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Research paper

Non-autonomous equations with unpredictable solutions

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ABSTRACT

To make research of chaos more amenable to investigating differential and discrete equations, we introduce the concepts of an unpredictable function and sequence. The topology of uniform convergence on compact sets is applied to define unpredictable functions [1,2]. The unpredictable sequence is defined as a specific unpredictable function on the set of integers. The definitions are convenient to be verified as solutions of differential and discrete equations. The topology is metrizable and easy for applications with integral operators. To demonstrate the effectiveness of the approach, the existence and uniqueness of the unpredictable solution for a delay differential equation are proved as well as for quasilinear discrete systems. As a corollary of the theorem, a similar assertion for a quasilinear ordinary differential equation is formulated. The results are demonstrated numerically, and an application to Hopfield neural networks is provided. In particular, Poincaré chaos near periodic orbits is observed. The completed research contributes to the theory of chaos as well as to the theory of differential and discrete equations, considering unpredictable solutions.

1. Introduction

From the applications point of view, the theory of differential and discrete equations focuses on equilibria, periodic, and almost periodic oscillations. They meet the needs of any real world problem related to mechanics, electronics, economics, biology, etc., if one searches for regular and stable dynamics of an isolated motion. However, they are not sufficient for many modern and prospective demands of robotics, computer techniques, and the internet, and chaotic dynamics comprise constructive properties for applications. This is the reason why it is important to join the power of deterministic chaos with the immensely rich source of methods for differential and discrete equations. We contributed to this in our studies [3–5] and the book [6], where a method of replication of chaos has been developed. It consists of the verification of ingredients of chaos such as sensitivity, transitivity, proximality and the existence of infinitely many unstable regular motions [7–9] for solutions of an equation with chaotic perturbation. This approach gives a very effective instrument for application of the accumulated knowledge in chaos research. Nevertheless, we are not glad with the necessity to check the presence of several ingredients. Therefore, in our opinion, unpredictable functions have become an instrument for the simplification of chaos analysis through differential and discrete equations.

In this paper, another step in the adaptation of unpredictable functions to the theory of differential equations has been made. We apply the uniform convergence on compact subsets of the real axis to determine unpredictable functions for two reasons. The first reason is that the topology is easily metrizable, in particular, to the metric for Bebutov dynamical

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system [10], and consequently, the unpredictable functions and solutions immediately imply the presence of Poincaré chaos according to our results in [11]. The second one is the easy verification of the convergence. Thus, the present study is useful for the theory of differential equations as well as chaos researches. For the construction of unpredictable functions we have applied the results on the equivalence of discrete dynamics obtained in papers [12,13]. Moreover, an application to Hopfield neural networks [14,15] is provided.

In the instrumental sense, discreteness has been the main object in chaos investigation. To check this, it is sufficient to recall the definitions of chaos [7–9], which are based on sequences and maps, as well as Smale Horseshoe and logistic maps, Bernoulli shift [9], which are in the core of the chaos theory. One can say that stroboscopic observation of a motion was the single way to indicate the irregularity in continuous dynamics. The definitions of chaos for continuous dynamics, which are not related to discreteness [4,16–18], are requested for embedding the research to the theory of differential equations. The research as well as the origin of the chaos [19] gave us strong arguments for the development of motions in classical dynamical systems theory [20] by proceeding behind Poisson stable points to unpredictable points [11]. Then, the dynamics have been specified such that a function that is bounded on the real axis is an unpredictable point [1,2]. In the papers [1,2], we have demonstrated that unpredictable functions are easy to be analyzed as solutions of differential equations. This paradigm is not completed, if one does not consider discrete equations. Therefore, in the present paper we also deliver discrete analogues for unpredictable functions, calling them unpredictable sequences, and prove assertions on the existence and uniqueness of unpredictable solutions of discrete equations for the first time in the literature. The results can be useful for applications and theoretical analyses, in particular, for the modern development of computer technologies, software, and robotics [21,22].

2. Quasilinear differential equations

Let us introduce the following definition.

Definition 2.1. A uniformly continuous and bounded function $\vartheta : \mathbb{R} \to \mathbb{R}^m$ is unpredictable if there exist positive numbers ϵ_0, δ and sequences $\{t_n\}$, $\{u_n\}$ both of which diverge to infinity such that $\|\vartheta(t+t_n) - \vartheta(t)\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|\vartheta(t+t_n) - \vartheta(t)\| \geq \epsilon_0$ for each $t \in [u_n - \delta, u_n + \delta]$ and $n \in \mathbb{N}$.

To create Poincaré chaos [11], uniform continuity is not a necessary condition for an unpredictable function $\vartheta(t)$, and instead of the condition $\|\vartheta(t+t_n) - \vartheta(t)\| \geq \epsilon_0$ for each $t \in [u_n - \delta, u_n + \delta]$ and $n \in \mathbb{N}$, one can request that $\|\vartheta(t_n + u_n) - \vartheta(u_n)\| \geq \epsilon_0$ for each $n \in \mathbb{N}$. For the needs of verification of theorems on the existence of unpredictable solutions of differential equations we apply Definition 2.1, but for the future studies the following definitions may also be beneficial.

Definition 2.2. A continuous and bounded function $\vartheta : \mathbb{R} \to \mathbb{R}^m$ is unpredictable if there exist a positive number ϵ_0 and sequences $\{t_n\}$, $\{u_n\}$ both of which diverge to infinity such that $\|\vartheta(t+t_n) - \vartheta(t)\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|\vartheta(t_n + u_n) - \vartheta(u_n)\| \geq \epsilon_0$ for each $n \in \mathbb{N}$.

Definition 2.3. A continuous and bounded function $\vartheta : \mathbb{R} \to \mathbb{R}^m$ is unpredictable if there exist a positive number ϵ_0 and sequences $\{t_n\}$, $\{u_n\}$ both of which diverge to infinity such that $\|\vartheta(t_n) - \vartheta(0)\| \to 0$ as $n \to \infty$ and $\|\vartheta(t_n + u_n) - \vartheta(u_n)\| \geq \epsilon_0$ for each $n \in \mathbb{N}$.

The main object of the present section is the following system of delay differential equations,

$$x'(t) = Ax(t) + f(x(t-\tau)) + g(t), \tag{2.1}$$

where τ is a positive number, the eigenvalues of the matrix $A \in \mathbb{R}^{m\times m}$ have negative real parts, $f : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function, and $g : \mathbb{R} \to \mathbb{R}^m$ is a uniformly continuous and bounded function. Our purpose is to prove that system (2.1) possesses a unique unpredictable solution which is uniformly exponentially stable, provided that the function $g(t)$ is unpredictable in accordance with Definition 2.1.

In the remaining parts of the paper, we will make use of the usual Euclidean norm for vectors and the norm induced by the Euclidean norm for square matrices.

Since the eigenvalues of the matrix A in system (2.1) have negative real parts, there exist numbers $K \geq 1$ and $\omega > 0$ such that $\left\|e^{At}\right\| \leq Ke^{-\omega t}$ for $t \geq 0$.

The following conditions are required.

- **(C1)** There exists a positive number M_f such that $\sup_{x\in\mathbb{R}^m} \|f(x)\| \leq M_f$;
- **(C2)** There exists a positive number L_f such that $\|f(x_1) - f(x_2)\| \leq L_f \|x_1 - x_2\|$ for all $x_1, x_2 \in \mathbb{R}^m$;
- **(C3)** $\omega - 2\, KL_f e^{\omega\tau/2} > 0$.

The following theorem is concerned with the unpredictable solution of system (2.1).

Theorem 2.1. *Suppose that conditions $(C1) - (C3)$ are valid. If the function $g(t)$ is unpredictable, then system (2.1) possesses a unique uniformly exponentially stable unpredictable solution.*

Proof. Under the conditions $(C1)-(C3)$, one can verify using the techniques for delay differential equations [23] that there exists a unique solution $\phi(t)$ of (2.1) which is bounded on the whole real axis and satisfies the relation

$$\phi(t)=\int_{-\infty}^{t} e^{A(t-s)}[f(\phi(s-\tau))+g(s)]ds.$$

It is clear that $\sup_{t\in\mathbb{R}}\|\phi(t)\|\le M_\phi$, where $M_\phi=\dfrac{K(M_f+M_g)}{\omega}$ and $M_g=\sup_{t\in\mathbb{R}}\|g(t)\|$. According to results of [23], the solution $\phi(t)$ is uniformly exponentially stable. We will show that the solution $\phi(t)$ is unpredictable.

Since $g(t)$ is an unpredictable function, there exist a positive number ϵ_0 and sequences $\{t_n\}$, $\{u_n\}$ both of which diverge to infinity such that $\|g(t+t_n)-g(t)\|\to 0$ as $n\to\infty$ on compact subsets of $\mathbb{R}$, and $\|g(t_n+u_n)-g(u_n)\|\ge\epsilon_0$ for each $n\in\mathbb{N}$.

Fix an arbitrary $\epsilon>0$, and denote $R_1=\dfrac{2\omega M_\phi K}{\omega-2\,KL_f e^{\omega\tau/2}}$, $R_2=\dfrac{K}{\omega-KL_f}$. Condition $(C3)$ implies that both R_1 and R_2 are positive numbers. Take a positive number γ satisfying $\gamma<\dfrac{1}{R_1+R_2}$, and suppose that E is a positive number such that $E\ge\dfrac{2}{\omega}\ln\left(\dfrac{1}{\gamma\epsilon}\right)$.

Let α and β be fixed real numbers with $\beta>\alpha$. There exists a natural number n_0 such that for each $t\in[\alpha-E,\beta]$ and each $n\ge n_0$ the inequality $\|g(t+t_n)-g(t)\|<\gamma\epsilon$ holds.

Fix an arbitrary natural number $n\ge n_0$, and define the function $\xi(t)=\phi(t)-\phi(t+t_n)$. This function satisfies the delay equation

$$\xi'(t)=A\xi(t)+f(\xi(t-\tau)+\phi(t+t_n-\tau))-f(\phi(t+t_n-\tau))+g(t)-g(t+t_n). \tag{2.2}$$

By applying arguments of equivalent integral equations for the last system, one can find that for $t\ge\alpha-E$, $\xi(t)$ satisfies the relation

$$\begin{aligned}\xi(t)=\ & e^{A(t-\alpha+E)}[\phi(\alpha-E)-\phi(t_n+\alpha-E)]\\ &+\int_{\alpha-E}^{t} e^{A(t-s)}[f(\xi(s-\tau)+\phi(s+t_n-\tau))-f(\phi(s+t_n-\tau))]ds\\ &+\int_{\alpha-E}^{t} e^{A(t-s)}[g(s)-g(s+t_n)]ds.\end{aligned}$$

Let us denote by $\mathscr{C}$ the set of continuous functions $\xi(t)$ defined on $\mathbb{R}$ such that

$$\|\xi(t)\|\le R_1e^{-\omega(t-\alpha+E)/2}+R_2\gamma\epsilon$$

for $\alpha-E-\tau\le t\le\beta$ and $\|\xi\|_\infty\le 2K(M_\phi+\dfrac{M_f+M_g}{\omega})$, where $\|\xi\|_\infty=\sup_{t\in\mathbb{R}}\|\xi(t)\|$.

Define on $\mathscr{C}$ an operator Π by

$$\Pi\xi(t)=\begin{cases}\phi(t)-\phi(t+t_n),\ t<\alpha-E,\\ e^{A(t-\alpha+E)}[\phi(\alpha-E)-\phi(t_n+\alpha-E)]+\displaystyle\int_{\alpha-E}^{t} e^{A(t-s)}[g(s)-g(s+t_n)]ds\\ +\displaystyle\int_{\alpha-E}^{t} e^{A(t-s)}[f(\xi(s-\tau)+\phi(s+t_n-\tau))-f(\phi(s+t_n-\tau))]ds,\ t\ge\alpha-E.\end{cases}$$

First of all, we will show that $\Pi(\mathscr{C})\subseteq\mathscr{C}$. If $\xi(t)$ belongs to $\mathscr{C}$, then we have for $t\in[\alpha-E,\beta]$ that

$$\begin{aligned}\|\Pi\xi(t)\|\le\ & Ke^{-\omega(t-\alpha+E)}\|\phi(\alpha-E)-\phi(t_n+\alpha-E)\|+\int_{\alpha-E}^{t}K\gamma\epsilon e^{-\omega(t-s)}ds\\ &+\int_{\alpha-E}^{t}KL_fe^{-\omega(t-s)}\|\xi(s-\tau)\|ds\\ &<\left(2M_\phi K+\frac{2KL_fR_1e^{\omega\tau/2}}{\omega}\right)e^{-\omega(t-\alpha+E)/2}+\frac{K\gamma\epsilon(1+L_fR_2)}{\omega}\\ &=R_1e^{-\omega(t-\alpha+E)/2}+R_2\gamma\epsilon.\end{aligned}$$

The inequality $\|\Pi\xi(t)\|<R_1e^{-\omega(t-\alpha+E)/2}+R_2\gamma\epsilon$ is valid also for $t\in[\alpha-E-\tau,\alpha-E)$ since $R_1>2M_\phi$. On the other hand, if $\xi(t)$ belongs to $\mathscr{C}$, then one can confirm that $\|\Pi\xi\|_\infty\le 2K(M_\phi+\dfrac{M_f+M_g}{\omega})$. Hence, $\Pi(\mathscr{C})\subseteq\mathscr{C}$.

Now, let us take two functions $\xi(t),\overline{\xi}(t)\in\mathscr{C}$. Clearly, $\Pi\xi(t)-\Pi\overline{\xi}(t)=0$ for each $t<\alpha-E$. It can be verified for $t\ge\alpha-E$ that

$$\left\|\Pi\xi(t)-\Pi\overline{\xi}(t)\right\|\le\int_{\alpha-E}^{t}KL_fe^{-\omega(t-s)}\left\|\xi(s-\tau)-\overline{\xi}(s-\tau)\right\|ds$$

$$\leq \frac{KL_f}{\omega}\Big(1 - e^{-\omega(t-\alpha+E)}\Big) \sup_{t \geq \alpha - E - \tau} \left\| \xi(t) - \overline{\xi}(t) \right\|.$$

The last inequality yields $\|\Pi\xi - \Pi\overline{\xi}\|_\infty \leq \frac{KL_f}{\omega}\|\xi - \overline{\xi}\|_\infty$. Thus, the operator Π is contractive by means of condition $(C3)$.

According to the uniqueness of solutions for (2.2), $\xi(t) = \phi(t) - \phi(t+t_n)$ is the unique fixed point of the operator Π. Consequently, the sequence $\xi_k(t)$, $\xi_{k+1} = \Pi(\xi_k)$, $k = 0, 1, 2, \ldots,$ where

$$\xi_0(t) = \begin{cases} \phi(t) - \phi(t+t_n), & t < \alpha - E, \\ \phi(\alpha - E) - \phi(\alpha - E + t_n), & t \geq \alpha - E, \end{cases}$$

which belongs to $\mathscr{C}$, is converging to $\phi(t) - \phi(t+t_n)$ on $\mathbb{R}$. Therefore,

$$\|\phi(t+t_n) - \phi(t)\| \leq R_1 e^{\omega(t-\alpha+E)/2} + R_2\gamma\epsilon$$

for $t \in [\alpha - E, \beta]$.

Since the number E is sufficiently large such that $E \geq \frac{2}{\omega}\ln(\frac{1}{\gamma\epsilon})$, we have for each $t \in [\alpha, \beta]$ that

$$\|\phi(t+t_n) - \phi(t)\| \leq (R_1 + R_2)\gamma\epsilon < \epsilon.$$

Hence, $\|\phi(t+t_n) - \phi(t)\| \to 0$ as $n \to \infty$ uniformly on the compact interval $[\alpha, \beta]$.

In the remaining part of the proof, we will show the existence of a sequence $\{\overline{u}_n\}$, $\overline{u}_n \to \infty$ as $n \to \infty$, and positive numbers $\overline{\epsilon}_0$, δ such that $\|\phi(t+t_n) - \phi(t)\| \geq \overline{\epsilon}_0$ for $t \in \left[\overline{u}_n - \delta, \overline{u}_n + \delta\right]$.

Since the function $g(t)$ is uniformly continuous, there exists a positive number $\widetilde{\delta}$, which is independent of t_n and u_n, $n \in \mathbb{N}$, such that both of the inequalities

$$\|g(t+t_n) - g(t_n+u_n)\| \leq \frac{\epsilon_0}{4\sqrt{m}}$$

and

$$\|g(t) - g(u_n)\| \leq \frac{\epsilon_0}{4\sqrt{m}}$$

hold for every $t \in [u_n - \widetilde{\delta}, u_n + \widetilde{\delta}]$ and $n \in \mathbb{N}$.

Fix an arbitrary natural number n, and suppose that $g(t) = (g_1(t), g_2(t), \ldots, g_m(t))$, where each $g_k(t)$, $1 \leq k \leq m$, is a real valued function. One can confirm that there exists an integer j_n, $1 \leq j_n \leq m$, such that

$$\left|g_{j_n}(t_n+u_n) - g_{j_n}(u_n)\right| \geq \frac{\epsilon_0}{\sqrt{m}}.$$

Therefore, using the last inequality, we obtain for $t \in [u_n - \widetilde{\delta}, u_n + \widetilde{\delta}]$ that

$$\begin{aligned} \left|g_{j_n}(t+t_n) - g_{j_n}(t)\right| &\geq \left|g_{j_n}(t_n+u_n) - g_{j_n}(u_n)\right| - \left|g_{j_n}(t+t_n) - g_{j_n}(t_n+u_n)\right| \\ &\quad - \left|g_{j_n}(t) - g_{j_n}(u_n)\right| \\ &\geq \frac{\epsilon_0}{2\sqrt{m}}. \end{aligned} \tag{2.3}$$

There exist numbers $s_1^n, s_2^n, \ldots, s_m^n \in \left[u_n - \widetilde{\delta}, u_n + \widetilde{\delta}\right]$ such that

$$\left\| \int_{u_n-\widetilde{\delta}}^{u_n+\widetilde{\delta}} (g(s+t_n) - g(s))ds \right\| = 2\widetilde{\delta}\left[\sum_{i=1}^{m} \left(g_i(s_i^n + t_n) - g_i(s_i^n)\right)^2\right]^{1/2}.$$

Accordingly, the inequality (2.3) implies that

$$\left\| \int_{u_n-\widetilde{\delta}}^{u_n+\widetilde{\delta}} (g(s+t_n) - g(s))ds \right\| \geq 2\widetilde{\delta}\left|g_{j_n}(s_{j_n}^n + t_n) - g_{j_n}(s_{j_n}^n)\right| \geq \frac{\widetilde{\delta}\epsilon_0}{\sqrt{m}}.$$

Now, using the relation

$$\begin{aligned} \phi\big(t_n + u_n + \widetilde{\delta}\big) - \phi\big(u_n + \widetilde{\delta}\big) = \phi\big(t_n + u_n - \widetilde{\delta}\big) - \phi\big(u_n - \widetilde{\delta}\big) + \int_{u_n-\widetilde{\delta}}^{u_n+\widetilde{\delta}} A[\phi(s+t_n) - \phi(s)]ds \\ + \int_{u_n-\widetilde{\delta}}^{u_n+\widetilde{\delta}} [f(\phi(s+t_n-\tau)) - f(\phi(s-\tau))]ds \\ + \int_{u_n-\widetilde{\delta}}^{u_n+\widetilde{\delta}} [g(s+t_n) - g(s)]ds, \end{aligned}$$

one can verify that

$$\left\|\phi\left(t_n+u_n+\widetilde{\delta}\right)-\phi\left(u_n+\widetilde{\delta}\right)\right\| \geq \frac{\widetilde{\delta}\epsilon_0}{\sqrt{m}}-\left(1+2\widetilde{\delta}\|A\|\right)\sup_{t\in\left[u_n-\widetilde{\delta},u_n+\widetilde{\delta}\right]}\|\phi(t+t_n)-\phi(t)\|$$
$$-2\widetilde{\delta}L_f\sup_{t\in\left[u_n-\widetilde{\delta}-\tau,u_n+\widetilde{\delta}-\tau\right]}\|\phi(t+t_n)-\phi(t)\|.$$

Hence, we have

$$\sup_{t\in[u_n-\widetilde{\delta}-\tau,u_n+\widetilde{\delta}]}\|\phi(t+t_n)-\phi(t)\| \geq \frac{\widetilde{\delta}\epsilon_0}{2\sqrt{m}\left(1+\widetilde{\delta}\|A\|+\widetilde{\delta}L_f\right)}.$$

Let $\bar{u}_n$ be a point that belongs to the interval $\left[u_n-\widetilde{\delta}-\tau,u_n+\widetilde{\delta}\right]$ satisfying

$$\sup_{t\in[u_n-\widetilde{\delta}-\tau,u_n+\widetilde{\delta}]}\|\phi(t+t_n)-\phi(t)\| = \|\phi(t_n+\bar{u}_n)-\phi(\bar{u}_n)\|.$$

Define the numbers

$$\bar{\epsilon}_0=\frac{\widetilde{\delta}\epsilon_0}{4\sqrt{m}\left(1+\widetilde{\delta}\|A\|+\widetilde{\delta}L_f\right)}$$

and

$$\delta=\frac{\widetilde{\delta}\epsilon_0}{8\sqrt{m}\left(1+\widetilde{\delta}\|A\|+\widetilde{\delta}L_f\right)\left(M_\phi\|A\|+M_\phi L_f+M_g\right)}.$$

If t belongs to the interval $[\bar{u}_n-\delta,\bar{u}_n+\delta]$, then it can be obtained that

$$\begin{aligned}\|\phi(t+t_n)-\phi(t)\| &\geq \|\phi(t_n+\eta_n)-\phi(\eta_n)\|-\left|\int_{\eta_n}^t\|A\|\|\phi(s+t_n)-\phi(s)\|ds\right|\\ &\quad-\left|\int_{\eta_n}^t L_f\|\phi(s+t_n-\tau)-\phi(s-\tau)\|ds\right|\\ &\quad-\left|\int_{\eta_n}^t\|g(s+t_n)-g(s)\|ds\right|\\ &\geq \frac{\widetilde{\delta}\epsilon_0}{2\sqrt{m}\left(1+\widetilde{\delta}\|A\|+\widetilde{\delta}L_f\right)}-2\delta\left(M_\phi\|A\|+M_\phi L_f+M_g\right)\\ &=\bar{\epsilon}_0.\end{aligned}$$

Hence, $\|\phi(t+t_n)-\phi(t)\|\geq\bar{\epsilon}_0$ for each t from the intervals $[\bar{u}_n-\delta,\bar{u}_n+\delta]$, $n\in\mathbb{N}$. Clearly, $\bar{u}_n\to\infty$ as $n\to\infty$. Consequently, the bounded solution $\phi(t)$ is unpredictable. □

Remark 2.1. The result of Theorem 2.1 is valid also for the case $\tau=0$. More precisely, if $(C1)$, $(C2)$ are valid and $\omega-KL_f>0$, then the system

$$x'(t)=Ax(t)+f(x(t))+g(t)$$

possesses a unique uniformly exponentially stable unpredictable solution provided that $g(t)$ is an unpredictable function.

3. Unpredictable solutions of discrete equations

The definition of an unpredictable sequence is as follows.

Definition 3.1. A bounded sequence $\{\kappa_i\}$, $i\in\mathbb{Z}$, in $\mathbb{R}^p$ is called unpredictable if there exist a positive number ϵ_0 and sequences $\{\zeta_n\}$, $\{\eta_n\}$, $n\in\mathbb{N}$, of positive integers both of which diverge to infinity such that $\left\|\kappa_{i+\zeta_n}-\kappa_i\right\|\to 0$ as $n\to\infty$ for each i in bounded intervals of integers and $\left\|\kappa_{\zeta_n+\eta_n}-\kappa_{\eta_n}\right\|\geq\epsilon_0$ for each $n\in\mathbb{N}$.

Definition 3.1 is of main use in the present section. It is requested by the method of the proof. Nevertheless, in future analyses, there may be needs for the following other definition, which can be considered as an analogue of Definition 2.3.

Definition 3.2. A bounded sequence $\{\kappa_i\}$, $i\in\mathbb{Z}$, in $\mathbb{R}^p$ is called unpredictable if there exist a positive number ϵ_0 and sequences $\{\zeta_n\}$, $\{\eta_n\}$, $n\in\mathbb{N}$, of positive integers both of which diverge to infinity such that $\left\|\kappa_{\zeta_n}-\kappa_0\right\|\to 0$ as $n\to\infty$ and $\left\|\kappa_{\zeta_n+\eta_n}-\kappa_{\eta_n}\right\|\geq\epsilon_0$ for each $n\in\mathbb{N}$.

It is worth noting that the topologies in Definitions 2.1 and 3.1 are metrizable [10]. Consequently, the existence of an unpredictable sequence in the sense of Definition 3.1 indicates the presence of Poincaré chaos [11]. Throughout the section, an unpredictable sequence and an unpredictable solution are understood as mentioned in Definition 3.1.

In this section, we will consider the following discrete equation,

$$z_{i+1} = Bz_i + h(z_i) + \psi_i, \tag{3.4}$$

where $i \in \mathbb{Z}$, $B \in \mathbb{R}^{p\times p}$ is a nonsingular matrix, $h : \mathbb{R}^p \to \mathbb{R}^p$ is a continuous function, and $\{\psi_i\}$, $i \in \mathbb{Z}$, is an unpredictable sequence.

The following assumptions on Eq. (3.4) are required.

(C4) There exists a positive number M_h such that $\sup\limits_{x\in\mathbb{R}^p} \|h(x)\| \le M_h$;

(C5) There exists a positive number L_h such that $\|h(x) - h(y)\| \le L_h\|x - y\|$ for all $x, y \in \mathbb{R}^p$;

(C6) $\|B\| + L_h < 1$.

According to the results of [24], if conditions $(C4) - (C6)$ hold, then Eq. (3.4) possesses a unique bounded solution $\{\varphi_i\}$, $i \in \mathbb{Z}$, which satisfies the relation

$$\varphi_i = \sum_{j=-\infty}^{i} B^{i-j}\big(h(\varphi_{j-1}) + \psi_{j-1}\big). \tag{3.5}$$

One can show under the same conditions that the bounded solution attracts all other solutions of (3.4). More precisely, the inequality

$$\|z_i - \varphi_i\| \le (\|B\| + L_h)^{(i-i_0)}\left\|z^0 - \varphi_{i_0}\right\|$$

is valid for all $i \ge i_0$, where $\{z_i\}$, $i \in \mathbb{Z}$, is a solution of (3.4) with $z_{i_0} = z^0$ for some integer i_0 and $z^0 \in \mathbb{R}^p$.

The following theorem is concerned with the existence of an unpredictable solution of the discrete Eq. (3.4).

Theorem 3.1. *The bounded solution $\{\varphi_i\}$, $i \in \mathbb{Z}$, of Eq. (3.4) is unpredictable under the conditions $(C4) - (C6)$.*

Proof. Fix an arbitrary positive number ϵ, and suppose that γ is a positive number satisfying

$$\gamma \le \left[\frac{1}{1 - \|B\| - L_h} + \frac{2(M_h + M_\psi)}{1 - \|B\|}\right]^{-1}.$$

Let i_1 and i_2 be integers such that $i_2 > i_1$, and take a natural number E with

$$E \ge \frac{\ln(\gamma\epsilon)}{\ln(\|B\| + L_h)} - 1. \tag{3.6}$$

Since $\{\psi_i\}$, $i \in \mathbb{Z}$, is an unpredictable sequence, there exist a positive number ϵ_0 and sequences $\{\zeta_n\}$, $\{\eta_n\}$, $n \in \mathbb{N}$, of positive integers, both of which diverge to infinity, such that $\left\|\psi_{i+\zeta_n} - \psi_i\right\| \to 0$ as $n \to \infty$ for each i with $i_1 - E - 1 \le i \le i_2 - 1$ and $\left\|\psi_{\zeta_n+\eta_n} - \psi_{\eta_n}\right\| \ge \epsilon_0$ for each $n \in \mathbb{N}$.

First of all, we will show that $\left\|\varphi_{i+\zeta_n} - \varphi_i\right\| \to 0$ as $n \to \infty$ for each i with $i_1 \le i \le i_2$. There exists a natural number n_0, independent of i, such that for each $n \ge n_0$ the inequality $\left\|\psi_{i+\zeta_n} - \psi_i\right\| < \gamma\epsilon$ is valid whenever $i_1 - E - 1 \le i \le i_2 - 1$.

Fix an arbitrary integer $n \ge n_0$. One can obtain using the relation (3.5) that

$$\varphi_{i+\zeta_n} - \varphi_i = \sum_{j=-\infty}^{i} B^{i-j}\big(h(\varphi_{j+\zeta_n-1}) - h(\varphi_{j-1}) + \psi_{j+\zeta_n-1} - \psi_{j-1}\big).$$

Therefore, for $i_1 - E \le i \le i_2$, we have

$$\begin{aligned}\left\|\varphi_{i+\zeta_n} - \varphi_i\right\| < {} & \frac{2(M_h + M_\psi)}{1 - \|B\|}\|B\|^{i-i_1+E+1} + \frac{\gamma\epsilon}{1 - \|B\|}\big(1 - \|B\|^{i-i_1+E+1}\big) \\ & + L_h \sum_{j=i_1-E}^{i} \|B\|^{i-j}\left\|\varphi_{j+\zeta_n-1} - \varphi_{j-1}\right\|.\end{aligned} \tag{3.7}$$

Let us denote

$$r_i = \|B\|^{-i}\left\|\varphi_{i+\zeta_n} - \varphi_i\right\|$$

and

$$q_i = \frac{2(M_h + M_\psi)}{1 - \|B\|}\|B\|^{-i_1+E+1} + \frac{\gamma\epsilon}{1 - \|B\|}\big(\|B\|^{-i} - \|B\|^{-i_1+E+1}\big).$$

The inequality (3.7) yields

$$r_i < q_i + \frac{L_h}{\|B\|} \sum_{j=i_1-E}^{i} r_{j-1}.$$

It can be verified by applying the discrete analogue of Gronwall inequality that

$$r_i \le q_i + \frac{L_h}{\|B\|} \sum_{j=i_1-E}^{i} q_{j-1} \left(1 + \frac{L_h}{\|B\|}\right)^{i-j}.$$

Thus, for $i_1 - E \le i \le i_2$, we have

$$r_i \le \frac{2(M_h + M_\psi)}{1 - \|B\|} \|B\|^{-i} (\|B\| + L_h)^{i-i_1+E+1} + \frac{\gamma\epsilon}{1 - \|B\| - L_h} \|B\|^{-i} \big[1 - (\|B\| + L_h)^{i-i_1+E+1}\big].$$

The last inequality implies that

$$\left\|\varphi_{i+\zeta_n} - \varphi_i\right\| < \frac{2(M_h + M_\psi)}{1 - \|B\|} (\|B\| + L_h)^{i-i_1+E+1} + \frac{\gamma\epsilon}{1 - \|B\| - L_h}.$$

One can confirm using (3.6) that $(\|B\| + L_h)^{i-i_1+E+1} \le \gamma\epsilon$ for $i_1 \le i \le i_2$. Therefore, for each $n \ge n_0$, the inequality

$$\left\|\varphi_{i+\zeta_n} - \varphi_i\right\| < \left[\frac{2(M_h + M_\psi)}{1 - \|B\|} + \frac{1}{1 - \|B\| - L_h}\right]\gamma\epsilon \le \epsilon$$

is valid for $i_1 \le i \le i_2$. Hence, $\left\|\varphi_{i+\zeta_n} - \varphi_i\right\| \to 0$ as $n \to \infty$ for each i with $i_1 \le i \le i_2$.

Next, we will show the existence of a positive number $\overline{\epsilon}_0$ and a sequence $\{\widetilde{\eta}_n\}$ with $\widetilde{\eta}_n \to \infty$ as $n \to \infty$ such that $\left\|\varphi_{\zeta_n+\widetilde{\eta}_n} - \varphi_{\widetilde{\eta}_n}\right\| \ge \overline{\epsilon}_0$ for each $n \in \mathbb{N}$.

Using the relations

$$\varphi_{\zeta_n+\eta_n+1} = B\varphi_{\zeta_n+\eta_n} + h(\varphi_{\zeta_n+\eta_n}) + \psi_{\zeta_n+\eta_n}$$

and

$$\varphi_{\eta_n+1} = B\varphi_{\eta_n} + h(\varphi_{\eta_n}) + \psi_{\eta_n}$$

we obtain for $n \in \mathbb{N}$ that

$$\left\|\varphi_{\zeta_n+\eta_n+1} - \varphi_{\eta_n+1}\right\| \ge \epsilon_0 - (\|B\| + L_h)\left\|\varphi_{\zeta_n+\eta_n} - \varphi_{\eta_n}\right\|.$$

Therefore,

$$\max\left\{\left\|\varphi_{\zeta_n+\eta_n+1} - \varphi_{\eta_n+1}\right\|, \left\|\varphi_{\zeta_n+\eta_n} - \varphi_{\eta_n}\right\|\right\} \ge \overline{\epsilon}_0, \tag{3.8}$$

where $\overline{\epsilon}_0 = \dfrac{\epsilon_0}{1 + \|B\| + L_h}$.

For each $n \in \mathbb{N}$, let us take $\widetilde{\eta}_n = \eta_n + 1$ if $\left\|\varphi_{\zeta_n+\eta_n+1} - \varphi_{\eta_n+1}\right\| \ge \left\|\varphi_{\zeta_n+\eta_n} - \varphi_{\eta_n}\right\|$, and we set $\widetilde{\eta}_n = \eta_n$ otherwise. Clearly, $\widetilde{\eta}_n \to \infty$ as $n \to \infty$. According to inequality (3.8), we have $\left\|\varphi_{\zeta_n+\widetilde{\eta}_n} - \varphi_{\widetilde{\eta}_n}\right\| \ge \overline{\epsilon}_0$ for each $n \in \mathbb{N}$. Consequently, the bounded solution $\{\varphi_i\}$, $i \in \mathbb{Z}$, of (3.4) is unpredictable. □

A possible way to obtain a different unpredictable sequence from a given one is mentioned in the following theorem.

Theorem 3.2. *Suppose that $\{\kappa_i\}$, $i \in \mathbb{Z}$, is an unpredictable sequence such that $\kappa_i \in \Lambda$ for each i, where Λ is a bounded subset of $\mathbb{R}^p$. If $\Phi : \Lambda \to \mathbb{R}^q$ is a function such that there exist positive numbers L_1 and L_2 with $L_1\|s_1 - s_2\| \le \|\Phi(s_1) - \Phi(s_2)\| \le L_2\|s_1 - s_2\|$ for all $s_1, s_2 \in \Lambda$, then the sequence $\{\overline{\kappa}_i\}$ defined through the equation $\overline{\kappa}_i = \Phi(\kappa_i)$, $i \in \mathbb{Z}$, is also unpredictable.*

Proof. Since $\{\kappa_i\}$, $i \in \mathbb{Z}$, is an unpredictable sequence, there exist a positive number ϵ_0 and sequences $\{\zeta_n\}$, $\{\eta_n\}$, $n \in \mathbb{N}$, of positive integers, both of which diverge to infinity, such that $\left\|\kappa_{i+\zeta_n} - \kappa_i\right\| \to 0$ as $n \to \infty$ for each i in bounded intervals of integers and $\left\|\kappa_{\zeta_n+\eta_n} - \kappa_{\eta_n}\right\| \ge \epsilon_0$ for each $n \in \mathbb{N}$.

Fix an arbitrary positive number ϵ, and let i_1 and i_2 be any two integers such that $i_2 > i_1$. One can find a natural number n_0, which does not depend on i, such that for each $n \ge n_0$ we have $\left\|\kappa_{i+\zeta_n} - \kappa_i\right\| < \epsilon/L_2$ whenever $i_1 \le i \le i_2$. Therefore, the inequality

$$\left\|\overline{\kappa}_{i+\zeta_n} - \overline{\kappa}_i\right\| \le L_2\left\|\kappa_{i+\zeta_n} - \kappa_i\right\| < \epsilon$$

is satisfied for each $n \ge n_0$ and each i with $i_1 \le i \le i_2$. This shows that $\left\|\overline{\kappa}_{i+\zeta_n} - \overline{\kappa}_i\right\| \to 0$ as $n \to \infty$ on bounded intervals of integers. On the other hand, for each $n \in \mathbb{N}$, we have that

$$\left\|\overline{\kappa}_{\zeta_n+\eta_n} - \overline{\kappa}_{\eta_n}\right\| \ge L_1\left\|\kappa_{\zeta_n+\eta_n} - \kappa_{\eta_n}\right\| \ge L_1\epsilon_0.$$

Consequently, $\{\overline{\kappa}_i\}$, $i \in \mathbb{Z}$, is an unpredictable sequence. □

4. A continuous unpredictable function via the logistic map

Consider the space $\Sigma_2 = \left\{s = (s_0 s_1 s_2 \ldots) \mid s_j = 0 \text{ or } 1\right\}$ of infinite sequences of 0's and 1's with the metric

$$d(s, \bar{s}) = \sum_{k=0}^{\infty} \frac{|s_k - \bar{s}_k|}{2^k},$$

where $s = (s_0 s_1 s_2 \ldots)$, $\bar{s} = (\bar{s}_0 \bar{s}_1 \bar{s}_2 \ldots) \in \Sigma_2$. The Bernoulli shift $\sigma\colon \Sigma_2 \to \Sigma_2$ is defined as $\sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 s_3 \ldots)$. The map σ is continuous and Σ_2 is a compact metric space [8,9].

Through the proof of Lemma 3.1 [1], we constructed an element $s^{**} = (s_0^{**} s_1^{**} s_2^{**} \ldots)$ of Σ_2 which is unpredictable by placing all blocks of 0's and 1's in a specific order without any repetitions and extending it to the left hand side by appropriately choosing the terms s_i^{**} for negative values of i.

Let us take into account the logistic map

$$\lambda_{i+1} = F_\mu(\lambda_i), \tag{4.9}$$

where $i \in \mathbb{Z}$ and $F_\mu(s) = \mu s(1-s)$. The interval [0, 1] is invariant under the iterations of (4.9) for $\mu \in (0, 4]$ [25].

It was proved by Shi and Yu [13] that for each $\mu \in [3 + (2/3)^{1/2}, 4]$, there exist a natural number $m_0 > 4$ and a Cantor set $\Lambda_0 \subset [0, 1]$ such that the map $F_\mu^{m_0}$ on Λ_0 is topologically conjugate to the Bernoulli shift σ on Σ_2. Using the results of paper [13] and Lemma 3.1 [1], a property of the logistic map (4.9) was given in Theorem 4.1 of paper [1]. According to Theorem 4.1 [1], for each $\mu \in [3 + (2/3)^{1/2}, 4]$, the logistic map (4.9) possesses an unpredictable solution.

Now, let us denote by $\{\psi_i\}$, $i \in \mathbb{Z}$, an unpredictable solution of the logistic map (4.9) with $\mu = 3.91$ inside the unit interval [0, 1], and consider the function

$$\Theta(t) = \int_{-\infty}^{t} e^{-2(t-s)} \Omega(s) ds \tag{4.10}$$

where the function $\Omega(t)$ is defined by $\Omega(t) = \psi_i$ for $t \in [i, i+1)$, $i \in \mathbb{Z}$.

It can be verified that the function $\Theta(t)$ is the unique globally exponentially stable solution of the differential equation

$$v'(t) = -2v(t) + \Omega(t). \tag{4.11}$$

Moreover, $\Theta(t)$ is bounded on the whole real axis such that $\sup_{t \in \mathbb{R}} |\Theta(t)| \le \frac{1}{2}$, and it is uniformly continuous since its derivative is bounded.

Because the sequence $\{\psi_i\}$, $i \in \mathbb{Z}$, is unpredictable, there exist a positive number ϵ_0 and sequences $\{\zeta_n\}$, $\{\eta_n\}$ both of which diverge to infinity such that $\left|\psi_{i+\zeta_n} - \psi_i\right| \to 0$ as $n \to \infty$ for each i in bounded intervals of integers and $\left|\psi_{\zeta_n + \eta_n} - \psi_{\eta_n}\right| \ge \epsilon_0$ for each n.

Let us fix an arbitrary positive number ϵ and take an arbitrary compact subset $[\alpha, \beta] \subset \mathbb{R}$. Suppose that N is a sufficiently large positive integer satisfying $N \ge \frac{1}{2} \ln\left(\frac{3}{2\epsilon}\right)$. There exists a natural number n_0 such that for each $n \ge n_0$ the inequality

$$\left|\psi_{i+\zeta_n} - \psi_i\right| < \frac{2\epsilon}{3}$$

is valid for $i = \lfloor\alpha\rfloor - N, \lfloor\alpha\rfloor - N + 1, \ldots, \lfloor\beta\rfloor$, where $\lfloor\alpha\rfloor$ and $\lfloor\beta\rfloor$ denote the largest integers which are not greater than α and β, respectively.

Fix a natural number $n \ge n_0$. Using the relation

$$\Theta(t + \zeta_n) - \Theta(t) = e^{-2(t - \lfloor\alpha\rfloor + N)}\left(\Theta(\lfloor\alpha\rfloor - N + \zeta_n) - \Theta(\lfloor\alpha\rfloor - N)\right) + \int_{\lfloor\alpha\rfloor - N}^{t} e^{-2(t-s)} [\Omega(s + \zeta_n) - \Omega(s)] ds,$$

one can verify for $t \in [\lfloor\alpha\rfloor, \lfloor\beta\rfloor + 1)$ that $|\Theta(t + \zeta_n) - \Theta(t)| < \epsilon$. Hence, $\Theta(t + \zeta_n) \to \Theta(t)$ as $n \to \infty$ uniformly on $[\alpha, \beta]$.

On the other hand, one can show that $\sup_{t \in [\eta_n, \eta_n + 1]} |\Theta(t + \zeta_n) - \Theta(t)| \ge \frac{\epsilon_0}{4}$ for each $n \in \mathbb{N}$. The last inequality implies that the function $\Theta(t)$ is unpredictable.

It is still difficult to simulate this unpredictable solution, but there is chaos because of unpredictability, and this is why in what follows we visualize the chaotic behavior.

Let us take into account the differential equation

$$v'(t) = -2v(t) + \widetilde{\Omega}(t), \tag{4.12}$$

where the function $\widetilde{\Omega}(t)$ is defined by $\widetilde{\Omega}(t) = \lambda_i$ for $t \in [i, i+1)$, $i \in \mathbb{Z}$, in which $\{\lambda_i\}$ is the solution of (4.9) with $\lambda_0 = 0.4$.

We depict in Fig. 1 the solution of (4.12) corresponding to the initial data $v(0) = 0.37$. The choice of the parameter $\mu = 3.91$ of the logistic map (4.9) and the value $\lambda_0 = 0.4$ were considered for shadowing in the paper [26]. It is seen in Fig. 1 that the dynamics of (4.12) is chaotic, and this supports that the function $\Theta(t)$ is unpredictable.

Illustrative examples that support the theoretical results are provided in the next section.

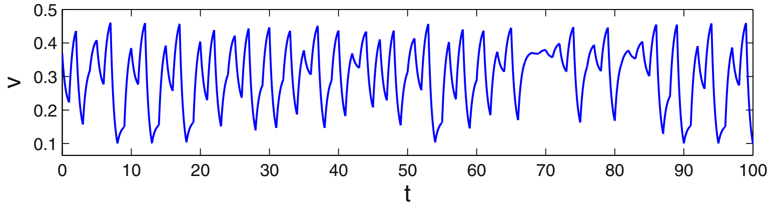

Fig. 1. Chaotic behavior of equation (4.12). The initial data $\nu(0) = 0.37$ is utilized.

5. Examples

5.1. Example 1

In this example, we take into account the retarded non-autonomous differential equation

$$x''(t) + 4x'(t) + 1.5x(t) + 0.02x^2(t-0.1) = \Theta(t), \tag{5.13}$$

where $\Theta(t)$ is the unpredictable function defined by (4.10).

Using the variables $x_1(t) = x(t)$ and $x_2(t) = x'(t)$, Eq. (5.13) can be written as

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= -1.5x_1(t) - 4x_2(t) - 0.02x_1^2(t-0.1) + \Theta(t). \end{aligned} \tag{5.14}$$

System (5.14) is in the form of (2.1) with $\tau = 0.1, f(x_1, x_2) = \left(0, -0.02x_1^2\right)$, and $A = \begin{pmatrix} 0 & 1 \\ -1.5 & -4 \end{pmatrix}$. The eigenvalues of the matrix A are $-2 + \sqrt{10}/2$ and $-2 - \sqrt{10}/2$. One can show that

$$e^{At} = P \begin{pmatrix} e^{(-2+\sqrt{10}/2)t} & 0 \\ 0 & e^{(-2-\sqrt{10}/2)t} \end{pmatrix} P^{-1},$$

where $P = \begin{pmatrix} 1 & (-4+\sqrt{10})/3 \\ (-4+\sqrt{10})/2 & 1 \end{pmatrix}$. Thus, the inequality $\left\| e^{At} \right\| \le Ke^{-\omega t}$ is valid for $t \ge 0$ with $K = \|P\| \left\|P^{-1}\right\| \approx 2.0685$ and $\omega = 2 - \sqrt{10}/2$.

One can verify numerically that the solutions of (5.14) eventually enter the compact region

$$\mathscr{D} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \ 0.14 \le x_1 \le 0.26, \ -0.06 \le x_2 \le 0.05 \right\}$$

as t increases. Therefore, it is reasonable to consider the conditions $(C1)$ and $(C2)$ inside the region $\mathscr{D}$.

Conditions $(C1) - (C3)$ are valid for system (5.14) with $M_f = 0.001352$ and $L_f = 0.0104$. According to Theorem 2.1, system (5.14) possesses a unique uniformly exponentially stable unpredictable solution.

To demonstrate the chaos appearance due to the unpredictable function $\Theta(t)$, let us consider the system

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= -1.5x_1(t) - 4x_2(t) - 0.02x_1^2(t-0.1) + \nu(t), \end{aligned} \tag{5.15}$$

where $\nu(t)$ is the solution of (4.12) represented in Fig. 1.

The x_1 and x_2-coordinates of the solution of (5.15) corresponding to the initial conditions $x_1(t) = 0.18$, $x_2(t) = 0.01$, $t \in [-0.1, 0]$, are shown in Fig. 2. The figure supports the result of Theorem 2.1 such that (5.14) possesses an unpredictable solution, and it reveals that the dynamics of (5.15) is chaotic. Moreover, the trajectory of the same solution is depicted in Fig. 3, and this simulation also confirms the presence of chaos in system (5.15).

5.2. Example 2

We take into account the discrete system

$$\begin{aligned} x_{i+1} &= \frac{x_i}{2} - \frac{y_i}{7} + 3\psi_i^3 \\ y_{i+1} &= -\frac{x_i}{8} + \frac{y_i}{3} + \frac{x_i^{2/3}}{12} + 4\psi_i, \end{aligned} \tag{5.16}$$

where $\{\psi_i\}$ is an unpredictable solution of (4.9) with $\mu = 3.91$. Theorem 3.2 implies that the sequence $\left\{\overline{\psi}_i\right\}$, $i \in \mathbb{Z}$, defined by $\overline{\psi}_i = (3\psi_i^3, \ 4\psi_i) \in \mathbb{R}^2$ is also unpredictable.

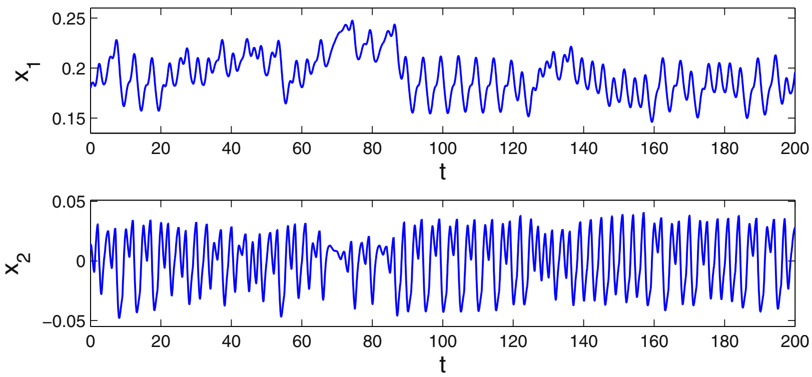

Fig. 2. Time series of the x_1 and x_2-coordinates of system (5.15). Chaotic behavior in both coordinates is observable in the figure.

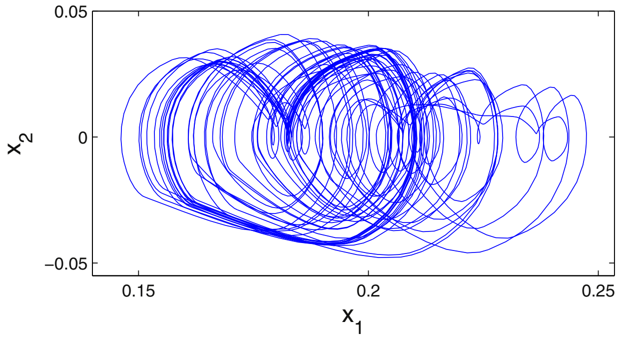

Fig. 3. The trajectory of system (5.15). The figure manifests that the dynamics of (5.15) is chaotic.

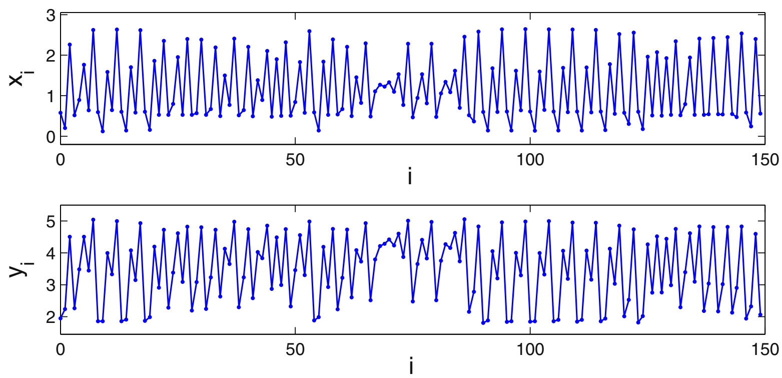

Fig. 4. The solution of (5.17) with the initial data $\lambda_0 = 0.4$, $x_0 = 0.58$, and $y_0 = 1.95$. The figure supports the result of Theorem 3.1 such that (5.16) possesses an unpredictable solution.

In order to demonstrate the chaotic behavior of (5.16), we consider the system

$$\begin{aligned} x_{i+1} &= \frac{x_i}{2} - \frac{y_i}{7} + 3\lambda_i^3 \\ y_{i+1} &= -\frac{x_i}{8} + \frac{y_i}{3} + \frac{x_i^{2/3}}{12} + 4\lambda_i, \end{aligned} \tag{5.17}$$

where $\{\lambda_i\}$ is a solution of (4.9) inside the interval [0,1], again with $\mu = 3.91$. One can numerically verify that for each $\{\lambda_i\}$, the bounded solutions of (5.17) take place inside the compact region

$$\{(x, y) \in \mathbb{R}^2 : \ 0.1 \le x \le 2.7, \ \ 1.7 \le y \le 5.1\}.$$

Therefore, the conditions $(C4) - (C6)$ are satisfied for system (5.16), and there exists a unique unpredictable solution of (5.16) in accordance with Theorem 3.1.

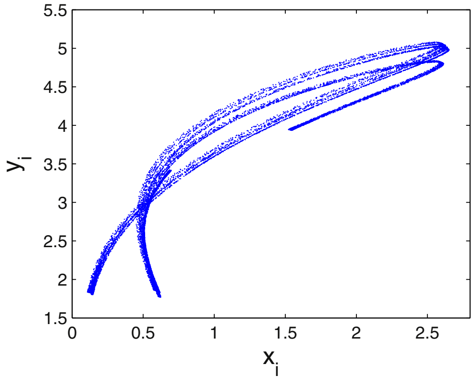

Fig. 5. The trajectory of the discrete system (5.17) corresponding to the initial data $\lambda_0 = 0.4,\ x_0 = 0.58,$ and $y_0 = 1.95$. The figure indicates the chaotic behavior of system (5.16).

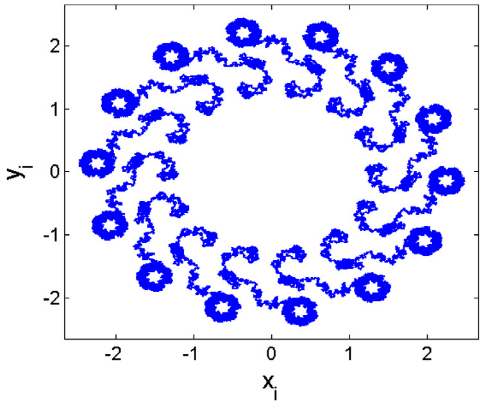

Fig. 6. The orbit of system (5.19) with $\lambda_0 = 0.4,\ x_0 = 1,$ and $y_0 = 1$. The figure reveals that the orbit behaves chaotically near the 14-periodic orbit of (5.18).

Fig. 4 shows the first and second coordinates of the solution of system (5.17) with the initial data $\lambda_0 = 0.4,\ x_0 = 0.58,$ and $y_0 = 1.95$. Moreover, we represent in Fig. 5 the two dimensional trajectory of the same solution. Both Figs. 4 and 5 support the result of Theorem 3.1 such that an unpredictable sequence takes place in the dynamics of the discrete system (5.16) and the behaviour of the system is chaotic.

5.3. Example 3 (Poincaré chaos near periodic orbits)

In this example, we will demonstrate the appearance of irregular behavior near periodic orbits of discrete systems. For that purpose, let us consider the system

$$\begin{aligned} x_{i+1} &= \cos(\omega_0)x_i + \sin(\omega_0)y_i \\ y_{i+1} &= -\sin(\omega_0)x_i + \cos(\omega_0)y_i, \end{aligned} \tag{5.18}$$

where ω_0 is a real parameter. It is shown in the book [25] that the system (5.18) admits a stable periodic orbit whenever the value $\omega_0/2\pi$ is rational. Taking $\mu = 3.86$ in the logistic map (4.9) and perturbing system (5.18) with solutions of (4.9), we set up the system

$$\begin{aligned} x_{i+1} &= \cos(\omega_0)x_i + \sin(\omega_0)y_i + 0.001\lambda_i \\ y_{i+1} &= -\sin(\omega_0)x_i + \cos(\omega_0)y_i + 0.001\lambda_i, \end{aligned} \tag{5.19}$$

where $\{\lambda_i\}$ is a solution of (4.9).

Let us take $\omega_0 = \pi/7$ so that the non-perturbed system (5.18) possesses a one parameter family of stable 14-periodic orbits. We depict in Fig. 6 the trajectory of (5.19) corresponding to the initial data $\lambda_0 = 0.4,\ x_0 = 1,$ and $y_0 = 1$. The total number of iterations used in the simulation is 65×10^6. The utilized parameter value $\mu = 3.86$ of the logistic map (4.9) and the initial point $\lambda_0 = 0.4$ were analyzed for shadowing in paper [26]. It is seen in Fig. 6 that the applied perturbation makes system (5.19) behave chaotically near the 14-periodic orbit of (5.18). It is worth noting that Fig. 6 represents a single orbit. The fractal structure of the orbit is also observable in the simulation. Fig. 6 manifests the appearance of chaos near the periodic orbit of (5.18).

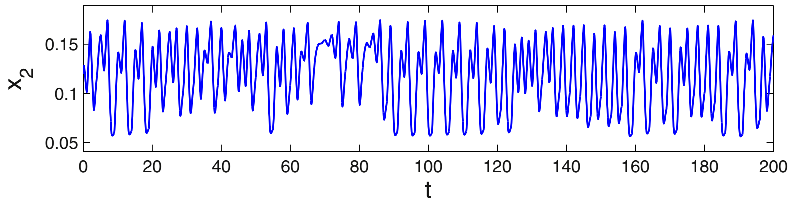

Fig. 7. The x_2-coordinate of the output of network (6.21) corresponding to the initial data $x_1(t) = 0.34,\ x_2(t) = 0.12,\ x_3(t) = 0.19,\ t \in [-\tau_1, 0]$. The simulation result demonstrates the presence of chaos.

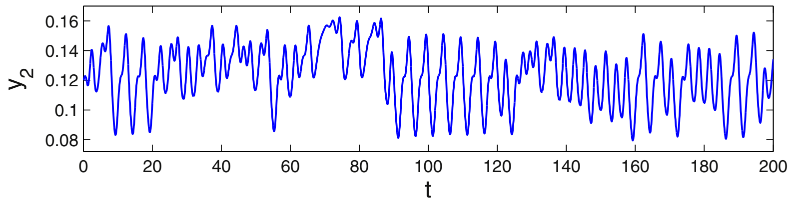

Fig. 8. The y_2-coordinate of the output of network (6.23) corresponding to the initial data $y_1(t) = 0.15,\ y_2(t) = 0.12,\ y_3(t) = 0.09,\ t \in [-\tau_2, 0]$. The figure confirms the extension of Poincaré chaos by the Hopfield neural network (6.22).

6. An application to Hopfield neural networks

This section is devoted to an application of our results to neural networks. Let us, first, consider the Hopfield neural network with delay and unpredictable input

$$
\begin{aligned}
x_1'(t) &= -1.7x_1(t) + 0.01\tanh(x_1(t-\tau_1)) - 0.02\tanh(x_2(t-\tau_1)) \\
&\quad +0.04\tanh(x_3(t-\tau_1)) + 2.1\Theta(t) \\
x_2'(t) &= -3.5x_2(t) + 0.06\tanh(x_1(t-\tau_1)) + 0.03\tanh(x_2(t-\tau_1)) \\
&\quad +0.08\tanh(x_3(t-\tau_1)) + 1.3\Theta(t) \\
x_3'(t) &= -2.8x_3(t) - 0.05\tanh(x_1(t-\tau_1)) - 0.06\tanh(x_2(t-\tau_1)) \\
&\quad +0.03\tanh(x_3(t-\tau_1)) + 1.9\Theta(t),
\end{aligned}
\tag{6.20}
$$

where $\tau_1 = 0.3$ and $\Theta(t)$ is the unpredictable function defined by (4.10).

The coefficients of the nonlinear terms in (6.20) are sufficiently small in absolute value such that the conditions of Theorem 2.1 hold, and accordingly, network (6.20) possesses a unique uniformly exponentially stable unpredictable solution $\widetilde{\Theta}(t) = \big(\widetilde{\Theta}_1(t), \widetilde{\Theta}_2(t), \widetilde{\Theta}_3(t)\big)$.

To demonstrate the chaotic dynamics of (6.20) numerically, in a similar way to the example in Section 5.1, we utilize the network

$$
\begin{aligned}
x_1'(t) &= -1.7x_1(t) + 0.01\tanh(x_1(t-\tau_1)) - 0.02\tanh(x_2(t-\tau_1)) \\
&\quad +0.04\tanh(x_3(t-\tau_1)) + 2.1v(t) \\
x_2'(t) &= -3.5x_2(t) + 0.06\tanh(x_1(t-\tau_1)) + 0.03\tanh(x_2(t-\tau_1)) \\
&\quad +0.08\tanh(x_3(t-\tau_1)) + 1.3v(t) \\
x_3'(t) &= -2.8x_3(t) - 0.05\tanh(x_1(t-\tau_1)) - 0.06\tanh(x_2(t-\tau_1)) \\
&\quad +0.03\tanh(x_3(t-\tau_1)) + 1.9v(t),
\end{aligned}
\tag{6.21}
$$

where $v(t)$ is the solution of (4.12) shown in Fig. 1. We represent in Fig. 7 the x_2-coordinate of (6.21) using the initial data $x_1(t) = 0.34,\ x_2(t) = 0.12,\ x_3(t) = 0.19,\ t \in [-\tau_1, 0]$. Fig. 7 reveals that the dynamics of the network (6.20) is Poincaré chaotic.

Next, we will discuss the extension of Poincaré chaos by Hopfield neural networks. For that purpose, we use the unpredictable output $\widetilde{\Theta}(t)$ of (6.20) as an external input for another Hopfield neural network and set up the system

$$
\begin{aligned}
y_1'(t) &= -2.3y_1(t) + 0.05\tanh(y_1(t-\tau_2)) - 0.07\tanh(y_3(t-\tau_2)) + \widetilde{\Theta}_1(t), \\
y_2'(t) &= -y_2(t) + 0.02\tanh(y_1(t-\tau_2)) - 0.08\tanh(y_2(t-\tau_2)) \\
&\quad +0.09\tanh(y_3(t-\tau_2)) + \widetilde{\Theta}_2(t), \\
y_3'(t) &= -1.5y_3(t) + 0.04\tanh(y_2(t-\tau_2)) - 0.05\tanh(y_3(t-\tau_2)) + \widetilde{\Theta}_3(t),
\end{aligned}
\tag{6.22}
$$

where $\tau_2 = 0.2$.

Using the result of Theorem 2.1 one more time, it can be confirmed that (6.22) admits a unique unpredictable output, which is exponentially stable. In order to simulate the extension of unpredictability, we consider the network

$$\begin{aligned}
y_1'(t) &= -2.3y_1(t) + 0.05\tanh(y_1(t-\tau_2)) - 0.07\tanh(y_3(t-\tau_2)) + x_1(t),\\
y_2'(t) &= -y_2(t) + 0.02\tanh(y_1(t-\tau_2)) - 0.08\tanh(y_2(t-\tau_2))\\
&\quad +0.09\tanh(y_3(t-\tau_2)) + x_2(t),\\
y_3'(t) &= -1.5y_3(t) + 0.04\tanh(y_2(t-\tau_2)) - 0.05\tanh(y_3(t-\tau_2)) + x_3(t),
\end{aligned} \tag{6.23}$$

where $(x_1(t), x_2(t), x_3(t))$ is the output of (6.21) whose second coordinate is depicted in Fig. 7.

The time series of the y_2-coordinate of the Hopfield neural network (6.23) is shown in Fig. 8, where the chaotic behavior is observable. The initial data $y_1(t) = 0.15,\ y_2(t) = 0.12,\ y_3(t) = 0.09,\ t \in [-\tau_2, 0]$, are used in the simulation. Fig. 8 confirms the extension of the unpredictable behavior of (6.20) by the network (6.22).

7. Conclusions

The starting point for the present research is the unpredictable point [11], a new object for the dynamical systems theory founded by Poincaré and Birkhoff [19,20]. In paper [11], we developed the Poisson stability of a motion to unpredictability such that a new type of chaos, the Poincaré chaos, has been obtained. It has become clear that the concept can be easily extended to the object of analysis in the theory of differential equations, considering unpredictable functions as points moving by shifts of the time argument [1,2]. Thus, an unpredictable function was defined as an unpredictable point of the Bebutov dynamics, and the first theorems on the existence of unpredictable solutions were proved in [1,2]. The metric of the Bebutov dynamics is not convenient for applications, since it is hard to verify. For this reason, in the present study, we apply the topology of uniform convergence on compact sets to define unpredictable functions. The topology is metrizable and easy for applications with integral operators. Thus, one can accept that we lay a corner stone to the foundation of differential equations theory related to unpredictable solutions, and consequently, chaos. Therefore, in our opinion, a new field to analyze in the theory of differential equations has been discovered. Since many results of differential equations have their counterparts in discrete equations [24], one can suppose that theorems on the existence of unpredictable solutions can be proved for discrete equations. The present paper is a one that realizes the both paradigms. The existence and uniqueness theorems for quasilinear delay and ordinary differential equations and difference equations have been proved, when the perturbation is an unpredictable function or sequence. This is visualized as Poincaré chaos in simulations.

We emphasize the meaning of our results for the development of theory of differential and discrete equations issuing from the general character of considered systems. In the next research, one can investigate the existence of unpredictable solutions, and consequently, chaos in discrete equations by applying well developed techniques such as averaging method, method of integral manifolds, method of asymptotic integrations, second Lyapunov method, and others [27].

We hope that the constructions of unpredictable functions and sequences suggested in the present study will be developed to more larger classes of functions, enlarging the applicability meaning of our results. This method of chaos appearance and of its consequent control cannot be underestimated in neuroscience [28–30]. Our approach suggests very effective applications and analysis methods of generation and control of chaos in neural networks through a single function, an unpredictable one. This is why we can say that the motivation for our results is strong. In our research we provide unpredictable functions as external inputs which are not discontinuous but obtained from discontinuous functions by integration. This may enrich application variance for chaos analysis of neural network dynamics.

Our results can trigger further extension of the theory for discrete dynamical systems which can be defined as iterated maps. For this reason, we expect that the introduction of unpredictability in the discrete dynamics will be beneficial for new researches in hyperbolic dynamics, strange attractors, and ergodic theory [9,27,31–34].

Unpredictable functions are compulsorily accompanied by Poincaré chaos, and this is considered in our previous papers [1,2], too. It is significant that the unpredictable motion is still a Poisson stable one. On the basis of unpredictable functions one immediately considers a new type of chaos, Poincaré chaos, and consequently, the next question is how the chaos is related to the previously known types of chaos. Developing this line of research, one will increase the practical role of the unpredictable functions and motions as much as of the previously known types of chaos.

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