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# Li-Yorke chaos generation by SICNNs with chaotic/almost periodic postsynaptic currents

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ABSTRACT

We consider shunting inhibitory cellular neural networks with inputs and outputs that are chaotic in a modified Li-Yorke sense. The original Li-Yorke definition of chaos has been modified such that infinitely many periodic motions separated from the motions of the scrambled set are now replaced with almost periodic ones. Another principal novelty of the paper is that chaos is obtained as solutions of differential equations (neural networks) which are *perturbed chaotically*. To construct the original chaos, the special set of piecewise continuous postsynaptic currents is applied. It is shown that a control can be realized for the extended chaos in an effective way. This is the first time in the theory of neural networks that the Ott–Grebogi–Yorke control method is used to stabilize almost periodic motions. Our techniques can be performed to investigate chaotic dynamics in human brain activities, communication security, combinatorial optimization problems and control of legged robots. The results of this paper were announced in the 5th International Conference on Nonlinear Science and Complexity (August 4–9, 2014, Xi'an, China) and in an International Conference on Nonlinear Dynamics and Complexity (May 11–15, 2015, La Manga, Spain).

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## 1. Introduction

The roles of chaos for brain behavior have been investigated within the scope of many papers. Different kinds of electroencephalogram signals were observed in the experiments of Skarda and Freeman [1] when known and unknown odorants were given to a rabbit. According to the experimental results, it was proposed that deterministic chaos is utilized in neural activities for learning new sensory patterns as well as ensuring continual access to previously learned ones. The obtained signals were *chaotically cyclic* for unknown odorants, and they were in the form of a *limit cycle* for known ones. It is natural to presume that the chaotic behavior in a brain activity is not only of *periodic* type as it was learned by the experiments [1], but also of *quasi-periodic* or more general *almost periodic* type. Our main result is the approvement of Li-Yorke chaos [2] generation by shunting inhibitory cellular neural networks with infinitely many *almost periodic motions* in basis instead of *periodic motions*.

It was demonstrated by Watanabe et al. [3] that chaotic dynamics works as means to learn new patterns and increases the *memory capacity* of neural networks. Considering infinitely many almost periodic motions as a basis for the developed Li-Yorke chaos, our results provide dynamics with a *higher complexity*, and one can suppose that this may provide a memory with a *larger capacity* than that with periodic motions.

According to Guevara et al. [4] chaos may be responsible for dynamical diseases such as schizophrenia, insomnia, epilepsy and dyskinesia. From another point of view, synchronization of neurons may cause Parkinson's disease, essential tremor and epilepsy [5,6]. Consequently, our study on the *extension of chaos* through neural networks and subsequent *control* of it may be important for researches of the central nervous system diseases as well as their treatment [7].

Motivated, first of all, by the deficiency of mathematical methods for chaos recognition in neural networks and the importance of irregular behavior for effective brain activities, we suggest the results of the present paper. To prove the presence of chaos, analysis of special type of maps (logistic, Horseshoe, Hénon), symbolic dynamics, Lyapunov exponents and bifurcation diagrams have been used. Distinctively, in this study, we apply the special type of chaos indication [8–13] considering inputs being chaotic and proving that outputs are of the same type of chaoticity. We rigorously prove the ingredients of Li-Yorke chaos [2], proximality and frequent separation features, which may play an essential role in the brain dynamics. The importance of these ingredients can be supported by means of the experimental analyses of Freeman and other neurobiologists [1,3,4,14–19], and through the development of their researches.

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A neuron is an information-processing unit that is fundamental to the operation of a neural network. Neural signals consist of short electrical pulses, called action potentials or spikes. A spike train is a chain of action potentials emitted by a single neuron, i.e. a sequence of stereotyped events which occur at regular or irregular intervals. The potential difference between the interior of a cell and its surroundings is called the membrane potential. A neuron, when it is at rest, has a constant membrane potential. After the arrival of a spike, the potential changes and finally decays back to the resting potential. A synapse, which is a junction between two neurons, is said to be excitatory if the potential change is positive, and it is called inhibitory if the change is negative. Because of their strategic position, a few inhibitory input spikes can *shunt* the whole input that is gathered by the dendritic tree from hundreds of excitatory synapses. This phenomenon is called *shunting inhibition*. It is common to refer to the sending neuron as the presynaptic cell and to the receiving neuron as the postsynaptic cell [20].

A neuron may regularly fire infinitely many spikes (tonic spiking) or finitely many spikes (phasic spiking) [21]. Many meritorious information about the diverse dynamics of neurons can be found in the book [22]. Regular (tonic) spiking will be considered in our investigations.

Information-processing hinges not only on the electrophysical features of neurons but also on their dynamical features. Even though two neurons in the same part of the nervous system are equipped with similar electrophysical properties, they may respond to the identical synaptic input in various ways because of bifurcation dynamics of cells [22]. Such bifurcations may induce a chaos in a system [22–24].

Cellular neural networks (CNNs) are widely used in signal processing, image processing, pattern recognition, and many other areas. Bouzerdoum and Pinter [25] introduced a new class of CNNs, namely shunting inhibitory cellular neural networks (SICNNs), which have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing [26–31].

In the paper [32], the dynamics range compression and contrast enhancement properties of SICNNs on color image enhancement were investigated. It has been observed that besides performing contrast enhancement, SICNNs also improve the color constancy of images as well as their sharpness. According to Beare and Bouzerdoum [33], there is an important evidence to suggest that the mechanism responsible for motion detection in some organisms is structurally inhibitory. In the study [33], a SICNN with an identity activation function has been applied to model an inhibitory motion detector.

Let us describe the model of SICNNs in its most original form [25]. Consider a two dimensional grid of processing cells arranged into $m$ rows and $n$ columns, and let $C_{ij}$ denote the cell at the $(i,j)$ position of the lattice, where $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$. In SICNNs, neighboring cells exert mutual inhibitory interactions of the shunting type. Define the $r$-neighborhood of the cell $C_{ij}$ as

$$N_r(i,j)=\{C_{kl}:\max(|k-i|,|l-j|)\le r,\ 1\le k\le m,\ 1\le l\le n\}.$$

The dynamics of a cell $C_{ij}$ are described by the following nonlinear ordinary differential equation:

$$\frac{dx_{ij}}{dt}=-a_{ij}x_{ij}-\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}f(x_{kl}(t))x_{ij}+\mathcal{L}_{ij}(t), \tag{1.1}$$

where $x_{ij}$ is the activity of the cell $C_{ij}$; $\mathcal{L}_{ij}(t)$ is the external input to the cell $C_{ij}$; the constant $a_{ij}>0$ represents the passive decay rate of the cell activity; $C_{ij}^{kl}\ge 0$ is the coupling strength of post-synaptic activity of the cell $C_{kl}$ transmitted to the cell $C_{ij}$; the positive and continuous function $f(x_{kl})$ represents the output or firing rate of the cell $C_{kl}$.

Eq. (1.1) describes the activity of individual shunting neurons that are arranged in a cellular fashion, where each neuron receives a single external excitatory input and the weighted output of neurons in a predefined neighborhood are fed back, through the nonlinear activation function, as inhibitory inputs. This cellular form is a recurrent (or feedback) network. These networks have been successfully applied to vision and image processing tasks [25].

The presence of chaos in neural networks is useful for separating image segments [34], information processing [35] and synchronization of the networks [36–38]. Besides, the synchronization phenomenon is also observable in the dynamics of coupled chaotic CNNs [39,40]. Chaotic dynamics can improve the performance of CNNs on problems that have local minima in energy (cost) functions, since chaotic behavior of CNNs can help the network avoid local minima and reach the global optimum [41]. Moreover, chaotic dynamics in CNNs is an important tool for the studies of chaotic communication [42–44] and combinatorial optimization problems [45].

Infinitely many periodic solutions which are separated from solutions of a scrambled set [2,8] and almost periodic solutions [46] can serve as a basis of discrete chaos, and Poisson stable orbits of dynamical systems [47] can serve as a dense basis of chaos. In this study almost periodic solutions are considered as a basis of the continuous Li-Yorke chaotic attractor, and this phenomenon has never been reported before in the literature.

To initiate the chaos, the special set of piecewise continuous postsynaptic currents, which will be explained in the next section, is applied. In Section 6, by the numerical analysis we not only confirm the rigorously proved results of chaos generation, but also demonstrate that the generation can be performed by SICNNs with different topologies of connections. One of the possible cases is considered for that. Moreover, it is shown that a control can be realized for the extended chaos in an effective way. The control is valid not only for periodic solutions (as it is known [48–51]), but also for almost periodic (quasi-periodic) motions. This control result was utilized also in our paper [10], and it relates strongly to the method of generation. Another novelty of the analysis of neural networks is that we give a detailed description of continuous Li-Yorke chaos. In the literature [52–55] as well as in the pioneer manuscript [2], the discrete version was considered. We started the presentation for the continuous chaos in the papers [8–13].

In our previous study [13], we used only continuous external inputs and considered Li-Yorke chaotic SICNNs with infinitely many periodic solutions as a basis of chaos. However, in the present paper, we make use of piecewise continuous as well as continuous external inputs and take into account almost periodic solutions as a basis of chaos. The piecewise continuous external inputs were also approved by investigations in neuroscience [20]. The demonstration of the extension of chaos in neural systems and its controllability are also distinctive features of the present paper. The existence of a chaotic attractor in impulsive SICNNs was numerically observed in [56] without a theoretical support. On the other hand, in this study we theoretically prove the presence of chaos with a precise type. The almost periodicity phenomenon for SICNNs was investigated in [57–62], but chaotic dynamics was not considered in these papers.

The remaining parts of the paper are organized as follows. In the next section, we will introduce some necessary notations and preliminaries that will be used later. Additionally, some information about two different types of postsynaptic currents will be given. To describe the structure of our manuscript, we have to remind that any chaotic attractor consists of infinitely many *bounded* solutions. Moreover, the peculiarity of our investigation is that the attractor contains infinitely many *almost periodic* *motions*, which are bounded. That is why we especially consider Section 3 about bounded and almost periodic solutions. This section contains definitions, lemmas about the almost periodicity of rectangular input currents (RICs) and exponential decaying input currents (EDICs), and the proof concerning the existence of a unique almost periodic solution. In the fourth section, the extended definition of Li-Yorke chaos is represented, and the main result about the existence of Li-Yorke chaos is mentioned. Moreover, an illustrative example is given in this part. The fifth section covers SICNNs with EDICs. One of the chaos extension possibilities is presented in Section 6, and some concluding remarks are provided in Section 7. Finally, the proof for the presence of Li-Yorke chaos is given in the Appendix.

## 2. Preliminaries

In our investigations, we will make use of two types of piecewise continuous functions as input currents. One of them is EDIC and the other one is RIC. These currents, as well as others, are described in [20].

### 2.1. Exponential Decaying Input Currents (EDICs)

Let us describe an EDIC for a fixed cell $C_{ij}$. An input current is a spike train which acts on the cell. According to Gerstner and Kistler [20], an EDIC can be defined through a postsynaptic current $\alpha(s) = \mathrm{e}^{-\overline{\beta} s} H(s)$, where $\overline{\beta}$ is a positive constant and $H$ is the Heaviside step function, i.e. $H(s) = 1$ for $s > 0$, and $H(s) = 0$ otherwise. An EDIC is described as

$$K(t,\xi) = \sum_{i \in \mathbb{Z}} \kappa_i \alpha(t - \xi_i), \tag{2.2}$$

where $\kappa = \{\kappa_i\}_{i \in \mathbb{Z}}$ is a sequence of real numbers and the sequence $\xi = \{\xi_i\}_{i \in \mathbb{Z}}$ of spike moments is strictly increasing with $|\xi_i| \to \infty$ as $|i| \to \infty$.

The total input current applied to a fixed cell $C_{ij}$ can be computed as

$$K_{ij}(t,\theta) = \sum_{r=1}^{r_k} K_r(t,\theta^{(r)}), \tag{2.3}$$

where $K_r(t,\theta^{(r)})$, $r = 1,2,\ldots,r_k$, are functions of type (2.2), $r_k$ is a positive integer, and $\theta^{(r)} = \{\theta_i^{(r)}\}$ are the sequences of discontinuity moments for the functions $K_r(t,\theta^{(r)})$. The number of input currents, $r_k$, is finite because the number of neurons possibly connected to a fixed cell $C_{ij}$ is finite. The sequence $\theta = \bigcup_{r=1}^{r_k} \theta^{(r)}$ consists of discontinuity moments of the function $K_{ij}(t,\theta)$.

### 2.2. Rectangular Input Currents (RICs)

A formulation of a RIC can be described as

$$P(t,\xi) = \sum_{i \in \mathbb{Z}} p_i H(t - \xi_{2i}) H(\xi_{2i+1} - t), \tag{2.4}$$

where $\{p_i\}_{i \in \mathbb{Z}}$ is a sequence of real numbers, the sequence $\xi = \{\xi_i\}_{i \in \mathbb{Z}}$ of spike moments is strictly increasing with $|\xi_i| \to \infty$ as $|i| \to \infty$, and $H$ denotes the Heaviside step function, which is defined in the previous subsection. In Fig. 1, a sample RIC is presented.

We consider the total input current for a cell $C_{ij}$ as

$$P_{ij}(t,\theta) = \sum_{r=1}^{r_p} P_r(t,\theta^{(r)}), \tag{2.5}$$

where $P_r(t,\theta^{(r)})$, $r = 1,\ldots,r_p$ are functions of type (2.4), $r_p$ is a positive integer and $\theta^{(r)} = \{\theta_i^{(r)}\}$ are the sequences of discontinuity moments for the functions $P_r(t,\theta^{(r)})$. The sequence $\theta = \bigcup_{r=1}^{r_p} \theta^{(r)}$ is composed of the discontinuity moments of the function $P_{ij}(t,\theta)$.

The discontinuity moments of an input current are well separated [20]. One can find the discontinuity moments of a total input current for a fixed cell $C_{ij}$ by uniting the discontinuity moments of all input currents received by the cell. In this manner, the discontinuity moments of a total input current may not be well separated. In the present study, we will consider the discontinuity moments to be common for all input currents gathered by a single cell $C_{ij}$. Thus, the discontinuity moments of the total input current for $C_{ij}$ will be well separated.

### 2.3. The external input, $\mathcal{L}_{ij}(t)$, of the cell $C_{ij}$

In this subsection, we focus on the description of the total input of a single cell. The total input of a cell $C_{ij}$ can be comprehended as a combination of two different types of inputs. One of them is explicitly defined by presynaptic activities, $x_{kl}$, of neighbor cells of SICNNs. In Eq. (1.1), this input is expressed by the term $\sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(t)) x_{ij}$, and it is determined by the weights $C_{ij}^{kl}$ and a well-defined scalar activation function $f$. Another contribution to the total input is made by activities that are not gathered from presynaptic spikes. They are provided from postsynaptic spikes through the function $\mathcal{L}_{ij}(t)$. In the literature, the external input, denoted as $\mathcal{L}_{ij}(t)$, is usually not specified, and it is assumed to be a bounded, periodic or an almost periodic function. The input currents are observable and computable as postsynaptic parameters of a neuron. In other words, the presynaptic action potentials and information of the structure of SICNNs can be used to evaluate the term $\sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(t)) x_{ij}$. One of the essential parts of a neural network analysis is to determine the output when the input is bounded, periodic, or almost periodic. It is also significant to understand the output of the system when a chaotic input $\mathcal{L}_{ij}(t)$ is applied to it.

To produce a chaos in network (1.1), we consider the function $\mathcal{L}_{ij}(t)$ as the sum of a chaotic *RIC* $P_{ij}(t,\theta)$ and a continuous almost periodic external input $L_{ij}(t)$. In that case, Eq. (1.1) takes the form

$$\frac{dx_{ij}}{dt} = -a_{ij} x_{ij} - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(t)) x_{ij} + L_{ij}(t) + P_{ij}(t,\theta). \tag{2.6}$$

The insertion of the term $\mathcal{L}_{ij}(t)$ makes SICNNs more convenient and universal to obtain desired dynamics. We propose to use the opportunity and consider $\mathcal{L}_{ij}(t)$ as consisting of two parts, almost periodic and chaotic. For this reason, we have specified the external input to obtain chaos from the system with infinitely many almost periodic solutions, which are separated from the solutions of a scrambled set. Moreover, it is plausible to consider $\mathcal{L}_{ij}(t)$ as the sum of all external inputs. Thus, it is admissible to suppose that $\mathcal{L}_{ij}(t) = L_{ij}(t) + P_{ij}(t,\theta)$ is the total input current determined from the postsynaptic stage.

Eq. (2.6) is an ordinary differential equation with discontinuous right hand side such that the functions $P_{ij}(t,\theta)$ are piecewise constant. Since the system is Lipschitzian in the intervals of constancy of $P_{ij}(t,\theta)$, the local existence and uniqueness of solutions are valid, and at the switching moments of the constancy, one can use the continuity condition of solutions to proceed for the next interval of constancy of $P_{ij}(t,\theta)$. Thus, one can be sure about the local existence and uniqueness of solutions for (2.6) and continuation of them for maximal interval of existence. For more information about the existence and uniqueness of solutions of differential equations with discontinuous right hand side, one can look in [63,64]. In our paper, we discuss the global existence and boundedness of solutions on the whole real axis, as well as almost periodic solutions and chaotic attractor with RICs, by making use of the equivalent integral equation which has been widely applied

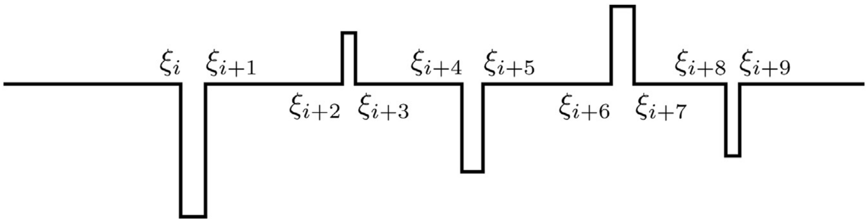

**Fig. 1.** The graph of a sample RIC.

[64,65]. Similar results for EDICs are formulated without verification in Section 5.

We have introduced two different types of postsynaptic input currents by adhering to the nature of them. In order to verify the chaos generation, we investigate the existence of bounded, almost periodic and chaotic solutions in SICNNs. The generation of Li-Yorke chaos through almost periodic solutions is crucial for the development of not only the neural networks theory, but also the chaos theory. Especially, it is important for systems that are not capable of performing chaos, almost periodic solutions and quasi-periodic solutions without implementing external inputs.

In the remaining parts of the paper, we will assume that the network (2.6) has the following properties. The chaotic piecewise constant external input will be considered as $P_{ij}(t,\theta)=p_{ij}^k$, $i=1,2,\dots,m$, $j=1,2,\dots,n$, on the interval $\theta_k<t\le\theta_{k+1}$, $k\in\mathbb{Z}$, where $p_{ij}^k$ are real numbers and $\theta=\{\theta_k\}_{k\in\mathbb{Z}}$ is a strictly increasing sequence such that $|\theta_k|\to\infty$ as $|k|\to\infty$. For each $k\in\mathbb{Z}$, let us denote by $M_k$ the matrix $(p_{ij}^k)_{m\times n}$. Each of the functions $P_{ij}(t,\theta)$ can be considered as a total input current of the type (2.5). Thus, the discontinuity moments $\theta_k$, $k\in\mathbb{Z}$, can be comprehended as spike moments, and it is plausible to use the network (2.6) in modeling biological neural systems.

Throughout the paper, the norm $\|u\|=\max_{(i,j)}|u_{ij}|$ will be used, where $u=\{u_{ij}\}\in\mathbb{R}^{m\times n}$. The following conditions are required.

(C1) $\gamma=\min_{(i,j)}a_{ij}>0$.
(C2) There exists a positive number $L_f$ such that $|f(s_1)-f(s_2)|\le L_f|s_1-s_2|$ for each $s_1,s_2\in\mathbb{R}$.
(C3) There exists a positive number $M_f$ such that $\sup_{s\in\mathbb{R}}|f(s)|\le M_f$.
(C4) There exist positive numbers $\overline{L}_{ij}$ such that $\sup_{t\in\mathbb{R}}|L_{ij}(t)|\le\overline{L}_{ij}$.
(C5) $M_f\delta_0<1$, where $\delta_0=\max_{(i,j)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}/a_{ij}$.
(C6) There exist positive numbers $\overline{p}_{ij}$ such that $\sup_{k\in\mathbb{Z}}\left|p_{ij}^k\right|\le\overline{p}_{ij}$.

In what follows, let us denote $\delta_1=\max_{(i,j)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}$, $\overline{L}=\max_{(i,j)}(\overline{L}_{ij}+\overline{p}_{ij})/a_{ij}$ and $P_0=\overline{L}/(1-M_f\delta_0)$.

The following conditions are also needed.

(C7) $\gamma-\delta_1(M_f+L_fP_0)>0$.
(C8) There exists a positive number $\underline{\theta}$ such that $\theta_{k+1}-\theta_k\ge\underline{\theta}$ for each $k\in\mathbb{Z}$.
(C9) There exists a positive number $\underline{m}_p$ such that $\max_{(i,j)}\inf_{k\in\mathbb{Z}}|p_{ij}^{k+1}-p_{ij}^k|\ge\underline{m}_p$.

The condition (C1) is one of the properties of SICNNs such that the positive constant $a_{ij}$ represents the ratio of the resting conductance to the membrane capacitance in the electrical equivalent circuit of the corresponding cell $C_{ij}$ [25]. The conditions (C2) and (C3) are needed for the proofs of the existence, uniqueness and almost periodicity of bounded solutions, as well as the presence of chaos. Likewise condition (C2), the activation function $f$ in (1.1) is also assumed to be Lipschitzian in the papers [57,58]. Additionally, different types of activation functions such as sigmoid functions are used in the literature, and (C2), (C3) can be easily verified for these type of functions. The boundedness of the external inputs, (C4), is a direct consequence of almost periodicity [66]. Conditions (C5)–(C7) are similar to the ones used in [57,58]. (C8) is needed for the analyses of almost periodicity of solutions and piecewise continuous inputs, and chaotic dynamics. The condition (C9) is needed for the verification of the chaos generation.

## 3. Almost periodic solutions

According to Pasemann et al. [67], periodic and quasi-periodic solutions have many fundamental importances in biological and artificial systems, as they are associated with central pattern generators, establishing stability properties and bifurcations (leading to the discovery of periodic solutions). Moreover, for discrete-time dynamical systems, the sinusoidal shape of neural output signals is, in general, associated with appropriate quasi-periodic attractors. It is shown in paper [67] that the frequency of oscillators can be controlled by a single parameter. Signals from the neurons have a phase shift of $\pi/2$ and may be useful for various kinds of applications; for instance, controlling the gait of legged robots [68]. An alternative discrete time model of coupled quasi-periodic and chaotic neural network oscillators were studied by Wang [69]. Besides, Izhikevich [22] considered the dynamics of the brain activity as a system of many coupled oscillators with different incommensurable periods. Therefore, one can accept that quasi-periodic oscillations have an important place in the theory of neural networks.

### 3.1. *Almost periodicity of EDICs and RICs*

Let $\{a_i\}_{i\in\mathbb{Z}}$ be a sequence in $\mathbb{R}^n$. An integer $p$ is an $\epsilon-$almost period of the sequence $\{a_i\}$, if the inequality $\|a_{i+p}-a_i\|<\epsilon$ holds for all $i\in\mathbb{Z}$. On the other hand, a set $\mathcal{R}\subset\mathbb{R}$ is said to be relatively dense if there exists a number $l>0$ such that $[a,a+l]\cap\mathcal{R}\neq\varnothing$ for all $a\in\mathbb{R}$. Moreover, $\{a_i\}$ is *almost periodic*, if for any $\epsilon>0$, there exists a relatively dense set of its $\epsilon-$almost periods.

Let $\xi_i^j=\xi_{i+j}-\xi_i$ for any integers $i$ and $j$. We call the family of sequences $\{\xi_i^j\}$, $j\in\mathbb{Z}$, equipotentially almost periodic if for an arbitrary $\epsilon>0$, there exists a relatively dense set of $\epsilon-$almost periods, common for all sequences $\{\xi_i^j\}$, $j\in\mathbb{Z}$ [70].

**Definition 3.1.** [70] A continuous function $\mathcal{F}(t)$ is said to be almost periodic, if for any $\epsilon>0$ the set $T(\mathcal{F},\epsilon)=\{\omega:\|\mathcal{F}(t+\omega)-\mathcal{F}(t)\|<\epsilon$ for all $t\in\mathbb{R}\}$ is relatively dense, i.e., for any $\epsilon>0$ there exists $l>0$ such that for any interval with length $l$ there exists a number $\omega$ in this interval satisfying $\|\mathcal{F}(t+\omega)-\mathcal{F}(t)\|<\epsilon$ for all $t\in\mathbb{R}$.

Piecewise continuous almost periodic functions play a very important role for discontinuous dynamics [64,70]. The following definition is also needed.

**Definition 3.2.** [64] Let a function $\mathcal{G}(t)$ be defined on $\mathbb{R}$, piecewise continuous with the first kind discontinuities at the points of a fixed sequence $\{\theta_k\}_{k\in\mathbb{Z}}$. Assume that $\left\{\theta_k^l\right\}$, $l\in\mathbb{Z}$, is a family of equipotentially almost periodic sequences.

The function $\mathcal{G}(t)$ is called piecewise continuous almost periodic if

(i) for any $\epsilon>0$, there exists a positive number $\delta=\delta(\epsilon)$ such that if the points $t', t''$ belong to the same interval of continuity and $|t'-t''|<\delta$, then $\|\mathcal{G}(t')-\mathcal{G}(t'')\|<\epsilon$;
(ii) for any $\epsilon>0$, there exists a relatively dense set $\Gamma$ of $\epsilon-$almost periods such that if $\omega\in\Gamma$, then $\|\mathcal{G}(t+\omega)-\mathcal{G}(t)\|<\epsilon$ for all $t\in\mathbb{R}$ which satisfy the condition $|t-\theta_k|>\epsilon$ for $k\in\mathbb{Z}$.

In Definition 3.2, part (i), it is stated that the functions should be uniformly continuous on the interval of continuity.

The following assertion was first proved in [71].

**Lemma 3.1.** *Suppose that $\Lambda(t)$ is an almost periodic $m\times n$ matrix function, $\{M_k\}_{k\in\mathbb{Z}}$ is an almost periodic sequence of $m\times n$ matrices and $\left\{\theta_k^l\right\}$, $l\in\mathbb{Z}$, is a family of equipotentially almost periodic sequences. Moreover, assume that the condition (C8) holds. Then, for arbitrary $\eta>0$ and $0<\nu<\eta$, there exist relatively dense sets of real numbers $\Omega$ and integers $Q$ such that*

(i) $\|\Lambda(t+\omega)-\Lambda(t)\|<\eta$, $t\in\mathbb{R}$;
(ii) $\|M_{k+q}-M_k\|<\eta$, $k\in\mathbb{Z}$;
(iii) $|\theta_k^q-\omega|<\nu$, $k\in\mathbb{Z}$, $\omega\in\Omega$, $q\in Q$.

The following assertion is about the almost periodicity of the function $K(t,\xi)$, which is defined by Eq. (2.2).

**Lemma 3.2.** *Suppose that $\{\kappa_i\}_{i\in\mathbb{Z}}$ is an almost periodic sequence, $\left\{\xi_i^j\right\}$, $j\in\mathbb{Z}$, is a family of equipotentially almost periodic sequences, and there exists a positive number $\underline{\xi}$ such that $\xi_{i+1}-\xi_i\geq\underline{\xi}$ for all $i\in\mathbb{Z}$. Then, $K(t,\xi)$ is a piecewise continuous almost periodic function.*

**Proof.** Because the sequence $\kappa=\{\kappa_i\}_{i\in\mathbb{Z}}$ is almost periodic, there exists a positive number $N_\kappa$ such that $\sup_{i\in\mathbb{Z}}|\kappa_i|\leq N_\kappa$. By means of the condition $\xi_{i+1}-\xi_i\geq\underline{\xi}$, one can verify for $t\neq\xi_i$ that

$$|K'(t,\xi)|\leq\sum_{i\in\mathbb{Z}}\overline{\beta}|\kappa_i|e^{-\overline{\beta}(t-\xi_i)}H(t-\xi_i)\leq\sum_{k=0}^{\infty}\overline{\beta}N_\kappa e^{-\overline{\beta}\underline{\xi}k}=\frac{\overline{\beta}N_\kappa}{1-e^{-\overline{\beta}\underline{\xi}}}.$$

Therefore, part $(i)$ of Definition 3.2 is valid for the function $K(t,\xi)$.

Next, we will verify the second part of Definition 3.2. Fix an arbitrary positive number $\epsilon$. Let $\eta$ be a sufficiently small positive number such that $\eta/(1-e^{-\overline{\beta}\underline{\xi}})+N_\kappa(e^{\overline{\beta}\eta}-1)/(1-e^{-\overline{\beta}\underline{\xi}})<\epsilon$, and take a number $\nu$ satisfying $0<\nu<\eta$. Suppose that $\omega$ and $q$ are numbers as mentioned in Lemma 3.1 such that $\left|\kappa_{i+q}-\kappa_i\right|<\eta$ and $\left|\xi_i^q-\omega\right|<\nu$, $i\in\mathbb{Z}$. For each $t$ with $\left|t-\xi_i\right|>\epsilon$, $i\in\mathbb{Z}$, we have that

$$\begin{aligned}|K(t+\omega,\xi)-K(t,\xi)|&\leq\sum_{i\in\mathbb{Z}}\Big|\kappa_{i+q}e^{-\overline{\beta}(t+\omega-\xi_{i+q})}H(t+\omega-\xi_{i+q})\\&\quad-\kappa_i e^{-\overline{\beta}(t-\xi_i)}H(t-\xi_i)\Big|\\&\leq\sum_{i\in\mathbb{Z}}\left|\kappa_{i+q}-\kappa_i\right|e^{-\overline{\beta}(t+\omega-\xi_{i+q})}H(t+\omega-\xi_{i+q})\\&\quad+\sum_{i\in\mathbb{Z}}|\kappa_i|e^{-\overline{\beta}(t-\xi_i)}\left|e^{-\overline{\beta}(\omega-\xi_{i+q}+\xi_i)}H(t+\omega-\xi_{i+q})-H(t-\xi_i)\right|\\&<\sum_{k=0}^{\infty}\eta e^{-\overline{\beta}\underline{\xi}k}+\sum_{k=0}^{\infty}N_\kappa e^{-\overline{\beta}\underline{\xi}k}\left(e^{\overline{\beta}\nu}-1\right)\\&=\frac{\eta}{1-e^{-\overline{\beta}\underline{\xi}}}+\frac{N_\kappa(e^{\overline{\beta}\nu}-1)}{1-e^{-\overline{\beta}\underline{\xi}}}\\&<\epsilon.\end{aligned}$$

Consequently, $K(t,\xi)$ is a piecewise continuous almost periodic function. □

It is worth noting that the total input current $K_{ij}(t,\theta)$ defined in (2.3) is a piecewise continuous almost periodic function since it is the sum of a finite number of piecewise continuous almost periodic functions [70].

The almost periodicity of the function $P(t,\xi)$, which is defined by Eq. (2.4), is discussed in the next assertion.

**Lemma 3.3.** *If $\{p_i\}_{i\in\mathbb{Z}}$ is an almost periodic sequence, $\left\{\xi_i^j\right\}$, $j\in\mathbb{Z}$, is a family of equipotentially almost periodic sequences and there exists a positive number $\underline{\xi}$ such that $\xi_{i+1}-\xi_i\geq\underline{\xi}$ for all $i\in\mathbb{Z}$, then $P(t,\xi)$ is a piecewise constant almost periodic function.*

One can prove Lemma 3.3 by applying the method of the proof of almost periodicity for piecewise constant functions, which can be found in [64,70,71]. Additionally, one can affirm that the function $P_{ij}(t,\theta)$ is a piecewise constant almost periodic function, as a sum of a finite number of piecewise constant almost periodic functions [70,71].

### 3.2. Almost periodic solutions

In the original paper of Li and Yorke [2], infinitely many periodic solutions, which are separated from the elements of a scrambled set, is introduced. In the present study, we will use infinitely many almost periodic solutions, which are separated from the elements of the scrambled set, to generate the Li-Yorke chaos. Thus, the existence of the almost periodic solution should be verified.

By means of the theory of quasi-linear equations [72], one can verify that a bounded on $\mathbb{R}$ function $x(t)=\{x_{ij}(t)\}$ is a solution of the network (2.6) if and only if the following integral equation is satisfied

$$x_{ij}(t)=-\int_{-\infty}^{t}e^{-a_{ij}(t-s)}\left[\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}f(x_{kl}(s))x_{ij}(s)-L_{ij}(s)-P_{ij}(s,\theta)\right]ds.$$

Suppose that the conditions (C1)–(C7) are valid. By using the technique of Lemma 2.1 [13], it can be verified that for each sequence $\theta=\{\theta_k\}_{k\in\mathbb{Z}}$ there exists a unique bounded on $\mathbb{R}$ solution $\phi_\theta(t)=\{\phi_\theta^{ij}(t)\}$ of the network (1.1) such that $\sup_{t\in\mathbb{R}}\|\phi_\theta(t)\|\leq P_0$. On the other hand, under the same conditions, one can prove in a very similar way to Theorem 2 [58] that for a fixed sequence $\theta$, all other solutions of SICNN (2.6) converge exponentially to the unique bounded on $\mathbb{R}$ solution $\phi_\theta(t)$.

Now, let us show that the bounded solution is almost periodic.

**Theorem 3.1.** *Assume that $\left\{\theta_k^l\right\}$, $l\in\mathbb{Z}$, is a family of equipotentially almost periodic sequences, $\{M_k\}_{k\in\mathbb{Z}}$ is an almost periodic sequence of $m\times n$ matrices, $L_{ij}(t)$, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$, are continuous almost periodic functions and conditions (C1)–(C8) are valid. Then, $\phi_\theta(t)=\{\phi_\theta^{ij}(t)\}$ is the unique almost periodic solution of the network (2.6).*

**Proof.** Consider the set $\mathcal{C}_0$ of almost periodic functions $u(t)=\{u_{ij}(t)\}$, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$, such that $\|u\|_\infty\leq P_0$, where $\|u\|_\infty=\sup_{t\in\mathbb{R}}\|u(t)\|$. Define the operator $\Pi$ on $\mathcal{C}_0$ as follows:

$$(\Pi u(t))_{ij}=-\int_{-\infty}^{t}e^{-a_{ij}(t-s)}\left[\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}f(u_{kl}(s))u_{ij}(s)-L_{ij}(s)-P_{ij}(s,\theta)\right]ds. \quad (3.7)$$

Let us denote

$$H_0=\max_{(i,j)}\left[(M_f+L_fP_0)\frac{\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}}{a_{ij}}+\frac{2}{a_{ij}}+\frac{4\overline{p}_{ij}}{1-e^{-a_{ij}\underline{\theta}}}\right].$$

Take a function $u(t)$ from the set $\mathcal{C}_0$ and fix an arbitrary positive number $\epsilon$. According to uniform continuity of $\Pi u$, there exists a positive number $\eta$ satisfying $\eta<\underline{\theta}/5$, $\eta\leq\epsilon/3H_0$ such that $\|\Pi u(t')-\Pi u(t'')\|<\epsilon/3$ whenever $|t'-t''|<4\eta$. Let $\nu$ be a number such that $0<\nu<\eta$ and consider the numbers $\omega$ and $q$ as mentioned in Lemma 3.1 such that: (i) $\|u(t+\omega)-u(t)\|<\eta,\ t\in\mathbb{R}$; (ii) $|L_{ij}(t+\omega)-L_{ij}(t)|<\eta,\quad i=1,2,\dots,m,\quad j=1,2,\dots,n,\quad t\in\mathbb{R}$; (iii) $\|M_{k+q}-M_k\|<\eta,\ k\in\mathbb{Z}$; (iv) $|\theta_k^q-\omega|<\nu,\ k\in\mathbb{Z}$.

Assume that $t\in(\theta_{\overline{k}}+\eta,\theta_{\overline{k}+1}-\eta)$ for some $\overline{k}\in\mathbb{Z}$. We obtain by using Eq. (3.7) that

$$
\begin{aligned}
&\left|(\varPi u(t+\omega))_{ij}-(\varPi u(t))_{ij}\right|\\
&\quad\le\int_{-\infty}^{t}\mathrm{e}^{-a_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}\left|f(u_{kl}(s+\omega))u_{ij}(s+\omega)-f(u_{kl}(s))u_{ij}(s)\right|ds\\
&\qquad+\int_{-\infty}^{t}\mathrm{e}^{-a_{ij}(t-s)}\left|L_{ij}(s+\omega)-L_{ij}(s)\right|ds\\
&\qquad+\int_{-\infty}^{t}\mathrm{e}^{-a_{ij}(t-s)}\left|P_{ij}(s+\omega,\theta)-P_{ij}(s,\theta)\right|ds.
\end{aligned}\tag{3.8}
$$

One can confirm that

$$
\begin{aligned}
&\int_{-\infty}^{t}\mathrm{e}^{-a_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}\left|f(u_{kl}(s+\omega))u_{ij}(s+\omega)-f(u_{kl}(s))u_{ij}(s)\right|\,ds\\
&\quad\le\int_{-\infty}^{t}\mathrm{e}^{-a_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}M_f\left|u_{ij}(s+\omega)-u_{ij}(s)\right|ds\\
&\qquad+\int_{-\infty}^{t}\mathrm{e}^{-a_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}L_fP_0\left|u_{kl}(s+\omega)-u_{kl}(s)\right|ds\\
&\quad<(M_f+L_fP_0)\eta\frac{\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}}{a_{ij}}.
\end{aligned}
$$

Besides, it can be verified that

$$
\int_{-\infty}^{t}\mathrm{e}^{-a_{ij}(t-s)}\left|L_{ij}(s+\omega)-L_{ij}(s)\right|ds<\frac{\eta}{a_{ij}}.
$$

The inequality $|P_{ij}(s+\omega,\theta)-P_{ij}(s,\theta)|<\eta$ holds for each $s$ that belongs to the intervals $(\theta_k+\nu,\theta_{k+1}-\nu),\ k\in\mathbb{Z}$. Thus, we have

$$
\begin{aligned}
&\int_{-\infty}^{t}\mathrm{e}^{-a_{ij}(t-s)}\left|P_{ij}(s+\omega,\theta)-P_{ij}(s,\theta)\right|ds\\
&\quad=\int_{\theta_{\overline{k}}+\nu}^{t}\mathrm{e}^{-a_{ij}(t-s)}\left|P_{ij}(s+\omega,\theta)-P_{ij}(s,\theta)\right|ds\\
&\qquad+\sum_{\sigma=0}^{\infty}\int_{\theta_{\overline{k}-\sigma-1}+\nu}^{\theta_{\overline{k}-\sigma}-\nu}\mathrm{e}^{-a_{ij}(t-s)}\left|P_{ij}(s+\omega,\theta)-P_{ij}(s,\theta)\right|ds\\
&\qquad+\sum_{\sigma=0}^{\infty}\int_{\theta_{\overline{k}-\sigma}-\nu}^{\theta_{\overline{k}-\sigma}+\nu}\mathrm{e}^{-a_{ij}(t-s)}\left|P_{ij}(s+\omega,\theta)-P_{ij}(s,\theta)\right|ds\\
&\quad<\int_{-\infty}^{t}\mathrm{e}^{-a_{ij}(t-s)}\eta\,ds+4\nu\overline{p}_{ij}\sum_{\sigma=0}^{\infty}\mathrm{e}^{-a_{ij}\sigma\underline{\theta}}\\
&\quad<\eta\left(\frac{1}{a_{ij}}+\frac{4\overline{p}_{ij}}{1-\mathrm{e}^{-a_{ij}\underline{\theta}}}\right).
\end{aligned}
$$

The inequality (3.8) implies that

$$
\left|(\varPi u(t+\omega))_{ij}-(\varPi u(t))_{ij}\right|<\eta\left[(M_f+L_fP_0)\frac{\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}}{a_{ij}}+\frac{2}{a_{ij}}+\frac{4\overline{p}_{ij}}{1-\mathrm{e}^{-a_{ij}\underline{\theta}}}\right].
$$

Therefore, we have that

$$
\left\|\varPi u(t+\omega)-\varPi u(t)\right\|<H_0\eta\le\frac{\epsilon}{3},
$$

for each $t$ from the intervals $(\theta_k+\eta,\theta_{k+1}-\eta),\ k\in\mathbb{Z}$.

Now, let $t\in[\theta_k-\eta,\theta_k+\eta]$ for some $k\in\mathbb{Z}$. Because $\eta$ satisfies the inequality $\eta<\underline{\theta}/5$, the number $t+3\eta$ belongs to the interval $(\theta_k+\eta,\theta_{k+1}-\eta)$. One can obtain that

$$
\begin{aligned}
\|\varPi u(t+\omega)-\varPi u(t)\|&\le\|\varPi u(t+\omega)-\varPi u(t+\omega+3\eta)\|\\
&\quad+\|\varPi u(t+\omega+3\eta)-\varPi u(t+3\eta)\|+\|\varPi u(t+3\eta)-\varPi u(t)\|\\
&<\epsilon.
\end{aligned}
$$

Thus, $\|\varPi u(t+\omega)-\varPi u(t)\|<\epsilon$ for all $t\in\mathbb{R}$, and accordingly $\varPi u(t)$ is almost periodic.

On the other hand, the inequality

$$
\begin{aligned}
\left|(\varPi u(t))_{ij}\right|&\le\int_{-\infty}^{t}\mathrm{e}^{-a_{ij}(t-s)}\left[\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}|f(u_{kl}(s))|\,|u_{ij}(s)|+|L_{ij}(s)|+|P_{ij}(s,\theta)|\right]ds\\
&\le\frac{1}{a_{ij}}\left(M_fP_0\sum_{C_{kl}\in N_r(i,j)}C_{ij}^{kl}+\overline{L}_{ij}+\overline{p}_{ij}\right)
\end{aligned}
$$

implies that $\|\varPi u\|_\infty\le M_fP_0\delta_0+\overline{L}=P_0$. Hence, $\varPi(\mathcal{C}_0)\subseteq\mathcal{C}_0$.

One can also verify for $u(t),\quad\overline{u}(t)\in\mathcal{C}_0$ that $\left\|\varPi u-\varPi\overline{u}\right\|_\infty\le(M_f+L_fP_0)\delta_0\|u-\overline{u}\|_\infty$. Condition (C7) implies that the operator $\varPi$ is contractive. Consequently, $\phi_\theta(t)$ is the unique almost periodic solution of the network (2.6).

## 4. The Li-Yorke chaos

The first mathematical definition of chaos was introduced by Li and Yorke [2], and it became one of the most discussed topics for the last several decades. Li and Yorke [2] proved that if a map on an interval has a point of period three, then it has points of all periods. Moreover, there exists an uncountable scrambled subset of the interval. While the existence of periodic solutions, as it was discovered later, is a particular case of Sharkovsky's Theorem, the scrambled set remains the feature that distinguishes the Li-Yorke chaos from other definitions [73]. Another fact which makes the chaos attractive for applications is that it can be developed for multidimensional systems [52–55,74,75].

In this section, we will deal with the presence of Li-Yorke chaos in SICNN (2.6) such that infinitely many almost periodic motions take place in the basis of chaos instead of periodic ones. We will show that all other ingredients of Li-Yorke chaos, proximality and frequent separation, as well as the scrambled set are present in the dynamics of (2.6).

Let us consider the sequence $\theta=\{\theta_k\}_{k\in\mathbb{Z}}$ in the following form:

$$
\theta_k=\tau_k+\zeta_k,\tag{4.9}
$$

where $\{\tau_k^l\}$, $l\in\mathbb{Z}$, is a family of equipotentially almost periodic sequences and

$$
\zeta_{k+1}=F(\zeta_k).\tag{4.10}
$$

Here, $F:J\to J$ is a continuous function and $J\subset\mathbb{R}$ is an interval.

Condition (C8) is valid, for example, if $\tau_{k+1}-\tau_k\ge2\underline{\theta}$ and there exist numbers $\alpha_0,\beta_0$ with $\beta_0-\alpha_0\le\underline{\theta}$ such that $J=[\alpha_0,\beta_0]$.

The map $F$ is Li-Yorke chaotic on $J$ if [2]: (i) for every natural number $p$, there exists a $p-$ periodic point of $F$ in $J$; (ii) there is an uncountable set $S\subset J$, the scrambled set, containing no periodic points, such that for every $s_1,s_2\in S$ with $s_1\ne s_2$, we have $\limsup_{n\to\infty}|F^n(s_1)-F^n(s_2)|>0$ and $\liminf_{n\to\infty}|F^n(s_1)-F^n(s_2)|=0$; (iii) for every $s\in S$ and a periodic point $\sigma\in J$, we have $\limsup_{n\to\infty}|F^n(s)-F^n(\sigma)|>0$.

Let $\Theta$ be the set of all sequences $\theta=\{\theta_k\}_{k\in\mathbb{Z}}$ obtained by Eq. (4.9). We say that a pair $\theta,\ \tilde{\theta}\in\Theta$ is proximal if $\liminf_{k\to\infty}|\theta_k-\tilde{\theta}_k|=0$. Moreover, the pair is called frequently separated if $\limsup_{k\to\infty}|\theta_k-\tilde{\theta}_k|>0$. These descriptions will be used for Lemmas A.1, Lemmas A.2 and the proof of Theorem 4.1, in the Appendix.

### 4.1. The Li-Yorke chaos in SICNNs

The description of Li-Yorke chaos for the SICNN (2.6) is as follows.

We say that a pair $\phi_\theta(t)$, $\phi_{\tilde{\theta}}(t)$ of bounded solutions of (2.6) is proximal if for arbitrary small $\epsilon > 0$ and arbitrary large $E > 0$, there exists an interval $I \subset \mathbb{R}$ with a length no less than $E$ such that $\|\phi_\theta(t) - \phi_{\tilde{\theta}}(t)\| < \epsilon$ for all $t \in I$. On the other hand, the pair $\phi_\theta(t)$, $\phi_{\tilde{\theta}}(t)$ is frequently $(\epsilon_0, \epsilon_1)$-separated if there exist positive numbers $\epsilon_0$, $\epsilon_1$ and infinitely many disjoint intervals $I_q \subset \mathbb{R}$, $q \in \mathbb{N}$, each with a length no less than $\epsilon_1$, such that $\|\phi_\theta(t) - \phi_{\tilde{\theta}}(t)\| > \epsilon_0$ for each $t$ from these intervals. Furthermore, a pair $\phi_\theta(t)$, $\phi_{\tilde{\theta}}(t)$ of solutions of (2.6) is called a Li-Yorke pair if it is proximal and frequently $(\epsilon_0, \epsilon_1)$-separated for some positive numbers $\epsilon_0, \epsilon_1$.

The network (2.6) is called Li-Yorke chaotic if: (i) there exists a countably infinite set $\mathcal{A}_0$ of almost periodic solutions of (2.6); (ii) there exists an uncountable set $\Sigma_0$, the scrambled set, consisting of bounded on $\mathbb{R}$ solutions of (2.6) such that the intersection of $\Sigma_0$ and $\mathcal{A}_0$ is empty, and any pair of different solutions inside $\Sigma_0$ is a Li-Yorke pair; (iii) for any solution $\phi_\theta(t) \in \Sigma_0$ and any almost periodic solution $\phi_{\tilde{\theta}}(t) \in \mathcal{A}_0$, the pair $\phi_\theta(t)$, $\phi_{\tilde{\theta}}(t)$ is frequently $(\epsilon_0, \epsilon_1)$-separated for some positive numbers $\epsilon_0, \epsilon_1$.

The main result of the present study is given in the next theorem, which indicates that the network (2.6) is chaotic, provided that the map $F$ is chaotic.

**Theorem 4.1.** *Assume that the map $F$ is Li-Yorke chaotic on $J$, $\{\tau_k^l\}$, $l \in \mathbb{Z}$, is a family of equipotentially almost periodic sequences, $\{M_k\}_{k \in \mathbb{Z}}$ is an almost periodic sequence of $m \times n$ matrices and $L_{ij}(t)$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, are continuous almost periodic functions. Moreover, assume that the conditions $(C1)-(C9)$ hold. Then, the network (2.6) is Li-Yorke chaotic.*

The proof of Theorem 4.1 is presented in the Appendix.

In the following example, we will use the map $F$ mentioned in Eq. (4.10) as the logistic map. That is, $\zeta_{k+1} = G_\lambda(\zeta_k)$, where

$$G_\lambda(u) = \lambda u(1-u). \tag{4.11}$$

For the values of the parameter $\lambda$ between 3.84 and 4, the logistic map (4.11) is chaotic in the sense of Li-Yorke [2]. Moreover, for these values of the parameter, the interval $[0, 1]$ is invariant under the iterations of the map [50]. In the next subsection, we will take into account a Li-Yorke chaotic SICNN that possesses infinitely many quasi-periodic solutions which are separated from the solutions of a scrambled set.

### *4.2. An example*

Consider the SICNN

$$\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} - \sum_{C_{kl} \in N_1(i,j)} C_{ij}^{kl} f(x_{kl}(t))x_{ij} + L_{ij}(t) + P_{ij}(t,\theta), \tag{4.12}$$

where $i, j = 1, 2, 3$, $f(s) = 2s^{1/3}$ and

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 2 & 5 & 4 \\ 3 & 6 & 7 \\ 10 & 8 & 3 \end{pmatrix}.$$

In (4.12), for each $i$ and $j$ we use the following coupling strengths,

$$\begin{pmatrix} C_{ij}^{11} & C_{ij}^{12} & C_{ij}^{13} \\ C_{ij}^{21} & C_{ij}^{22} & C_{ij}^{23} \\ C_{ij}^{31} & C_{ij}^{32} & C_{ij}^{33} \end{pmatrix} = \begin{pmatrix} 0.005 & 0.004 & 0.007 \\ 0.003 & 0.005 & 0.008 \\ 0.002 & 0.009 & 0.001 \end{pmatrix},$$

whenever the cells $C_{kl}$, $k, l = 1, 2, 3$, belong to $N_1(i,j)$. More precisely, for fixed $k$ and $l$, the coupling strengths $C_{ij}^{kl}$, $i, j = 1, 2, 3$, are taken to be equal to each other in cases where $C_{kl}$ belongs to $N_1(i,j)$. The sequence $\theta = \{\theta_k\}_{k \in \mathbb{Z}}$ is described through the equation $\theta_k = \tau_k + \zeta_k$, $k \in \mathbb{Z}$, such that $\tau_k = 2k + \frac{1}{8}\left|\sin(\sqrt{5}k) + 2\cos(k)\right|$, $\zeta_{k+1} = G_\lambda(\zeta_k)$, where $G_\lambda(u)$ is the logistic map defined by (4.11), and $\zeta_0 \in [0, 1]$. For each $i$ and $j$, we set $L_{ij}(t) = 3 + \cos(2t) + \cos(2\pi t)$, and $P_{ij}(t,\theta) = 1 + \left|\sin(2k) + \sin(\sqrt{2}k)\right|$ if $\theta_{2k} < t \le \theta_{2k+1}$ and $P_{ij}(t,\theta) = 0.5 - 0.25\left|\sin(2k) + \sin(\sqrt{2}k)\right|$ if $\theta_{2k-1} < t \le \theta_{2k}$, $k \in \mathbb{Z}$. One can confirm that $\sum_{C_{kl} \in N_1(1,1)} C_{11}^{kl} = 0.017$, $\sum_{C_{kl} \in N_1(1,2)} C_{12}^{kl} = 0.032$, $\sum_{C_{kl} \in N_1(1,3)} C_{13}^{kl} = 0.024$, $\sum_{C_{kl} \in N_1(2,1)} C_{21}^{kl} = 0.028$, $\sum_{C_{kl} \in N_1(2,2)} C_{22}^{kl} = 0.044$, $\sum_{C_{kl} \in N_1(2,3)} C_{23}^{kl} = 0.034$, $\sum_{C_{kl} \in N_1(3,1)} C_{31}^{kl} = 0.019$, $\sum_{C_{kl} \in N_1(3,2)} C_{32}^{kl} = 0.028$, $\sum_{C_{kl} \in N_1(3,3)} C_{33}^{kl} = 0.023$. The conditions of Theorem 4.1 hold for system (4.12) with $\gamma = 2$, $\delta_0 = 0.028/3$, $\delta_1 = 0.044$, $M_f = 3.04$, $L_f = 3.095$, $\underline{\theta} = 5/8$, $\overline{L} = 4$, $\underline{m}_p = 0.5$. Thus, the SICNN (4.12) possesses a chaotic attractor with infinitely many quasi-periodic solutions which are separated from solutions of a scrambled set provided that the map (4.11) is Li-Yorke chaotic.

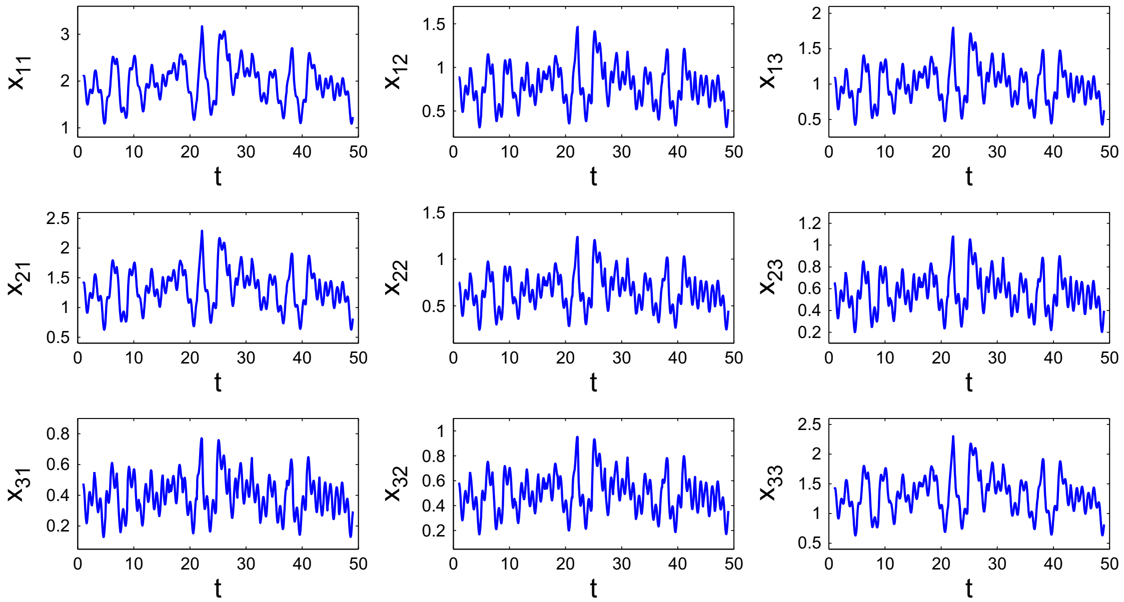

**Fig. 2.** The chaotic behavior of the SICNN (4.12).

Using $\lambda = 3.9$ and $\zeta_0 = 0.74$, we depict in Fig. 2 the solution of SICNN (4.12) with $x_{11}(t_0) = 2.098$, $x_{12}(t_0) = 0.883$, $x_{13}(t_0) = 1.081$, $x_{21}(t_0) = 1.405$, $x_{22}(t_0) = 0.749$, $x_{23}(t_0) = 0.656$, $x_{31}(t_0) = 0.476$, $x_{32}(t_0) = 0.583$, $x_{33}(t_0) = 1.412$, where $t_0 = 0.74$. Fig. 2 supports our theoretical results such that each cell of the SICNN (4.12) possesses chaotic motions.

## 5. The network with EDICs

All the proofs concerning chaotic dynamics are provided for rectangular input currents (RICs) in the present study. However, besides RICs, the significant role is assigned for exponential decaying input currents (EDICs) [20]. Consequently, it is also important to consider SICNNs with EDICs. In this section, we formulate the main result of the chaos extension for that kind of neural networks without proof.

If one considers the external input $\mathcal{L}_{ij}(t)$ in (1.1) as the sum of a continuous external input $L_{ij}(t)$ and an EDIC $K_{ij}(t,\theta)$, then the equation takes the form

$$\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(t))x_{ij} + L_{ij}(t) + K_{ij}(t,\theta). \tag{5.13}$$

The piecewise continuous external input current (2.3) can be represented as $K_{ij}(t,\theta) = \kappa_{ij}^k \alpha(t-\theta_k)$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ on the interval $\theta_k < t \le \theta_{k+1}$, $k \in \mathbb{Z}$, where $\kappa_{ij}^k$ are real numbers and $\theta = \{\theta_k\}_{k \in \mathbb{Z}}$ is a strictly increasing sequence such that $|\theta_k| \to \infty$ as $|k| \to \infty$. Here, we again suppose that the sequence $\theta$ is in the form of (4.9). For each $k \in \mathbb{Z}$, let $V_k$ denote the matrix $(\kappa_{ij}^k)_{m \times n}$. One can take into account each of the functions $K_{ij}(t,\theta)$ as a total input current of the form (2.3), and consider $\theta_k$, $k \in \mathbb{Z}$, as the spike moments. It is worth noting that a bounded on $\mathbb{R}$ function $x(t) = \{x_{ij}(t)\}$ is a solution of the network (5.13) if and only if the integral equation

$$x_{ij}(t) = -\int_{-\infty}^{t} e^{-a_{ij}(t-s)} \left[ \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl} f(x_{kl}(s))x_{ij}(s) - L_{ij}(s) - K_{ij}(s,\theta) \right] ds$$

is satisfied.

The following conditions are required.

- (C10) There exist positive numbers $\overline{\kappa}_{ij}$ such that $\sup_{k \in \mathbb{Z}} \left|\kappa_{ij}^k\right| \le \overline{\kappa}_{ij}$.
- (C11) $\gamma - \delta_1(M_f + L_f K_0) > 0$, where $K_0 = \max_{(i,j)}((\overline{L}_{ij} + \overline{\kappa}_{ij})/a_{ij})/(1 - M_f\delta_0)$.
- (C12) There exists a positive number $\underline{m}_\kappa$ such that $\max_{(i,j)} \inf_{k \in \mathbb{Z}} |\kappa_{ij}^k| \ge \underline{m}_\kappa$.

If the conditions (C1–C5), (C10) and (C11) are valid, then one can confirm that there exists a unique bounded on $\mathbb{R}$ solution $\overline{\phi}_\theta(t) = \{\overline{\phi}_\theta^{ij}(t)\}$ of SICNN (5.13) such that $\sup_{t \in \mathbb{R}} \|\overline{\phi}_\theta(t)\| \le K_0$. Under the same conditions, it can be verified in a similar way to Theorem 2 [58] that all other solutions of (5.13) converge exponentially to $\overline{\phi}_\theta(t)$ for a fixed sequence $\theta$. Moreover, if $\left\{\theta_k^l\right\}$, $l \in \mathbb{Z}$, is a family of equipotentially almost periodic sequences, $\{V_k\}_{k \in \mathbb{Z}}$ is an almost periodic sequence of $m \times n$ matrices, $L_{ij}(t)$ are continuous almost periodic functions and the condition (C8) holds in addition to (C1)–(C5), (C10) and (C11), then $\overline{\phi}_\theta(t)$ is the unique almost periodic solution of the network (5.13).

The following theorem can be proved in a similar way to the proof of Theorem 4.1, which is presented in the Appendix.

**Theorem 5.1.** *Assume that the map $F$ is Li-Yorke chaotic on $J$, $\{\tau_k^l\}$, $l \in \mathbb{Z}$, is a family of equipotentially almost periodic sequences, $\{V_k\}_{k \in \mathbb{Z}}$ is an almost periodic sequence of $m \times n$ matrices and $L_{ij}(t)$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, are continuous almost periodic functions. Moreover, assume that the conditions (C1)–(C5), (C8), (C10)–(C12) hold. Then, the network (5.13) is Li-Yorke chaotic.*

## 6. A chaos extension in SICNNs

In this section, we will deal with an application of the obtained results in neural systems consisting of SICNNs for chaos extension possibilities. It is feasible to find different connection topologies of chaos extension and we will take into account one of them. A neural system consisting of three SICNNs will be considered. As in the example presented in Section 4.2, again the logistic map (4.11) will be utilized as the main source of the chaotic behavior in the SICNNs. We will also show that a control can be realized for the extended chaos in an effective way. In the literature, control of chaos is understood as the stabilization of unstable periodic orbits embedded in a chaotic attractor [50,51]. However, we will demonstrate the stabilization of unstable almost periodic motions instead of periodic ones, and this is one of the distinctive features of our paper.

The literature on the control of chaos originated with Ott, Grebogi and Yorke [48]. The Ott–Grebogi–Yorke (OGY) control technique depends on the usage of small time-dependent perturbations in an accessible system parameter to stabilize an already existing periodic orbit, which is initially unstable [48]. One can select the most desirable unstable periodic orbit, wait until the system approaches it sufficiently and apply a slight nudge to an appropriate parameter to keep the system on that orbit. In the case of continuous chaotic flows, the implementation of the method requires the construction of a Poincaré section and the knowledge of a periodic orbit of the Poincaré map. Due to the high dimension of neural systems and the presence of non-linear terms, in many cases it is difficult to implement the OGY control method to the continuous-time dynamics. In the present paper, we apply the OGY algorithm [48,50,51] to the logistic map (4.11), which is the source of the chaos in the neural system. The performed simulations (see Fig. 6) confirm that the OGY method is an appropriate technique to stabilize the almost periodic motions, which are invisible in the chaotic attractor due to instability. Another well known technique for the control of continuous-time chaotic systems is the Pyragas method [49,51,76], which is known as time-delayed feedback control. To apply this method, the period of the target unstable periodic motion has to be previously known. In our case, there are almost periodic motions instead of periodic ones in the chaotic attractor, and this is the reason why the Pyragas control method cannot be applied directly to the continuous-time dynamics of the neural system under consideration. However, likewise the application of the OGY method to the logistic map, the Pyragas method can also be applied to the map to control the chaos of the neural system.

Let us introduce the SICNNs

$$\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} - \sum_{C_{kl} \in N_1(i,j)} C_{ij}^{kl} f(x_{kl}(t))x_{ij} + L_{ij}(t) + P_{ij}(t,\theta), \tag{6.14}$$

$$\frac{dy_{ij}}{dt} = -b_{ij}y_{ij} - \sum_{\overline{C}_{kl} \in N_1(i,j)} \overline{C}_{ij}^{kl} g(y_{kl}(t))y_{ij} + \overline{L}_{ij}(t) + \overline{P}_{ij}(t,\theta), \tag{6.15}$$

$$\frac{dz_{ij}}{dt} = -c_{ij}z_{ij} - \sum_{\overline{\overline{C}}_{kl} \in N_1(i,j)} \overline{\overline{C}}_{ij}^{kl} h(z_{kl}(t))z_{ij} + \mathcal{L}_{ij}(t), \tag{6.16}$$

in which $i,j = 1,2,3$, $f(s) = \frac{1}{2}s^3$, $g(s) = s^2$, $h(s) = \frac{1}{4}s^{3/2}$,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 9 & 4 & 12 \\ 3 & 8 & 7 \\ 2 & 5 & 9 \end{pmatrix},$$

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 4 & 9 & 3 \\ 2 & 2 & 4 \\ 7 & 8 & 6 \end{pmatrix},$$

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} 8 & 7 & 2 \\ 10 & 9 & 1 \\ 4 & 3 & 6 \end{pmatrix}.$$

For each $i$ and $j$, we will make use of the following coupling strengths,

$$\begin{pmatrix} C_{ij}^{11} & C_{ij}^{12} & C_{ij}^{13} \\ C_{ij}^{21} & C_{ij}^{22} & C_{ij}^{23} \\ C_{ij}^{31} & C_{ij}^{32} & C_{ij}^{33} \end{pmatrix} = \begin{pmatrix} 0.004 & 0.001 & 0.008 \\ 0 & 0.002 & 0.003 \\ 0.007 & 0.008 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \overline{C}_{ij}^{11} & \overline{C}_{ij}^{12} & \overline{C}_{ij}^{13} \\ \overline{C}_{ij}^{21} & \overline{C}_{ij}^{22} & \overline{C}_{ij}^{23} \\ \overline{C}_{ij}^{31} & \overline{C}_{ij}^{32} & \overline{C}_{ij}^{33} \end{pmatrix} = \begin{pmatrix} 0.001 & 0.004 & 0.002 \\ 0.002 & 0.008 & 0.006 \\ 0.004 & 0.003 & 0.009 \end{pmatrix},$$

$$\begin{pmatrix} \overline{\overline{C}}_{ij}^{11} & \overline{\overline{C}}_{ij}^{12} & \overline{\overline{C}}_{ij}^{13} \\ \overline{\overline{C}}_{ij}^{21} & \overline{\overline{C}}_{ij}^{22} & \overline{\overline{C}}_{ij}^{23} \\ \overline{\overline{C}}_{ij}^{31} & \overline{\overline{C}}_{ij}^{32} & \overline{\overline{C}}_{ij}^{33} \end{pmatrix} = \begin{pmatrix} 0.002 & 0 & 0.004 \\ 0.008 & 0.001 & 0.002 \\ 0.003 & 0.005 & 0.006 \end{pmatrix},$$

in the case that the cells $C_{kl}$, $\overline{C}_{kl}$, $\overline{\overline{C}}_{kl}$, $k,l=1,2,3$, belong to the 1-neighborhoods of $C_{ij}$, $\overline{C}_{ij}$, $\overline{\overline{C}}_{ij}$, respectively. For instance, $C_{11}^{11}=C_{12}^{11}=C_{21}^{11}=C_{22}^{11}=0.004$. Define the sequence $\theta=\{\theta_k\}_{k\in\mathbb{Z}}$ through the equation $\theta_k=\tau_k+\zeta_k$, where $\tau_k=1.1k+\frac{1}{32}|\ \sin(k)-\cos(\sqrt{2}k)|$, $\zeta_{k+1}=G_{3.9}(\zeta_k)$ with $\zeta_0\in[0,1]$, and $G_{3.9}(u)$ is the logistic map, which is mentioned in Eq. (4.11). It is worth noting that we make use of the same spike moments $\theta_k$, $k\in\mathbb{Z}$, in the networks (6.14) and (6.15). For each $i$ and $j$, we set $L_{ij}(t)=\sin(5t/2)$, $\overline{L}_{ij}(t)=\cos(\pi t)$,

$$P_{ij}(t,\theta)=\begin{cases} 1.7 & \text{if } \theta_{2k}<t\le\theta_{2k+1}, \\ 3.2 & \text{if } \theta_{2k-1}<t\le\theta_{2k}, \end{cases}$$

and

$$\overline{P}_{ij}(t,\theta)=\begin{cases} 3.8 & \text{if } \theta_{2k}<t\le\theta_{2k+1}, \\ 2.1 & \text{if } \theta_{2k-1}<t\le\theta_{2k}, \end{cases}$$

In accordance with Theorem 4.1, the SICNNs (6.14) and (6.15) exhibit Li-Yorke chaos with infinitely many quasi-periodic solutions, which are separated from the solutions of the corresponding scrambled sets.

Let us consider the nonlinear functions $\varphi(v)=\left\{\varphi_{ij}(v)\right\}$ and $\psi(v)=\left\{\psi_{ij}(v)\right\}$, which are defined through the equations $\varphi_{11}(v)=0.4e^{v_{21}}$, $\varphi_{12}(v)=\arctan v_{22}$, $\varphi_{13}(v)=(0.1+v_{23})^2$, $\varphi_{21}(v)=2v_{11}$, $\varphi_{22}(v)=1/(1+v_{12}^2)$, $\varphi_{23}(v)=2v_{13}+0.8\cos v_{13}$, $\varphi_{31}(v)=(0.5+v_{31})^3$, $\varphi_{32}(v)=(1.2+v_{32})^{1/3}$, $\varphi_{33}(v)=v_{33}+\tanh v_{33}$ and $\psi_{11}(v)=1+0.7\sin v_{33}$, $\psi_{12}(v)=\sqrt{1+v_{32}}$, $\psi_{13}(v)=\tanh v_{31}$, $\psi_{21}(v)=0.5v_{11}^2$, $\psi_{22}(v)=0.1v_{12}+0.6e^{v_{12}}$, $\psi_{23}(v)=1.5v_{13}+0.5\arctan v_{13}$, $\psi_{31}(v)=0.2v_{21}$, $\psi_{32}(v)=0.9v_{22}^2+1/v_{22}$, $\psi_{33}(v)=2v_{23}^{-1/3}$, where $v=\{v_{ij}\}$, $i,j=1,2,3$.

In order to make our results more applicable and to see more general connections between neural networks, let us apply the nonlinear functions $\varphi$ and $\psi$, and set $\mathcal{L}_{ij}(t)=\varphi_{ij}(x(t))+\psi_{ij}(y(t))$ in SICNN (6.16), where $x(t)=\{x_{ij}(t)\}$ and $y(t)=\left\{y_{ij}(t)\right\}$. That is, the external inputs $\mathcal{L}_{ij}(t)$ of (6.16) are provided through the outputs of (6.14) and (6.15). Fig. 3 depicts the connection topology of the neural system (6.14)–(6.15)–(6.16). In this figure, the cells of the SICNNs (6.14) and (6.15) are shown in blue color, while the cells of the SICNN (6.16) are represented in red color. Both of the networks (6.14) and (6.15) influence (6.16) in a unidirectional way. According

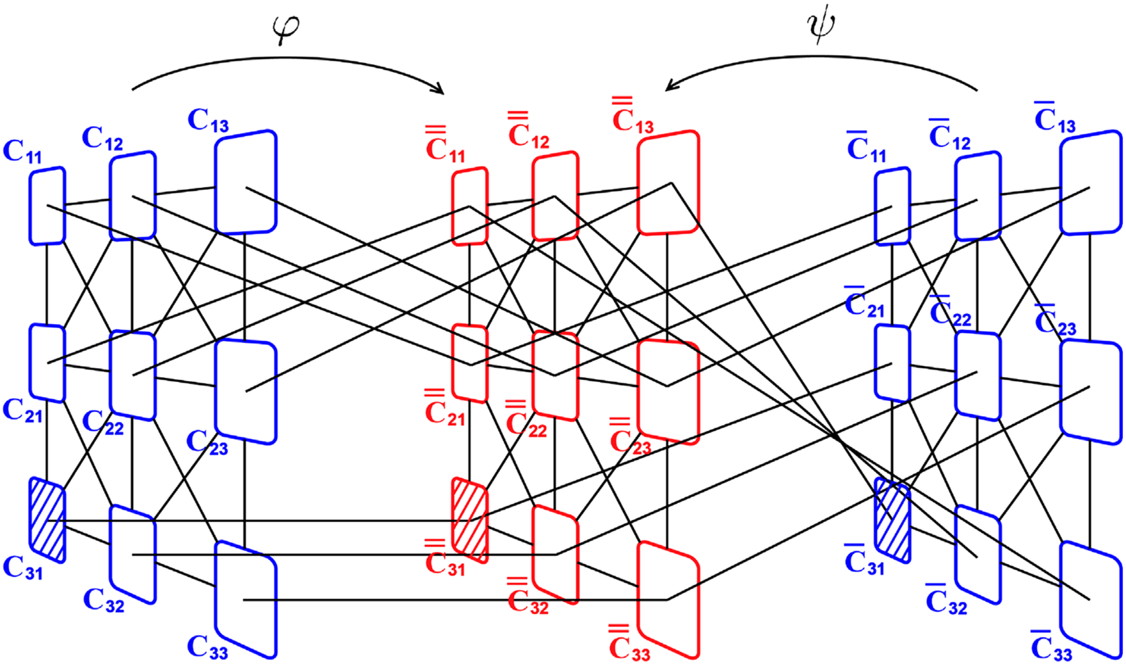

**Fig. 3.** The schematic diagram of the neural system (6.14)–(6.15)–(6.16). Blue color is used to represent the cells of the SICNNs (6.14) and (6.15), while the cells of the SICNN (6.16) are shown in red color. The networks (6.14) and (6.15) influence (6.16) by means of the functions $\varphi$ and $\psi$, respectively. The chaotic as well as stabilized quasi-periodic outputs of the cells $C_{31}$, $\overline{C}_{31}$ and $\overline{\overline{C}}_{31}$ will be depicted in later figures. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

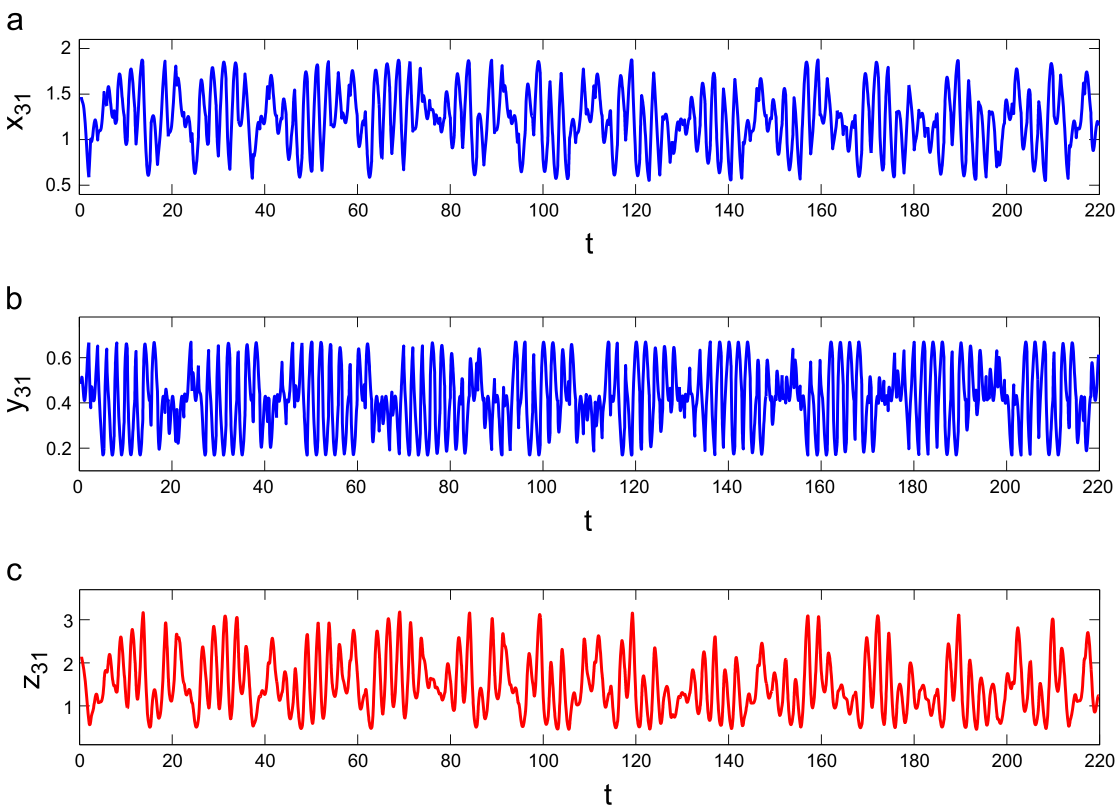

**Fig. 4.** Irregular behavior in the system (6.14)–(6.15)–(6.16). The graphs of the $x_{31}$, $y_{31}$ and $z_{31}$ coordinates are shown in (a), (b) and (c), respectively. It is seen that the SICNNs (6.14) and (6.15) lead to chaotic motions in the SICNN (6.16).

to the theoretical results of the paper [13], the functions $\varphi$ and $\psi$ give rise to the presence of chaos in (6.16) since they satisfy both the Lipschitz and inverse Lipschitz conditions.

In the neural system (6.14)–(6.15)–(6.16), we use $\zeta_0 = 0.41$, and represent in Fig. 4 the graphs of the $x_{31}$, $y_{31}$ and $z_{31}$ coordinates of the solution with $x_{11}(t_0) = 0.393$, $x_{12}(t_0) = 0.604$, $x_{13}(t_0) = 0.271$, $x_{21}(t_0) = 0.879$, $x_{22}(t_0) = 0.304$, $x_{23}(t_0) = 0.418$, $x_{31}(t_0) = 1.467$, $x_{32}(t_0) = 0.562$, $x_{33}(t_0) = 0.293$, $y_{11}(t_0) = 0.783$, $y_{12}(t_0) = 0.385$, $y_{13}(t_0) = 0.956$, $y_{21}(t_0) = 1.362$, $y_{22}(t_0) = 1.354$, $y_{23}(t_0) = 0.781$, $y_{31}(t_0) = 0.484$, $y_{32}(t_0) = 0.429$, $y_{33}(t_0) = 0.556$, $z_{11}(t_0) = 0.234$, $z_{12}(t_0) = 0.182$, $z_{13}(t_0) = 0.198$, $z_{21}(t_0) = 0.045$, $z_{22}(t_0) = 0.153$, $z_{23}(t_0) = 4.439$, $z_{31}(t_0) = 2.141$, $z_{32}(t_0) = 1.056$, $z_{33}(t_0) = 0.408$, where $t_0 = 0.41$. Fig. 4 shows that the SICNNs (6.14) and (6.15) exhibit chaos, and they give rise to chaotic motions in the SICNN (6.16). Chaos is also present in the remaining cells, which are not depicted here. On the other hand, the projection on the $x_{22} - y_{21} - z_{31}$ space of the trajectory with the same initial data is shown in Fig. 5. This figure also confirms that the overall dynamics of the neural system (6.14)–(6.15)–(6.16) is chaotic.

According to their instability, the existing quasi-periodic solutions of the aforementioned SICNNs are invisible in the simulations. We will illustrate a quasi-periodic solution of the neural system (6.14)–(6.15)–(6.16) with $\lambda = 3.9$ by means of the OGY control method [48,50,51] applied to the logistic map $G_\lambda(u)$. Let us explain the method briefly.

Suppose that the parameter $\lambda$ in the logistic map $G_\lambda(u)$ is allowed to vary in the range $[3.9-\varepsilon, 3.9+\varepsilon]$, where $\varepsilon$ is a given small positive number. Consider an arbitrary solution $\{\zeta_k\}$, $\zeta_0 \in [0,1]$, of the map and denote by $\zeta^{(j)}$, $j = 1, 2, \ldots, p$, the target $p$-periodic orbit to be stabilized. In the OGY control method [50], at each iteration step $k$ after the control mechanism is switched on, we consider the logistic map

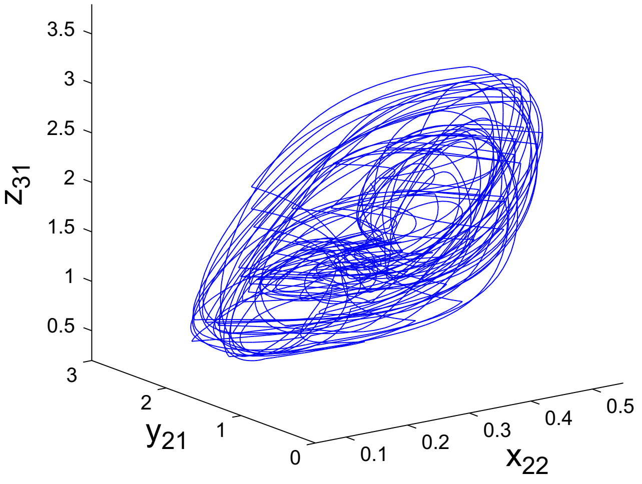

**Fig. 5.** Projection of the chaotic attractor of system (6.14)–(6.15)–(6.16) on the $x_{22} - y_{21} - z_{31}$ space.

with the parameter value $\lambda = \overline{\lambda}_k$, where

$$\overline{\lambda}_k = 3.9\left(1 + \frac{(2\zeta^{(j)} - 1)(\zeta_k - \zeta^{(j)})}{\zeta^{(j)}(1-\zeta^{(j)})}\right), \tag{6.17}$$

provided that the number on the right hand side of the formula (6.17) belongs to the interval $[3.9-\varepsilon, 3.9+\varepsilon]$. In other words, formula (6.17) is valid if the trajectory $\{\zeta_k\}$ is sufficiently close to the target periodic orbit. Otherwise, we take $\overline{\lambda}_k = 3.9$, so that the system evolves at its

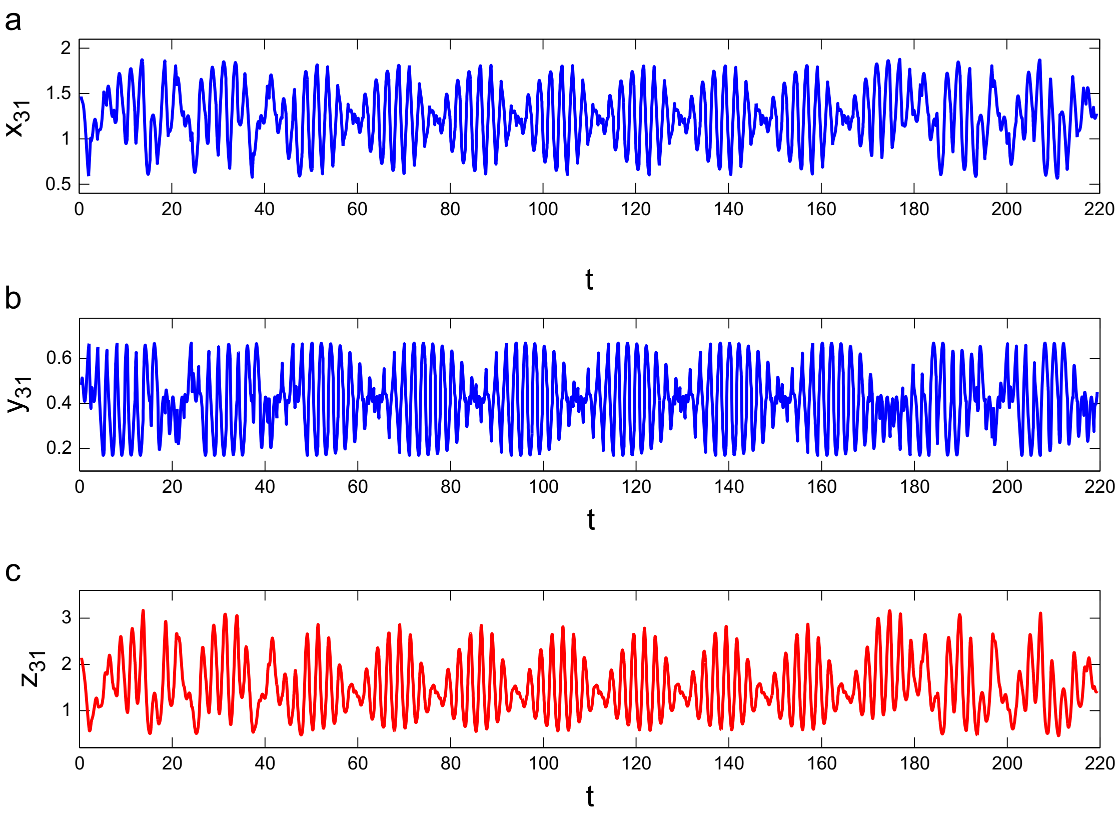

**Fig. 6.** The quasi-periodic solution of the system (6.14)–(6.15)–(6.16) is controlled by means of the OGY control method applied to the logistic map $G_{3.9}(u)$ around the fixed point 2.9/3.9. The value $\varepsilon = 0.085$ is used in the simulation. The control mechanism is switched on at $t=\theta_{40}$ and switched off at $t=\theta_{100}$.

original parameter value, and wait until the trajectory $\{\zeta_k\}$ enters in a sufficiently small neighborhood of the periodic orbit $\zeta^{(j)}, j=1,2,\ldots,p$, such that the inequality $-\varepsilon \le 3.9(2\zeta^{(j)}-1)(\zeta_k-\zeta^{(j)})/\zeta^{(j)}(1-\zeta^{(j)}) \le \varepsilon$ holds. If this is the case, the control of chaos is not achieved immediately after switching on the control mechanism. Instead, there is a transition time before the desired periodic orbit is stabilized. The transition time increases if the number $\varepsilon$ decreases [51].

The OGY control method is convenient to stabilize the quasi-periodic solutions of SICNNs (6.14) and (6.15). We consider the solution of system (6.14)–(6.15)–(6.16) with the same initial data as in the simulation presented in Fig. 4, and apply the OGY method around the fixed point 2.9/3.9 of the logistic map $G_{3.9}(u)$. Fig. 6(a), (b) and (c) shows the simulation results for the $x_{31}$, $y_{31}$ and $z_{31}$ coordinates, respectively. We used the value $\varepsilon = 0.085$. The control mechanism is switched on at $t=\theta_{40}$ and switched off at $t=\theta_{100}$. The control becomes dominant approximately at $t$=48 and its effect lasts approximately until $t=170$, after which the instability becomes dominant and irregular behavior develops again. It is seen that the quasi-periodic solutions of the SICNNs (6.14) and (6.15) are stabilized, and accordingly the chaos of the SICNN (6.16) is controlled.

## 7. Conclusions

In the present paper, the method of Li-Yorke chaos generation by SICNNs is suggested. The chaos is considered with infinitely many almost periodic motions, which are separated from the scrambled set, for the first time in the literature. Continuous and piecewise continuous postsynaptic currents are used as inputs. The stabilization of almost periodic (quasi-periodic) motions is another novelty in the theory of neural networks. It is also shown that there is an effective way to control collectives of connected SICNNs. Appropriate illustrations with chaotic and quasi-periodic motions are depicted to support the theoretical results.

The idea that unstable periodic motions as a basis of chaos can be replaced by more general types of regular motions stems from the investigations of Poincaré and Birkhoff [47]. In the paper [77], it was certified that, in general, in place of a countable set of periodic solutions to form chaos, one can take an uncountable collection of Poisson stable motions which are dense in a quasiminimal set. This can also be observed in the Horseshoe attractor [78].

In neural systems an input current is generated by the activity of presynaptic neurons, and presynaptic spikes are capable of generating postsynaptic current pulses [20]. In SICNNs (2.6) and (5.13) the formation of input currents are characterized by the firing times of the presynaptic neurons. From the biological point of view, similar to the case of spike response models [20], one can interpret the input currents of the form (2.2) and (2.4) as postsynaptic currents evoked by the firings of a single presynaptic neuron, while (2.3) and (2.5) are total postsynaptic currents caused by all presynaptic neurons. In general, the effects of these inputs can be either excitatory or inhibitory depending on their signs.

In an electrical equivalent circuit of a cell of SICNNs, the synaptic conductance of each inhibitory channel is proportional to the firing rate of the cell controlling it and the shunting conductance of the cell is the sum of the synaptic conductances of individual inhibitory channels [25]. Since continuous as well as discontinuous external inputs in the form of (2.3) and (2.5) are utilized, one may design electrical equivalent circuits of the neural networks under consideration by properly coupling appropriate pulse generating circuits [79,80] with the one mentioned in [25].

Our results can be applied to investigations of chaotic communication [42–44], combinatorial optimization problems [45] and control of legged robots [68]. Moreover, they can be useful for neural processes in which the size of the memory capacity is important as well as for understanding the chaotic behavior in the human brain [1,81].

## Acknowledgments

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## Appendix A. Verification of Theorem 4.1

Before the proof of Theorem 4.1, we present Lemmas A.1–A.3 which will be needed in the proof. The following lemma is about the proximality feature of the bounded solutions.

**Lemma A.1.** *Suppose that conditions* $(C1)$*–*$(C8)$ *hold. A pair* $\phi_\theta(t), \phi_{\tilde\theta}(t)$ *is proximal, provided that the same is true for the pair* $\theta, \tilde\theta \in \Theta$.

**Proof.** Fix an arbitrary small positive number $\epsilon$ and an arbitrary large positive number $E$ such that

$$E > \frac{2}{\gamma - \delta_1(M_f + L_f P_0)} \ln\left(\frac{2M_f P_0 \delta_0 + \beta_1}{\alpha_0 \epsilon}\right),$$

where

$$\alpha_0 = \frac{\gamma - \delta_1(M_f + L_f P_0)}{\gamma - \delta_1(M_f + L_f P_0) + \beta_2 \gamma}, \quad \beta_1 = 2\max_{(i,j)} \frac{\overline{p}_{ij}}{a_{ij}} \quad \text{and}$$
$$\beta_2 = 2\max_{(i,j)} \overline{p}_{ij}\left(1 + \frac{1}{1 - \mathrm{e}^{-a_{ij}\underline{\theta}}}\right).$$

Suppose that $\delta$ is a positive number such that $\delta \le \alpha_0 \epsilon$. Since the pair of sequences $\theta, \tilde\theta$ is proximal, there exists a number $R \in \mathbb{R}$ such that

$$\sup_{\theta_k, \tilde\theta_k \in [R, R+E]} |\theta_k - \tilde\theta_k| < \delta.$$

The solutions $\phi_\theta(t)$ and $\phi_{\tilde\theta}(t)$ satisfy the inequality

$$\begin{aligned}\left|\phi^{ij}_\theta(t) - \phi^{ij}_{\tilde\theta}(t)\right| \le & \int_{-\infty}^t \mathrm{e}^{-a_{ij}(t-s)} \sum_{C_{kl} \in N_r(i,j)} C^{kl}_{ij} \Big| f(\phi^{kl}_\theta(s))\phi^{ij}_\theta(s) \\ & - f(\phi^{kl}_{\tilde\theta}(s))\phi^{ij}_{\tilde\theta}(s)\Big| ds \\ & + \int_{-\infty}^t \mathrm{e}^{-a_{ij}(t-s)} \left|P_{ij}(s,\theta) - P_{ij}(s,\tilde\theta)\right| ds. \qquad (A.18)\end{aligned}$$

Let us denote by $(\widehat{\theta_k, \tilde\theta_k}]$ the oriented interval such that $(\widehat{\theta_k, \tilde\theta_k}] = (\theta_k, \tilde\theta_k]$, if $\theta_k < \tilde\theta_k$, and $(\widehat{\theta_k, \tilde\theta_k}] = (\tilde\theta_k, \theta_k]$, otherwise.

Since the values of $P_{ij}(t,\theta)$ and $P_{ij}(t,\tilde\theta)$ are possibly different for each $t$ from the intervals $(\widehat{\theta_k, \tilde\theta_k}]$, $k \in \mathbb{Z}$, one can show for each $i$ and $j$ that

$$\int_R^t \mathrm{e}^{-a_{ij}(t-s)} \left|P_{ij}(s,\theta) - P_{ij}(s,\tilde\theta)\right| ds \le 2\overline{p}_{ij}\left(1 + \frac{1}{1 - \mathrm{e}^{-a_{ij}\underline{\theta}}}\right)\delta, \quad t \in [R, R+E].$$

In view of (A.18), we have for $t \in [R, R+E]$ that

$$\begin{aligned}\left|\phi^{ij}_\theta(t) - \phi^{ij}_{\tilde\theta}(t)\right| \le & 2M_f P_0 \frac{\sum_{C_{kl} \in N_r(i,j)} C^{kl}_{ij}}{a_{ij}} \mathrm{e}^{-a_{ij}(t-R)} \\ & + \frac{2\overline{p}_{ij}}{a_{ij}} \mathrm{e}^{-a_{ij}(t-R)} + 2\overline{p}_{ij}\left(1 + \frac{1}{1 - \mathrm{e}^{-a_{ij}\underline{\theta}}}\right)\delta \\ & + \int_R^t \mathrm{e}^{-a_{ij}(t-s)} \sum_{C_{kl} \in N_r(i,j)} C^{kl}_{ij} L_f P_0 \left|\phi^{kl}_\theta(s) - \phi^{kl}_{\tilde\theta}(s)\right| ds \\ & + \int_R^t \mathrm{e}^{-a_{ij}(t-s)} \sum_{C_{kl} \in N_r(i,j)} C^{kl}_{ij} M_f \left|\phi^{ij}_\theta(s) - \phi^{ij}_{\tilde\theta}(s)\right| ds.\end{aligned}$$

The last inequality yields

$$\begin{aligned}\|\phi_\theta(t) - \phi_{\tilde\theta}(t)\| \le & (2M_f P_0 \delta_0 + \beta_1)\mathrm{e}^{-\gamma(t-R)} + \beta_2 \delta \\ & + \delta_1(M_f + L_f P_0) \int_R^t \mathrm{e}^{-\gamma(t-s)} \|\phi_\theta(s) - \phi_{\tilde\theta}(s)\| ds.\end{aligned}$$

Defining the function $u(t) = \mathrm{e}^{\gamma t}\|\phi_\theta(t) - \phi_{\tilde\theta}(t)\|$, we obtain for $t \in [R, R+E]$ that

$$u(t) \le (2M_f P_0 \delta_0 + \beta_1)\mathrm{e}^{\gamma R} + \beta_2 \delta \mathrm{e}^{\gamma t} + \delta_1(M_f + L_f P_0) \int_R^t u(s)\, ds.$$

By means of Gronwall's Lemma [82], one can verify that

$$\begin{aligned}u(t) \le & (2M_f P_0 \delta_0 + \beta_1)\mathrm{e}^{\gamma R}\mathrm{e}^{\delta_1(M_f + L_f P_0)(t-R)} + \beta_2 \delta \mathrm{e}^{\gamma t} \\ & + \frac{\beta_2 \delta \delta_1(M_f + L_f P_0)}{\gamma - \delta_1(M_f + L_f P_0)} \mathrm{e}^{\gamma t}\left(1 - \mathrm{e}^{[-\gamma + \delta_1(M_f + L_f P_0)](t-R)}\right).\end{aligned}$$

Multiplying both sides of the last inequality with $e^{-\gamma t}$, we attain that

$$\begin{aligned}\|\phi_\theta(t) - \phi_{\tilde\theta}(t)\| \le & \beta_2 \delta + (2M_f P_0 \delta_0 + \beta_1)\mathrm{e}^{-\gamma(t-R)}\mathrm{e}^{\delta_1(M_f + L_f P_0)(t-R)} \\ & + \frac{\beta_2 \delta \delta_1(M_f + L_f P_0)}{\gamma - \delta_1(M_f + L_f P_0)}\left(1 - \mathrm{e}^{[-\gamma + \delta_1(M_f + L_f P_0)](t-R)}\right) \\ \le & \frac{\beta_2 \delta \gamma}{\gamma - \delta_1(M_f + L_f P_0)} + (2M_f P_0 \delta_0 + \beta_1)\mathrm{e}^{[-\gamma + \delta_1(M_f + L_f P_0)](t-R)}.\end{aligned}$$

If $t$ belongs to the interval $[R+E/2, R+E]$, then we get $(2M_f P_0 \delta_0 + \beta_1)\mathrm{e}^{[-\gamma + \delta_1(M_f + L_f P_0)](t-R)} < \alpha_0 \epsilon$. Hence, the inequality

$$\|\phi_\theta(t) - \phi_{\tilde\theta}(t)\| < \left(1 + \frac{\beta_2 \gamma}{\gamma - \delta_1(M_f + L_f P_0)}\right)\alpha_0 \epsilon = \epsilon$$

holds for all $t \in [R+E/2, R+E]$. Consequently, the pair $\phi_\theta$, $\phi_{\tilde\theta}$ is proximal. □

In the following assertion, we continue with the second ingredient of Li-Yorke chaos.

**Lemma A.2.** *Suppose that the conditions* $(C1)$*–*$(C7)$ *and* $(C9)$ *hold. If a pair* $\theta, \tilde\theta \in \Theta$ *is frequently separated, then the pair* $\phi_\theta(t)$, $\phi_{\tilde\theta}(t)$ *is frequently* $(\epsilon_0, \epsilon_1)$*-separated for some positive numbers* $\epsilon_0$ *and* $\epsilon_1$.

**Proof.** Since the pair $\theta, \tilde\theta$ is frequently separated, there exist a positive number $\overline{\epsilon}_0$ and a sequence $\{k_q\}$, $k_q \to \infty$ as $q \to \infty$, such that $\left|\theta_{k_q} - \tilde\theta_{k_q}\right| > \overline{\epsilon}_0$ for each $q \in \mathbb{N}$.

Our aim is to determine positive numbers $\epsilon_0$, $\epsilon_1$ and infinitely many intervals $I_q$, $q \in \mathbb{N}$, each with length $\epsilon_1$ such that the inequality $\|\phi_\theta(t) - \phi_{\tilde\theta}(t)\| > \epsilon_0$ holds for each $t$ from these intervals.

Fix an arbitrary $q \in \mathbb{N}$. Let us denote by $[\widehat{\theta_{k_q}, \tilde\theta_{k_q}}]$ the interval $[\theta_{k_q}, \tilde\theta_{k_q}]$, if $\theta_{k_q} \le \tilde\theta_{k_q}$, and the interval $[\tilde\theta_{k_q}, \theta_{k_q}]$, otherwise.

For $t \in [\theta_{k_q}, \tilde\theta_{k_q}]$, using the relation

$$\begin{aligned}\phi^{ij}_\theta(\tilde\theta_{k_q}) - \phi^{ij}_{\tilde\theta}(\tilde\theta_{k_q}) = & \int_{\theta_{k_q}}^{\tilde\theta_{k_q}} \left(P_{ij}(s,\theta) - P_{ij}(s,\tilde\theta)\right) ds \\ & + (\phi^{ij}_\theta(\theta_{k_q}) - \phi^{ij}_{\tilde\theta}(\theta_{k_q})) - \int_{\theta_{k_q}}^{\tilde\theta_{k_q}} a_{ij}\left(\phi^{ij}_\theta(s) - \phi^{ij}_{\tilde\theta}(s)\right) ds \\ & - \int_{\theta_{k_q}}^{\tilde\theta_{k_q}} \left[\sum_{C_{kl} \in N_r(i,j)} C^{kl}_{ij} f(\phi^{kl}_\theta(s))\phi^{ij}_\theta(s) - \sum_{C_{kl} \in N_r(i,j)} C^{kl}_{ij} f(\phi^{kl}_{\tilde\theta}(s))\phi^{ij}_{\tilde\theta}(s)\right] ds,\end{aligned}$$

we attain that

$$\|\phi_\theta(\tilde\theta_{k_q}) - \phi_{\tilde\theta}(\tilde\theta_{k_q})\| \ge \max_{(i,j)} \left|\int_{\theta_{k_q}}^{\tilde\theta_{k_q}} \left(P_{ij}(s,\theta) - P_{ij}(s,\tilde\theta)\right) ds\right|$$

$$
\begin{aligned}
&-\|\phi_\theta(\theta_{k_q})-\phi_{\tilde\theta}(\theta_{k_q})\|-\max_{(i,j)}\left|\int_{\theta_{k_q}}^{\tilde\theta_{k_q}} a_{ij}\left(\phi^{ij}_\theta(s)-\phi^{ij}_{\tilde\theta}(s)\right)ds\right|\\
&-\max_{(i,j)}\left|\int_{\theta_{k_q}}^{\tilde\theta_{k_q}}\left[\sum_{C_{kl}\in N_r(i,j)} C^{kl}_{ij} f(\phi^{kl}_\theta(s))\phi^{ij}_\theta(s)-\right.\right.\\
&\left.\left.\sum_{C_{kl}\in N_r(i,j)} C^{kl}_{ij} f(\phi^{kl}_{\tilde\theta}(s))\phi^{ij}_{\tilde\theta}(s)\right]ds\right|. \qquad (A.19)
\end{aligned}
$$

By means of condition (C9), one can verify that

$$
\max_{(i,j)}\left|\int_{\theta_{k_q}}^{\tilde\theta_{k_q}}\left(P_{ij}(s,\theta)-P_{ij}(s,\tilde\theta)\right)ds\right| > \underline{m}_p\overline{\epsilon}_0. \qquad (A.20)
$$

Denote by $l_0$ the length of the interval $J$ on which the map (4.10) is defined, and set $\overline{\gamma}=\max_{(i,j)}a_{ij}$ and $M_p=\max_{(i,j)}\overline{p}_{ij}$. The inequalities (A.19) and (A.20) imply that

$$
\begin{aligned}
\max_{t\in\widehat{[\theta_{k_q},\tilde\theta_{k_q}]}}\|\phi_\theta(t)-\phi_{\tilde\theta}(t)\| &\geq \|\phi_\theta(\tilde\theta_{k_q})-\phi_{\tilde\theta}(\tilde\theta_{k_q})\|\\
&> \underline{m}_p\overline{\epsilon}_0-[1+l_0\overline{\gamma}+l_0\delta_1(M_f+L_fP_0)]\max_{t\in\widehat{[\theta_{k_q},\tilde\theta_{k_q}]}}\|\phi_\theta(t)-\phi_{\tilde\theta}(t)\|.
\end{aligned}
$$

Therefore, $\max_{t\in\widehat{[\theta_{k_q},\tilde\theta_{k_q}]}}\|\phi_\theta(t)-\phi_{\tilde\theta}(t)\| > a_0$, where $a_0=\dfrac{\underline{m}_p\overline{\epsilon}_0}{2+l_0\overline{\gamma}+l_0\delta_1(M_f+L_fP_0)}$.

Suppose that

$$
\max_{t\in\widehat{[\theta_{k_q},\tilde\theta_{k_q}]}}\|\phi_\theta(t)-\phi_{\tilde\theta}(t)\| = \|\phi_\theta(\eta_q)-\phi_{\tilde\theta}(\eta_q)\|
$$

for some $\eta_q\in\widehat{[\theta_{k_q},\tilde\theta_{k_q}]}$.

Define

$$
\epsilon_1=\min\left\{\frac{\overline{\epsilon}_0}{2},\frac{a_0}{4(\overline{\gamma}P_0+\delta_1M_fP_0+M_p)}\right\}
$$

and

$$
\theta^1_q=\begin{cases}\eta_q, & \eta_q\leq(\theta_{k_q}+\tilde\theta_{k_q})/2,\\ \eta_q-\epsilon_1, & \eta_q>(\theta_{k_q}+\tilde\theta_{k_q})/2,\end{cases}
$$

One can obtain for $t\in[\theta^1_q,\theta^1_q+\epsilon_1]$ that

$$
\begin{aligned}
\|\phi_\theta(t)-\phi_{\tilde\theta}(t)\| &\geq \|\phi_\theta(\eta_q)-\phi_{\tilde\theta}(\eta_q)\|-\max_{(i,j)}\left|\int_{\eta_q}^{t} a_{ij}\left(\phi^{ij}_\theta(s)-\phi^{ij}_{\tilde\theta}(s)\right)ds\right|\\
&\quad-\max_{(i,j)}\left|\int_{\eta_q}^{t}\left[\sum_{C_{kl}\in N_r(i,j)} C^{kl}_{ij} f(\phi^{kl}_\theta(s))\phi^{ij}_\theta(s)\right.\right.\\
&\quad\left.\left.-\sum_{C_{kl}\in N_r(i,j)} C^{kl}_{ij} f(\phi^{kl}_{\tilde\theta}(s))\phi^{ij}_{\tilde\theta}(s)\right]ds\right|\\
&\quad-\max_{(i,j)}\left|\int_{\eta_q}^{t}\left(P_{ij}(s,\theta)-P_{ij}(s,\tilde\theta)\right)ds\right|\\
&> a_0-2\epsilon_1\left(\overline{\gamma}P_0+\delta_1M_fP_0+M_p\right)\\
&\geq \frac{a_0}{2}.
\end{aligned}
$$

Thus, for each $t$ from the intervals $I_q=[\theta^1_q,\theta^1_q+\epsilon_1]$, $q\in\mathbb{N}$, the inequality $\|\phi_\theta(t)-\phi_{\tilde\theta}(t)\|>\epsilon_0$ holds, where $\epsilon_0=a_0/2$. Consequently, the pair $\phi_\theta(t)$, $\phi_{\tilde\theta}(t)$ is frequently $(\epsilon_0,\epsilon_1)$-separated. □

Another assertion that will be needed in the proof of Theorem 4.1 is the following one.

**Lemma A.3.** *If $\{\tau^l_k\}$, $l\in\mathbb{Z}$, is a family of equipotentially almost periodic sequences and $\{\zeta_k\}_{k\in\mathbb{Z}}$ is an almost periodic sequence, then $\left\{\theta^l_k\right\}$, $l\in\mathbb{Z}$, is a family of equipotentially almost periodic sequences.*

**Proof.** By applying the method of finding common almost periods of two sequences [70], one can confirm that there exists a respectively dense set of $\epsilon/3$-almost periods, for both the family of sequences $\{\tau^l_k\}$, $l\in\mathbb{Z}$, and the sequence $\{\zeta_k\}$. Then, for such an almost period $p$ and an arbitrarily fixed $k\in\mathbb{Z}$, it is true that

$$
\begin{aligned}
|\theta^l_{k+p}-\theta^l_k| &= |\tau_{k+l+p}+\zeta_{k+l+p}-\tau_{k+p}-\zeta_{k+p}-\tau_{k+l}-\zeta_{k+l}+\tau_k+\zeta_k|\\
&\leq |\tau_{k+l+p}-\tau_{k+p}-\tau_{k+l}+\tau_k|+|\zeta_{k+l+p}-\zeta_{k+l}|+|\zeta_{k+p}-\zeta_k|\\
&\leq |\tau^l_{k+p}-\tau^l_k|+|\zeta_{k+l+p}-\zeta_{k+l}|+|\zeta_{k+p}-\zeta_k|\\
&< \epsilon.
\end{aligned}
$$

Thus, $\left\{\theta^l_k\right\}$, $l\in\mathbb{Z}$, is a family of equipotentially almost periodic sequences. □

The proof of Theorem 4.1 is as follows.

**Proof of Theorem 4.1.** According to the chaoticity of the map $F$ in the sense of Li-Yorke, there exists a $p-$ periodic solution $\{\zeta_k\}$ of Eq. (4.10) for each natural number $p$.

Let us denote by $\mathcal{A}$ the set of all sequences $\theta=\{\theta_k\}_{k\in\mathbb{Z}}$ obtained by Eq. (4.9) in which $\{\zeta_k\}$ is periodic. One can conclude by Lemma A.3 that $\left\{\theta^l_k\right\}$, $l\in\mathbb{Z}$, is a family of equipotentially almost periodic sequences for each $\theta\in\mathcal{A}$.

On the other hand, denote by $\Sigma$ the set of all sequences obtained by (4.9) such that $\zeta_0\in S$, where $S$ is a scrambled set for the map $F$. It is worth noting that the intersection of $\mathcal{A}$ and $\Sigma$ is empty.

By means of these descriptions, it can be verified that each pair of different sequences inside $\Sigma$ is both proximal and frequently separated. Additionally, frequent separation feature holds also for each pair $\theta\in\Sigma$ and $\hat\theta\in\mathcal{A}$.

Making use of Theorem 3.1, one can confirm that the collection

$$
\mathcal{A}_0=\{\phi_\theta(t):\theta\in\mathcal{A}\}
$$

is a countably infinite set of almost periodic solutions of the network (2.6). Furthermore, the set

$$
\Sigma_0=\{\phi_\theta(t):\theta\in\Sigma\}
$$

is uncountable. Clearly, the intersection of $\Sigma_0$ and $\mathcal{A}_0$ is empty.

In accordance with Lemmas A.1 and A.2, $\Sigma_0$ is a scrambled set. That is, any pair of different solutions inside $\Sigma_0$ is a Li-Yorke pair. Besides, Lemma A.2 implies that for any solution $\phi_\theta(t)\in\Sigma_0$ and any almost periodic solution $\phi_{\hat\theta}(t)\in\mathcal{A}_0$, the pair $\phi_\theta(t)$, $\phi_{\hat\theta}(t)$ is frequently $(\epsilon_0,\epsilon_1)$-separated for some positive numbers $\epsilon_0$ and $\epsilon_1$. Consequently, the network (2.6) is Li-Yorke chaotic.

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