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Non-autonomous grazing phenomenon

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Abstract Non-autonomous grazing phenomenon is investigated through periodic systems and their solutions. The analysis is different than for autonomous systems in many aspects. Conditions for the existence of a linearization have been found. Stability of a periodic solution and its persistence under regular perturbations are investigated. Through examples, the theoretical results are visualized.

Keywords Non-autonomous impulsive system · Grazing point · Linearization around a grazing solution · Periodic grazing solution · Stability · Regular perturbations

1 Introduction

Vibrating systems are throughly investigated in the literature and there can be found many applications in mechanics, electronics and medicine [1–7]. The stability of the periodic solutions of such systems and the appearance of chaos became popular in the analysis of such systems. Then, some of the scientists keep on analyzing such systems with impacts. The vibrating systems which admit recurring impacts during their motions are called vibro-impacts systems. Such systems take the attention of many scientists through history [8–35]. It is considered in two different ways. First one is that the impact occurs whenever the vibrating mass coincides with the barrier transversely [8–21] and another is that the impact occurs when it coincides with the barrier with zero velocity or tangentially [22–35]. In the literature, the point when the oscillator meets the barrier tangentially or with zero velocity is called grazing point. For the first type of motions, the existing theory of the impact mechanisms are enough [12,37] to analyze the dynamical properties around the solutions. However, for systems with grazing the existing theory is inappropriate for the analysis. For this reason, scientists search some other conditions which facilitate the analysis of the dynamics around grazing point and it gets great attention for the theoretical analysis as well as applications [22–35]. In literature, grazing is understood as a particular case which makes the dynamics around it complicated such as the appearance of chaos through period adding [32]. Many investigations are conducted on grazing, and some of them are; in [34], the existence of periodic solution for the higher-dimensional mechanical systems is investigated, and in the studies [10,22,23], the authors define the grazing bifurcation for the systems of differential equations with discontinuous right-hand side and analyze the stability of periodic grazing solutions, and in [24], grazing

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is defined as a bounding case which divides the regions with quite different dynamical behaviors and point, and the system trajectory makes tangential contact with an event. It is observed that by finding smallest parameter alteration necessary to induce grazing, a basis for an optimization technique is obtained.

In the literature, two different approaches have been utilized to define grazing phenomenon. One of them is that grazing is the case when a trajectory meets with zero velocity to the surface of discontinuity [32–34]. The other is that the trajectory meets the surface of discontinuity tangentially [11,22,23,31]. In the present paper, to develop theory for non-autonomous systems with grazing points, we will take into account the comprehension of the authors who assert that the solutions intersect the surface of discontinuity tangentially at the grazing point. Our way of investigation is maximally close to the way of investigations in ordinary differential equations. In the paper [36], we present a new approach for the analysis of grazing phenomenon which is based on the fundamental definitions of the papers [10,11,22,23,31], but from researches we obtain a different procedure such as constructing a linearization by means of the near solutions around the periodic solution. This idea is effective not only for autonomous systems with impacts but also for other systems with discontinuities. In the present paper, in the light of these ideas, we extend our results for non-autonomous systems with impacts. This extension requests new definitions as well as different from [36] discussion to obtain constructive results. Our ideas is far more different than the existing results [22–35] since the previous results are mainly rely on the reduction of the dynamics to an analysis through mappings. In our case, the investigations has been done not for discrete moments of time; they have been conducted on the all time which includes the process duration. Additionally, by harmonizing the vector field, the jump function, and the surfaces of discontinuity, we have suppressed the singularity which is seen in the gradient at the grazing point in the system. Our results can be applied for various mechanical problems this approves how the way of the analysis are useful.

Some non-autonomous systems are taken into account through history of the grazing phenomenon. They consist of a non-autonomous vector field with the autonomous surfaces of discontinuity. It is the first time in literature that the systems with non-autonomous surfaces are taken into account for the grazing. In this present paper, we will take into account a special non-autonomous system with the non-autonomous surfaces of discontinuity such that the surfaces of impacts are defined as follows $t = \tau_i(x),\ i \in \mathbb{Z}$. For such systems, we introduce definitions such as a grazing point for the non-autonomous system, the continuous/ discontinuous grazing point, and a proper linearization for non-autonomous impulsive systems near the periodic solution which has grazing points constructed. Moreover, the theoretical results are supported by examples and simulations.

The rest of the paper is divided into six parts. The next section covers information about the non-autonomous systems, the definitions for a grazing point, continuous and discontinuous ones, a grazing solution, sufficient conditions for the existence, uniqueness of solutions of these systems, and the B equivalence method which is an important tool in the analysis of the stability of non-autonomous systems. The third section is the motivation of the present research. In the fourth section, we consider linearization around the grazing periodic solution of non-autonomous systems. The fifth one contains the examples with simulations which support the theoretical results. The next section covers the regular perturbations of the grazing solution, and the last one is the summary of our work and also presents future works related with our subject of discussion.

2 Preliminaries

Let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$ be the sets of all real numbers, natural numbers, and integers, respectively. Let $G \subset \mathbb{R}^n$ be an open, bounded, and connected set. Introduce the following system of differential equations with variable moments of impulses

$$x' = f(t, x),$$
$$\Delta x|_{t=\tau_i(x)} = J_i(x), \tag{1}$$

where $(t, i, x) \in \mathbb{R} \times \mathbb{Z} \times G$, the function $f(t, x)$ is continuous on $\mathbb{R} \times G$, continuously differentiable in x and T-periodic, i.e., $f(t + T, x) = f(t, x)$, functions J_i and $\tau_i(x)$, $i \in \mathbb{Z}$ are defined on G and continuously differentiable on G. The following equality $J_{i+p} = J_i$ for a natural number p is valid and $\tau_i(x)$ has (T, p)-property, i.e., $\tau_i(x)+T = \tau_{i+p}(x)$ for all $i \in \mathbb{Z}$, $x \in G$. Denote by $I_i(x) = J_i(x) + x$.

Consider a solution $x(t)$ of (1). Denote θ_i, $i \in \mathbb{Z}$, if $\theta_i = \tau_i(x(\theta_i))$. That is, $t = \theta_i$ is the moment of the intersection of the solution $x(t)$ with the surface $t = \tau_i(x)$. Regardless, if $x(t)$ has a discontinuity at the moment or not, we call the $t = \theta_i$ the *moment of discontinuity*.

Denote by $\nabla\tau_i(x) = \left(\frac{\partial\tau_i(x)}{\partial x_1}, \frac{\partial\tau_i(x)}{\partial x_2}, \ldots, \frac{\partial\tau_i(x)}{\partial x_n}\right)$ the gradient of the function $\tau_i(x)$ and let $\langle , \rangle$ be the usual dot product. Let us introduce the main object of the present discussion.

Definition 1 A point $(\theta_i, x(\theta_i))$, $i \in \mathbb{Z}$, is a grazing point if $\langle\nabla\tau_i(x(\theta_i)), f(\theta_i, x(\theta_i))\rangle = 1$. It is a continuous grazing point provided $I(x(\theta_i)) = 0$, otherwise it is discontinuous one.

Definition 2 A solution $x(t)$ of (1) is grazing if it has a grazing point $(\theta_i, x(\theta_i))$.

Assume that (1) admits a grazing T-periodic solution $\Psi(t)$ with discontinuity moments θ_i, $i \in \mathbb{Z}$, such that $\theta_{i+p} = \theta_i + T$, $i \in \mathbb{Z}$.

Consider the system of ordinary differential equations

$$x' = f(t, x), \tag{2}$$

which is a part of (1).

Let us formulate the following conditions.

(N1) For any $c \in G$, $i \in \mathbb{Z}$, the inequality $\tau_i(c + J_i(c)) < \tau_i(c)$ is valid;
(N2) for all $x \in G$, $\tau_i(x) < \tau_{i+1}(x)$.

In what follows, let $\|\cdot\|$ be the Euclidean norm, that is for a vector $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$, the norm is equal to $\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

(N3) There exist positive numbers C and N with $CN < 1$ such that

$$\max_{(t,x)\in\mathbb{R}\times G} \|f(t,x)\| \le C, \quad \max_{x\in G} \left\|\frac{\partial\tau_i(x)}{\partial x}\right\| \le N.$$

(N4) for all $x \in G$ and $i \in \mathbb{Z}$, $\max_{0\le\sigma\le 1}\left\langle \frac{\partial\tau_i(x+\sigma I_i(x))}{\partial x}, I_i(x)\right\rangle \le 0.$

If conditions (N1)–(N4) are fulfilled, then every solution $x(t) : I \to G$ of (1) intersects each of the surfaces of discontinuity $t = \tau_i(x)$, $i \in \mathbb{Z}$, at most once [37].

A function $\phi(t) : \mathbb{R} \to \mathbb{R}^n$, $n \in \mathbb{N}$, is from the set $PC(\mathbb{R}, \theta)$ if it : (i) is left continuous, (ii) is continuous, except, possibly, points of θ, where θ is the set of discontinuity points of the function $\phi(t)$ and it has discontinuities of the first kind at that points.

A function $\phi(t)$ is from the set $PC^1(\mathbb{R}, \theta)$ if $\phi(t)$, $\phi'(t) \in PC(\mathbb{R}, \theta)$, where the derivative at points of θ is assumed to be the left derivative. If $\phi(t)$ is a solution of (1), then it is required that it belongs to $PC^1(\mathbb{R}, \theta)$ [37].

Denote by $\widehat{[a, b]}$, $a, b \in \mathbb{R}$, the interval $[a, b]$, whenever $a \le b$ and $[b, a]$, otherwise. Let $x_1(t) \in PC(\mathbb{R}_+, \theta^1)$, $\theta^1 = \{\theta_i^1\}$, and $x_2(t) \in PC(\mathbb{R}_+, \theta^2)$, $\theta^2 = \{\theta_i^2\}$, be two different solutions of (1).

Definition 3 The solution $x_2(t)$ is in the ϵ-neighborhood of $x_1(t)$ on the interval I if

* $|\theta_i^1 - \theta_i^2| < \epsilon$ for all $\theta_i^1 \in \mathbb{R}$;
* the inequality $||x_1(t) - x_2(t)|| < \epsilon$ is valid for all t, which satisfy $t \in \mathbb{R} \setminus \cup_{\theta_i^1\in\mathbb{R}}(\theta_i^1 - \epsilon, \theta_i^1 + \epsilon)$.

The topology defined with the help of ϵ-neighborhoods is called the B-topology [37]. One can easily see that it is Hausdorff and it can be considered also if two solutions $x_1(t)$ and $x_2(t)$ are defined on a semi-axis or on the entire real axis.

To analyze the system with variable moments of impulses is hard. To make our investigations easier, an important method is presented in [37] which reduces the systems with variable moments of impulses to those with fixed moments of impulses. The system with fixed moment of impulses is named a *B-equivalent system* to the system with variable moments of impulses. In order to construct a B-equivalent system near the integral curve of $\Psi(t)$, we will consider the following way.

Consider a point $(\theta_i, x) \in \mathbb{R} \times G$ with a fixed ξ, on the periodic solution with a fixed $i \in \mathbb{Z}$. Let $\xi_i = \xi_i(x)$ be the meeting moment of the solution $x(t) = x(t, \theta_i, x)$ of (2) with the surface of discontinuity $t = \tau_i(x)$. Additionally, assume that the solution $x_1(t) = x(t, \theta_i, x(\theta_i))$ of (2) exists on $\widehat{[\theta_i, \xi_i]}$. The map $W : x \to x_1(\xi)$ can be constructed as

$$W_i(x) = \int_{\theta_i}^{\xi_i} f(u, x(u))\mathrm{d}u + J_i(x + \int_{\theta_i}^{\xi_i} f(u, x(u))\mathrm{d}u) + \int_{\xi_i}^{\theta_i} f(u, x_1(u))\mathrm{d}u. \tag{3}$$

Let us take into consideration the following system of differential equations with fixed moments of impulses

$$
\begin{aligned}
&y' = f(t, y),\\
&\Delta y|_{t=\theta_i} = W_i(y),
\end{aligned}
\tag{4}
$$

which is B-equivalent in $G \subset \mathbb{R}^n$ to (1). It is easy to see that $\Psi(t)$ is also a solution of (4) as well. In the following part, we will consider the system (4) to construct a linearization system around $\Psi(t)$.

Since of the way of construction of $W_i(x)$ systems (1) and (4) are B-equivalent in the neighborhood of $\Psi(t)$. That is, if $x(t) : U \to G$ is a solution of (1), then is coincides with a solution $y(t) : U \to G$ when $y(t_0) = x(t_0)$, for $t_0 \in U \setminus \cup_{i\in\mathbb{Z}}\widehat{[\theta_i, \xi_i]}$. Particularly, $x(\theta_i) = y(\theta_i+),\ x(\xi_i) = y(\xi_i),\ if\ \theta_i > \xi_i,\ x(\theta_i) = y(\theta_i),\ x(\xi_i+) = y(\xi_i),\ if\ \theta_i < \xi_i.$

3 Motivation

Vibro-impact system is the term used to present a system which is driven in some way and which also exhibits an intermittent or continuous sequence of contacts with limiting constraints of the motion. Vibro-impact systems involve multiple impact interactions in the form of jumps in the state space. The dynamics and properties of vibro-impact systems and specifications of nonlinear phenomena with discontinuity have been investigated in the literature for decades [8–13,16,19–21,23–27,39]. Compared with a single impact, the nonlinear dynamics of vibro-impact systems are more complicated. The trajectories of such systems have discontinuities, which are caused by the impacts, in phase space. Although the presence of nonlinearity and discontinuity complicates the dynamic analysis of such systems, they can be described theoretically and numerically with discontinuities in good agreement with reality. Such systems with impacts appear in a wide variety of engineering applications. The operation of vibration hammers, impact dampers, inertial shakers, pile drivers, milling and forming machines, and other vibro-impact systems is based on the impact action for moving bodies [8–10,12,13,16,19–21,26]. Machines with clearances, heat exchangers, steam generator tubes, fuel rods in nuclear power plants, rolling railway wheel set, piping systems, granular gases, gear transmissions, and other such systems perform impacts. In these mechanisms, there are two types of impacts: a) when one of the colliding parts is motionless; b) there is no stationary colliding parts. Mostly, the first case is considered in literature. But the second one can also be discussed, if the law of motions for some of the parts are known [9,18,21]. These vibro-impact systems are most difficult for dynamical analysis. Our present results are developed for their research, since they are non-autonomous, not necessary in differential equations and equations of impacts, but in surfaces of discontinuity. In mechanics, the most famous model is a ball bouncing on a moving table [9,18,21], and it is not surprising, that namely for this mechanism chaos presence was explored [15]. In this part of the paper, we will consider the model, developed with two vibrating tables and a *grazing point*, to exhibit motivation for our research.

We will take into account a bouncing bead model which moves between two vibrating tables. The motions of the bottom table and above table are governed by $X_1(t) = \sin(t)$ and $X_2(t) = 50 + \sin(t + 1)$, respectively. In the remaining part of the example denote by $x = (x_1, x_2)$. The motion of the bead can be formulated mathematically by using the differential equation with non-fixed moments of impulses

$$
\begin{aligned}
&x_1' = x_2,\\
&x_2' = -9.8,\\
&\Delta x_2|_{t=\tau_i(x)} = -(1 + R_1)x_2,\\
&\Delta x_2|_{t=\eta_i(x)} = (1 + R_2)[\cos(\tau_2(x) + 1) - x_2],
\end{aligned}
\tag{5}
$$

where $\tau_i(x) = \arcsin(x_1) + 2\pi i,\ \eta_i(x) = \arcsin(x_1 - 50) - 1 + 2\pi i,\ R_1 = 1$ and $R_2 = 0.9$

Consider a solution $\Psi(t)$ of system (5) which starts the motion with initial value $(\zeta_1, \Psi(\zeta_1)) = (\frac{\pi}{2}, 1, 9.8\pi)$. It can be shown numerically that the solution is periodic. The integral curve of the solution is seen in Fig. 1. One can get

$$
\begin{aligned}
&\langle \nabla\tau_0(\Psi(\zeta_1)), f(\zeta_1, \Psi(\zeta_1))\rangle\\
&\quad = \langle (\frac{\partial\tau_0(\Psi(\zeta_1))}{\partial x_1}, \frac{\partial\tau_i(\Psi(\zeta_1))}{\partial x_1}), (x_0^2, -9.8)\rangle\\
&\quad = \langle (\frac{1}{\sqrt{1-(x_0^1)^2}}, 0), (x_0^2, -9.8)\rangle\\
&\quad = \langle (1, 0), (9.8\pi, -9.8)\rangle = 9.8\pi \neq 1.
\end{aligned}
$$

Thus, at the point the solution meets with the surface of discontinuity $t = \tau_i(x)$ transversally. The solution meets the surface of discontinuity $t = \eta_i(x)$ at

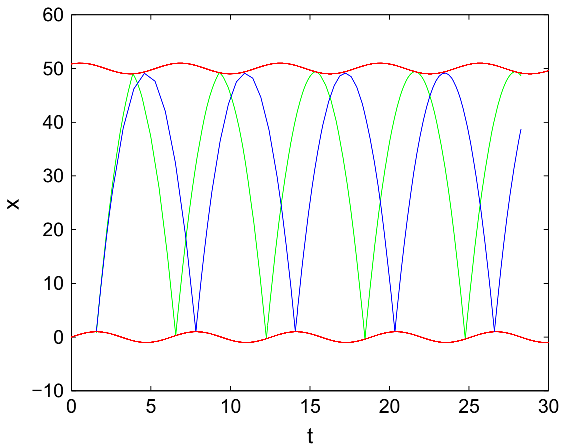

Fig. 1 The *green curve* is the solution of (5) which starts at the point $(\pi/2, 0.8, 32.2876)$ and the *blue one* is the grazing periodic solution. (Color figure online)

$(\zeta_2, \Psi(\zeta_2)) = (4.63127, 49.33, 0.795)$, and touches it with the same velocity as the surface has.

The integral curve is tangential to the surface, since

$$\langle \nabla\eta_0(\Psi(\zeta_2)), (\Psi_2(\zeta_2), -9.8)\rangle = \left\langle \left(\frac{1}{\sqrt{1-(\Psi_1(\zeta_2)-50)^2}}, 0\right), (\Psi_2(\zeta_2), -9.8) \right\rangle = 1.$$

Furthermore, there is no jump at the moment. Thus, we can conclude that, (4.63127, 49.33, 0.795), is a *continuous grazing point*. We have obtained numerically that the periodic solution is unstable. It is observed from Fig. 1.

4 Differentiability of the solutions

In this part of the paper, we will analyze the differential dependence of solutions on initial conditions for the differential equations with variable moments of impulses with emphasis on grazing points.

Denote by another solution $\bar{x}(t) = x(t, t_0, x_0 + \Delta x)$, $\Delta x = (\xi_1, \xi_2, \ldots, \xi_n)$, of (1) and η_i, $i \in \mathbb{Z}$ the moments of discontinuity of $\bar{x}(t)$. Denote by $B((t_0, x_0), \delta) \subset \mathbb{R} \times \mathbb{R}^n$ with center (t_0, x_0) and with a radius δ.

The solution $x(t)$ is B-differentiable with respect to x_0^j, $j = 1, 2, \ldots, n$, if there exist functions $u_{1i}(t) \in PC^1([t_0, T], \theta)$, $\{\theta_i\}_{i=1,2,\ldots,k}$ and constants ν_{1i}^l, $l = 1, 2, \ldots, k$ such that if $(t_0, x_0 + \Delta x) \in B((t_0, x_0), \delta) \cap G$ for a sufficiently small positive δ, then:

(A) there exist constants ν_{ij}, $i \in \mathbb{Z}$, such that

$$\theta_l - \eta_l = \sum_{i=0}^{n} \nu_{1i}^l \xi_i + o(\|\xi\|), \tag{6}$$

where $l = 1, 2, \ldots, k$;

(B) for all $t \notin \widehat{(\theta_i, \eta_i]}$, $i = 1, 2, \ldots, k$, the following equality is satisfied

$$\bar{x}(t) - x(t) = \sum_{i=0}^{n} u_{1i}(t)\xi_i + o(\|\xi\|), \tag{7}$$

where $u_{1i}(t) \in PC([t_0, T], \theta)$, $i=1, 2, \ldots, n$. The pair $\{u_{1i}(t), \{\nu_{1i}^l\}\}$ is said to be a B-derivative of $x(t)$ with respect to initial conditions.

Because of the complexity of analysis, which appears due to the grazing phenomenon, we will only discuss the linearization for periodic solutions.

The object of this section is to find conditions for the smoothness of the grazing solution. In other words, for the existence of linearization around a grazing periodic solution $\Psi(t)$ with a period T, and with discontinuity moments θ_i, $i = 1, 2, \ldots, p$, on the interval $[0, T]$.

We will construct the variational system in a neighborhood of the periodic solution $\Psi(t)$ as follows:

$$u' = A(t)u,$$
$$\Delta u|_{t=\theta_i} = D_i u(\theta_i), \tag{8}$$

where the matrix $A(t) \in \mathbb{R}^{n\times n}$ of the form $A(t) = \frac{\partial f(t,x)}{\partial x}|_{x=\Psi(t)}$. The matrices D_i, $i = 1, \ldots, n$ will be defined in the remaining part of the paper. Solutions of the variational Eq. (8) are the B-derivatives, $(u_j(t))$, $j = 1, 2, \ldots, n$. We will call the second equation in (8) a linearization at discontinuity moments, θ_i, $i \in \mathbb{Z}$. If the near solution meets the surfaces transversally, the differentiability properties of such systems are investigated widely in [37]. In the remainder, we will give a brief explanation for the transversal meeting and we seek to determine some sufficient conditions for the existence of the linearization around the grazing points.

4.1 Linearization at a transversal point

In this subsection, our aim is to give information about the matrices D_i and the gradient $\nabla\theta_i(x)$ if the discontinuity point $(\theta_i + jT, \Psi(\theta_i + jT))$, $j \in \mathbb{Z}$, is a transversal one which means $\nabla\tau_i(\Psi(\theta_i + jT)) f(\theta_i +$

$jT, \Psi(\theta_i + jT)) \neq 1$. In the following part of the paper, we will consider the discontinuity points, for $j = 0$, $(\theta_i, \Psi(\theta_i))$, as discontinuity moments. The linearization in these circumstances is described in [37].

Fix a transversal discontinuity point $(\theta_i, \Psi(\theta_i))$, $i = k+1, \ldots, p$. The following equation is driven by considering the equation $\theta_i(x) = \tau_i(x(\theta_i(x)))$, [37]

$$\nabla\theta_i(\Psi(\theta_i)) = \frac{\nabla\tau_i(\Psi(\theta_i))U(\theta_i)}{1 - \nabla\tau_i(\Psi(\theta_i))f(\theta_i, \Psi(\theta_i))}, \tag{9}$$

where $U(t)$, is a fundamental matrix of $u' = f_x(t, x(t))u$ with $U(\kappa) = I$, where I is $n \times n$ identity matrix.

By taking into account derivative of the B-map defined by (3) with respect to x, we can determine the matrix D_i as

$$D_i = \left[\frac{\partial W_i(\Psi(\theta_i))}{\partial x_1^0}, \frac{\partial W_i(\Psi(\theta_i))}{\partial x_2^0}, \ldots, \frac{\partial W_i(\Psi(\theta_i))}{\partial x_n^0}\right], \tag{10}$$

where the expression $\frac{\partial W_i(\Psi(\theta_i))}{\partial x_j^0}$ denotes the derivative of the map $W_i(x)$ with respect to j th component of initial value, x_j^0, $j = 1, 2, \ldots, n$ and calculated at the point $x = \Psi(\theta_i)$.

4.2 Linearization at a grazing point

Assume that the periodic solution $\Psi(t)$ intersects the surface of discontinuity $t = \tau_l(x)$ at the moment, $t = \theta_l$, $1 \leq l \leq k$, tangentially. That is, $(\theta_l + jT, \Psi(\theta_l + jT))$, $j \in \mathbb{Z}$, are grazing points of the periodic solution $\Psi(t)$.

Let us consider the grazing point $(\theta_l, \Psi(\theta_l))$. In the remaining par of the subsection, we will compute the derivatives of functions $\theta_l(x)$ and $W_l(x)$, at the grazing point which are described in the previous part of the paper. One can observe from the equality (9), there exists two different possibility for the $\nabla\theta_l(x)$, first is at least one of the coordinate of the gradient $\nabla\theta_l(x)$ is infinity or all its coordinates are finite numbers. The complexity arises when at least one of the coordinate is infinity. It can be observed that the singularity is caused by the vector field and the surface of discontinuity. In order to handle with the complexity, the following conditions should be asserted.

(A1) A grazing point is isolated.
(A2) The matrix $W_i(x)$ is a differentiable at $(\theta_l, \Psi(\theta_l))$.

In the present paper, we consider the case when the singularity appears at gradient $\nabla\theta_l(x)$ at the grazing point $(\theta_l, \Psi(\theta_l))$ and we consider that how the impact function eliminate the singularity of the gradient at the grazing point. To suppress the singularity, we harmonize the interaction of the impact law, the vector field and the surface of discontinuity such that it validates the condition $(A2)$. If these components do not work in an harmony, some complex situations may appear [10,14,32,35]. This complex situations are not taken into account in our paper.

4.3 Stability of the grazing periodic solution

Assume that the linearization of $\theta_l(x)$ at the grazing point, $(\theta_l, \Psi(\theta_l))$, exists in the above defined sense for each $l = 1, 2, \ldots, k$. Because of the previous discussion, the gradient, $\nabla\theta_l(x)$, depends on the solution $x_1(t) = x(t, t_0, x_0 + \Delta x)$ of (1), neighbor to $\Psi(t)$, with small $\|\Delta x\|$.

Let us formulate one of them. Other constructive conditions will be investigated in our future papers.

(N5) For each $\Delta x \in \mathbb{R}^n$, the linearization system around $\Psi(t)$ is

$$\begin{aligned} u' &= A(t)u, \\ \Delta u|_{t=\theta_i} &= D_i u, \end{aligned} \tag{11}$$

such that $D_{i+p} = D_i$.

The following assertions can be verified in the way of Theorem 6.1.1 in [37].

Theorem 1 *Assume that conditions (N1)–(N5) as well as the assumptions (A1) and (A2) are valid. Then, the solution $\Psi(t)$ of (1) for each finite interval $[0, a]$, $a > 0$, has B- derivatives with respect to initial conditions, $(u_j(t))$, which satisfies the variational Eq. (8) with initial values $e_j = (\underbrace{0, 0, \ldots, 1}_{j}, 0, \ldots, 0)$, $j = 1, 2, \ldots, n$.*

The system (11) is the variational system around the grazing periodic solution $\Psi(t)$. One can derive the matrix of monodromy, $U_j(T)$, and the corresponding Floquet multipliers ρ_i, $i = 1, 2, \ldots, n$. The next assumption is needed to verify the stability of the periodic solution, $\Psi(t)$.

(N6) $|\rho_i| < 1$, $i = 1, 2, \ldots, n$.

Theorem 2 *Assume that conditions (N1)–(N6) and assumptions (A1) and (A2) are valid. Then, T-periodic solution $\Psi(t)$ of (1) is asymptotically stable.*

The last theorem can be proved similarly to Theorem 7.2.1 in [37].

We will exhibit some examples to actualize our theoretical results in the following section.

5 Examples

Example 1 Consider the following one-dimensional system

$$x' = 4 - \sin(t),$$
$$\Delta x|_{t=\tau_i(x)} = -4\pi + 1 - \frac{17}{16}x^2, \qquad (12)$$

where $\tau_i(x) = \frac{1}{4}\arctan(x) + i\pi$, $i \in \mathbb{Z}$ and the domain is $G = (-16, 16)$.

It is easy to verify by substituting (12) that the following expression,

$$\Psi(t) = \begin{cases} 0 & \text{if} \quad t = 0, \\ 4t + \cos(t) - 4\pi + 1 & \text{if} \quad t \in (0, 1], \end{cases} \qquad (13)$$

defines a π-periodic solution of (12). The solution is simulated in Fig. 2.

For the point $(\zeta_1, \Psi((\zeta_1))) = (0, 0)$, one can derive the following equality $\langle \nabla\tau_0(\Psi((\zeta_1))), f(\zeta_1, \Psi((\zeta_1)))\rangle = \langle \frac{1}{4}, 4\rangle = 1$. That is, $(\zeta_1, \Psi((\zeta_1))) = (0, 0)$ is a grazing point. Denote the grazing point by $(t^*, x^*) = (0, 0)$.

The periodic solution $\Psi(t)$ has exactly one discontinuity point $(\zeta_1, \Psi((\zeta_1))) = (0, 0)$ in the period interval $[0, \pi)$, which is a grazing point. Our aim in this example is to verify that the solutions of (12) meets the discontinuity surfaces exactly once and derive the linearization for (12) around the grazing periodic orbit, $\Psi(t)$. The discontinuity moments $t = i\pi$, $i \in \mathbb{Z}$, are also grazing.

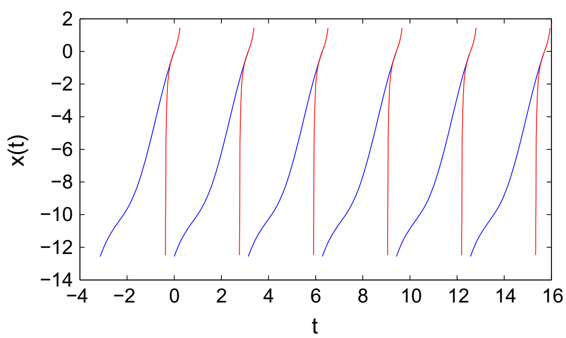

Fig. 2 The *blue curves* correspond to the periodic solution, $\Psi(t)$, of system (12) and *red ones* are the surfaces of discontinuity, $t = \tau_i(x)$, $i = 0, 1, 2, \ldots, 5$. (Color figure online)

Let us verify the conditions (N1)–(N4). For any $\tilde{x} \in G$, and $i \in \mathbb{Z}$, because $\tau_i(x)$ is increasing function and $\tilde{x} > \tilde{x} - 4\pi + 1 - \frac{17}{16}\tilde{x}^2$, it is true that $\tau_i(\tilde{x} - 4\pi + 1 - \frac{17}{16}\tilde{x}^2) < \tau_i(\tilde{x})$. This validates the condition (N1). It is apparent that (N2) is also true. There exist positive numbers $N = 1/4$ $C = 4$, with the inequality $CN < 1$, which is true for all point in the domain G except the grazing points $(i\pi, 0)$, $i \in \mathbb{Z}$. The differentiability in the grazing point will be expressed in details further. The following ones can be estimated as $\max_{(t,x)\in I\times G}\|f(t,x)\| = \max_{(t,x)\in\mathbb{R}\times G}\|4 - \sin(t)\| \leq 4$, $\max_{x\in G}\|\frac{\partial\tau_i(x)}{\partial x}\| = \max_{x\in G}\|\frac{1}{4(x^2+1)}\| \leq \frac{1}{4}$. So, (N3) is verified. For all $x \in G$ and $i \in \mathbb{Z}$, we obtain that $\max_{0\leq\sigma\leq 1}\langle\frac{\partial\tau_i(x+\sigma I_i(x))}{\partial x}, I_i(x)\rangle = \max_{0\leq\sigma\leq 1}\langle\frac{1}{1+(x+\sigma(-4\pi-1-\frac{17}{16}))^2}, -4\pi - 1 - \frac{17}{16}\rangle \leq 0$. Thus verifies condition (N4).

Now, we will continue with the linearization at the point $(\zeta_1, \Psi((\zeta_1))) = (0, 0)$. The grazing point is isolated as well. First, consider a near solution of (12) $\bar{x}(t) = x(t, 0, \Delta x)$ to $\Psi(t)$, which meets the surface $\tau_0(x) = \frac{1}{4}\arctan(x)$, at the point $(\frac{1}{4}\arctan(\bar{x}), \bar{x})$. Considering derivative of (3) with respect to a solution of (12) near to the periodic solution, we obtain that

$$\begin{aligned} \frac{\partial W_i(x)}{\partial x} &= \int_{\xi_i}^{\theta_i(x)} \frac{\partial f(u, x_0(u))}{\partial x}\frac{\partial x_0(u)}{\partial x}du \\ &+ f(\theta_i(x), x_0(\theta_i(x)))\frac{\partial\theta_i(x)}{\partial x} \\ &+ \frac{\partial J_i(x)}{\partial x}\Bigg(1 + \int_{\xi_i}^{\theta_i(x)} \frac{\partial f(u, x_0(u))}{\partial x}\frac{\partial x_0(u)}{\partial x}du \\ &+ f(\theta_i, x_0(\theta_i(x)))\frac{\partial\theta_i}{\partial x}\Bigg) \\ &+ \int_{\theta_i}^{\xi_i} \frac{\partial f(u, x_1(u))}{\partial x}\frac{\partial x_1(u)}{\partial x}du \\ &- f(\theta_i(x), x_1(\theta_i(x)))\frac{\partial\theta_i(x)}{\partial x}. \end{aligned} \qquad (14)$$

Substituting $(\frac{1}{4}\arctan(\bar{x}), \bar{x})$ to 14, we have

$$\frac{\partial W_i(x)}{\partial x} = f(\theta_i(\bar{x}), x_0(\theta_i(\bar{x})))\frac{\partial \theta_i(\bar{x})}{\partial x} + \frac{\partial J_i(\bar{x})}{\partial x} \times \left(1 + f(\theta_i(\bar{x}), x_0(\theta_i(\bar{x})))\frac{\partial \theta_i(\bar{x})}{\partial x}\right) - f(\theta_i(\bar{x}), x_1(\theta_i(\bar{x})))\frac{\partial \theta_i(\bar{x})}{\partial x}. \quad (15)$$

Next, we will evaluate the derivative $\frac{\partial \theta_i(\bar{x})}{\partial x}$. To do it, the formula (9) will be taken into account, and the derivative is calculated as $\frac{\partial \theta_i(\bar{x})}{\partial x} = \frac{1}{4\tan^2(4\bar{t})+2\sin(2\bar{t})}$. It is easy to see that as $\bar{t}$ tends to zero the fraction diverges to infinity. Moreover, it is easy to see that $\bar{x} = x(\bar{t})$ Thus, at the grazing point singularity appears, to cope with the singularity, we will utilize the compliance of vector field and the jump function, and we consider the equality (15), and we get

$$\frac{\partial W_i(\bar{x})}{\partial x} = \frac{4-\sin(\bar{t})}{4\tan^2(4\bar{t})+\sin(\bar{t})} - \frac{\tan(4\bar{t})}{16}\Big(1 + \frac{4-\sin(\bar{t})}{4\tan^2(4\bar{t})+2\sin(2\bar{t})}\Big) - \frac{4-2\sin(2\bar{t})}{4\tan^2(4\bar{t})+2\sin(2\bar{t})} = -\frac{4\bar{x}^3+4\bar{x}}{64\bar{x}^2+32\sin(0.5\tan(\bar{x}))}, \quad (16)$$

calculating above expression as $\bar{x}$ tends to the grazing point $x^* = 0$, we obtain that

$$\lim_{\bar{x}\to x^*} \frac{\partial W_i(\bar{x})}{\partial x} = \lim_{\bar{x}\to 0} -\frac{4\bar{x}^3+4\bar{x}}{64\bar{x}^2+16\sin(0.5\tan(\bar{x}))} = Z, \quad (17)$$

where $Z = -\frac{1}{2}$.

In order to obtain a linearization system around grazing periodic solution $\Psi(t)$, the differentiability of the functions $W_i(x)$ at the grazing point x^* should be verified. To accomplish it, we will verify the derivative of the function $W_i(x)$ exists at the point x^*. The derivative can be calculated as follows

$$W_{ix}(x^*) = \lim_{\bar{x}\to x^*} \frac{W_i(x) - W_i(x^*)}{x - x^*}, \quad (18)$$

above equation can be calculated by applying mean value theorem [38], we obtain that

$$W_{ix}(x^*) = \lim_{\bar{x}\to x^*} \frac{\frac{\partial W_i(\bar{x})}{\partial x}(x-x^*) - Z(x-x^*)}{x-x^*} + Z, \quad (19)$$

where $\bar{x}$ lies in the interval $(x^* - \epsilon, x^* + \epsilon)$, for some positive ϵ. By means of expressions (18) with (19) it is easy to obtain that

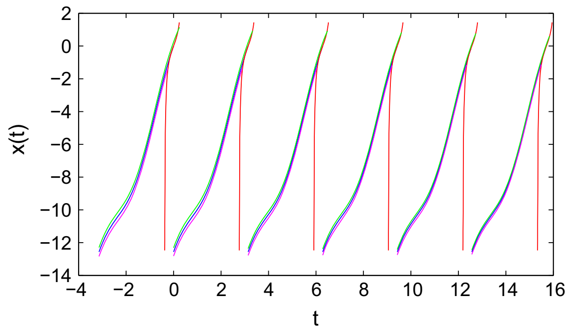

Fig. 3 The *green* and *magenta curves* are the solutions of (12) with initial values $(-\pi/16, -1)$ and $(\pi/16, 1)$, respectively. The *blue* one corresponds to the periodic solution $\Psi(t)$ and the *red curves* are the surfaces of discontinuity, $t = \tau_i(x)$, $i = 0, 1, 2, \ldots, 5$. (Color figure online)

$$W_{ix}(x^*) = Z. \quad (20)$$

Then, we can conclude that the linearization exists at the grazing point and the derivative is continuous as well. This verifies condition $(A2)$.

Next, we will continue to analyze the system in Example 1. We derive the linearization for $\theta(x)$ at the grazing point $(\theta_l, \Psi(\theta_l)) = (0, 0)$, there. Thus, the linearization for $\Psi(t)$ consists of a π-periodic system,

$$u' = 0,$$
$$\Delta u|_{t=\pi i} = Du, \quad (21)$$

where coefficient D, by the equality (20), is equal to $-\frac{1}{2}$. The multiplier of the variational system (21) is $\rho = \frac{1}{2}$. It is inside the unit circle and condition (N6) holds. The conditions (N1)–(N6) are valid, then by Theorem 2, the periodic solution $\Psi(t)$ of (12) is asymptotically stable. The stability of the solution, $\Psi(t)$, is pictured in Fig. 3 through simulations.

Example 2 In this example, we will consider the following system of differential equations with variable moments of impulse actions

$$x_1' = -x_1 + 4,$$
$$x_2' = -\cos(2\pi t) + 1,$$
$$\Delta x_1|_{t=\tau_i(x)} = -4(x_1 + 0.75x_1^2) - 1,$$
$$\Delta x_2|_{t=\tau_i(x)} = 1 - 0.25x_2, \quad (22)$$

where $\tau_i(x) = 0.25x_1 + i$. For this system, denote by $x = (x_1, x_2)$. Let the domain of the system be

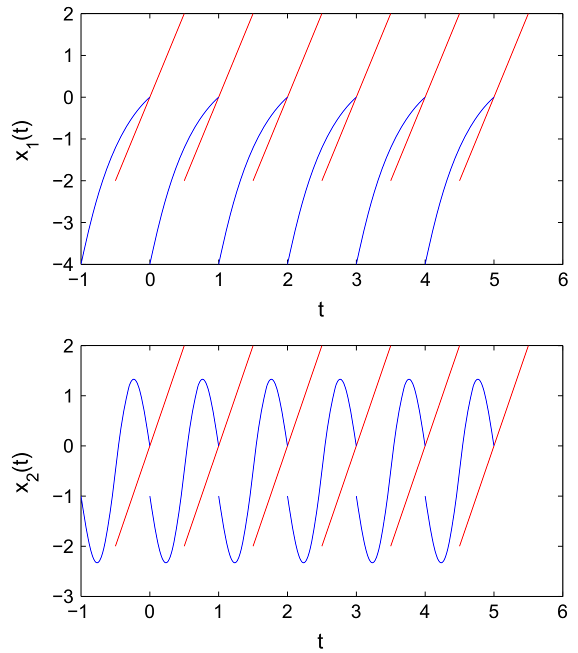

Fig. 4 The above figure is the first component $x_1(t)$ of the periodic solution, $\Psi(t)$, with grazing points at $(i, 0, 0)$, $i = 0, 1, \ldots, 5$, versus time, t and the below one is the second component $x_2(t)$ of the periodic solution, $\Psi(t)$, with grazing points at $(i, 0, 0)$, $i = 0, 1, \ldots, 5$, versus time, t

$G = \{(t, x)|\ t \in \mathbb{R},\ x_1 \in (-13, 13),\ x_2 \in (-3, 3)\}$. System (22) is of the type (1) with $f(t, x) = (-x_1 + 4, 2\pi \sin(2\pi t) + 1)$ and $J_i(x) = (-x_1 - 1, 1)$. It is easy to observe that $f(t, x)$ is a 1-periodic function.

It can be easily verified that the system admits a 1-periodic solution of the form (23)

$$\Psi(t) = \begin{cases} (0, 0), & t = 0, \\ (\exp(-t)(4t - \exp(1)), -\frac{1}{2\pi}\sin(2\pi t) + t - 1), & t \in (0, 1], \end{cases} \tag{23}$$

with discontinuity moments $\theta_i = i, i \in \mathbb{Z}$. It is easy to determine, utilizing the equality $\langle(4, 0), (0.25, -2)\rangle = 1$, that $(\theta_1, \Psi(\theta_1)) = (0, 0, 0)$ is a grazing point. Moreover, by means of the periodicity of $\Psi(t)$, we can conclude that all moments $t = i\pi, i \in \mathbb{Z}$, are grazing ones. The components of periodic solution is simulated in Fig. 4.

For ever point $(\tilde{x}_1, \tilde{x}_2) \in G$, the inequality $0.25(\tilde{x}_1 - 1) < 0.25\tilde{x}_1$ is true, this validates (N1). The condition (N2) is also valid because $\tau_i(x) = 0.25x_1 + i < 0.25x_1 + i + 1 = \tau_{i+1}(x)$. Due to the vector field, surface of discontinuity, and the jump function, it is easy to say that every solution which meets the surface of discontinuity in the neighborhood of the grazing periodic solution $\Psi(t)$ intersects the surface at most once. For this reason, there is no need to check (N3).

Next, we will continue with the linearization of the system (22) around the periodic solution $\Psi(t)$. To obtain it, first, we will consider the derivative of the formula (3), then we get

$$\begin{aligned} \frac{\partial W_i(x)}{\partial x_1^0} &= \int_{\xi_i}^{\theta_i(x)} \frac{\partial f(u, x_0(u))}{\partial x} \frac{\partial x_0(u)}{\partial x} \mathrm{d}u \\ &+ f(\theta_i(x), x_0(\theta_i(x))) \frac{\partial \theta_i(x)}{\partial x} + \frac{\partial J_i(x)}{\partial x} \Bigg(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &+ \int_{\xi_i}^{\theta_i(x)} \frac{\partial f(u, x_0(u))}{\partial x} \frac{\partial x_0(u)}{\partial x} du \\ &+ f(\theta_i, x_0(\theta_i(x))) \frac{\partial \theta_i}{\partial x} \Bigg) + \int_{\theta_i}^{\xi_i} \frac{\partial f(u, x_1(u))}{\partial x} \frac{\partial x_1(u)}{\partial x} du \\ &- f(\theta_i(x), x_1(\theta_i(x))) \frac{\partial \theta_i(x)}{\partial x}. \end{aligned} \tag{24}$$

Consider a near solution $\tilde{x}(t)$ of (22) to $\Psi(t)$. Assume that near solution meets the surface of discontinuity $t = \tau_i(x)$, at the point $\bar{x}$. Denote the meeting point by $\bar{x} = (\bar{x}_1, \bar{x}_2) = \tilde{x}(\tau_i(\bar{x}))$, substituting it to (24), we have

$$\begin{aligned} \frac{\partial W_i(\bar{x})}{\partial x} &= f(\theta_i(\bar{x}), \bar{x}(\theta_i(\bar{x}))) \frac{\partial \theta_i(\bar{x})}{\partial x} + \frac{\partial J_i(\bar{x})}{\partial x} \Bigg(1 \\ &+ f(\theta_i(\bar{x}), \bar{x}(\theta_i(\bar{x}))) \frac{\partial \theta_i(\bar{x})}{\partial x} \Bigg) \\ &+ f(\theta_i(\bar{x}), x_1(\theta_i(\bar{x}))) \frac{\partial \theta_i(\bar{x})}{\partial x}. \end{aligned} \tag{25}$$

Substitute the function $f(t, x)$ and the Jacobian $J_x(x)$ into (25), it is easy to obtain that

$$\begin{aligned} \frac{\partial W_i(\bar{x})}{\partial x_1^0} &= \begin{bmatrix} \bar{x}_1 + 4 \\ -\cos(2\pi t) + 1 \end{bmatrix} \frac{\partial \theta_i(\bar{x})}{\partial x_1^0} \\ &+ \frac{\partial J_i(\bar{x})}{\partial x} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 + 4 \\ -\cos(2\pi t) + 1 \end{bmatrix} \frac{\partial \theta_i(\bar{x})}{\partial x_1^0} \right) \\ &+ \begin{bmatrix} 0.75\bar{x}_1^2 - 1 + 4 \\ -\cos(2\pi t) + 1 \end{bmatrix} \frac{\partial \theta_i(\bar{x})}{\partial x_1^0}, \end{aligned} \tag{26}$$

In order to evaluate above expression, we need to find the derivative $\frac{\partial \theta_i(\bar{x})}{\partial x_1^0}$, by applying formula (9), we obtain that $\frac{\partial \theta_i(\bar{x})}{\partial x_1^0} = \frac{1}{\bar{x}_1}$. It is easy to see that at the grazing point the derivative is infinity. To handle with it, we will apply a special jump function and vector field. Substituting the derivative to (26), we get

$$\frac{\partial W_i(\bar{x})}{\partial x_1^0} = \begin{bmatrix} -4\bar{x}_1^2 \\ 0 \end{bmatrix} \frac{1}{\bar{x}_1} + \begin{bmatrix} 0.2x_1 & 0 \\ 0 & -0.25 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \bar{x}_1 + 4 \\ -\cos(2\pi t) + 1 \end{bmatrix} \frac{1}{\bar{x}_1} \right) = \begin{bmatrix} -0.8x_1 \\ 0 \end{bmatrix} \quad (27)$$

By applying similar technique one can determine that

$$\frac{\partial W_i(\bar{x})}{\partial x_2^0} = \begin{bmatrix} 0 \\ -0.25 \end{bmatrix}. \quad (28)$$

Consequently, it is easy to determine that $W_i(x)$ is differentiable at $(\theta_i, \Psi(\theta_i))$ by applying similar technique in the Example 3, then it can be determined as

$$D_i = W_{ix}(\Psi(\theta_i)) = \begin{bmatrix} 0 & 0 \\ 0 & -0.25 \end{bmatrix}.$$

Now, we will continue with the linearization around the grazing periodic solution $\Psi(t)$.

Depending on the position of the near solution, the variational system for the periodic solution, $\Psi(t)$, consists of two subsystems. They are

$$\begin{aligned} u_1' &= -u_1, \\ u_2' &= 0, \\ \Delta u|_{t=\pi i} &= D_i u, \end{aligned} \quad (29)$$

where $D_i \equiv \begin{bmatrix} 0 & 0 \\ 0 & -0.25 \end{bmatrix}$, and $\theta_i = i$, $i \in \mathbb{Z}$. System (29) is (1, 1)-periodic. The multipliers are equal to $\rho_1 = 0.3679$, $\rho_2 = 0.75$. All of them are inside the unit circle, and by Theorem 2 one can conclude that the periodic solution is asymptotically stable. Considering the near solutions with initial values $(-1, -3.2, -1.4)$ and $(-1, -2.8, -0.8)$, by using numerical simulation tools, we depicted the components of the near solutions to the components of $\Psi(t)$ in Fig. 5.

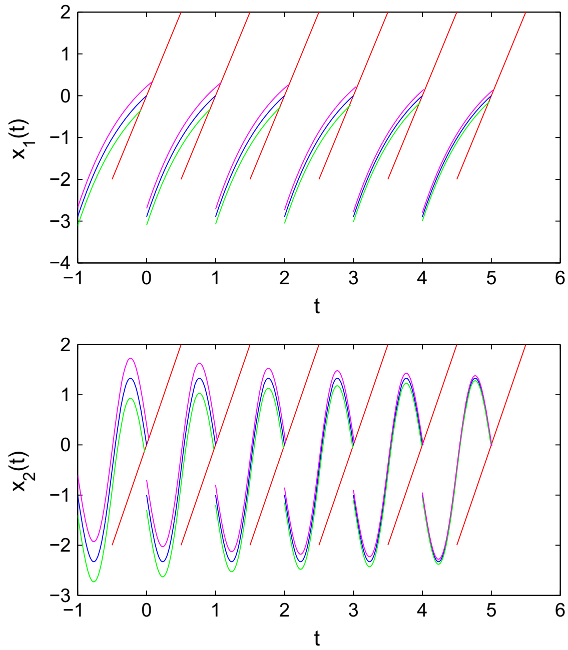

Fig. 5 The above figure is for the first component of the periodic solution, $\Psi(t)$. *Green curves* are the solution $x_1(t)$ of (22) with initial values $(-1, -3.2)$ and $(-1, -2.8)$, and *blue curves* are the grazing periodic solution, which have grazing points at $(i, 0, 0)$, $i = -0, 1, 2, 3, 4, 5$. The bottom one is for the second component $x_2(t)$ of the periodic solution, $\Psi(t)$. *Green curves* are the solution $x_2(t)$ of (22) with initial values $(-1, -1.4)$ and $(-1, -0.8)$. The *red curves* are surfaces of discontinuity $t = \tau_i(x)$, $i \in \mathbb{Z}$ in both figures. (Color figure online)

6 Regular perturbations around grazing periodic solution

In this chapter, we will seek the existence of the periodic solution under a parameter variation. The regular perturbations are widely investigated in literature [1,3,4,6,37]. The conditions (N1)–(N6) and assumptions (A1) and (A2) are also valid in this section.

Let D_x be a domain in $\mathbb{R}^n$ having compact closure, and let μ_0 be a fixed positive number. On the set

$$D = \{(x, t, i, \mu) | x \in D_x,\ t \in \mathbb{R},\ i \in \mathbb{Z}, \mu \in (-\mu_0, \mu_0)\}.$$

we take into account the following system,

$$\begin{aligned} x' &= f(t, x) + \mu\phi(t, x, \mu), \\ \Delta x|_{t=\tau_i(x)+\mu\eta_i(x,\mu)} &= I_i(x) + \mu\theta_i(x, \mu), \end{aligned} \quad (30)$$

where the functions I_i, τ_i, θ_i, and η_i have continuous partial derivatives of second order with respect to the variables μ, x_j, $j = 1, 2, \ldots, n$, $f \in C^{(0,2)}(D) \cap C^{(1,2)}(D_0)$, $\phi \in C^{(0,1,1)}(D) \cap C^{(1,2,2)}(D_0)$, where D_0 is the union of certain neighborhoods of the surfaces $t = \tau_i(x)$, $i \in \mathbb{Z}$. Moreover, we will assume that there exist a real number $T > 0$ and an integer $p > 0$ for which the following equalities are valid in the domain $D : f(t+T,x) = f(t,x)$, $\phi(t+T,x,\mu) = \phi(t,x,\mu)$, $I_{i+p} = I_i$, $\theta_{i+p} = \theta_i$, $\tau_{i+p} = \tau_i + T$ and $\eta_{i+p} = \eta_i$.

The generating system is of the form

$$x' = f(t,x),$$
$$\Delta x|_{t=\tau_i(x)} = I_i(x). \tag{31}$$

Assume that system (31) has a periodic solution $\Psi(t)$ with period T and satisfies the conditions (N1)–(N5) and assumptions (A1) and (A2) are valid. If $|\mu|$ is sufficiently small, then (30) admits a T-periodic solution which converges $\Psi(t)$ as $|\mu|$ tends to zero.

The next examples are presented to actualize our theoretical results and the increment of the periodic solution is demonstrated through simulation.

Example 3 Let us consider the following one-dimensional system with variable moments of impulses

$$x' = 4 - 2\sin(2t) + \mu\phi(t,x,\mu),$$
$$\Delta x|_{t=\tau_i(x)+\mu\kappa_i(x,\mu)} = -4\pi - 1 - \frac{15}{16}x + \mu\eta_i(x,\mu), \tag{32}$$

where μ is a sufficiently small parameter. The system is of the form (12) for $\mu = 0$. Considering the system (30), the functions and matrices can be determined as $A = 0$, $f(t) = 4 - 2\sin(2t)$, which is π-periodic, $I_i = -4\pi - 1 - \frac{15}{16}x$, $\mu = \frac{1}{32}$ and $\eta_i(x,\mu) = 2(x^2 - \tan^2(0.04) + \frac{15}{16}x)$.

The generating system can be determined as in the form

$$x' = 4 - 2\sin(2t),$$
$$\Delta x_2|_{t=\tau_i(x_1)} = -4\pi - 1 - \frac{15}{16}x. \tag{33}$$

The eigenvalue of the matrix of monodromy for it can be determined as ρ, which is not equal to one, so we can say that the system (33) has a unique T-periodic solution, $\Psi_\mu(t)$ for μ sufficiently small.

Example 4 Let us consider the following one-dimensional system with variable moments of impulses

$$x_1' = -x_1 + 4 + \mu x_1^2,$$
$$x_2' = 2\pi\sin(2\pi t) + 1 + \mu x_2^2,$$
$$\Delta x_1|_{t=\tau_i(x)} = -4(x_1 + 0.75x_1^2) - 1,$$
$$\Delta x_2|_{t=\tau_i(x)} = 1 - 0.25x_2, \tag{34}$$

where μ is a sufficiently small parameter.

The generating system can be determined as in the form

$$x_1' = -x_1 + 4,$$
$$x_2' = -\cos(2\pi t) + 1,$$
$$\Delta x_1|_{t=\tau_i(x)} = -4(x_1 + 0.75x_1^2) - 1,$$
$$\Delta x_2|_{t=\tau_i(x)} = 1 - 0.25x_2. \tag{35}$$

It is considered in Example 2 and the multipliers are inside the unit circle, so it is easy to conclude that the system (34) has a unique T-periodic solution, $\Psi_\mu(t)$ for μ sufficiently small.

7 Conclusion

This paper includes information about non-autonomous system with non-fixed moments of impulses whose solutions have grazing points. By applying a novel technique, we construct a linearization system around the grazing periodic solution. Concrete models are demonstrated and some simulations are presented to visualize theoretical results. By applying regular perturbations, existence of periodic solution of these systems are investigated and exemplified.

In the present paper we have not found linearization around the periodic solution $\Psi(t)$ of the model (5) for the bouncing bead which moves between two vibrating tables. That is, the singularity caused by the grazing can not be suppressed by methods of our research. We suppose that one can arrange the linearization by applying perturbations [1,3] and, moreover, stabilize the periodic solution if methods similar to that of the paper [39] will be utilized.

References

1. Nayfeh, A.H., Pai, P.F.: Linear and Nonlinear Structural Mechanics. Wiley-Interscience, Hoboken, NY (2004)
2. Nayfeh, A.H.: Nonlinear Interactions: Analytical, Computational, and Experimental Methods. Wiley, New York (2000)
3. Nayfeh, A.H., Balachandran, B.: Applied Nonlinear Dynamics, Analytical, Computational, and Experimental Methods. Wiley, New York (1995)
4. Nayfeh, A.H.: Introduction to Perturbation Techniques. Wiley, New York (1981)
5. Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer, New York (1983)

6. Farkas, M.: Periodic Motions. Springer, New York (1998)
7. Vestroni, F., Luongo, A., Paolone, A.: A perturbation method for evaluating nonlinear normal modes of a piecewise linear two-degrees-of-freedom system. Nonlinear Dynamics **54**, 379–393 (2008)
8. Awrejcewicz, J., Lamarque, C.H.: Bifurcation and Chaos in Nonsmooth Mechanical Systems. World Scientific Series on Nonlinear Science, Singapore (2003)
9. Babitsky, V.I.: Theory of Vibro-Impact System and Applications. Springer, Berlin (1998)
10. di Bernardo, M., Budd, C.J., Champneys, A.R., Kowalczyk, P.: Piecewise-Smooth Dynamical Systems Theory and Applications. Springer, London (2008)
11. di Bernardo, M., Hogan, S.J.: Discontinuity-induced bifurcations of piecewise smooth dynamical systems. Philos. Trans. R. Soc. A **368**, 4915–4935 (2010)
12. Brogliato, B.: Nonsmooth Mechanics. Springer, London (1999)
13. Burton, R.: Vibration and Impact. Addison-Wesley Publishing Company, New York (1958)
14. Foale, S., Bishop, S.R.: Bifurcations in impact oscillations. Nonlinear Dyn. **6**, 285–299 (1994)
15. Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields. Springer, New York (1997)
16. Ibrahim, R.A.: Vibro-Impact Dynamics Modeling, Mapping and Applications. Springer, Berlin (2009)
17. Luo, A.C.J.: Singularity and Dynamics on Discontinuous Vector Fields. Elsevier, Amsterdam (2006)
18. Nagaev, R.F.: Mechanical Processes with Repeated Attenuated Impacts. World Scientific Publishing, Singapore (1985)
19. Stronge, W.J.: Impact Mechanics. Cambridge University Press, Cambridge (2000)
20. Tung, P.C., Shaw, S.W.: The dynamics of an impact print hammer. J. Vib. Acoust. Stress Reliab. Des. **110**, 193–200 (1988)
21. Vogel, S., Linz, S.J.: Regular and chaotic dynamics in bouncing ball models. Int. J. Bifurc. Chaos **21**, 869–884 (2011)
22. di Bernardo, M., Budd, C.J., Champneys, A.R.: Grazing bifurcations in n-dimensional piecewise-smooth dynamical systems. Phys. D **160**, 222–254 (2001)
23. di Bernardo, M., Budd, C.J., Champneys, A.R.: Grazing, skipping and sliding: analysis of the nonsmooth dynamics of the DC/DC buck converter. Nonlinearity **11**, 858–890 (1998)
24. Donde, V., Hiskens, I.A.: Shooting methods for locating grazing phenomena in hybrid systems. Int. J. Bifur. Cation Chaos **16**(3), 671–692 (2006)
25. Junca, S., Tong, L.: Limitation on the method of strained coordinates for vibrations with weak grazing unilateral contact. Nonlinear Dyn. **80**, 197–207 (2015)
26. Hosa, C., Champneys, A.R.: Grazing bifurcations and chatter in a pressure relief valve model. Phys. D **241**, 2068–2076 (2012)
27. Kapitaniak, T., Wiercigroch, M.: Dynamics of impact oscillators: an introduction. Chaos Solitons Fractals **11**, 2411–2412 (2000)
28. Kryzhevich, S., Wiercigroch, M.: Topology of vibro-impact systems in the neighborhood of grazing. Phys. D Nonlinear Phenom. **241**(22), 919–1931 (2012)
29. Luo, A.C.J., Gegg, B.C.: Grazing phenomena in a periodically forced, friction-induced, linear oscillator. Commun. Nonlinear Sci. Numer. Simul. **11**, 777–802 (2006)
30. Luo, A.C.J.: Discontinuous Dynamical Systems on Time-Varying Domains. Higher Education Press, Beijing (2009)
31. Luo, A.C.J.: A theory for non-smooth dynamical systems on connectible domains. Commun. Nonlinear Sci. Numer. Simul. **10**, 1–55 (2005)
32. Nordmark, A.B.: Non-periodic motion caused by grazing incidence in an impact oscillator. J. Sound Vib. **145**, 279–297 (1991)
33. Nordmark, A.B.: Existence of periodic orbits in grazing bifurcations of impacting mechanical oscillators. Nonlinearity **14**, 1517–1542 (2001)
34. Nordmark, A.B., Kowalczyk, P.: A codimension-two scenario of sliding solutions in grazing–sliding bifurcations. Nonlinearity **19**, 1–26 (2006)
35. Piiroinen, P.T., Virgin, L.N., Champneys, A.R.: Chaos and period-adding: experimental and numerical verification of the grazing bifurcation. J. Nonlinear Sci. **14**, 383–404 (2004)
36. Akhmet, M.U., Kıvılcım, A.: Discontinuous dynamics with grazing points. Commun. Nonlinear Sci. Numer. Simul. **38**, 218–242 (2016)
37. Akhmet, M.U.: Principles of Discontinuous Dynamical Systems. Springer, New York (2010)
38. Apostol, T.M.: Calculus, Volume 1, One-Variable Calculus with an Introduction to Linear Algebra. Wiley, New York (1967)
39. Arkhipova, I.M., Luongo, A.: Stabilization via parametric excitation of multi-dof statically unstable systems. Commun. Nonlinear Sci. Numer. Simul. **19**, 3913–3926 (2014)