\|Z(t+\omega,\theta_{k+1+q})-Z(t,\theta_{k+1})\|\\ \leq&\|Z(t+\omega,\theta_{k+1+q})-Z(t+\omega,\theta_{k+1}+\omega)\|+\|Z(t+\omega,\theta_{k+1}+\omega)-Z(t,\theta_{k+1})\|.\end{aligned}$$

Then

$$\begin{aligned}\|Z(t+\omega,\theta_{k+1+q})-Z(t+\omega,\theta_{k+1}+\omega)\|\leq&\|Z(t+\omega,\theta_{k+1+q})\|\|I-Z(\theta_{k+1+q},\theta_{k+1}+\omega)\|\\ \leq&KR_3(\eta)\mathrm{e}^{-\alpha(t+\omega-\theta_{k+1+q})},\end{aligned}$$

where $R_3(\eta) \to 0$ as $\eta \to 0$. Moreover,

$$\|Z(t+\omega, \theta_{k+1}+\omega) - Z(t, \theta_{k+1})\| \le R(\eta) \mathrm{e}^{-\frac{\alpha}{2}(t-\theta_{k+1})},$$

since of Lemma 5.3.

Thus, it is true that

$$\|Z(t+\omega, \theta_{k+1+q}) - Z(t, \theta_{k+1})\| \le R_4(\eta) \mathrm{e}^{-\frac{\alpha}{2}(t-\theta_{k+1})}.$$

There exist numbers $\nu_1, \nu_2, |\nu_j| < \eta, j = 1, 2,$ such that $\zeta_{k+q+1} = \zeta_{k+1} + \omega + \nu_2$ and $\zeta_{k+q} = \zeta_k + \omega + \nu_1$. Let us make the following transformations,

$$\begin{aligned}
&\int_{\zeta_{k+q}}^{\zeta_{k+1+q}} X(\theta_{k+1+q}, s) f(s, y_s, y_{\gamma(s)})\, ds - \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1}, s) f(s, y_s, y_{\gamma(s)})\, ds \\
&= \int_{\zeta_k+\omega+\nu_1}^{\zeta_k+\omega} X(\theta_{k+1+q}, s) f(s, y_s, y_{\gamma(s)})\, ds - \int_{\zeta_{k+1}+\omega+\nu_2}^{\zeta_{k+1}+\omega} X(\theta_{k+1+q}, s) f(s, y_s, y_{\gamma(s)})\, ds \\
&\quad + \int_{\zeta_k}^{\zeta_{k+1}} [X(\theta_{k+1+q}, s+\omega) f(s+\omega, y_{s+\omega}, y_{\gamma(s+\omega)}) - X(\theta_{k+1}, s) f(s, y_s, y_{\gamma(s)})]\, ds
\end{aligned}$$

Apply to the last expressions discussion similar to that made above to obtain that

$$\| \int_{\zeta_{k+q}}^{\zeta_{k+1+q}} X(\theta_{k+1+q}, s) f(s, y_s, y_{\gamma(s)}) ds - \int_{\zeta_k}^{\zeta_{k+1}} X(\theta_{k+1}, s) f(s, y_s, y_{\gamma(s)}) ds \| \le R_5(\eta),$$

where $R_5(\eta) \to 0$, as $\eta \to 0$.

Write $\tilde{M} = \sup_{\mathbb{Z}} \| \int_{\zeta_{k+q}}^{\zeta_{k+1+q}} X(\theta_{k+1+q}, s) f(s, y_s, y_{\gamma(s)})\, ds\| < \infty$, and obtain that

$$\begin{aligned}
&\|\Pi y(t+\omega) - \Pi y(t)\| \\
\le &R_2(\eta)\bar{\theta} + \sum_{k=-\infty}^{j-1} \{R_4(\eta)\tilde{M} \mathrm{e}^{-\frac{\alpha}{2}(t-\theta_{k+1})} + R_5(\eta) K \mathrm{e}^{-\alpha(t-\theta_{k+1})}\} \\
\le &R_0(\eta) + R_2(\eta)\bar{\theta} + R_4(\eta)\tilde{M} \frac{\mathrm{e}^{\frac{\alpha}{2}\bar{\theta}}}{1-\mathrm{e}^{-\frac{\alpha}{2}\underline{\theta}}} + R_5(\eta) K \frac{\mathrm{e}^{\alpha\bar{\theta}}}{1-\mathrm{e}^{-\alpha\underline{\theta}}}.
\end{aligned}$$

From the properties of functions R_j, it follows that if η sufficiently small, then

$$\|\Pi y(t+\omega) - \Pi y(t)\| \le \epsilon/2,$$

if $\theta_i + \eta < t < \theta_{i+1} - \eta$. Now, use uniform continuity of Πy, and take η so small that $\|\Pi y(t') - \Pi y(t'')\| < \epsilon/2$, if $|t' - t''| < \eta$. Then ω is an $\epsilon-$almost period of $\Pi y(t)$.

The theorem is proved. □

6. **Examples.**

Example 1. Let us give examples of the function $f(t, x_t, x_{\gamma(t)})$ in (1).

1.
$$\sum_{j=1}^m A_j(t) x(t-\tau_j) + \sum_{i=1}^k B_i(t) x(\gamma(t) - \omega_j),$$

where τ_j and ω_i are fixed positive numbers. The linear function is with constant delays and alternate constancy of argument;

2.
$$\sum_{j=1}^{m} A_j(t)x(t-\tau_j(t)) + \sum_{i=1}^{k} B_i(t)x(\gamma(t)-\omega_j(t)),$$
where $\tau_j(t)$ and $\omega_i(t)$ are fixed bounded positive functions. This linear function is with variable delays and alternate constancy of argument;

3.
$$\int_{-\tau}^{0} K(s,x(t+s))ds + \int_{-\gamma(t)}^{0} L(s,x(\beta(t)+s))ds.$$
The function is with bounded distributed delay and constancy of argument is of two types, alternate and retarded.

4.
$$\int_{-\infty}^{0} K(s,x(t+s))ds + \int_{-\infty}^{0} L(s,x(\beta(t)+s))ds.$$
The function is with unbounded distributed delay and retarded constancy of argument.

One can easily see that the Lipschitz condition is valid for the last examples, if coefficient-functions A_j and B_i are bounded, and functions K and L are Lipschitzian. Finally, one can remark that functions of the form $f(t,x(t),x(\gamma(t)))$ are also particular case of the functional $f(t,x_t,x_{\gamma(t)})$.

Example 2. Consider the system
$$z'(t) = A_0 z(t) + A_1 z(\gamma(t)) + f(t,z_t,z_{\gamma(t)}), \tag{19}$$
where $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, and $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$, $A_1 = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}$.

We find that $X(t,s) = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{a(t-s)} \end{pmatrix}$, and
$$M_i(\theta_i) = \begin{pmatrix} 1+k(\theta_i-\zeta_i) & 0 \\ 0 & \mathrm{e}^{a(\theta_i-\zeta_i)} \end{pmatrix},$$
$$M_{i-1}(\theta_i) = \begin{pmatrix} 1+k(\theta_i-\zeta_{i-1}) & 0 \\ 0 & \mathrm{e}^{a(\theta_i-\zeta_{i-1})} \end{pmatrix}, i \in \mathbb{Z}.$$

Next, we have that
$$G = M_i^{-1}(\theta_i)M_{i-1}(\theta_i) = \begin{pmatrix} \frac{1+k(\theta_i-\zeta_{i-1})}{1+k(\theta_i-\zeta_i)} & 0 \\ 0 & \mathrm{e}^{a(\zeta_i-\zeta_{i-1})} \end{pmatrix}.$$

From (5), it implies that the zero solution of (3) is uniformly exponentially stable, if
$$\sup_i \left|\frac{1+k(\theta_i-\zeta_{i-1})}{1+k(\theta_i-\zeta_i)}\right| < 1, \sup_i \mathrm{e}^{a(\zeta_i-\zeta_{i-1})} < 1.$$

Assume that the last two inequalities are correct. Then system (19) admits uniformly asymptotically stable bounded, periodic or almost periodic solution, if the Lipschitz constant is sufficiently small and sequences θ,ζ and functional f satisfy appropriate properties of periodicity and almost periodicity mentioned for Theorems 3.2 to 4.1.

Let us discuss more specific cases. Assume, first, that $\theta_i = \zeta_i = i, i \in \mathbb{Z}$. Then matrix
$$G = \begin{pmatrix} 1+k & 0 \\ 0 & \mathrm{e}^{a} \end{pmatrix}, i \in \mathbb{Z}.$$

Denote by $\rho_j, j = 1, 2$, eigenvalues of matrix G. They are *multipliers* of system (19). From (5), it implies that the zero solution of (3) is exponentially stable, if and only if absolute values of both multiplies less than one. Easy to see that $|\rho_1| = |1 + k|, |\rho_2| = \mathrm{e}^a$, and sufficient conditions for uniform asymptotic stability of the bounded solution are $a < 0, -2 < k < 0$.

Now, suppose that the piecewise constant argument is advanced, $\theta_{i+1} = \zeta_i = i+1$. This time $\rho_1 = (1 - k)^{-1}, \rho_2 = \mathrm{e}^a$, and sufficient conditions for asymptotic stability are $a < 0, k > 0$ or $k < -2$.

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REFERENCES

[1] A. Alonso, J. Hong and R. Obaya, *Almost periodic type solutions of differential equations with piecewise constant argument via almost periodic type sequences*, Appl. Math. Lett., **13** (2000), 131–137.

[2] M. U. Akhmet, *On the integral manifolds of the differential equations with piecewise constant argument of generalized type*, in "Proceedings of the Conference on Differential and Difference Equations at the Florida Institute of Technology," August 1-5, 2005, Melbourne, Florida, Editors: R.P. Agarval and K. Perera, Hindawi Publishing Corporation, 2006, 11–20.

[3] M. U. Akhmet, *Integral manifolds of differential equations with piecewise constant argument of generalized type*, Nonlinear Anal., **66** (2007), 367–383.

[4] M. U. Akhmet, *On the reduction principle for differential equations with piecewise constant argument of generalized type*, J. Math. Anal. Appl., **336** (2007), 646–663.

[5] M. U. Akhmet, "Nonlinear Hybrid Continuous/Discrete Time Models," Amsterdam, Paris, Antlantis Press, 2011.

[6] M. U. Akhmet, *Stability of differential equations with piecewise constant argument of generalized type*, Nonlinear Analysis: TMA, **68** (2008), 794–803.

[7] M. U. Akhmet, *Exponentially dichotomous linear systems of differential equations with piecewise constant argument*, Discontinuity, Nonlinearity and Complexity, **1** (2012), 337–352.

[8] M. U. Akhmet and D. Aruğaslan, *Lyapunov-Razumikhin method for differential equations with piecewise constant argument*, Discrete Contin. Dyn. Syst., **25** (2009), 457–466.

[9] M. U. Akhmet, D. Aruğaslan and E. Yılmaz, *Method of Lyapunov functions for differential equations with piecewise constant delay*, J. Comput. Appl. Math., **235** (2011), 4554–4560.

[10] M. U. Akhmet, D. Aruğaslan and E. Yılmaz, *Stability analysis of recurrent neural networks with piecewise constant argument of generalized type*, Neural Networks, **23** (2010), 805–811.

[11] M. U. Akhmet, D. Aruğaslan and E. Yılmaz, *Method of Lyapunov functions for differential equations with piecewise constant delay*, J. Comput. Appl. Math., **235** (2011), 4554–4560.

[12] M. U. Akhmet and C. Buyukadali, *Differential equations with a state-dependent piecewise constant argument*, Nonlinear Analysis: TMA, **72** (2010), 4200–4210.

[13] M. U. Akhmet and C. Buyukadali, *Periodic solutions of the system with piecewise constant argument in the critical case*, Comput. Math. Appl., **56** (2008), 2034–2042.

[14] M. U. Akhmetov, N. A. Perestyuk and A. M. Samoilenko, *Almost-periodic solutions of differential equations with impulse action*, (Russian) Akad. Nauk Ukrain. SSR Inst., Mat. Preprint, **26** (1983), 49.

[15] G. Bao, S. Wen and Zh. Zeng, *Robust stability analysis of interval fuzzy CohenGrossberg neural networks with piecewise constant argument of generalized type*, Neural Networks, **33** (2012), 32–41.

[16] S. Busenberg and K. L. Cooke, *Models of vertically transmitted diseases with sequential-continuous dynamics*, Nonlinear Phenomena in Mathematical Sciences, Academic Press, New York, (1982), 179–187.

[17] T. A. Burton, "Stability and Periodic Solutions of Ordinary and Functional Differential Equations," Academic Press, Orlando, Florida, 1985.

[18] K. L. Cooke and J. Wiener, *Neutral differential equations with piecewise constant argument*, Boll. Un. Mat. Ital., **7** (1987), 321–346.

[19] L. Dai, "Nonlinear Dynamics of Piecewise Constant Systems and Implementation of Piecewise Constant Arguments," World Scientific, Hackensack, NJ, 2008.

[20] A. Halanay and D. Wexler, "Qualitative Theory of Impulsive Systems," (Russian), Moscow, Mir, 1971.

[21] J. Hale , "Functional Differential Equations," Springer, New-York, 1971.

[22] A. M. Fink, "Almost-periodic Differential Equations," Lecture Notes in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[23] Y. Kuang, "Delay Differential Equations with Applications in Population Dynamics," Academic Press, Boston, New Yorke, 1993.

[24] M. Pinto, *Asymptotic equivalence of nonlinear and quasi linear differential equations with piecewise constant arguments*, Math. Comput. Modelling, **49** (2009), 1750–1758.

[25] A. Samoilenko and N. Perestyuk, "Impulsive Differential Equations," World Scientific, Singapore, 1995.

[26] S. M. Shah and J. Wiener, *Advanced differential equations with piecewise constant argument deviations*, Int. J. Math. Math. Sci., **6** (1983), 671–703.

[27] G. Seifert, *Second-order neutral delay-differential equations with piecewise constant time dependence*, J. Math. Anal. Appl., **281** (2003), 1–9.

[28] G. Seifert, *Almost periodic solutions of certain differential equations with piecewise constant delays and almost periodic time dependence*, J. Differential Equations, **164** (2000), 451–458.

[29] G. Wang, *Periodic solutions of a neutral differential equation with piecewise constant arguments*, J. Math. Anal. Appl., **326** (2007), 736–747.

[30] G. Q. Wang and S. S. Cheng, *Note on the set of periodic solutions of a delay differential equation with piecewise constant argument*, Int. J. Pure Appl. Math., **9** (2003), 139–143.

[31] L. Wang, R. Yuan and C. Zhang, *A spectrum relation of almost periodic solution of second order scalar functional differential equations with piecewise constant argument*, Acta Mathematica Sinica, English Series, **27** (2011), 2275–284.

[32] Y. Wang and J. Yan, *A necessary and sufficient condition for the oscillation of a delay equation with continuous and piecewise constant arguments*, Acta Math. Hungar., **79** (1998), 229–235.

[33] Y. Wang and J. Yan, *Necessary and sufficient condition for the global attractivity of the trivial solution of a delay equation with continuous and piecewise constant arguments*, Appl. Math. Lett., **10** (1997), 91–96.

[34] D. Wexler, *Solutions périodiques et presque-périodiques des systémes d'équations différetielles linéaires en distributions*, J. Differential Equations., **2** (1966), 12–32.

[35] J. Wiener, "Generalized Solutions of Functional Differential Equations," World Scientific, Singapore, 1993.

[36] R. Yuan, *The existence of almost periodic solutions of retarded differential equations with piecewise argument*, Nonlinear Analysis, Theory, Methods and Applications, **48** (2002), 1013–1032.

[37] R. Yuan, *On the spectrum of almost periodic solution of second order scalar functional differential equations with piecewise constant argument*, J. Math. Anal. Appl., **303** (2005), 103–118.

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