



Li–Yorke Chaos in Hybrid Systems on a Time Scale

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By using the reduction technique to impulsive differential equations [Akhmet & Turan, 2006], we rigorously prove the presence of chaos in dynamic equations on time scales (DETS). The results of the present study are based on the Li–Yorke definition of chaos. This is the first time in the literature that chaos is obtained for DETS. An illustrative example is presented by means of a Duffing equation on a time scale.

Keywords: Li–Yorke chaos; dynamic equations on time scales; proximality; frequent separation; Duffing equation; hybrid systems.

1. Introduction

The concept of chaos has been one of the attractive topics among scientists since the studies of Poincaré [Andersson, 1994], Cartwright and Littlewood [1945], Levinson [1949], Lorenz [1963] and Ueda [1978]. Another subject that is also popular is the theory of time scales, which was first presented by Hilger [1988]. Both concepts have many applications in various disciplines such as mechanics, electronics, neural networks, population models and economics. See, for instance, [Barrio *et al.*, 2014; Bohner & Peterson, 2001; Fečkan, 2011; Grebogi & Yorke, 1997; Luo, 2014; Owens *et al.*, 2013; Thamilmaran *et al.*, 2004; Tisdell & Zaidi, 2008; Zhang *et al.*, 2010] and the references therein.

Dynamic equations on time scales (DETS) have been extensively investigated in the literature [Bohner & Peterson, 2001; Lakshmikantham *et al.*, 1996]. However, to the best of our knowledge, the presence of chaos has never been achieved in DETS.

Motivated by the deficiency of mathematical methods for the investigation of chaos in such equations, we suggest the results of the present study.

The first mathematical definition of chaos was introduced by Li and Yorke [1975] for discrete dynamical systems in a compact interval of the real line. The presence of an uncountable scrambled set is one of the main features of the Li–Yorke chaos. The original definition of Li and Yorke was extended to dimensions greater than one by Marotto [1978]. According to Marotto [1978], a multidimensional continuously differentiable map possesses generalized Li–Yorke chaos if it has a snap-back repeller. The existence of Li–Yorke chaos in a spatiotemporal chaotic system was proved in [Li *et al.*, 2007] by means of Marotto’s theorem, and generalizations of Li–Yorke chaos to mappings in Banach spaces and complete metric spaces were provided in [Kloeden & Li, 2006; Shi & Chen, 2004, 2005]. It was shown by Kuchta and Smítal [1989]

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that if a map on a compact interval has a two-point scrambled set, then it possesses an uncountable scrambled set. Blanchard *et al.* [2002] proved that the presence of positive topological entropy implies chaos in the sense of Li–Yorke. Moreover, Li–Yorke chaos on several spaces in connection with the cardinality of its scrambled sets was studied within the scope of the paper [Guirao & Lampart, 2005]. Besides, Li–Yorke sensitivity, which links the Li–Yorke chaos with the notion of sensitivity, was studied in the paper [Akin & Kolyada, 2003]. The studies [Akhmet, 2009a, 2009b, 2009c; Akhmet & Fen, 2012a, 2012b, 2013, 2014] were concerned with the extension of chaos in continuous-time systems that possess asymptotically stable and hyperbolic equilibria as well as orbitally stable limit cycles. It was found in these papers that the solutions admit the same type of chaos as the perturbations. The paper [Akhmet, 2009c] deals with the general technique of dynamical synthesis, which was developed in [Brown & Chua, 1993, 1996, 1997; Brown *et al.*, 2001]. In the present study, we develop the concept of Li–Yorke chaos for DETS and prove its existence rigorously. Our results are appropriate to obtain chaotic DETS with arbitrary high dimensions.

Throughout the paper, we will denote by \mathbb{R} , \mathbb{Z} and \mathbb{N} the sets of real numbers, integers and natural numbers, respectively. In this study, we consider the following equation,

$$y^\Delta(t) = Ay(t) + f(t, y(t)) + g(t, \zeta), \quad t \in \mathbb{T}_0, \quad (1)$$

where A is a constant $n \times n$ real valued matrix, the function $f : \mathbb{T}_0 \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is rd-continuous and the function $g(t, \zeta)$ is defined through the equation $g(t, \zeta) = \zeta_k$ for $t \in [\theta_{2k-1}, \theta_{2k}]$, $k \in \mathbb{Z}$, such that $\zeta = \{\zeta_k\}$ is a sequence generated by the map

$$\zeta_{k+1} = F(\zeta_k), \quad (2)$$

where $\zeta_0 \in \Lambda$, $F : \Lambda \rightarrow \Lambda$ is a continuous function and Λ is a compact subset of \mathbb{R}^n . In Eq. (1) the time scale \mathbb{T}_0 is defined as $\mathbb{T}_0 = \bigcup_{k=-\infty}^{\infty} [\theta_{2k-1}, \theta_{2k}]$ in which $\{\theta_k\}$ is a strictly increasing sequence of real numbers such that $|\theta_k| \rightarrow \infty$ as $|k| \rightarrow \infty$ and $\sum_{-\infty}^{\infty} (\theta_{2k} - \theta_{2k-1}) = \infty$, $\sum_{-\infty}^{\infty} (\theta_{2k} - \theta_{2k-1}) = \infty$.

In the present paper, we investigate the existence of chaos in the dynamics of Eq. (1). The system under discussion is a hybrid one, since it combines the continuous dynamics on the time scale with the discrete equation used in the right-hand side of the system. We theoretically prove that chaos exists in (1) provided that the map (2) is

chaotic. For that purpose, we make use of the reduction technique to impulsive differential equations, which was presented by Akhmet and Turan [2006]. As far as we know, there is no paper on chaos in dynamics on time scales. The reason is that the dynamics is essentially nonautonomous and it is difficult to verify the ingredients of chaos for unspecified time scales. That is why we utilize the time scale introduced in the papers [Akhmet & Turan, 2006, 2009] and the method of reduction of the dynamics to impulsive differential equations [Akhmet & Turan, 2006].

The rest of the paper is organized as follows. In Sec. 2, some preliminary results as well as basic concepts about DETS are mentioned. Section 3 is devoted to the bounded solutions of (1). In Sec. 4, we give the description of the chaos of Eq. (1) and prove its presence rigorously. An example concerning Duffing equations on a time scale is presented in Sec. 5 to support the theoretical results. Finally, some concluding remarks are given in Sec. 6.

2. Preliminaries

The basic concepts that are needed in the present paper about differential equations on time scales are as follows [Bohner & Peterson, 2001; Lakshmikantham *et al.*, 1996; Lakshmikantham & Vatsala, 2002; Lakshmikantham & Devi, 2006]. A time scale is a nonempty closed subset of \mathbb{R} . On a time scale \mathbb{T} , the forward and backward jump operators are defined as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, respectively. We say that a point $t \in \mathbb{T}$ is right-scattered if $\sigma(t) > t$ and right-dense if $\sigma(t) = t$. In a similar way, if $\rho(t) < t$, then $t \in \mathbb{T}$ is called left-scattered, and otherwise it is called left-dense. Besides, a function $h : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called rd-continuous if it is continuous at each $(t, u) \in \mathbb{T} \times \mathbb{R}^n$ with right-dense t , and the limits $\lim_{(r, \nu) \rightarrow (t^-, u)} h(r, \nu) = h(t^-, u)$ and $\lim_{\nu \rightarrow u} h(t, \nu) = h(t, u)$ exist at each (t, u) with left-dense t . At a right-scattered point $t \in \mathbb{T}$, the Δ -derivative of a continuous function ϑ is defined as $\vartheta^\Delta(t) = \frac{\vartheta(\sigma(t)) - \vartheta(t)}{\sigma(t) - t}$. On the other hand, at a right-dense point t , we have $\vartheta^\Delta(t) = \lim_{r \rightarrow t} \frac{\vartheta(t) - \vartheta(r)}{t - r}$ provided that the limit exists.

It is worth noting that on the time scale \mathbb{T}_0 used in system (1) the points θ_{2k-1} , $k \in \mathbb{Z}$, are left-scattered and right-dense, and the points θ_{2k} , $k \in \mathbb{Z}$, are right-scattered and left-dense. Moreover, $\sigma(\theta_{2k}) = \theta_{2k+1}$, $\rho(\theta_{2k+1}) = \theta_{2k}$, $k \in \mathbb{Z}$, and

$\sigma(t) = \rho(t) = t$ for any $t \in \mathbb{T}_0$ except at the points $\theta_k, k \in \mathbb{Z}$.

Suppose that the time scale \mathbb{T}_0 used in the description of Eq. (1) satisfies the ω -property. That is, there exists a number $\omega > 0$ such that $t + \omega \in \mathbb{T}_0$ whenever $t \in \mathbb{T}_0$. In this case, there exists a natural number p such that $\delta_{k+p} = \delta_k$ for all $k \in \mathbb{Z}$, where $\delta_k = \theta_{2k+1} - \theta_{2k}$ [Akhmet & Turan, 2006]. Suppose that p is the minimal among those numbers.

We assume without loss of generality that $\theta_{-1} < 0 < \theta_0$. Define on the set $\mathbb{T}'_0 = \mathbb{T}_0 \setminus \bigcup_{k=-\infty}^{\infty} \{\theta_{2k-1}\}$ the ψ -substitution [Akhmet & Turan, 2006] as

$$\psi(t) = \begin{cases} t - \sum_{0 < \theta_{2k} < t} \delta_k, & t \geq 0, \\ t + \sum_{t \leq \theta_{2k} < 0} \delta_k, & t < 0. \end{cases} \quad (3)$$

The function $\psi(t)$ is one-to-one, $\psi(0) = 0, \psi(\mathbb{T}'_0) = \mathbb{R}$ and $\lim_{t \rightarrow \infty, t \in \mathbb{T}'_0} \psi(t) = \infty$. According to the results of the paper [Akhmet & Turan, 2006], $d\psi(t)/dt = 1, t \in \mathbb{T}'_0$, and $d\psi^{-1}(s)/ds = 1$ provided that $s \neq s_k, k \in \mathbb{Z}$, where

$$\psi^{-1}(s) = \begin{cases} s + \sum_{0 < s_k < s} \delta_k, & s \geq 0, \\ s - \sum_{s \leq s_k < 0} \delta_k, & s < 0, \end{cases} \quad (4)$$

and the sequence $\{s_k\}, k \in \mathbb{Z}$, is defined through the equation $s_k = \psi(\theta_{2k})$. The function ψ^{-1} is piecewise continuous with discontinuities of the first kind at the points $s_k, k \in \mathbb{Z}$, such that $\psi^{-1}(s_{k+}) - \psi^{-1}(s_k) = \delta_k$, where $\psi^{-1}(s_{k+}) = \lim_{s \rightarrow s_k^+} \psi^{-1}(s)$, the sequence $\{s_k\}$ is $(\psi(\omega), p)$ -periodic, i.e. $s_{k+p} = s_k + \psi(\omega)$ for all $k \in \mathbb{Z}$, and $\psi(t + \omega) = \psi(t) + \psi(\omega), t \in \mathbb{T}'_0$. Moreover, if a function $h(t)$ is ω -periodic on \mathbb{T}_0 , then $h(\psi^{-1}(s))$ is $\psi(\omega)$ -periodic, and vice versa.

Let us denote by $\mathcal{C}_{rd}(\mathbb{T}_0)$ the set of all functions which are rd-continuous on \mathbb{T}_0 , and let $\mathcal{C}_{rd}^1(\mathbb{T}_0) \subset \mathcal{C}_{rd}(\mathbb{T}_0)$ be the set of all continuously differentiable functions on \mathbb{T}_0 , assuming that the functions have a one sided derivative at $\theta_k, k \in \mathbb{Z}$. On the other hand, we say that a function defined on \mathbb{R} is an element of the set \mathcal{PC}_0 if it is left-continuous on \mathbb{R} and continuous on $\mathbb{R} \setminus \bigcup_{k=-\infty}^{\infty} \{s_k\}$, and it has discontinuities of the first kind at the points $s_k, k \in \mathbb{Z}$. Moreover, a function $h : \mathbb{R} \rightarrow \mathbb{R}^n$ belongs to the set \mathcal{PC}_0^1 if both h and h' are elements of \mathcal{PC}_0 , where

$h'(s_k) = \lim_{s \rightarrow s_k^-} \frac{h(s) - h(s_k)}{s - s_k}, k \in \mathbb{Z}$. It was shown by Akhmet and Turan [2006] that a function $\vartheta(t)$ belongs to $\mathcal{C}_{rd}(\mathbb{T}_0)$ ($\mathcal{C}_{rd}^1(\mathbb{T}_0)$) if and only if $\vartheta(\psi^{-1}(s))$ belongs to \mathcal{PC}_0 (\mathcal{PC}_0^1).

In accordance with the equation $y^\Delta(\theta_{2k}) = \frac{y(\theta_{2k+1}) - y(\theta_{2k})}{\theta_{2k+1} - \theta_{2k}}, k \in \mathbb{Z}$, system (1) can be written as

$$\begin{aligned} y'(t) &= Ay(t) + f(t, y(t)) \\ &+ g(t, \zeta), \quad t \in \mathbb{T}_0, \\ y(\theta_{2k+1}) &= \delta_k Ay(\theta_{2k}) + f(\theta_{2k}, y(\theta_{2k}))\delta_k \\ &+ \zeta_k \delta_k + y(\theta_{2k}). \end{aligned} \quad (5)$$

Applying the transformation $s = \psi(t)$ to (5) we obtain the following impulsive system,

$$\begin{aligned} x'(s) &= Ax(s) + f(\psi^{-1}(s), x(s)) \\ &+ g(\psi^{-1}(s), \zeta), \quad s \neq s_k, \end{aligned} \quad (6)$$

$$\begin{aligned} \Delta x|_{s=s_k} &= \delta_k Ax(s_k) + f(\psi^{-1}(s_k), x(s_k))\delta_k \\ &+ \zeta_k \delta_k, \end{aligned}$$

where $x(s) = y(\psi^{-1}(s)), \Delta x|_{s=s_k} = x(s_{k+}) - x(s_k), k \in \mathbb{Z}$, and $x(s_{k+}) = \lim_{s \rightarrow s_k^+} x(s)$.

In what follows, we will make use of the usual Euclidean norm for vectors and the norm induced by the Euclidean norm for square matrices [Horn & Johnson, 1992].

The following conditions are required throughout the paper.

- (C1) $\det(I + \delta_k A) \neq 0$ for all $k \in \mathbb{Z}$, where I is the $n \times n$ identity matrix;
- (C2) all eigenvalues of the matrix $e^{\psi(\omega)A} \prod_{j=0}^{p-1} (I + \delta_j A)$ lie inside the unit circle;
- (C3) there exists a positive number M_f such that $\sup_{t \in \mathbb{T}_0, y \in \mathbb{R}^n} \|f(t, y)\| \leq M_f$;
- (C4) there exists a positive number L_f such that $\|f(t, y_1) - f(t, y_2)\| \leq L_f \|y_1 - y_2\|$ for all $t \in \mathbb{T}_0$ and $y_1, y_2 \in \mathbb{R}^n$.

Let us denote by $X(s, r)$ the transition matrix of the linear homogeneous system

$$x'(s) = Ax(s), \quad s \neq s_k, \quad (7)$$

$$\Delta x|_{s=s_k} = \delta_k Ax(s_k).$$

Under the conditions (C1) and (C2) there exist positive numbers N and λ such that $\|X(s, r)\| \leq Ne^{-\lambda(s-r)}$ for $s \geq r$ [Akhmet, 2010; Samoilenko & Perestyuk, 1995].

The following conditions are also needed.

- (C5) $NL_f(\frac{1}{\lambda} + \frac{p\bar{\delta}}{1-e^{-\lambda\psi(\omega)}}) < 1$, where $\bar{\delta} = \max_{0 \leq k \leq p-1} \delta_k$;
 (C6) $-\lambda + NL_f + \frac{p}{\psi(\omega)} \ln(1 + NL_f\bar{\delta}) < 0$;
 (C7) $f(t + \omega, y) = f(t, y)$ for all $(t, y) \in \mathbb{T}_0 \times \mathbb{R}^n$.

The next section is devoted to the bounded solutions of system (1).

3. Bounded Solutions

Under the conditions (C1)–(C5), one can verify by using the results of [Akhmet, 2010; Samoilenko & Perestyuk, 1995] that for a fixed sequence $\zeta = \{\zeta_k\}$, $k \in \mathbb{Z}$, there exists a unique bound on \mathbb{R} solution $\phi_\zeta(s)$ of (6), which satisfies the relation

$$\begin{aligned} \phi_\zeta(s) &= \int_{-\infty}^s X(s, r)[f(\psi^{-1}(r), \phi_\zeta(r)) \\ &\quad + g(\psi^{-1}(r), \zeta)]dr + \sum_{-\infty < s_k < s} X(s, s_k+) \\ &\quad \times [f(\psi^{-1}(s_k), \phi_\zeta(s_k)) + \zeta_k]\delta_k. \end{aligned} \tag{8}$$

Moreover, $\sup_{s \in \mathbb{R}} \|\phi_\zeta(s)\| \leq K_0$, where $K_0 = N(M_f + M_F)(\frac{1}{\lambda} + \frac{p\bar{\delta}}{1-e^{-\lambda\psi(\omega)}})$ and $M_F = \max_{\eta \in \Lambda} \|F(\eta)\|$. Therefore, for a fixed sequence $\zeta = \{\zeta_k\}$, the function $\varphi_\zeta(t) = \phi_\zeta(\psi(t))$ satisfying $\varphi_\zeta(\theta_{2k+1}) = \phi_\zeta(s_k+)$, $k \in \mathbb{Z}$, is the unique solution of (5), and hence of (1), which is bounded on \mathbb{T}_0 such that $\sup_{t \in \mathbb{T}_0} \|\varphi_\zeta(t)\| \leq K_0$.

We say that the bounded solution $\varphi_\zeta(t)$ attracts a solution $y(t)$ of (1) if $\|y(t) - \varphi_\zeta(t)\| \rightarrow 0$ as $t \rightarrow \infty$, $t \in \mathbb{T}_0$. The attractiveness feature of the bounded solutions of (1) is mentioned in the next assertion.

Lemma 1. *If the conditions (C1)–(C6) are valid, then for a fixed sequence ζ , the bounded solution $\varphi_\zeta(t)$ attracts all other solutions of (1).*

Proof. Consider an arbitrary solution $y(t)$, $y(t^0) = y_0$, of (1) for some $t^0 \in \mathbb{T}_0$ and $y_0 \in \mathbb{R}^n$. Assume without loss of generality that $t^0 \neq \theta_{2k-1}$ for any $k \in \mathbb{Z}$. Let $s^0 = \psi(t^0)$ and $x(s) = y(\psi^{-1}(s))$. The relation

$$\begin{aligned} x(s) - \phi_\zeta(s) &= X(s, s^0)(y_0 - \phi_\zeta(s^0)) \\ &\quad + \int_{s^0}^s [f(\psi^{-1}(r), x(r)) - f(\psi^{-1}(r), \phi_\zeta(r))]dr \end{aligned}$$

$$\begin{aligned} &+ \sum_{s^0 \leq s_k < s} X(s, s_k+)[f(\psi^{-1}(s_k), x(s_k)) \\ &\quad - f(\psi^{-1}(s_k), \phi_\zeta(s_k))]\delta_k \end{aligned}$$

implies for $s \geq s^0$ that

$$\begin{aligned} \|x(s) - \phi_\zeta(s)\| &\leq Ne^{-\lambda(s-s^0)}\|y_0 - \phi_\zeta(s^0)\| \\ &\quad + \int_{s^0}^s NL_f e^{-\lambda(s-r)}\|x(r) - \phi_\zeta(r)\|dr \\ &\quad + \sum_{s^0 \leq s_k < s} NL_f \bar{\delta} e^{-\lambda(s-s_k)}\|x(s_k) - \phi_\zeta(s_k)\|. \end{aligned}$$

Applying the Gronwall–Bellman lemma for piecewise continuous functions [Akhmet, 2010] to the last inequality, one can obtain that

$$\begin{aligned} \|x(s) - \phi_\zeta(s)\| &\leq N(1 + NL_f \bar{\delta})^p \|y_0 - \phi_\zeta(s^0)\| \\ &\quad \times e^{[-\lambda + NL_f + p \ln(1 + NL_f \bar{\delta})/\psi(\omega)](s-s^0)}, \quad s \geq s^0. \end{aligned}$$

Therefore, we have for $t \geq t^0$, $t \in \mathbb{T}_0$, that

$$\begin{aligned} \|y(t) - \varphi_\zeta(t)\| &\leq N(1 + NL_f \bar{\delta})^p \|y_0 - \varphi_\zeta(t^0)\| \\ &\quad \times e^{[-\lambda + NL_f + p \ln(1 + NL_f \bar{\delta})/\psi(\omega)](\psi(t) - \psi(t^0))}. \end{aligned}$$

Consequently, $\|y(t) - \varphi_\zeta(t)\| \rightarrow 0$ as $t \rightarrow \infty$, $t \in \mathbb{T}_0$. ■

In the next section, we will deal with the presence of chaos in system (1).

4. The Chaotic Dynamics

The map (2) is called Li–Yorke chaotic on Λ if [Akin & Kolyada, 2003; Aulbach & Kieninger, 2001; Kolyada, 2004; Li & Yorke, 1975; Li, 1993]: (i) For every natural number p_0 , there exists a p_0 -periodic point of F in Λ ; (ii) there is an uncountable set $\mathcal{S} \subset \Lambda$, the scrambled set, containing no periodic points, such that for every $\zeta^1, \zeta^2 \in \mathcal{S}$ with $\zeta^1 \neq \zeta^2$, we have $\limsup_{k \rightarrow \infty} \|F^k(\zeta^1) - F^k(\zeta^2)\| > 0$ and $\liminf_{k \rightarrow \infty} \|F^k(\zeta^1) - F^k(\zeta^2)\| = 0$; (iii) for every $\zeta^1 \in \mathcal{S}$ and a periodic point $\zeta^2 \in \Lambda$, we have $\limsup_{k \rightarrow \infty} \|F^k(\zeta^1) - F^k(\zeta^2)\| > 0$.

Let us denote by Θ the set of all sequences $\zeta = \{\zeta_k\}$, $k \in \mathbb{Z}$, obtained by Eq. (2). A pair of sequences $\zeta = \{\zeta_k\}$, $\tilde{\zeta} = \{\tilde{\zeta}_k\} \in \Theta$ is proximal if

$\liminf_{k \rightarrow \infty} \|\zeta_k - \tilde{\zeta}_k\| = 0$. Moreover, the pair is frequently separated if $\limsup_{k \rightarrow \infty} \|\zeta_k - \tilde{\zeta}_k\| > 0$.

We say that a pair $\varphi_\zeta(t), \varphi_{\tilde{\zeta}}(t)$ of bounded solutions of (1) is proximal if for an arbitrary small real number $\epsilon > 0$ and an arbitrary large natural number E , there exists an integer m such that $\|\varphi_\zeta(t) - \varphi_{\tilde{\zeta}}(t)\| < \epsilon$ for all $t \in [\theta_{2m-1}, \theta_{2(m+E)}] \cap \mathbb{T}_0$. On the other hand, the pair $\varphi_\zeta(t), \varphi_{\tilde{\zeta}}(t)$ is frequently (ϵ_0, Δ) -separated if there exist numbers $\epsilon_0 > 0, \Delta > 0$ and infinitely many disjoint intervals $J_q \subset \mathbb{T}_0, q \in \mathbb{N}$, each with a length no less than Δ , such that $\|\varphi_\zeta(t) - \varphi_{\tilde{\zeta}}(t)\| > \epsilon_0$ for each t from these intervals. Furthermore, a pair $\varphi_\zeta(t), \varphi_{\tilde{\zeta}}(t)$ of solutions of (1) is called a Li-Yorke pair if it is proximal and frequently (ϵ_0, Δ) -separated for some positive numbers ϵ_0 and Δ .

Let \mathcal{A} be the collection of all bounded solutions $\varphi_\zeta(t)$ of (1) such that $\zeta \in \Theta$. The description of Li-Yorke chaos for system (1) is as follows.

Definition 4.1. System (1) is called Li-Yorke chaotic if:

- (i) There exists an $m\omega$ -periodic solution of (1) for each $m \in \mathbb{N}$.
- (ii) There exists an uncountable set $\Sigma \subset \mathcal{A}$, the scrambled set, which does not contain any periodic solution, such that any pair of different solutions of (1) inside Σ is a Li-Yorke pair.
- (iii) For any $\varphi_\zeta(t) \in \Sigma$ and any periodic solution $\varphi_{\tilde{\zeta}}(t) \in \mathcal{A}$, the pair $\varphi_\zeta(t), \varphi_{\tilde{\zeta}}(t)$ is frequently (ϵ_0, Δ) -separated for some positive numbers ϵ_0 and Δ .

One can verify that the sequence $\{\kappa_k\}$ defined through the equation $\kappa_k = \theta_{2k} - \theta_{2k-1}, k \in \mathbb{Z}$, is p -periodic. In what follows, we will denote $\underline{\kappa} = \min_{0 \leq k \leq p-1} \kappa_k$ and $\bar{\kappa} = \max_{0 \leq k \leq p-1} \kappa_k$. Moreover, let $i((a_0, b_0))$ be the number of terms of the sequence $\{s_k\}$ that belong to the interval (a_0, b_0) , where $a_0, b_0 \in \mathbb{R}$ with $a_0 < b_0$. One can verify that $i((a_0, b_0)) \leq p + \frac{p}{\psi(\omega)}(b_0 - a_0)$.

The next assertion is about the proximality feature of bounded solutions of Eq. (1).

Lemma 2. Suppose that the conditions (C1)–(C6) are fulfilled. If a pair of sequences $\zeta, \tilde{\zeta} \in \Theta$ is proximal, then the same is true for the pair $\varphi_\zeta(t), \varphi_{\tilde{\zeta}}(t) \in \mathcal{A}$.

Proof. Set $R_1 = 2N(M_f + M_F)(\frac{1}{\lambda} + \frac{p\bar{\delta}}{1 - e^{-\lambda\psi(\omega)}})$ and $\alpha = \lambda - NL_f - \frac{p}{\psi(\omega)} \ln(1 + NL_f\bar{\delta})$. Suppose that γ

is a real number which satisfies the inequality

$$\begin{aligned} \gamma \geq & 1 + N \left(\frac{1}{\lambda} + \frac{\bar{\delta}p}{1 - e^{-\lambda\psi(\omega)}} \right) \\ & \times \left(1 + \frac{NL_f(1 + NL_f\bar{\delta})^p}{\alpha} \right. \\ & \left. + \frac{NL_f\bar{\delta}p(1 + NL_f\bar{\delta})^p}{1 - e^{-\alpha\psi(\omega)}} \right). \end{aligned}$$

Fix an arbitrary small number $\epsilon > 0$ and an arbitrary large natural number E such that

$$E \geq \frac{1}{\alpha \underline{\kappa}} \ln \left(\frac{\gamma R_1 (1 + NL_f\bar{\delta})^p}{\epsilon} \right).$$

Assume without loss of generality that $\epsilon \leq 2M_F$. Since the pair $\zeta, \tilde{\zeta}$ is proximal, there exists an integer k_0 such that $\|g(t, \zeta) - g(t, \tilde{\zeta})\| < \epsilon/\gamma$ for $t \in [\theta_{2k_0-1}, \theta_{2(k_0+2E)}] \cap \mathbb{T}_0$. In this case, $\|g(\psi^{-1}(s), \zeta) - g(\psi^{-1}(s), \tilde{\zeta})\| < \epsilon/\gamma$ for $s \in (s_{k_0-1}, s_{k_0+2E}]$.

The bounded solutions $\phi_\zeta(s) = \varphi_\zeta(\psi^{-1}(s))$ and $\phi_{\tilde{\zeta}}(s) = \varphi_{\tilde{\zeta}}(\psi^{-1}(s))$ of (6) satisfy the relation

$$\begin{aligned} & \phi_\zeta(s) - \phi_{\tilde{\zeta}}(s) \\ &= \int_{-\infty}^s X(s, r) [f(\psi^{-1}(r), \phi_\zeta(r)) \\ & \quad - f(\psi^{-1}(r), \phi_{\tilde{\zeta}}(r)) + g(\psi^{-1}(r), \zeta) \\ & \quad - g(\psi^{-1}(r), \tilde{\zeta})] dr + \sum_{-\infty < s_k < s} X(s, s_k +) \\ & \quad \times [f(\psi^{-1}(s_k), \phi_\zeta(s_k)) \\ & \quad - f(\psi^{-1}(s_k), \phi_{\tilde{\zeta}}(s_k)) + \zeta_k - \tilde{\zeta}_k] \delta_k. \end{aligned}$$

Thus, for $s \in (s_{k_0-1}, s_{k_0+2E}]$, we have that

$$\begin{aligned} & \|\phi_\zeta(s) - \phi_{\tilde{\zeta}}(s)\| \\ & \leq R_1 e^{-\lambda(s-s_{k_0-1})} + \frac{N\epsilon}{\gamma\lambda} (1 - e^{-\lambda(s-s_{k_0-1})}) \\ & \quad + \frac{N\bar{\delta}p\epsilon}{\gamma(1 - e^{-\lambda\psi(\omega)})} (1 - e^{-\lambda(s-s_{k_0-1}+\psi(\omega))}) \\ & \quad + \int_{s_{k_0-1}}^s NL_f e^{-\lambda(s-r)} \|\phi_\zeta(r) - \phi_{\tilde{\zeta}}(r)\| dr \\ & \quad + \sum_{s_{k_0-1} < s_k < s} NL_f \bar{\delta} e^{-\lambda(s-s_k)} \|\phi_\zeta(s_k) - \phi_{\tilde{\zeta}}(s_k)\|. \end{aligned} \tag{9}$$

Let us define the functions

$$u(s) = e^{\lambda s} \|\phi_\zeta(s) - \phi_{\tilde{\zeta}}(s)\| \quad \text{and}$$

$$v(s) = \beta_1 + \beta_2 e^{\lambda s},$$

where

$$\beta_1 = R_1 e^{\lambda s_{k_0-1}} - \frac{N\epsilon}{\gamma\lambda} e^{\lambda s_{k_0-1}} - \frac{N\bar{\delta}p\epsilon}{\gamma(1 - e^{-\lambda\psi(\omega)})} e^{\lambda(s_{k_0-1} - \psi(\omega))} \quad \text{and}$$

$$\beta_2 = \frac{N\epsilon}{\gamma} \left(\frac{1}{\lambda} + \frac{\bar{\delta}p}{1 - e^{-\lambda\psi(\omega)}} \right).$$

One can confirm by means of (9) that

$$u(s) \leq v(s) + \int_{s_{k_0-1}}^s NL_f u(r) dr + \sum_{s_{k_0-1} < s_k < s} NL_f \bar{\delta} u(s_k).$$

It can be shown by applying the analogue of the Gronwall's lemma for piecewise continuous functions that

$$u(s) \leq v(s) + \int_{s_{k_0-1}}^s NL_f (1 + NL_f \bar{\delta})^{i((r,s))} e^{NL_f(s-r)} v(r) dr + \sum_{s_{k_0-1} < s_k < s} NL_f \bar{\delta} (1 + NL_f \bar{\delta})^{i((s_k,s))} e^{NL_f(s-s_k)} v(s_k).$$

Accordingly, the inequality

$$u(s) \leq \beta_1 (1 + NL_f \bar{\delta})^p e^{(\lambda-\alpha)(s-s_{k_0-1})} + \beta_2 e^{\lambda s} + \frac{NL_f \beta_2 (1 + NL_f \bar{\delta})^p}{\alpha} e^{\lambda s} (1 - e^{-\alpha(s-s_{k_0-1})}) + \frac{NL_f \bar{\delta} p \beta_2 (1 + NL_f \bar{\delta})^p}{1 - e^{-\alpha\psi(\omega)}} e^{\lambda s} (1 - e^{-\alpha(s-s_{k_0-1} + \psi(\omega))}).$$

is valid. Therefore,

$$\|\phi_\zeta(s) - \phi_{\tilde{\zeta}}(s)\| < R_1 (1 + NL_f \bar{\delta})^p e^{-\alpha(s-s_{k_0-1})} + \frac{N\epsilon}{\gamma} \left(\frac{1}{\lambda} + \frac{\bar{\delta}p}{1 - e^{-\lambda\psi(\omega)}} \right) \times \left(1 + \frac{NL_f (1 + NL_f \bar{\delta})^p}{\alpha} + \frac{NL_f \bar{\delta} p (1 + NL_f \bar{\delta})^p}{1 - e^{-\alpha\psi(\omega)}} \right)$$

for $s \in (s_{k_0-1}, s_{k_0+2E}]$.

Suppose that s belongs to the interval $(s_{k_0-1+E}, s_{k_0+2E}]$. Because the number E is sufficiently large such that $E \geq \frac{1}{\alpha\kappa} \ln\left(\frac{\gamma R_1 (1 + NL_f \bar{\delta})^p}{\epsilon}\right)$ and $s - s_{k_0-1} > E\kappa$, we have

$$R_1 (1 + NL_f \bar{\delta})^p e^{-\alpha(s-s_{k_0-1})} < \frac{\epsilon}{\gamma}.$$

Hence,

$$\|\phi_\zeta(s) - \phi_{\tilde{\zeta}}(s)\| < \frac{\epsilon}{\gamma} + \frac{N\epsilon}{\gamma} \left(\frac{1}{\lambda} + \frac{\bar{\delta}p}{1 - e^{-\lambda\psi(\omega)}} \right) \left(1 + \frac{NL_f (1 + NL_f \bar{\delta})^p}{\alpha} + \frac{NL_f \bar{\delta} p (1 + NL_f \bar{\delta})^p}{1 - e^{-\alpha\psi(\omega)}} \right) \leq \epsilon.$$

The last inequality yields $\|\varphi_\zeta(t) - \varphi_{\tilde{\zeta}}(t)\| < \epsilon$ for $t \in [\theta_{2(k_0+E)-1}, \theta_{2(k_0+2E)}] \cap \mathbb{T}_0$. Consequently, the couple $\varphi_\zeta(t), \varphi_{\tilde{\zeta}}(t)$ is proximal. ■

The frequent separation feature of the bounded solutions of (1) is presented in the next lemma.

Lemma 3. Under the conditions (C1)–(C5), if a pair of sequences $\zeta, \tilde{\zeta} \in \Theta$ is frequently separated, then the pair of solutions $\varphi_\zeta(t), \varphi_{\tilde{\zeta}}(t) \in \mathcal{A}$ is frequently (ϵ_0, Δ) -separated for some positive numbers ϵ_0 and Δ .

Proof. Because the pair of sequences $\zeta, \tilde{\zeta}$ is frequently separated, there exists a positive number $\bar{\epsilon}_0$ and a sequence $\{k_q\}$ of integers satisfying $k_q \rightarrow \infty$ as $q \rightarrow \infty$ such that $\|\zeta_{k_q} - \tilde{\zeta}_{k_q}\| > \bar{\epsilon}_0$ for each $q \in \mathbb{N}$.

Let us fix a natural number q . For $s \in (s_{k_q-1}, s_{k_q}]$, the solutions $\phi_\zeta(s) = \varphi_\zeta(\psi^{-1}(s))$ and $\phi_{\tilde{\zeta}}(s) = \varphi_{\tilde{\zeta}}(\psi^{-1}(s))$ of (6) satisfy the relations

$$\begin{aligned} \phi_\zeta(s) &= \phi_\zeta(s_{k_q-1}+) + \int_{s_{k_q-1}}^s [A\phi_\zeta(r) \\ &\quad + f(\psi^{-1}(r), \phi_\zeta(r)) + \zeta_{k_q}]dr \end{aligned}$$

and

$$\begin{aligned} \phi_{\tilde{\zeta}}(s) &= \phi_{\tilde{\zeta}}(s_{k_q-1}+) + \int_{s_{k_q-1}}^s [A\phi_{\tilde{\zeta}}(r) \\ &\quad + f(\psi^{-1}(r), \phi_{\tilde{\zeta}}(r)) + \tilde{\zeta}_{k_q}]dr, \end{aligned}$$

respectively. Therefore, one can obtain that

$$\begin{aligned} &\|\phi_\zeta(s_{k_q}) - \phi_{\tilde{\zeta}}(s_{k_q})\| \\ &> \bar{\epsilon}_0 \underline{\kappa} - \|\phi_\zeta(s_{k_q-1}+) - \phi_{\tilde{\zeta}}(s_{k_q-1}+)\| \\ &\quad - \int_{s_{k_q-1}}^{s_{k_q}} (\|A\| + L_f)\|\phi_\zeta(r) - \phi_{\tilde{\zeta}}(r)\|dr \\ &\geq \bar{\epsilon}_0 \underline{\kappa} - [1 + (\|A\| + L_f)\bar{\kappa}] \\ &\quad \times \sup_{s \in (s_{k_q-1}, s_{k_q})} \|\phi_\zeta(s) - \phi_{\tilde{\zeta}}(s)\|. \end{aligned}$$

The last inequality implies that

$$\sup_{s \in (s_{k_q-1}, s_{k_q})} \|\phi_\zeta(s) - \phi_{\tilde{\zeta}}(s)\| > \frac{\bar{\epsilon}_0 \underline{\kappa}}{2 + (\|A\| + L_f)\bar{\kappa}}.$$

Define the number

$$\Delta = \min \left\{ \frac{\underline{\kappa}}{2}, \frac{\bar{\epsilon}_0 \underline{\kappa}}{4[2 + (\|A\| + L_f)\bar{\kappa}](K_0\|A\| + M_f + M_F)} \right\}.$$

At first, suppose that $\sup_{s \in (s_{k_q-1}, s_{k_q})} \|\phi_\zeta(s) - \phi_{\tilde{\zeta}}(s)\| = \|\phi_\zeta(\eta) - \phi_{\tilde{\zeta}}(\eta)\|$ for some $\eta \in (s_{k_q-1}, s_{k_q}]$, and let

$$\nu_q = \begin{cases} \eta, & \text{if } \eta \leq \frac{s_{k_q-1} + s_{k_q}}{2}, \\ \eta - \Delta, & \text{if } \eta > \frac{s_{k_q-1} + s_{k_q}}{2}. \end{cases}$$

It can be verified for $s \in \tilde{J}_q = [\nu_q, \nu_q + \Delta]$ that

$$\begin{aligned} &\|\phi_\zeta(s) - \phi_{\tilde{\zeta}}(s)\| \\ &\geq \|\phi_\zeta(\eta) - \phi_{\tilde{\zeta}}(\eta)\| \\ &\quad - \left| \int_\eta^s \|A\| \|\phi_\zeta(r) - \phi_{\tilde{\zeta}}(r)\| dr \right| \\ &\quad - \left| \int_\eta^s \|f(\psi^{-1}(r), \phi_\zeta(r)) \right. \\ &\quad \left. - f(\psi^{-1}(r), \phi_{\tilde{\zeta}}(r))\| dr \right| - \left| \int_\eta^s \|\eta_{k_q} - \tilde{\eta}_{k_q}\| dr \right| \\ &> \frac{\bar{\epsilon}_0 \underline{\kappa}}{2[2 + (\|A\| + L_f)\bar{\kappa}]}. \end{aligned}$$

On the other hand, the inequality

$$\|\phi_\zeta(s) - \phi_{\tilde{\zeta}}(s)\| > \frac{\bar{\epsilon}_0 \underline{\kappa}}{2[2 + (\|A\| + L_f)\bar{\kappa}]}$$

is true also for $s \in \tilde{J}_q = (s_{k_q-1}, s_{k_q-1} + \Delta]$ in the case that $\sup_{s \in (s_{k_q-1}, s_{k_q})} \|\phi_\zeta(s) - \phi_{\tilde{\zeta}}(s)\| = \|\phi_\zeta(s_{k_q-1}+) - \phi_{\tilde{\zeta}}(s_{k_q-1}+)\|$.

Thus, $\|\varphi_\zeta(t) - \varphi_{\tilde{\zeta}}(t)\| > \epsilon_0$ for each t from the intervals J_q , $q \in \mathbb{N}$, where $\epsilon_0 = \frac{\bar{\epsilon}_0 \underline{\kappa}}{2[2 + (\|A\| + L_f)\bar{\kappa}]}$ and $J_q = \psi^{-1}(\tilde{J}_q)$. Consequently, the pair $\varphi_\zeta(t), \varphi_{\tilde{\zeta}}(t) \in \mathcal{A}$ is frequently (ϵ_0, Δ) -separated. ■

The main result of the present study is mentioned in the following theorem.

Theorem 1. *Assume that the conditions (C1)–(C7) are fulfilled. If the map (2) is Li-Yorke chaotic on Λ , then system (1) is chaotic in the sense of Definition 4.1.*

Proof. Suppose that $\zeta = \{\zeta_k\}$ is a p_0 -periodic solution of (2) for some $p_0 \in \mathbb{N}$. In this case, the function $g(t, \zeta)$, which is used on the right-hand side of Eq. (1), is $m\omega$ -periodic, where $m = \text{lcm}\{p_0, p\}/p$. Making use of the conditions (C5) and (C7), one can verify that the bounded solution $\varphi_\zeta(t)$ of (1) is $m\omega$ -periodic. Therefore, (1) possesses an $m\omega$ -periodic solution for each $m \in \mathbb{N}$.

Let us denote by Σ the set consisting of bounded solutions $\varphi_\zeta(t)$ of (1) for which the initial value ζ_0 of the sequence $\zeta = \{\zeta_k\}$ belongs to the scrambled set \mathcal{S} of the map (2). Because the set \mathcal{S} is uncountable, Σ is also uncountable. Moreover,

Σ does not contain any periodic solutions, since no periodic points of F take place inside \mathcal{S} .

According to Lemmas 2 and 3, any pair of different solutions inside Σ is a Li–Yorke pair, i.e. Σ is a scrambled set. Besides, Lemma 3 implies that for any solution $\varphi_\zeta(t) \in \Sigma$ and any periodic solution $\varphi_{\hat{\zeta}}(t) \in \mathcal{A}$, the pair $\varphi_\zeta(t), \varphi_{\hat{\zeta}}(t)$ is frequently (ϵ_0, Δ) -separated for some positive numbers ϵ_0 and Δ . Consequently, system (1) is Li–Yorke chaotic. ■

In the next section, a Duffing equation on a time scale will be utilized to illustrate the theoretical results.

5. An Example

Let us take into account the following forced Duffing equation,

$$y^{\Delta\Delta}(t) + 5y^\Delta(t) + \frac{35}{2}y(t) + 0.02y^3(t) = 0.1 \cos\left(\frac{\pi}{3}t\right) + g(t, \zeta), \quad t \in \mathbb{T}_0, \quad (10)$$

where $\mathbb{T}_0 = \bigcup_{k=-\infty}^{\infty} [\theta_{2k-1}, \theta_{2k}]$ and $\theta_k = 3k + \frac{1}{2}(1 + (-1)^k)$, $k \in \mathbb{Z}$. The function $g(t, \zeta)$ is defined through the equation $g(t, \zeta) = \zeta_k$ for $t \in [\theta_{2k-1}, \theta_{2k}]$, $k \in \mathbb{Z}$, in which the sequence $\zeta = \{\zeta_k\}$, $\zeta_0 \in [0, 1]$, is generated by the logistic map

$$\zeta_{k+1} = 3.9\zeta_k(1 - \zeta_k). \quad (11)$$

The time scale \mathbb{T}_0 satisfies the ω -property with $\omega = 6$, and one can confirm that $\psi(\omega) = 4$ and $\delta_k = 2$ for all $k \in \mathbb{Z}$, where $\delta_k = \theta_{2k+1} - \theta_{2k}$. According to the results of the paper [Li & Yorke, 1975], the map (11) possesses Li–Yorke chaos. It is worth noting that the unit interval $[0, 1]$ is invariant under the iterations of the map [Hale & Koçak, 1991].

By using the variables $y_1 = y$ and $y_2 = y^\Delta$, Eq. (10) can be reduced to the system

$$\begin{aligned} y_1^\Delta(t) &= y_2(t), \\ y_2^\Delta(t) &= -\frac{35}{2}y_1(t) - 5y_2(t) - 0.02y_1^3(t) \\ &\quad + 0.1 \cos\left(\frac{\pi}{3}t\right) + g(t, \zeta), \end{aligned} \quad (12)$$

which is in the form of (1), where

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{35}{2} & -5 \end{pmatrix}$$

and

$$f(t, y_1, y_2) = \begin{pmatrix} 0 \\ -0.02y_1^3 + 0.1 \cos\left(\frac{\pi}{3}t\right) \end{pmatrix}.$$

One can show that

$$e^{At} = e^{-\frac{5}{2}t} Q \begin{pmatrix} \cos\left(\frac{3\sqrt{5}}{2}t\right) & -\sin\left(\frac{3\sqrt{5}}{2}t\right) \\ \sin\left(\frac{3\sqrt{5}}{2}t\right) & \cos\left(\frac{3\sqrt{5}}{2}t\right) \end{pmatrix} Q^{-1},$$

where

$$Q = \begin{pmatrix} 0 & 1 \\ \frac{3\sqrt{5}}{2} & -\frac{5}{2} \end{pmatrix}$$

and the eigenvalues of the matrix $e^{4A}(I + 2A)$ are inside the unit circle, where I is the 2×2 identity matrix.

Due to the fact that the coefficient of the non-linear term $y_1^3(t)$ in (12) is sufficiently small in absolute value, it can be numerically verified for $\zeta_0 \in [0, 1]$ that the bounded solutions of system (12)

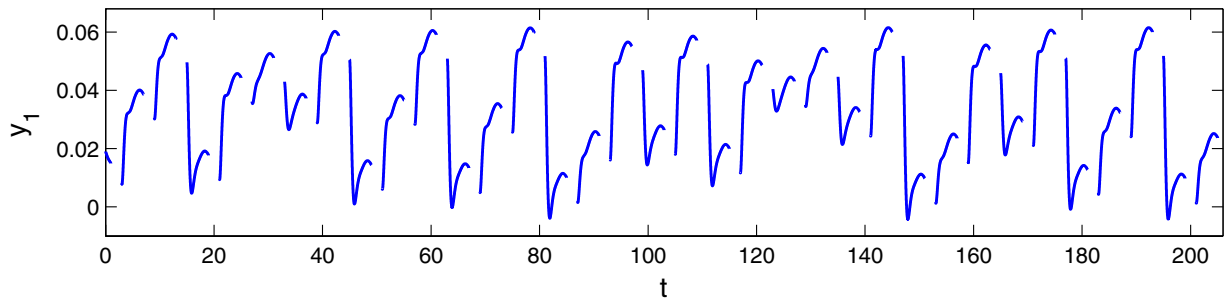


Fig. 1. The chaotic behavior in the solution of system (12).

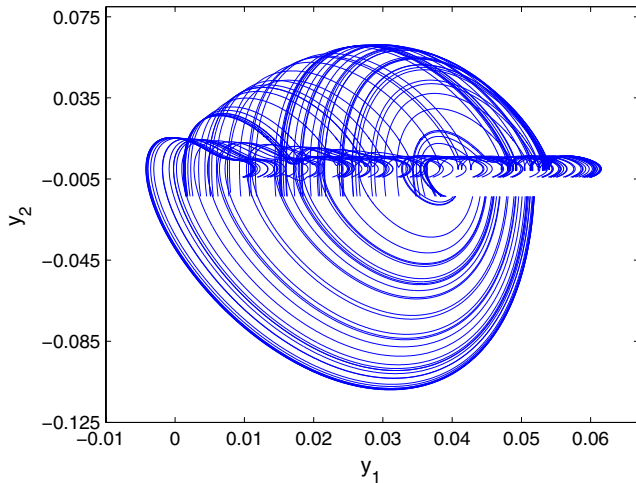


Fig. 2. The chaotic trajectory of system (12).

lie inside the region $\mathcal{D} = \{(y_1, y_2) \in \mathbb{R}^2 : -0.01 \leq y_1 \leq 0.07, -0.12 \leq y_2 \leq 0.07\}$. Therefore, it is reasonable to consider the dynamics of (12) inside \mathcal{D} .

The conditions (C5) and (C6) hold for (12) with $N = 193$, $\lambda = 1.6$, $p = 1$, $\bar{\delta} = 2$ and $L_f = 0.000294$. In accordance with Theorem 1, system (12) is Li–Yorke chaotic. It is worth noting that the chaoticity of the logistic map (11) gives rise to the presence of chaos in (12). Moreover, Lemma 1 implies that for a fixed solution $\zeta = \{\zeta_k\}$ of (11) the unique bounded solution of (12) attracts all other solutions of the system.

Let us use the solution $\zeta = \{\zeta_k\}$ of (11) with $\zeta_0 = 0.19$ in system (12). We depict in Fig. 1 the y_1 -coordinate of the solution of (12) corresponding to the initial data $y_1(0) = 0.019$ and $y_2(0) = -0.004$. Figure 1 supports the result of Theorem 1 such that system (12) possesses chaos. Moreover, the trajectory of the same solution in the y_1 – y_2 plane is represented in Fig. 2, which reveals the existence of a chaotic attractor in the dynamics of (12).

6. Conclusion

We rigorously prove the existence of chaos in dynamic equations on time scales, where the right-hand side of the equations depends on a chaotic map. The reduction technique to impulsive differential equations presented in the paper [Akhmet & Turan, 2006] is used in our investigations. A mathematical description of chaos in the sense of Li–Yorke is provided for DETS, and the ingredients of the Li–Yorke chaos, proximality and frequent separation, are theoretically proved. The results can be used

to obtain chaotic mechanical systems and electrical circuits on time scales without any restriction in the dimension.

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