

## Homoclinical structure of the chaotic attractor

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### ABSTRACT

In the reference [Akhmet MU. Devaney chaos of a relay system. Commun Nonlinear Sci Numer Simulat 2009;14:1486–93.], a relay system was introduced, which admits a chaotic attractor with Devaney's ingredients. Now, we prove that the attractor consists of homoclinic solutions. A simulation of the attractor is provided for a pendulum equation. Similar results for impulsive differential equations were announced in the plenary talk [Akhmet MU. Hyperbolic sets of impact systems. Dyn Contin Discrete Impuls Syst Ser A Math Anal 2008;15(Suppl. S1):1–2. Proceedings of the 5th international conference on impulsive and hybrid dynamical systems and applications, Beijing: Watan Press; 2008.].

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## 1. Introduction and preliminaries

The famous homoclinical structure of H. Poincaré initiated the fundamental investigations of complex dynamical systems by G. Birkhoff, M.L. Cartwright, J.E. Littlewood, N. Levinson, S. Smale [3,4] and their followers.

In this paper we investigate the structure of the attractor, which consists of bounded solutions [1]. The main object of our investigation is the following special initial value problem:

$$\begin{aligned} z'(t) &= Az(t) + f(t, t_0), \\ z(t_0) &= z_0, (t_0, z_0) \in A \times \mathbb{R}^n, \end{aligned} \quad (1)$$

where  $z \in \mathbb{R}^n$ ,  $t \in \mathbb{R}$ ,  $i \in \mathbb{Z}$ ,  $\mathbb{R}$  and  $\mathbb{Z}$  are sets of all real numbers and integers, respectively,

$$f(t, t_0) = \begin{cases} m_0 & \text{if } \zeta_{2i}(t_0) < t \leq \zeta_{2i+1}(t_0), i \in \mathbb{Z}, \\ m_1 & \text{if } \zeta_{2i-1}(t_0) < t \leq \zeta_{2i}(t_0), i \in \mathbb{Z}, \end{cases}$$

where  $m_0, m_1 \in \mathbb{R}^n$  are fixed vectors. Cantor set  $A \subset [0, 1]$  and the strictly increasing sequence  $\zeta(t_0) = \{\zeta_i(t_0)\}$ ,  $i \in \mathbb{Z}$ ,  $i \leq \zeta_i(t_0) \leq i + 1$ , will be described in the next section.

We shall use the following definition [1] of solutions of (1). It coincides with the definition for differential equations with piecewise constant arguments of generalized type [5], see also [6].

A function  $z(t)$ ,  $z(t_0) = z_0$ , is a solution of (1) on  $\mathbb{R}$  if: (i)  $z(t)$  is continuous on  $\mathbb{R}$ ; (ii) the derivative  $z'(t)$  exists at each point  $t \in \mathbb{R}$  with the possible exception of the points  $\zeta_i(t_0)$ ,  $i \in \mathbb{Z}$ , where one-sided derivatives exist; (iii) Eq. (1) is satisfied on each interval  $(\zeta_i(t_0), \zeta_{i+1}(t_0))$ ,  $i \in \mathbb{Z}$ .

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It can be easily verified that problem (1) has a unique solution for each  $t_0 \in A, z_0 \in \mathbb{R}^n$ .

In what follows we denote by  $z(t, \zeta, \nu), \zeta \in \mathbb{R}, \nu \in \mathbb{R}^n$ , a solution of (1) with  $t_0 = \zeta, z_0 = \nu$ .

We assume that  $A$  has the box-diagonal form so that  $A = \text{diag}\{A_-, A_+\}$ , where all eigenvalues of  $A_+$  have positive real part, and all eigenvalues of  $A_-$  have negative real part. Consequently, there exist positive numbers  $N$  and  $\omega$  such that

$$\|e^{A_- t}\| \leq Ne^{-\omega t}, \quad t \geq 0, \quad \|e^{A_+ t}\| \leq Ne^{\omega t}, \quad t \leq 0. \tag{2}$$

Let us denote  $Z(t, s) = \text{diag}\{Z_-(t, s), Z_+(t, s)\}, Z_-(t, s) = e^{A_-(t-s)}, Z_+(t, s) = e^{A_+(t-s)}, t, s \in \mathbb{R}$ . Using the standard technique one can verify that for every  $t_0 \in A$  there exists a unique vector  $\nu_0 \in \mathbb{R}^n$  such that  $z(t, t_0, \nu_0)$  is a bounded on  $\mathbb{R}$  solution of (1). Denote  $z(t, t_0) = z(t, t_0, \nu_0)$ , and  $z(t, t_0) = (u(t, t_0), \nu(t, t_0)), u \in \mathbb{R}^m, \nu \in \mathbb{R}^{n-m}$ . If we denote  $f(t, t_0) = (f_-(t, t_0), f_+(t, t_0))$ , then

$$u(t, t_0) = \int_{-\infty}^t Z_-(t, s) f_-(s, t_0) ds, \\ \nu(t, t_0) = - \int_t^{\infty} Z_+(t, s) f_+(s, t_0) ds,$$

and  $\|z(t, t_0)\| < \frac{2N \max\{\|m_0\|, \|m_1\|\}}{\omega}, t \in \mathbb{R}$ . Denote  $\mathcal{CB} = \{z(t, t_0) : t_0 \in A\}$ . Thus, we have that all bounded on  $\mathbb{R}$  solutions of (1) are placed in the tube with the radius  $M_1 = \frac{2N \max\{\|m_0\|, \|m_1\|\}}{\omega}$ .

One can easily verify (see also [5]) that a solution  $z(t)$  of (1) is bounded on  $[0, \infty)$  if and only if  $z(t) = (u(t), \nu(t)) = z(t, t_0, z_0), z_0 = (u_0, \nu_0)$ ,

$$u(t) = Z_-(t, t_0)u_0 + \int_{t_0}^t Z_-(t, s) f_-(s, t_0) ds, \\ \nu(t) = - \int_t^{\infty} Z_+(t, s) f_+(s, t_0) ds. \tag{3}$$

We denote the solutions defined by (3) as  $z(t, t_0, u_0)$ . Then  $\mathcal{C} = \{z(t, t_0, u_0) : t_0 \in A, u_0 \in \mathbb{R}^m\}$  is the set of all solutions of (1) bounded on  $[0, \infty)$ . One can confirm that

$$\|z(t, t_0, u_0) - z(t, t_0)\| < Ne^{-\omega(t-t_0)} (\|u_0\| + \max\{\|m_0\|, \|m_1\|\} / \omega), \quad t \geq t_0. \tag{4}$$

That is, every solution  $z(t, t_0, u_0) \in \mathcal{C} \setminus \mathcal{CB}$  is attracted by a bounded solution  $z(t, t_0) \in \mathcal{CB}$ . These solutions have a common set of discontinuity points  $\zeta(t_0)$ . Thus,  $\mathcal{CB}$  is an attractor with the basin  $\mathcal{C}$ . Obviously,  $\mathcal{CB} \subset \mathcal{C}$ . In [1] it was proved that  $\mathcal{CB}$  admits all Devaney's ingredients of chaos. In what follows we shall show that the attractor has the homoclinic structure.

### 2. Hyperbolic sets of maps

Consider the sequence space [7]  $\Sigma_2 = \{s = (s_0 s_1 s_2 \dots) : s_j = 0 \text{ or } 1\}$ , with the metric

$$d[s, t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i},$$

where  $t = (t_0 t_1 \dots) \in \Sigma_2$ , and the shift map  $\sigma : \Sigma_2 \rightarrow \Sigma_2$ , such that  $\sigma(s) = (s_1 s_2 \dots)$ . The semidynamics  $(\Sigma_2, \sigma)$  is the symbolic dynamics [8].

We assume that there exist a homeomorphism  $S$  between  $\Sigma_2$  and a set  $A \subset I, I = [0, 1]$ , and a map  $h : A \rightarrow A$ , such that  $S \circ h = \sigma \circ S$ . That is  $h$  and  $\sigma$  are topologically conjugate. It is known that  $\Sigma_2$  and  $A$  are Cantor sets [8]. Obviously, they are compact.

One of the most popular examples of the map  $h$  is the logistic map  $\mu x(1 - x), \mu > 4$ , considered on a subset of  $[0, 1]$ , [7].

For every  $t_0 \in A$  one can construct a sequence  $\kappa(t_0)$  of real numbers  $\kappa_i, i \in \mathbb{Z}$ , in the following way. If  $i \geq 0$ , then  $\kappa_{i+1} = h(\kappa_i)$  and  $\kappa_0 = t_0$ . Let us show, how the sequence is defined for negative  $i$ . Denote  $s^0 = S(t_0), s^0 = (s_0^0 s_1^0 \dots)$ . Consider elements  $\underline{s} = (0s_0^0 s_1^0 \dots), \bar{s} = (1s_0^0 s_1^0 \dots)$  of  $\Sigma_2$ , such that  $\sigma(\underline{s}) = s^0$  and  $\underline{s} = S^{-1}(s^0), \bar{s} = S^{-1}(\bar{s})$ . The homeomorphism implies that  $h(\bar{s}) = h(\underline{s}) = t_0$ . Set  $h^{-1}(t_0)$  may consist of not more than two elements  $\bar{t}, \underline{t} \in A$ . Each of these two values can be chosen as  $\kappa_{-1}(t_0)$ . Obviously, one can continue the process to  $-\infty$ , choosing always one element from the set  $h^{-1}$ . We have finalized the construction of the sequence, and, moreover, it is proved that  $\kappa(t_0) \subset A$ .

Thus, infinitely many sequences  $\kappa(t_0)$  can be constructed for a given  $t_0$ . However, each of this type of sequence is unique for an increasing  $i$ . Fix one of the sequences and define a sequence  $\zeta(t_0) = \{\zeta_i\}, \zeta_i = i + \kappa_i, i \in \mathbb{Z}$ . If we denote by  $\Pi$  the set of all such sequences  $\zeta = \{\zeta_i\}, i \in \mathbb{Z}$ , then a multivalued functional  $w : A \rightarrow \Pi$  such that  $\zeta(t_0) = w(t_0)$  is defined. Thus, the sequence of discontinuities,  $\zeta(t_0)$ , in (1) is an element of  $\Pi$ .

Denote by  $\Sigma_2^2 = \{s = (\dots s_{-2} s_{-1} \cdot s_0 s_1 s_2 \dots) : s_j = 0 \text{ or } 1\}$  the space of bi-sequences. The discussion indicates that homeomorphism  $S$  arranges one-to-one correspondence  $\Phi$  between  $\Sigma_2^2$  and the elements of space  $\Pi$  such that if  $\Phi(s) = \zeta$ , then  $\zeta_j = j + S((s_j s_{j+1} \dots)), j \in \mathbb{Z}$ .

Fix  $s^* \in \Sigma_2^2$ . We denote  $W^s(s^*)$  a set of all elements  $s \in \Sigma_2^2$ , which entries agree with those of  $s^*$  to the right of some entry of  $s^*$ . Set  $W^s(s^*)$  is the stable set of  $s^*$ . Similarly, we denote  $W^u(s^*)$  a set of all elements  $s \in \Sigma_2^2$ , which entries agree with those of  $s^*$  to the left of some entry of  $s^*$ . Set  $W^u(s^*)$  is the unstable set of  $s^*$ . A point  $s \in \Sigma_2^2$ , whose entries agree with those of  $s^*$  to the right and to the left of some entries of  $s^*$  is a homoclinic sequence to  $s^*$ . It is clear that the homoclinic sequence satisfies

$s \in W^s(s^*) \cap W^u(s^*)$ . If  $s \in W^s(s^*) \cap W^u(s^{**})$ , where  $s, s^*, s^{**} \in \Sigma_2^2$ , then  $s$  is a heteroclinic sequence to  $s^*$  and  $s^{**}$ . Since each point of  $\Sigma_2^2$  is equipped with a stable and unstable set we call it the hyperbolic set.

Consider the set  $\Pi$  again. We say that  $\eta \in \Pi$  is *homoclinic* to  $\zeta \in \Pi$  if  $|\eta_i - \zeta_i| \rightarrow 0$  as  $i \rightarrow \pm\infty$ , and  $\eta \in \Pi$  is *heteroclinic* to  $\zeta \in \Pi, \phi \in \Pi$  if  $|\eta_i - \zeta_i| \rightarrow 0$  as  $i \rightarrow \infty, |\eta_i - \phi_i| \rightarrow 0$  as  $i \rightarrow -\infty$ . For a given  $\zeta \in \Pi, W^s(\zeta) = \{\eta : |\eta_i - \zeta_i| \rightarrow 0 \text{ as } i \rightarrow \infty, \}, W^u(\zeta) = \{\eta : |\eta_i - \zeta_i| \rightarrow 0 \text{ as } i \rightarrow -\infty\}$ .

**Theorem 2.1.** *The following assertions are valid.*

1. Set  $\Pi$  is hyperbolic. That is, sets  $W^s(\zeta), W^u(\zeta)$  are non-empty for each  $\zeta \in \Pi$ .
2. Each sequence from  $\Pi$  is homoclinic to a sequence from  $\Pi$ .
3.  $W^s(\zeta) \cap W^u(\zeta) \neq \emptyset$  for all  $\zeta \in \Pi$ .

**Proof.** By using the compactness of sets  $\Lambda$  and  $\Sigma_2$ , the homeomorphism of  $S$  and the definition of  $\Phi$  one can obtain that for a given  $\zeta \in \Pi, W^s(\zeta) = \{\eta : \Phi^{-1}(\eta) \in W^s(\Phi^{-1}(\zeta))\}, W^u(\zeta) = \{\eta : \Phi^{-1}(\eta) \in W^u(\Phi^{-1}(\zeta))\}$ . Moreover, a sequence  $\eta \in \Pi$  is homoclinic to  $\zeta \in \Pi$  if and only if  $\Phi^{-1}(\eta)$  is homoclinic to  $\Phi^{-1}(\zeta)$ , and  $\eta \in \Pi$  is heteroclinic to  $\zeta, \phi \in \Pi$  if and only if  $\Phi^{-1}(\eta)$  is heteroclinic to  $\Phi^{-1}(\zeta)$  and  $\Phi^{-1}(\phi)$ . Similarly on can verify that the third assertion is valid. The theorem is proved.  $\square$

### 3. The main result

We shall say that a solution  $z(t, t_1) \in \mathcal{CB}$  belongs to the stable (unstable) set  $W^s(z(t, t_0))(W^u(z(t, t_0)))$  of solution  $z(t, t_0) \in \mathcal{CB}, t_1 \neq t_0$ , if for an arbitrary  $\epsilon > 0$  there exists a number  $T(\epsilon)$  such that  $\|z(t, t_1) - z(t, t_0)\| < \epsilon$ , for all  $t$  from  $(T, \infty)((-\infty, T))$ .

A solution  $z(t, t_1) \in \mathcal{CB}$  is *homoclinic* to solution  $z(t, t_0) \in \mathcal{CB}, t_1 \neq t_0$ , if  $z(t, t_1) \in W^s(z(t, t_0)) \cap W^u(z(t, t_0))$ .

A solution  $z(t, t_2) \in \mathcal{CB}$  is *heteroclinic* to solutions  $z(t, t_0), z(t, t_1) \in \mathcal{CB}, t_2 \neq t_0, t_2 \neq t_1$ , if  $z(t, t_2) \in W^s(z(t, t_0)) \cap W^u(z(t, t_1))$ .

Set  $\mathcal{CB}$  is *hyperbolic* if each solution of this set has stable and unstable sets.

**Theorem 3.1.** *The following assertions are valid.*

1. Each element of  $\mathcal{CB}$  is homoclinic to an element of  $\mathcal{CB}$ .
2. Set  $\mathcal{CB}$  is hyperbolic.
3.  $W^s(z(t, t_0)) \cap W^u(z(t, t_0)) \neq \emptyset$  for all  $t_0 \in \Lambda$ .

**Proof.** We prove only the first assertion of the theorem as the verification of other assertions is very similar. Consider a solution  $z_1(t) = z(t, t_1) \in \mathcal{CB}$ . By **Theorem 2.1** there exists  $t_0 \in \Lambda$ , such that sequence  $\zeta(t_1) \in \Pi$  is homoclinic to  $\zeta(t_0) \in \Pi$ . We shall prove that  $z(t, t_1)$  is homoclinic to  $z_0(t) = z(t, t_0) \in \mathcal{CB}$ .

Assume, without loss of generality, that  $\zeta_i(t_0) \leq \zeta_i(t_1) \forall i \in \mathbb{Z}$ . Fix a positive number  $\epsilon$ , and a positive number  $\delta$  sufficiently small so that  $8N(\|m_0\| + \|m_1\|)\delta \frac{e^{2\omega}}{1 - e^{-2\omega}} < \epsilon$ . Moreover, we can find a number  $\bar{T} > 0$ , such that  $|\zeta(t_1) - \zeta(t_0)| < \delta$  if  $\bar{T} < \zeta_i(t_0) \leq \zeta_i(t_1)$ .

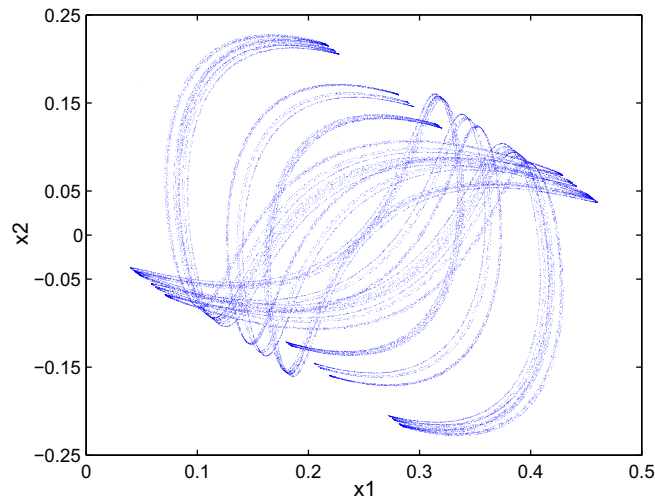
If  $z(t, \xi) = (u(t), v(t)) \in \mathcal{CB}$ , then for arbitrary  $t, \tau \in \mathbb{R}, t > \tau$ ,

$$u(t) = Z_-(t, \tau)u(\tau) + \int_{\tau}^t Z_-(t, s)f_-(s, \xi)ds,$$

$$v(t) = - \int_{\tau}^{\infty} Z_+(t, s)f_+(s, \xi)ds.$$

By using the last equations we have, for  $z_1(t) = (u_1, v_1), z_0(t) = (u_0, v_0)$ , that

$$\begin{aligned} \|z_1(t) - z_0(t)\| &= \|u_1(t) - u_0(t)\| + \|v_1(t) - v_0(t)\| \\ &\leq \|u_1(\bar{T}) - u_0(\bar{T})\| \|Z_-(t, \bar{T})\| + \int_{\bar{T}}^t \|Z_-(t, s)\| \|f_-(s, t_0) - f_-(s, t_1)\| ds + \int_{-\infty}^{\bar{T}} \|Z_-(t, s)\| \|f_-(s, t_0)\| ds \\ &\leq 2Ne^{-\omega(t-\bar{T})} M_1 + \sum_{\bar{T} \leq \zeta_j(t_1) < t} \left[ \int_{\zeta_{2j}(t_0)}^{\zeta_{2j}(t_1)} 2Ne^{-\omega(t-s)} (\|m_0\| + \|m_1\|) ds + \int_{\zeta_{2j+1}(t_0)}^{\zeta_{2j+1}(t_1)} 2Ne^{-\omega(t-s)} (\|m_0\| + \|m_1\|) ds \right] \\ &\quad + \int_{-\infty}^{\bar{T}} e^{-\omega(t-s)} (\|m_0\| + \|m_1\|) ds \\ &\leq N[2M_1 e^{-\omega(t-\bar{T})} + 4\delta \frac{e^{2\omega} (\|m_0\| + \|m_1\|)}{1 - e^{-2\omega}} + (\|m_0\| + \|m_1\|) e^{-\omega(t-\bar{T})} 1/\omega]. \end{aligned}$$



**Fig. 1.** The chaotic attractor by a stroboscopic sequence  $(x_1(n), x_2(n))$ ,  $1 \leq n \leq 50,000$ , is observable.

Now, if  $\tilde{T} \geq \bar{T}$  is sufficiently large so that

$$2NM_1 e^{-\omega(\tilde{T}-\bar{T})} + N(\|m_0\| + \|m_1\|) e^{-\omega(\tilde{T}-\bar{T})} \frac{1}{\omega} < \frac{\epsilon}{2},$$

then  $\|z_1(t) - z_0(t)\| < \epsilon$  for all  $t > \tilde{T}$ . Similarly, one can show that  $z_1(t) \in W^u(z(t), t_0)$ . The theorem is proved.  $\square$

**Remark 3.1.** By applying the technique of paper [9] one can extend the main result of the present manuscript for quasilinear relay systems. Its analogue for impulsive systems is proved in Ref. [2].

**Example 3.1.** Let the following pendulum equation be given

$$x'' + 3x' + 2x = f_2(t, t_0), \quad (5)$$

where  $f_2(t, t_0)$  is a scalar pulse function with  $m_0 = 1, m_1 = 0$ . Using new variables  $x_1 = x, x_2 = x'$ , one can reduce (5) to the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -2x_1 - 3x_2 + f_2(t, t_0). \end{aligned} \quad (6)$$

Both eigenvalues of the matrix of coefficients have negative real parts. Consider the sequence  $\zeta_i = i + \kappa_i, \kappa_i = 4\kappa_{i-1}(1 - \kappa_{i-1}), \kappa_0 = t_0 = 0.5$  and take a solution  $(x_1(t), x_2(t))$  of the last system with the initial condition  $x_1(t_0) = 0.02, x_2(t_0) = -0.025$ . In Fig. 1 the chaotic attractor is shown by using points  $(x_1(n), x_2(n)), n = 1, 2, 3, \dots, 50,000$ , in  $x_1, x_2$ -plane. In fact, the stroboscopic sequence of a solution from  $\mathcal{CB}$  is obtained.

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