Mapping Fatou-Julia Iterations

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ABSTRACT
A new method of fractals construction based on Fatou-Julia iteration is proposed. We develop a non-ordinary way to map fractal into a new fractal, where the mapping function is involved in the dynamics of the generated formula such that the modification does not violate the Julia and Fatou recipe for creating fractals. The method is applied for both Mandelbrot and Julia sets, and we follow the same technique of determining the fractal set using in the original iteration. The results of this paper are expected to have beneficial effects on many fields such as computer graphics and 3D printing technologies.

Keywords
Fatou-Julia iterations; Julia and Mandelbrot sets; Fractal mapping theorem

1. INTRODUCTION
French mathematicians P. Fatou and G. Julia invented a special iteration in the complex plane [1, 2] such that new geometrical objects with unusual properties can be built. One of the famous is the Julia set. Later B. Mandelbrot suggested to call them fractals [3].

The most important fractals for our present study are Julia sets which were discovered in 1917-1918 by Julia and Fatou, both of whom independently studied the iteration of rational functions in the complex plane [4]. They established the fundamental results on the dynamics of complex functions published in their papers [1, 2]. In 1979, Mandelbrot visualized Julia sets.

Julia and Mandelbrot fractals are very great achievements for set theory, topology, functions theory, chaos, and real world problems. Therefore, studies in the area has to be at the frontiers of modern sciences and applications. Additionally, the development of researches on the basis of fractals is of significant importance to any possible direction starting from the basis of set theory.

In general, a fractal is defined as a set that displays self-similarity and repeats the same patterns at every scale [5]. Mandelbrot [6] defined fractals as sets for which the Hausdorff dimension is either fractional or strictly exceeds the topological dimension. Based on this, self-similarity can occur in fractals, but, in general, a fractal need not be self-similar. The easiest way to indicate that a set has fractional dimension is through self-similarity [7].

There are two sides of the fractal research related to the present paper. The first one is the Fatou-Julia Iteration (FJI) and the second one is the proposal by Mandelbrot to consider dimension as a criterion for fractals. In our study, both factors are crucial as we apply the FJI for the construction of the sets and the dimension factor to confirm that the built sets are fractals. In previous studies the iteration and the dimension factors were somehow separated, since self-similarity provided by the iteration has been self-sufficient to recognize fractals, but in our research the similarity is not true in general. We have to emphasize that there is a third player on the scene, the modern state of computers' power. Their roles are important for the realization of the resulting images. One can say that the instrument is at least of the same importance for application of our idea to fractals as for realization of Fatou-Julia iteration in Mandelbrot and Julia sets. Nevertheless, we expect that the present study will significantly increase the usage of computers for fractal analysis. Moreover, beside differential equations, our suggestions will affect the software development for fractals investigation and applications [8, 9, 10, 11].

2. FATOU-JULIA ITERATIONS
The core of the present study is built through FJI and the quadratic map $P: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined by

$$P(z,c) = z^2 + c, \qquad (1)$$

where $\mathbb{C}$ denotes the set of complex numbers.

Consider the equation $z_{n+1} = P(z_n),\ n \geq 0$. The points $z_0 \in \mathbb{C}$ are included in a fractal $\mathcal{F}$ depending on the boundedness of the sequence z_n, and we say that the fractal $\mathcal{F}$ is constructed by FJI. The so-called filled-in Julia set, $\mathcal{K}_c$, is constructed by including only the points $z_0 \in \mathbb{C}$ such that the sequence z_n is bounded [12]. Moreover, in the simulation, those points $z_0 \in \mathbb{C}$ where $\{z_n\}$ is divergent are colored in a different way, correspondingly to the rate of divergence. The term Julia set $\mathcal{J}_c$, usually denotes the boundary of the filled Julia set, i.e., $\mathcal{J}_c = \partial\mathcal{K}_c$. Figure 1 shows a Julia set with $c = -0.175 - 0.655i$.

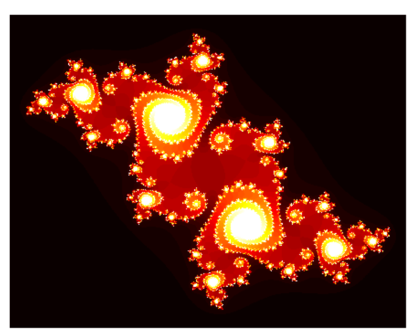

Figure 1. Julia set.

The Mandelbrot set, $\mathcal{M}$, is also generated by a Fato-Julia iteration. In this case, we consider the equation $z_{n+1} = p(z_n, c)$, and include in the set $\mathcal{M}$ the points $c \in \mathbb{C}$ such that $\{z_n(c)\}$, $z_0(c) = 0$, is bounded. Here again, the points $c \in \mathbb{C}$ corresponding to divergent sequences z_n are plotted in various colors depending on the rate of the divergence. Figure 2 shows the Mandelbrot set.

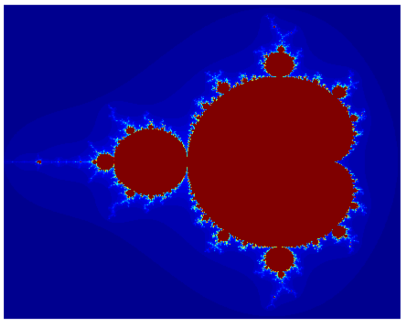

Figure 2. The Mandelbrot set.

3. MAPPING ITERATION

To describe our way for mapping of fractals, let us consider a fractal set $\mathcal{F} \subseteq A \subset \mathbb{C}$, constructed by the following FJI,

$$z_{n+1} = F(z_n), \quad (2)$$

where $F: A \to A$ is not necessarily a rational map. We suggest that the original fractal $\mathcal{F}$ can be transformed "recursively" into a new fractal set. For that purpose, we modify the FJI, and consider iterations to be of the form

$$f^{-1}(z_{n+1}) = F(f^{-1}(z_n)), \quad (3)$$

Or explicitly

$$z_{n+1} = f\left(F(f^{-1}(z_n))\right), \quad (4)$$

where f is a one-to-one map on A. Next, we examine the convergence of the sequence $\{z_n\}$ for each $z_0 \in f(A)$. Denote by $\mathcal{F}_f$ the set which contains only the points z_0 corresponding to the bounded sequences. Moreover, other points can be plotted in different colors depending on the rate of the divergence of $\{z_n\}$.

To distinct the iterations (4) from the Fatou-Julia iterations let us call the first ones *Fractals Mapping Iterations* (FMI). It is clear that FJI is a particular FMI, when the function is the identity map. The mapping of fractals is a difficult problem which depends on infinitely long iteration processes, and has to be accompanied with sufficient conditions to ensure that the image is again fractal.

The next theorem is the main instrument for the detection of fractal mappings. Accordingly, we call it *Fractal Mapping Theorem* (FMT).

Theorem 3.1 If f is a bi-Lipschitz function, i.e. there exist numbers $l_1, l_2 > 0$ such that

$$l_1|u - v| \le |f(u) - f(v)| \le l_2|u - v| \quad (5)$$

for all $u, v \in A$, then $\mathcal{F}_f = f(F)$.

Proof. Fix an arbitrary $w \in \mathcal{F}_f$. There exists a bounded sequence $\{w_k\}$ such that $w_0 = w$ and $f^{-1}(w_{k+1}) = F(f^{-1}(w_k))$. Let us denote $z_k = f^{-1}(w_k)$. Our purpose is to show that $\{z_k\}$ is a bounded sequence. Indeed

$$|z_k - z_0| = |f^{-1}(w_k) - f^{-1}(w_0)| \le \frac{1}{l_1}|w_k - w_0|$$

Hence, the boundedness of $\{w_k\}$ implies the same property for $\{z_k\}$, and therefore, we have $z_0 = f^{-1}(w) \in \mathcal{F}$.

Now, assume that $w \in f(\mathcal{F})$. There is $z \in \mathcal{F}$ such that $f(z) = w$ and a bounded sequence $\{z_k\}$ such that $z_0 = z$ and $z_{k+1} = F(z_k)$. Consider, $w_0 = w$ and $w_k = f(z_k), k \ge 0$. It is clear that the sequence $\{w_k\}$ satisfies the iteration (3) and moreover

$$|w_k - w_0| = |f(z_k) - f(z_0)| \le l_2|z_k - z_0|$$

Consequently, $\{w_k\}$ is bounded, and $w \in \mathcal{F}_f$. □

The bi-Lipschitz maps also preserve Hausdorff dimension. It is proven that [13], If f is a bi-Lipschitz function, then

$$\dim_H f(A) = \dim_H A,$$

where $\dim_H$ denotes the Hausdorff dimension. Furthermore, It can be verified that If $f: A \to \mathbb{C}$ is a homeomorphism, then it maps the boundary of A onto the boundary of $f(A)$. It is clear that a bi-Lipschitz function is a homeomorphism.

Shishikura [14] proved that the Hausdorff dimension of the boundary of the Mandelbrot set is 2. Moreover, he showed that the Hausdorff dimension of the Julia set corresponding to $c \in \partial\mathcal{M}$ is also 2.

It implies from the above discussions that if f is a bi-Lipschitz function and $\mathcal{F}$ is either a Julia set or the boundary of the Mandelbrot set, then their images $\mathcal{F}_f$ are fractals. In what follows, we will mainly use functions, which are bi-Lipschitzian except possibly in neighborhoods of single points.

Now, we apply FMI to a Julia set $\mathcal{J}$, and the iteration will be in the form

$$f^{-1}(z_{n+1}) = [f^{-1}(z_n)]^2 + c, \quad (6)$$

with various functions f and values of c. The resulting fractals $\mathcal{J}_f = f(\mathcal{J})$ are depicted in Figs. 3, 4 and 5. They are mapped by $f(z) = \cos^{-1}\left(\frac{1}{z} - 1\right)$, $f(z) = (\cos^{-1} z^3)^{\frac{1}{4}}$ and $f(z) = (\sin^{-1} z)^{\frac{1}{5}}$ from the Julia sets with $c = -0.7589 + 0.0735i$, $c = -0.75$ and $c = 0.135 + 0.628i$ respectively.

For mapping of the Mandelbrot set, we propose the FMI

$$z_{n+1} = z_n^2 + f^{-1}(c), \quad (7)$$

Along the lines of the proof of Theorem 3.1, one can show that if the map f is bi-Lipschitzian, then the iteration (7) defines the relation $f(\mathcal{M}) = \mathcal{M}_f$, where $\mathcal{M}_f$ is a new fractal. Figure 6 shows two fractals mapped from the Mandelbrot set in Fig. 2. The fractal shown in the above part of Fig. 6 is mapped by $f(c) = \sqrt{c + \frac{1}{4}} - \frac{1}{2}$ whereas the below one is mapped by $f(c) = \left(\frac{1}{c} - 1\right)^{\frac{1}{2}}$.

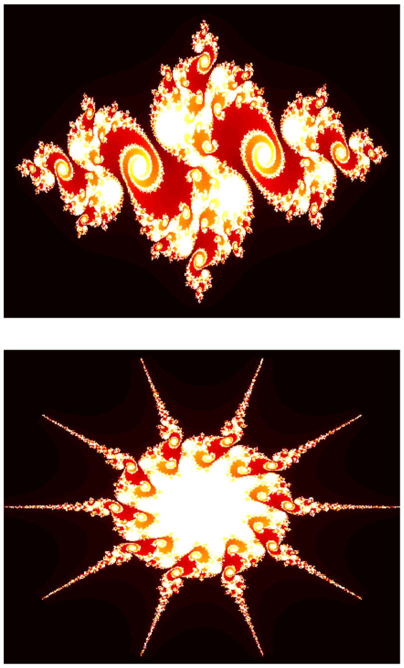

Figure 3. Julia set, $c = -0.7589 + 0.0735i$ and its image by $f(z) = (\sin^{-1} z)^{\frac{1}{5}}$.

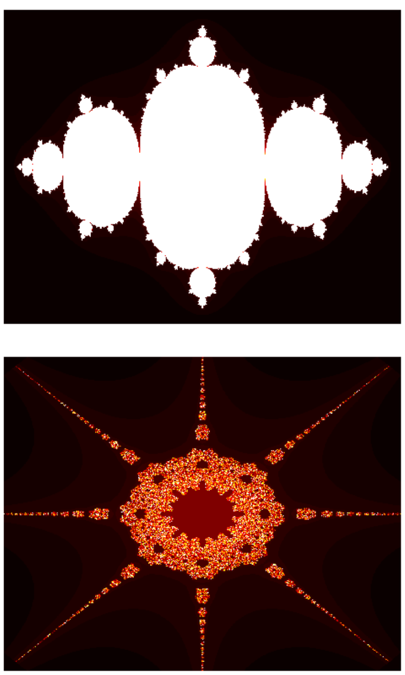

Figure 4. Julia set, $c = -0.75$ and its image by $f(z) = (\cos^{-1} z^3)^{\frac{1}{4}}$.

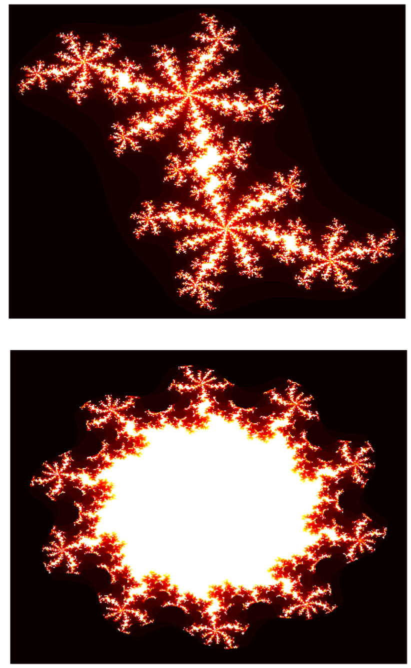

Figure 5. Julia set, $c = 0.135 + 0.628i$ and its image by $f(z) = (\sin^{-1} z)^{\frac{1}{5}}$.

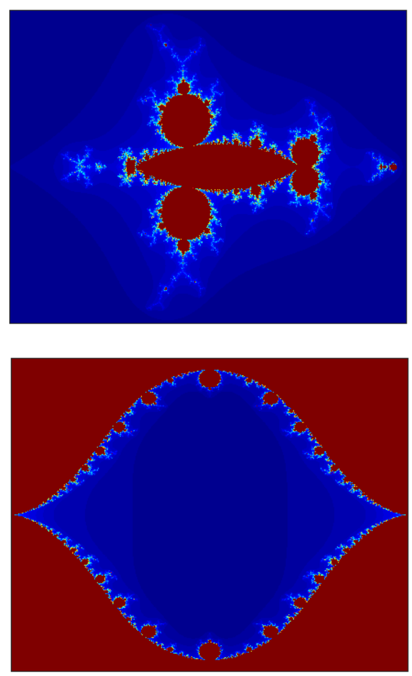

Figure 6. Images of Mandelbrot set by $f(c) = \sqrt{c + \frac{1}{4}} - \frac{1}{2}$ and $f(c) = \left(\frac{1}{c} - 1\right)^{\frac{1}{2}}$.

Despite the intensive research of fractals lasts more than 35 years [3], there are still no results on mapping of the sets, and our paper is the first one to consider the problem. To say about mathematical challenges connected to our suggestions, let us start with topological equivalence of fractals and consequently, normal forms. Differential and discrete equations will be analyzed with new methods of fractal dynamics joined with dimension analysis. Next, the theory for dynamical systems which is defined as iterated maps can be developed. Therefore, mapping of fractals will be beneficial for new researches in hyperbolic dynamics, strange attractors, and ergodic theory [15, 16]. The developed approach will enrich the methods for the discovery and construction of fractals in the real world and industry such as nano-fiber engineering, 3D printing, biotechnologies, and genetics [17, 18, 19, 20, 21].

4. ACKNOWLEDGMENTS

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