DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **21**, Number **4**, June **2016**

doi:10.3934/dcdsb.2016.21.1119

pp. **1119–1148**

IMPULSIVE SICNNS WITH CHAOTIC POSTSYNAPTIC CURRENTS

Mehmet Onur Fen* and Marat Akhmet

Department of Mathematics
Middle East Technical University
06800, Ankara, Turkey

(Communicated by Janet Best)

Abstract. In the present study, we investigate the dynamics of shunting inhibitory cellular neural networks (SICNNs) with impulsive effects. We give a mathematical description of the chaos for the multidimensional dynamics of impulsive SICNNs, and prove its existence rigorously by taking advantage of the external inputs. The Li-Yorke definition of chaos is used in our theoretical discussions. In the considered model, the impacts satisfy the cell and shunting principles. This enriches the applications of SICNNs and makes the analysis of impulsive neural networks deeper. The technique is exceptionally useful for SICNNs with arbitrary number of cells. We make benefit of unidirectionally coupled SICNNs to exemplify our results. Moreover, the appearance of cyclic irregular behavior observed in neuroscience is numerically demonstrated for discontinuous dynamics of impulsive SICNNs.

1. **Introduction.** Bouzerdoum and Pinter [26] introduced and analyzed a class of cellular neural networks (CNNs), namely the shunting inhibitory cellular neural networks (SICNNs), which have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision and image processing [23, 24, 25, 32, 35, 41, 50, 80]. The layers in SICNNs are arranged into two-dimensional arrays of processing units, called cells, where each cell is coupled to its neighboring units only. The interactions among cells within a single layer is mediated via the biophysical mechanism of recurrent shunting inhibition, where the shunting conductance of each cell is modulated by voltages of neighboring cells [26].

The model of SICNNs according to the most original formulation [26] is as follows. Consider a two-dimensional grid of processing cells, and let C_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, denote the cell at the (i, j) position of the lattice. The $r-$neighborhood of C_{ij} is defined as

$$N_r(i,j) = \{C_{hl} : \max\{|h-i|, |l-j|\} \leq r,\ 1 \leq h \leq m,\ 1 \leq l \leq n\}.$$

2010 *Mathematics Subject Classification.* Primary: 34C28, 92B20; Secondary: 37D45.

Key words and phrases. Shunting inhibitory cellular neural networks, discontinuous Li-Yorke chaos, chaotic inputs and outputs, impacts subject to cell and shunting principles, near-periodic discontinuous chaos, chaotification of impulsive SICNNs.

The first author is supported by the 2219 scholarship programme of TÜBİTAK, the Scientific and Technological Research Council of Turkey.

* Corresponding author.

In SICNNs, neighboring cells exert mutual inhibitory interactions of the shunting type. The dynamics of the cell C_{ij} is described by the nonlinear ordinary differential equation

$$\frac{dx_{ij}(t)}{dt} = -a_{ij}x_{ij}(t) - \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} f(x_{hl}(t))x_{ij}(t) + L_{ij}(t), \tag{1}$$

where x_{ij} is the activity of the cell C_{ij}; $L_{ij}(t)$ is the external input to C_{ij}; the constant $a_{ij} > 0$ represents the passive decay rate of the cell activity; $C_{ij}^{hl} \geq 0$ is the connection or coupling strength of the postsynaptic activity of the cell C_{hl} transmitted to the cell C_{ij}; and the activation function $f(x_{hl})$ is a positive continuous function representing the output or firing rate of the cell C_{hl}.

Because of the switching phenomenon, frequency changes or other sudden noises, the states of the electronic networks are often subject to instantaneous perturbations and experience abrupt changes at certain instants [14, 67, 78, 105, 107]. In other words, they exhibit impulsive effects. Therefore, neural network models with impulsive effects are more accurate to describe the evolutionary processes of the systems.

In the present study, we will consider impulsive SICNNs in the form

$$\begin{aligned} \frac{dx_{ij}(t)}{dt} &= -a_{ij}x_{ij}(t) - \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} f(x_{hl}(t))x_{ij}(t) + L_{ij}(t),\ t \neq \theta_k, \\ \Delta x_{ij}|_{t=\theta_k} &= b_{ij}x_{ij}(\theta_k) + \sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl} g(x_{hl}(\theta_k))x_{ij}(\theta_k) + I_{ij}^k, \end{aligned} \tag{2}$$

where $b_{ij} \neq -1$ for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, the sequence $\{\theta_k\}$, $k = 0, \pm 1, \pm 2, \ldots,$ of impact moments is strictly increasing, $\Delta x_{ij}|_{t=\theta_k} = x_{ij}(\theta_k+) - x_{ij}(\theta_k)$ and $x_{ij}(\theta_k+) = \lim_{t\to\theta_k+} x_{ij}(t)$. In SICNN (2), the couple $\big(L_{ij}(t), I_{ij}^k\big)$ is the external input to the cell C_{ij}. Similarly to the continuous interactions of neural networks through synapses, one can say about impact type of interactions [67, 105, 107]. We will say that the impact in the SICNN (2) is subject to the cell and shunting principles owing to the term $\displaystyle\sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl} g(x_{hl}(\theta_k))x_{ij}(\theta_k)$, where $D_{ij}^{hl} \geq 0$ is the impact coupling strength of the postsynaptic activity of the cell C_{hl} transmitted to the cell C_{ij} and the impact activation function $g(x_{hl})$ represents the output localized at a moment of impact of the cell C_{hl}. In the theoretical discussions, the functions $L_{ij}(t)$ will be assumed to be continuous. Our main objective is to verify chaos in the dynamics of the network (2), provided that the external inputs $L_{ij}(t)$ behave chaotically.

According to Chua and Yang [36], the cellular structure makes cells of a CNN communicate with each other directly only through its neighbors, and because of the local interconnection feature, CNNs are much more amenable to VLSI implementation than general neural networks. In the previous analyses of impulsive SICNNs [44, 61, 95, 102], the cell principle [36, 37] and the shunting phenomenon [26] have not been applied to the impacts of neurons. Contrarily, in the present study, we make benefit of the cell and shunting principles. In other words, the term $\displaystyle\sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl} g(x_{hl}(\theta_k))x_{ij}(\theta_k)$ is inserted in the SICNN (2). The arguments for the insertion are the same as in the original papers [26, 36, 37]. We suppose that the usage of the aforementioned principles in the impulsive part of the networks enriches the applications of SICNNs due to the arguments provided in the papers [26, 37]. These arguments are image processing, pattern recognition, visual selectivity for small objects and edges, adaptation of the organization of the spatial respective field and of the contrast sensitivity function, enhancement of edges and contrast, noise removal, feature extraction and the mediation of directional selectivity.

The dynamics of CNNs with impulsive effects have been widely investigated in the literature [2, 20, 58, 78, 93, 94, 99, 101, 102, 106]. The problem of global exponential stability for CNNs with time-varying delays and fixed moments of impulses was considered in the studies [2, 101] by means of the Lyapunov functions and the Razumikhin technique. Wang and Liu [99] used the method of variation of parameters and Lyapunov functionals to obtain sufficient conditions for the exponential stability of impulsive CNNs with time delays. Besides, Song et al. [93] dealt with the exponential stability of distributed delayed and impulsive CNNs with partially Lipschitz continuous activation functions. Li et al. [58] investigated impulsive CNNs with time-varying and distributed delays and obtained some sufficient conditions that ensure the existence, uniqueness and global exponential stability of the equilibrium point. Global stability of CNNs with time delays were also investigated in the paper [28] by means of the Lyapunov stability theorem. Taking advantage of piecewise continuous Lyapunov functions and the Razumikhin technique combined with Young's inequality, the stability of impulsive CNNs were analyzed by Stamova and Ilarionov [94]. On the other hand, contraction mapping principle and Krasnoselski's fixed point theorem were utilized by Pan and Cao [78] to verify the existence of anti-periodic solutions of delayed cellular neural networks with impulsive effects. Moreover, Yang and Cao [106] considered the global exponential stability as well as the existence of a periodic solution for delayed cellular neural networks with impulsive effects based on the Halanay inequality, mathematical induction and a fixed point theorem.

Chaotic dynamics is an object of great interest in the theory of neural networks [3, 4, 22, 40, 43, 53, 56, 73, 74, 82, 91, 92, 97, 98], and CNNs are not excluded [51, 66, 103, 113, 114]. The presence of chaotic attractors was observed in two-cell non-autonomous and three-cell autonomous CNNs in the studies [113, 114]. Yan et al. [103] proposed algebraic conditions for the control of multiple time-delayed chaotic CNNs. Moreover, the effect of variable thresholds in chaotic CNNs were investigated by Liu and Wang [66].

The presence of chaos in neural networks is useful for separating image segments [91], information processing [73, 74] and synchronization [29, 65, 68, 110]. Besides, the synchronization phenomenon is also observable in the dynamics of coupled chaotic CNNs [49, 84]. Chaotic dynamics can improve the performance of CNNs on problems that have local minima in energy (cost) functions, since chaotic behavior of CNNs can help the network avoid local minima and reach the global optimum [96]. Furthermore, chaotic dynamics in CNNs is an important tool for the studies of chaotic communication [31, 57, 109] and combinatorial optimization problems [75].

As a mathematical notion, the term chaos has first been used by Li and Yorke [62] for one dimensional discrete equations. According to Marotto [69], a multidimensional continuously differentiable map exhibits chaos in the sense of Li-Yorke, provided that it has a snap-back repeller. Marotto's Theorem was utilized in [59] to prove the existence of Li-Yorke chaos in a spatiotemporal chaotic system. This theorem is also a powerful tool in the theory of neural networks. For instance, it was used by Lin and Ruan [63] to determine the existence of chaos in a pacemaker neuron type integrate-and-fire circuit having two states with a periodic pulse-train input. Moreover, in the study [52], the chaos was approved by virtue of the Marotto's Theorem in discrete time delayed Hopfield neural networks. Li-Yorke sensitivity, which links the Li-Yorke chaos with the notion of sensitivity, was studied in [19], and generalizations of Li-Yorke chaos to mappings in Banach spaces and complete metric spaces were provided in [55, 88, 89]. Impulsive systems can be used as an appropriate source of chaotic motions and there are several studies on the subject [7, 9, 10, 16, 44]. In the present paper, we develop the concept of Li-Yorke chaos to the multidimensional dynamics of impulsive SICNNs, and prove its presence rigorously.

Many results concerning the dynamics of SICNNs have been published in the last two decades. The existence and stability of periodic, almost periodic and anti-periodic outputs for SICNNs with delay have been studied in the papers [27, 33, 34, 38, 47, 60, 64, 77, 79, 87, 111, 112] by using external inputs with the same type of regularity. The existence and global attractiveness of almost periodic solutions of SICNNs with delay were studied in [33] by means of the Banach fixed point theorem. Similar results for SICNNs with time-varying delay based on the Halanay inequality technique were obtained by Huang and Cao [47]. On the other hand, some sufficient conditions for the existence and local exponential stability of the almost periodic solutions were established in the paper [27] for SICNNs with time-varying delays without assuming the global Lipschitz and boundedness conditions of activation functions. Besides, the existence and stability of periodic and almost periodic solutions for impulsive SICNNs without the cell and shunting principles in the impacts were considered by Sun [95] and Xia et al. [102], respectively. However, in the present study, we make use of chaotic external inputs and obtain chaos in the outputs of impulsive SICNNs with impacts subject to the cell and shunting principles. In the paper [44], the existence of a chaotic attractor in SICNNs with impulses was numerically demonstrated without a theoretical support, and the chaos type was not indicated. Contrarily, in this paper, we rigorously prove the presence of chaos in impulsive SICNNs with a precise type of chaos. The method of replication of chaos was considered in the papers [12, 13, 16] and summarized in the book [18]. In the present study, we apply the method for impulsive SICNNs. The paper [12] was concerned with chaotic dynamics of SICNNs without impulses. The main novelty of the present paper is the discussion of the problem with impulsive effects, which requests a more sophisticated analysis and a new approach of the proofs. In [13], the chaos extension in continuous-time dynamics was considered without impulsive effects. The aforementioned technique was also utilized for attraction of chaos in differential equations with impulses and applied to mechanical problems (by means of Duffing oscillators) in the study [16]. Our results can be extended for the different classes of neural networks by applying the linear matrix inequality technique [28, 46, 90, 100].

One of the advantages of our results is the suitability to obtain high dimensional neural systems. A possible chaos extension mechanism is represented in Figure 1. N pieces of SICNNs are shown in Figure 1 such that SICNN 1 is the source of chaotic motions, and the other networks, SICNN 2, SICNN 3, $\ldots$, SICNN N, are influenced by the outputs of SICNN 1. The couplings between SICNN 1 and the remaining networks are all unidirectional. According to our theoretical results, the neural system consisting of N pieces of SICNNs possesses chaos under the conditions that will be presented in the next section. We call this type of chaos extension process as the "core" mechanism. In Section 4, we will make use of the core mechanism to demonstrate how to obtain high dimensional neural systems.

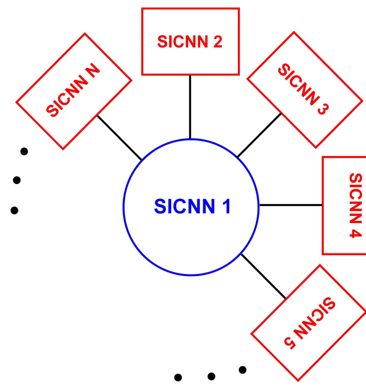

FIGURE 1. The core mechanism of chaos extension.

Another possible mechanism that can be used to obtain high dimensional chaotic neural systems is the "chain" mechanism, which is shown in Figure 2. The figure represents consecutively connected N pieces of SICNNs such that the couplings between the networks are unidirectional. In the first coupling, we take into account SICNN 1 as the source of chaos. The chaotic outputs of SICNN 1, which are used as inputs for SICNN 2, gives rise to the appearance of chaos in the latter. In the next coupling, SICNN 2 is considered as the source of chaotic motions with respect to the third network. That is, SICNN 2 changes its role in the process. Similarly, in the remaining couplings, the role of the previously chaotically influenced SICNN changes and we start to use it as the source of chaotic external inputs for the next network. As a result, all of the networks become chaotic as well as the whole neural system consisting of N pieces of SICNNs. It is worth noting that the type of the chaos is preserved in this procedure.

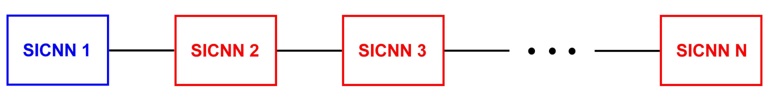

FIGURE 2. The chain mechanism of chaos extension.

In the mechanisms, one can take the number of networks, N, arbitrarily large, even with the possibility of infinite number of networks in the core mechanism. Other mechanisms are also possible, for example, by means of the "composition" of the proposed ones.

In their study, Skarda and Freeman [92] reported the formation of periodic and chaotic EEG signals when a rabbit was given known and unknown odorants, respectively. Additionally, Yao and Freeman [108] observed the presence of chaotic behavior near periodic motions in a model of the olfactory system. The emergence of near-periodic chaos in continuous-time systems without impulses was considered in the studies [15, 17] by means of weak chaotic perturbations applied to systems that possess stable periodic solutions. In the study [48], the brain units such as neurons, cortical columns and neuronal modules were supposed to be weakly connected. The presence of weak synaptic connections in the hippocampals cells and between neurons in the cortex was experimentally observed by McNaughton et al. [70] and Abeles [1], respectively. In the present study, by establishing weak connections between SICNNs, we numerically demonstrate the appearance of near-periodic discontinuous chaos.

The rest of the paper is organized as follows. In Section 2, we introduce the description of Li-Yorke chaos for impulsive SICNNs, and prove the existence, uniqueness and attractiveness feature of the bounded solutions. The main result of the present study is indicated in Section 3, where we prove the presence of chaos in the dynamics of the impulsive SICNNs (2). Illustrative examples are presented in Section 4, and Section 5 is devoted for conclusions. Finally, the proofs of the proximality and frequent separation features are provided in the Appendix.

2. **Preliminaries.**

Throughout the paper, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ will stand for the sets of real numbers, integers and natural numbers, respectively. We will use the norm $\|w\| = \max\limits_{(i,j)} |w_{ij}|$, where $w = \{w_{ij}\} = (w_{11}, \ldots, w_{1n}, \ldots, w_{m1} \ldots, w_{mn}) \in \mathbb{R}^{m\times n}$.

We say that a function $\psi(t) = \{\psi_{ij}(t)\}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, which is defined on $\mathbb{R}$, belongs to the set $\mathcal{PC}(\mathbb{R})$ if it is left-continuous and continuous except, possibly, at the points where it has discontinuities of the first kind. The definition of a Li-Yorke chaotic set of piecewise continuous functions that will be used in the present study is as follows.

Suppose that $\mathcal{B}$ is a set of uniformly bounded functions $\psi(t) = \{\psi_{ij}(t)\}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, which belong to $\mathcal{PC}(\mathbb{R})$ and have common points of discontinuity.

We say that a couple $\left(\psi(t), \widetilde{\psi}(t)\right) \in \mathcal{B}\times\mathcal{B}$ is proximal if for arbitrary small $\epsilon > 0$ and arbitrary large $E > 0$, there exists an interval J with a length no less than E such that $\left\|\psi(t) - \widetilde{\psi}(t)\right\| < \epsilon$ for $t \in J$. Besides, a couple $\left(\psi(t), \widetilde{\psi}(t)\right) \in \mathcal{B}\times\mathcal{B}$ is called frequently $(\epsilon_0, \Delta)-$separated if there exist positive numbers ϵ_0, Δ and infinitely many disjoint intervals, each with a length no less than Δ, such that $\left\|\psi(t) - \widetilde{\psi}(t)\right\| > \epsilon_0$ for each t from these intervals, and each of these intervals contains at most one discontinuity point of both $\psi(t)$ and $\widetilde{\psi}(t)$. It is worth noting that the numbers ϵ_0 and Δ depend on the functions $\psi(t)$ and $\widetilde{\psi}(t)$.

A couple $\left(\psi(t), \widetilde{\psi}(t)\right) \in \mathcal{B}\times\mathcal{B}$ is a Li$-$Yorke pair if it is proximal and frequently $(\epsilon_0, \Delta)-$separated for some positive numbers ϵ_0 and Δ. Moreover, an uncountable set $\mathcal{C} \subset \mathcal{B}$ is called a scrambled set if $\mathcal{C}$ does not contain any periodic functions and each couple of different functions inside $\mathcal{C}\times\mathcal{C}$ is a Li$-$Yorke pair.

We say that the collection $\mathcal{B}$ is a Li$-$Yorke chaotic set if: (i) There exists a positive number T_0 such that $\mathcal{B}$ admits a periodic function of period mT_0, for any $m \in \mathbb{N}$; (ii) $\mathcal{B}$ possesses a scrambled set $\mathcal{C}$; (iii) For any function $\psi(t) \in \mathcal{C}$ and any periodic function $\widetilde{\psi}(t) \in \mathcal{B}$, the couple $\left(\psi(t), \widetilde{\psi}(t)\right)$ is frequently $(\epsilon_0, \Delta)-$separated for some positive numbers ϵ_0 and Δ.

Let us describe a method for obtaining a new Li-Yorke chaotic set of piecewise continuous functions from a given one. Suppose that $\varphi : \mathbb{R}^{m\times n} \to \mathbb{R}^{\overline{m}\times\overline{n}}$ is a function which satisfies for all $u, v \in \mathbb{R}^{m\times n}$ that

$$L_1 \|u - v\| \le \|\varphi(u) - \varphi(v)\| \le L_2 \|u - v\|, \tag{3}$$

where L_1 and L_2 are some positive numbers. In this case, if $\mathcal{B}$ is a Li-Yorke chaotic set, then the collection $\mathcal{B}_\varphi$ whose elements are of the form $\varphi(\psi(t))$, $\psi(t) \in \mathcal{B}$, is also Li-Yorke chaotic.

For any interval I_0, we will denote by $i(I_0)$ the number of elements of the sequence $\{\theta_k\}$, $k \in \mathbb{Z}$, that belong to I_0. Let us denote $u_{ij}(t,s) = e^{-a_{ij}(t-s)}(1+b_{ij})^{i([s,t))}$, where $t \geq s$ [9].

The following conditions are required in the paper:

(C1) There exist a positive number T and a natural number p such that $\theta_{k+p} = \theta_k + T$ for all $k \in \mathbb{Z}$;

(C2) $\lambda = \min_{(i,j)} \lambda_{ij} > 0$, where $\lambda_{ij} = a_{ij} - \dfrac{p}{T} \ln |1 + b_{ij}|$;

(C3) There exist positive numbers M and M_{ij} such that $\sup_{t\in\mathbb{R}} |f(t)| + \sup_{t\in\mathbb{R}} |g(t)| \leq M$ and $\sup_{t\in\mathbb{R}} |L_{ij}(t)| + \sup_{k\in\mathbb{Z}} \left|I_{ij}^k\right| \leq M_{ij}$;

(C4) There exists a positive number L_0 such that $|f(s_1) - f(s_2)| + |g(s_1) - g(s_2)| \leq L_0 |s_1 - s_2|$ for all $s_1, s_2 \in \mathbb{R}$.

In view of the condition $(C1)$, we have $\left| i([s,t)) - \dfrac{p}{T}(t-s) \right| \leq p$ for all $t \geq s$. Moreover, the conditions $(C1)$ and $(C2)$ imply the existence of a positive number K such that for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, the inequality $|u_{ij}(t,s)| \leq K e^{-\lambda_{ij}(t-s)}$ holds for all $t \geq s$. We suppose that $KM\delta < 1$, where

$$\delta = \max_{(i,j)} \left(\frac{\sum_{C_{hl} \in N_r(i,j)} C_{ij}^{hl}}{\lambda_{ij}} + \frac{p \sum_{C_{hl} \in N_r(i,j)} D_{ij}^{hl}}{1 - e^{-\lambda_{ij} T}} \right).$$

The notations $H_0 = \dfrac{K}{1 - KM\delta} \max_{(i,j)} \left(\dfrac{M_{ij}}{\lambda_{ij}} + \dfrac{pM_{ij}}{1 - e^{-\lambda_{ij}T}} \right)$, $b_0 = \min_{(i,j)} \Big(|1 + b_{ij}| - M \sum_{C_{hl} \in N_r(i,j)} D_{ij}^{hl} \Big)$, $\overline{c} = \max_{(i,j)} \sum_{C_{hl} \in N_r(i,j)} C_{ij}^{hl}$, and $\overline{d} = \max_{(i,j)} \sum_{C_{hl} \in N_r(i,j)} D_{ij}^{hl}$ will be used in the paper.

The following conditions are also needed.

(C5) $K(M + L_0 H_0)\delta < 1$;

(C6) $-\lambda + K\overline{c}(M + L_0 H_0) + \dfrac{p}{T} \ln(1 + K\overline{d}(M + L_0 H_0)) < 0$;

(C7) $b_0 - L_0 H_0 \overline{d} > 0$;

(C8) $I_{ij}^{k+p} = I_{ij}^k$ for each $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and $k \in \mathbb{Z}$.

The conditions $(C1)$ and $(C8)$ are essentially required for the existence of infinitely many periodic solutions, which is one of the features of the Li-Yorke chaos. Similar conditions were also used in the studies [44, 95]. In SICNNs of the form (1), for each i and j, the passive decay rate of the cell activity a_{ij} is assumed to be positive, since it represents the ratio of the resting conductance to the membrane capacitance, which are connected in parallel to each other, in the electrical equivalent circuit of a cell [26]. The condition $(C2)$ is natural for impulsive SICNNs (2), since it is the counterpart of the aforementioned feature of SICNNs without impulses, and a similar condition was required by Xia et al. [102]. The Lipschitz continuity and boundedness of the activation functions were used in the studies [44, 47, 60, 79], and such conditions are specified in $(C3)$ and $(C4)$ for the functions f and g. The conditions $(C5)$ and $(C6)$ can be achieved by means of the smallness of the coupling strengths C_{ij}^{hl} and D_{ij}^{hl}. Similar conditions were used in the paper

[47]. The condition $(C7)$ can also be attained by the smallness of the constants D_{ij}^{hl} and it is needed for the frequent separation feature.

We say that a left-continuous function $x(t) = \{x_{ij}(t)\}$ is a solution of (2) if: (i) It has discontinuities only at the points $\theta_k,\ k \in \mathbb{Z}$, and these discontinuities are of the first kind; (ii) The derivatives $\dfrac{dx_{ij}(t)}{dt}$ exist at each point $t \in \mathbb{R} \setminus \{\theta_k\}$, and the left-sided derivatives exist at the points $\theta_k,\ k \in \mathbb{Z}$; (iii) The differential equations are satisfied by $x_{ij}(t)$ on $\mathbb{R} \setminus \{\theta_k\}$, and they hold for the left derivatives of $x_{ij}(t)$ at every point $\theta_k,\ k \in \mathbb{Z}$; (iv) The jump equations are satisfied by $x_{ij}(t)$ for every $k \in \mathbb{Z}$.

According to the results of [9, 85], if conditions $(C1)-(C3)$ hold, then a bounded on $\mathbb{R}$ function $x(t) = \{x_{ij}(t)\}$ is a solution of the network (2) if and only if the relation

$$\begin{aligned} x_{ij}(t) = & -\int_{-\infty}^{t} u_{ij}(t,s)\Big[\sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} f(x_{hl}(s))x_{ij}(s) - L_{ij}(s)\Big]ds \\ & + \sum_{-\infty<\theta_k<t} u_{ij}(t,\theta_k+)\Big[\sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl} f(x_{hl}(s))x_{ij}(s) + I_{ij}^{k}\Big] \end{aligned} \tag{4}$$

is satisfied for each i and j.

The next lemma is about the existence and uniqueness of the bounded on $\mathbb{R}$ solutions of network (2).

Lemma 2.1. *If the conditions $(C1) - (C5)$ are fulfilled, then for any $L(t) = \{L_{ij}(t)\},\ i = 1,2,\ldots,m,\ j = 1,2,\ldots,n$, there exists a unique bounded on $\mathbb{R}$ solution $\phi_L(t) = \left\{\phi_L^{ij}(t)\right\}$ of the network (2) such that $\sup_{t\in\mathbb{R}} \|\phi_L(t)\| \le H_0$.*

Proof. Fix an arbitrary function $L(t) = \{L_{ij}(t)\}$, and consider the set $\mathcal{S}_0$ of functions $w(t) = \{w_{ij}(t)\} \in \mathcal{PC}(\mathbb{R})$ which have discontinuities at the points $\theta_k,\ k \in \mathbb{Z}$, such that $\|w\|_1 \le H_0$, where $\|w\|_1 = \sup_{t\in\mathbb{R}} \|w(t)\|$. The set $\mathcal{S}_0$ is complete [9].

Define on $\mathcal{S}_0$ the operator Π as

$$\begin{aligned} (\Pi w(t))_{ij} \equiv & -\int_{-\infty}^{t} u_{ij}(t,s)\Big[\sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} f(w_{hl}(s))w_{ij}(s) - L_{ij}(s)\Big]ds \\ & + \sum_{-\infty<\theta_k<t} u_{ij}(t,\theta_k+)\Big[\sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl} g(w_{hl}(\theta_k))w_{ij}(\theta_k) + I_{ij}^{k}\Big], \end{aligned}$$

where $\Pi w(t) = \{(\Pi w(t))_{ij}\}$.

First, we shall show that $\Pi(\mathcal{S}_0) \subseteq \mathcal{S}_0$. If $w(t)$ belongs to $\mathcal{S}_0$, then it is easy to verify that

$$\begin{aligned} |(\Pi w(t))_{ij}| \le & \int_{-\infty}^{t} K e^{-\lambda_{ij}(t-s)}\Big(MH_0 \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} + M_{ij}\Big)ds \\ & + \sum_{-\infty<\theta_k<t} K e^{-\lambda_{ij}(t-\theta_k)}\Big(MH_0 \sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl} + M_{ij}\Big). \end{aligned}$$

Making use of the inequality $\displaystyle\sum_{-\infty<\theta_k<t} e^{-\lambda_{ij}(t-\theta_k)} \le \frac{p}{1-e^{-\lambda_{ij}T}}$, one can obtain that

$$|(\Pi w(t))_{ij}| \le \frac{K}{\lambda_{ij}}\Big(MH_0 \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} + M_{ij}\Big)$$

$$+\frac{Kp}{1-e^{-\lambda_{ij}T}}\Big(MH_0\sum_{C_{hl}\in N_r(i,j)}D_{ij}^{hl}+M_{ij}\Big).$$

Accordingly, the inequality $\|\Pi w\|_1 \le KMH_0\delta + K\max_{(i,j)}\Big(\frac{M_{ij}}{\lambda_{ij}}+\frac{pM_{ij}}{1-e^{-\lambda_{ij}T}}\Big)=H_0$ holds. Therefore, $\Pi(\mathcal{S}_0)\subseteq\mathcal{S}_0$.

Next, we will verify that the operator Π is a contraction. For any $w(t),\overline{w}(t)\in\mathcal{S}_0$, one can attain that

$$\begin{aligned}
&(\Pi w(t))_{ij}-(\Pi\overline{w}(t))_{ij}=-\int_{-\infty}^{t}u_{ij}(t,s)\sum_{C_{hl}\in N_r(i,j)}C_{ij}^{hl}\\
&\times\left[f(w_{hl}(s))w_{ij}(s)-f(\overline{w}_{hl}(s))\overline{w}_{ij}(s)\right]ds\\
&+\sum_{-\infty<\theta_k<t}u_{ij}(t,\theta_k+)\sum_{C_{hl}\in N_r(i,j)}D_{ij}^{hl}\left[g(w_{hl}(\theta_k))w_{ij}(\theta_k)-g(\overline{w}_{hl}(\theta_k))\overline{w}_{ij}(\theta_k)\right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&|(\Pi w(t))_{ij}-(\Pi\overline{w}(t))_{ij}|\le\int_{-\infty}^{t}Ke^{-\lambda_{ij}(t-s)}\sum_{C_{hl}\in N_r(i,j)}C_{ij}^{hl}\\
&\times\Big(|f(w_{hl}(s))|\,|w_{ij}(s)-\overline{w}_{ij}(s)|+|\overline{w}_{ij}(s)|\,|f(w_{hl}(s))-f(\overline{w}_{hl}(s))|\Big)ds\\
&+\sum_{-\infty<\theta_k<t}Ke^{-\lambda_{ij}(t-\theta_k)}\sum_{C_{hl}\in N_r(i,j)}D_{ij}^{hl}\Big(|g(w_{hl}(\theta_k))|\,|w_{ij}(\theta_k)-\overline{w}_{ij}(\theta_k)|\\
&+|\overline{w}_{ij}(\theta_k)|\,|g(w_{hl}(\theta_k))-g(\overline{w}_{hl}(\theta_k))|\Big)ds\\
&\le K(M+L_0H_0)\left(\frac{\sum_{C_{hl}\in N_r(i,j)}C_{ij}^{hl}}{\lambda_{ij}}+\frac{p\sum_{C_{hl}\in N_r(i,j)}D_{ij}^{hl}}{1-e^{-\lambda_{ij}T}}\right)\|w-\overline{w}\|_1.
\end{aligned}$$

The last inequality yields $\|\Pi w-\Pi\overline{w}\|_1\le K(M+L_0H_0)\delta\,\|w-\overline{w}\|_1$. Hence, in accordance with condition $(C5)$, the operator Π is contractive. Consequently, there exists a unique bounded on $\mathbb{R}$ solution $\phi_L(t)=\left\{\phi_L^{ij}(t)\right\}$ of the network (2) such that $\sup_{t\in\mathbb{R}}\|\phi_L(t)\|\le H_0$. $\square$

As mentioned in Lemma 2.1, in the remaining parts of the paper, we will denote by $\phi_L(t)=\left\{\phi_L^{ij}(t)\right\}$ the unique bounded on $\mathbb{R}$ solution of the impulsive SICNN (2). Moreover, for a given external input $L(t)=\{L_{ij}(t)\}$ and initial data $x_0\in\mathbb{R}^{m\times n}$, let us denote by $x_L(t,x_0)=\left\{x_L^{ij}(t,x_0)\right\}$ the unique solution of (2) with $x_L(0,x_0)=x_0$. We note that the solution $x_L(t,x_0)$ is not necessarily bounded on $\mathbb{R}$.

Consider the collection $\mathscr{L}$ whose elements are equicontinuous functions of the form $L(t)=\{L_{ij}(t)\}$ such that $\sup_{t\in\mathbb{R}}|L_{ij}(t)|+\sup_{k\in\mathbb{Z}}\left|I_{ij}^k\right|\le M_{ij}$ for each $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$. Suppose that $\mathscr{A}$ denotes the set of bounded on $\mathbb{R}$ solutions $\phi_L(t)$ of the network (2), where $L(t)$ belongs to $\mathscr{L}$.

The following lemma is about the attractiveness of the set $\mathscr{A}$.

Lemma 2.2. *Suppose that the conditions $(C1)-(C6)$ are valid. Then for any $x_0\in\mathbb{R}^{m\times n}$ and $L(t)=\{L_{ij}(t)\}$, we have $\|x_L(t,x_0)-\phi_L(t)\|\to 0$ as $t\to\infty$.*

Proof. Fix an arbitrary $x_0 \in \mathbb{R}^{m\times n}$ and an arbitrary function $L(t) = \{L_{ij}(t)\}$. For $t \geq 0$, making use of the relations

$$\begin{aligned}
&x_L^{ij}(t,x_0) = u_{ij}(t,0)x_L^{ij}(0,x_0)\\
&-\int_0^t u_{ij}(t,s)\Big[\sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} f(x_L^{hl}(s,x_0))x_L^{ij}(s,x_0) - L_{ij}(s)\Big]ds\\
&+\sum_{0\leq\theta_k<t} u_{ij}(t,\theta_k+)\Big[\sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl} g(x_L^{hl}(\theta_k,x_0))x_L^{ij}(\theta_k,x_0) + I_{ij}^k\Big]
\end{aligned}$$

and

$$\begin{aligned}
&\phi_L^{ij}(t) = u_{ij}(t,0)\phi_L^{ij}(0)\\
&-\int_0^t u_{ij}(t,s)\Big[\sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} f(\phi_L^{hl}(s))\phi_L^{ij}(s) - L_{ij}(s)\Big]ds\\
&+\sum_{0\leq\theta_k<t} u_{ij}(t,\theta_k+)\Big[\sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl} g(\phi_L^{hl}(\theta_k))\phi_L^{ij}(\theta_k) + I_{ij}^k\Big],
\end{aligned}$$

we obtain that

$$\begin{aligned}
&\left|x_L^{ij}(t,x_0) - \phi_L^{ij}(t)\right| \leq Ke^{-\lambda_{ij}t}\left|x_L^{ij}(0,x_0) - \phi_L^{ij}(0)\right|\\
&+\int_0^t Ke^{-\lambda_{ij}(t-s)}\sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl}\Big(\left|f(x_L^{hl}(s,x_0))\right|\left|x_L^{ij}(s,x_0) - \phi_L^{ij}(s)\right|\\
&+\left|\phi_L^{ij}(s)\right|\left|f(x_L^{hl}(s,x_0)) - f(\phi_L^{hl}(s))\right|\Big)ds\\
&+\sum_{0\leq\theta_k<t} Ke^{-\lambda_{ij}(t-\theta_k)}\sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl}\Big(\left|g(x_L^{hl}(\theta_k,x_0))\right|\left|x_L^{ij}(\theta_k,x_0) - \phi_L^{ij}(\theta_k)\right|\\
&+\left|\phi_L^{ij}(\theta_k)\right|\left|g(x_L^{hl}(\theta_k,x_0)) - g(\phi_L^{hl}(\theta_k))\right|\Big)\\
&\leq Ke^{-\lambda_{ij}t}\left\|x_0 - \phi_L(0)\right\| + \int_0^t K(M+L_0H_0)e^{-\lambda_{ij}(t-s)}\\
&\times\sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl}\left\|x_L(s,x_0) - \phi_L(s)\right\|ds\\
&+\sum_{0\leq\theta_k<t} K(M+L_0H_0)e^{-\lambda_{ij}(t-\theta_k)}\sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl}\left\|x_L(\theta_k,x_0) - \phi_L(\theta_k)\right\|.
\end{aligned}$$

The last inequality implies for $t \geq 0$ that

$$\begin{aligned}
&\left\|x_L(t,x_0) - \phi_L(t)\right\| \leq Ke^{-\lambda t}\left\|x_0 - \phi_L(0)\right\|\\
&+K\bar{c}(M+L_0H_0)\int_0^t e^{-\lambda(t-s)}\left\|x_L(s,x_0) - \phi_L(s)\right\|ds\\
&+K\bar{d}(M+L_0H_0)\sum_{0\leq\theta_k<t} e^{-\lambda(t-\theta_k)}\left\|x_L(\theta_k,x_0) - \phi_L(\theta_k)\right\|.
\end{aligned}$$

Let us define the function $u(t) = e^{\lambda t}\left\|x_L(t,x_0) - \phi_L(t)\right\|$. Then we have that

$$u(t) \leq K\left\|x_0 - \phi_L(0)\right\| + K\bar{c}(M+L_0H_0)\int_0^t u(s)ds$$

$$+K\overline{d}(M+L_0H_0)\sum_{0\le\theta_k<t}u(\theta_k).$$

With the aid of the Gronwall-Bellman Lemma for piecewise continuous functions, one can verify that

$$\begin{aligned}u(t) &\le K\left\|x_0-\phi_L(0)\right\|e^{K\overline{c}(M+L_0H_0)t}[1+K\overline{d}(M+L_0H_0)]^{i([0,t))}\\ &\le K[1+K\overline{d}(M+L_0H_0)]^p\left\|x_0-\phi_L(0)\right\|\\ &\times e^{[K\overline{c}(M+L_0H_0)+(p/T)\ln(1+K\overline{d}(M+L_0H_0))]t}.\end{aligned}$$

Thus, the inequality

$$\begin{aligned}\|x_L(t,x_0)-\phi_L(t)\| &\le K[1+K\overline{d}(M+L_0H_0)]^p\left\|x_0-\phi_L(0)\right\|\\ &\times e^{[-\lambda+K\overline{c}(M+L_0H_0)+(p/T)\ln(1+K\overline{d}(M+L_0H_0))]t}\end{aligned}$$

holds for all $t\ge 0$. Consequently, in accordance with condition $(C6)$, we have that $\|x_L(t,x_0)-\phi_L(t)\|\to 0$ as $t\to\infty$. □

The next section is devoted for the chaotic dynamics of the network (2).

3. The existence of chaos.

In this part of the paper, we will rigorously prove that if the collection $\mathscr{L}$ is chaotic in the sense of Li-Yorke, then the same is true for the collection $\mathscr{A}$. Before the main result of the present study that will be stated in Theorem 3.3, we will mention about the main ingredients of Li-Yorke chaos, proximality and frequent separation features, in Lemma 3.1 and Lemma 3.2, respectively. The lemmas are as follows.

Lemma 3.1. *Suppose that the conditions $(C1)-(C6)$ hold. If a couple of functions $\left(L(t),\overline{L}(t)\right)\in\mathscr{L}\times\mathscr{L}$ is proximal, then the same is true for the couple $(\phi_L(t),\phi_{\overline{L}}(t))\in\mathscr{A}\times\mathscr{A}$.*

Lemma 3.2. *Suppose that the conditions $(C1)-(C5)$, $(C7)$ are fulfilled. If a couple $\left(L(t),\overline{L}(t)\right)\in\mathscr{L}\times\mathscr{L}$ is frequently $(\epsilon_0,\Delta)-$separated for some positive numbers ϵ_0 and Δ, then there exist positive numbers ϵ_1 and $\overline{\Delta}$ such that the couple $(\phi_L(t),\phi_{\overline{L}}(t))\in\mathscr{A}\times\mathscr{A}$ is frequently $(\epsilon_1,\overline{\Delta})-$separated.*

The proofs of Lemma 3.1 and Lemma 3.2 will be provided in the Appendix. The main result of the present study is given in the following theorem.

Theorem 3.3. *Suppose that the conditions $(C1)-(C8)$ are valid. If $\mathscr{L}$ is a Li-Yorke chaotic set which possesses a $\rho T-$periodic function for each natural number ρ, then the set $\mathscr{A}$ is also Li-Yorke chaotic.*

Proof. Using the conditions $(C1)-(C5)$ and $(C8)$ one can show that if $L(t)\in\mathscr{L}$ is an $\rho T-$periodic function for some natural number ρ, then the bounded on $\mathbb{R}$ solution $\phi_L(t)\in\mathscr{A}$ is also a periodic function with the same period, and vice versa. Therefore, the collection $\mathscr{A}$ contains $\rho T-$periodic functions for each natural number ρ.

Suppose that the set $\mathcal{C}_{\mathscr{L}}$ is a scrambled set inside $\mathscr{L}$. Define the set

$$\mathcal{C}_{\mathscr{A}}=\left\{\phi_L(t)\mid L(t)\in\mathcal{C}_{\mathscr{L}}\right\}.$$

There is a one-to-one correspondence between the sets $\mathcal{C}_{\mathscr{L}}$ and $\mathcal{C}_{\mathscr{A}}$. Because the set $\mathcal{C}_{\mathscr{L}}$ is uncountable, the same is true for $\mathcal{C}_{\mathscr{A}}$. Moreover, no periodic functions exist inside $\mathcal{C}_{\mathscr{A}}$, since there are no such functions inside $\mathcal{C}_{\mathscr{L}}$.

Lemma 3.1 and Lemma 3.2 together imply that the set $\mathcal{C}_{\mathscr{A}}$ is a scrambled set. On the other hand, according to Lemma 3.2, for any function $\phi_L(t) \in \mathcal{C}_{\mathscr{A}}$ and any periodic function $\phi_{\overline{L}}(t) \in \mathscr{A}$, there exist positive numbers ϵ_1 and $\overline{\Delta}$ such that the pair $(\phi_L(t), \phi_{\overline{L}}(t))$ is frequently $(\epsilon_1, \overline{\Delta})-$separated. Consequently, the set $\mathscr{A}$ is Li-Yorke chaotic. □

The results of the present section reveal that if the external inputs $L_{ij}(t)$ are chaotic, then the impulsive SICNNs (2) behave chaotically. Accordingly, to illustrate our results, we need external inputs which are ensured to be chaotic in the Li-Yorke sense. In the next section, to obtain such external inputs, we will consider SICNNs in the form of (1) whose external inputs are relay functions with chaotically changing switching moments [5, 6, 8, 9, 11, 12, 13, 16]. Moreover, we will take advantage of the core mechanism, which is represented in Figure 1, in order to set up a neural system consisting of three SICNNs.

4. **Examples.** Each neuron in a neural network is capable of receiving input signals, processing them and sending an output signal. Neural signals consist of short electrical pulses called action potentials or spikes. A chain of action potentials emitted by a single neuron is called a spike train. Action potentials in a spike train are usually well separated, and it is impossible to excite a second spike during or immediately after a first one [42]. That is why the discontinuity phenomenon is a natural property of neural networks. In this section, we take into account an example of a neural system consisting of three SICNNs. Discontinuous external inputs in a rectangular form are used in the first SICNN to provide the chaos.

Let us consider the SICNN

$$\frac{dx_{ij}(t)}{dt} = -a_{ij}x_{ij}(t) - \sum_{C_{hl}\in N_1(i,j)} C_{ij}^{hl} f(x_{hl}(t))x_{ij}(t) + L_{ij}(t), \tag{5}$$

in which $i, j = 1, 2, 3,$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 6 & 8 & 10 \\ 1 & 9 & 4 \\ 12 & 7 & 5 \end{pmatrix},$$

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} 0.004 & 0.002 & 0 \\ 0.006 & 0.008 & 0.005 \\ 0.009 & 0.007 & 0.003 \end{pmatrix}.$$

In the network (5), we set $L_{ij}(t) = R_{ij}(t,\zeta)$, where the relay function $R_{ij}(t,\zeta)$ is defined by the equation

$$R_{ij}(t,\zeta) = \begin{cases} \alpha_{ij}, & \text{if} \quad \zeta_{2q} < t \leq \zeta_{2q+1}, \\ \beta_{ij}, & \text{if} \quad \zeta_{2q-1} < t \leq \zeta_{2q}, \end{cases} .$$

Here, the numbers ζ_q, $q \in \mathbb{Z}$, denote the switching moments and they are the same for all i and j. The sequence $\zeta = \{\zeta_q\}$ is defined through the formula $\zeta_q = q + \vartheta_q$, $q \in \mathbb{Z}$, where the sequence $\{\vartheta_q\}$, $\vartheta_0 \in [0,1]$, is generated by the logistic map $\vartheta_{q+1} = 3.9\vartheta_q(1-\vartheta_q)$, which is chaotic in the Li-Yorke sense [62]. We note that the interval $[0,1]$ is invariant under the iterations of the map [45]. More information about the dynamics of relay systems can be found in the studies [5, 6, 8, 9, 11, 12, 13, 16].

We consider the SICNN (5) with $f(s) = 0.6\sqrt{s}$ and $\alpha_{ij} = 1.5$, $\beta_{ij} = 0.4$ for all i, j. According to the results of [5, 12], the SICNN (5) exhibits chaotic motions for $\zeta_0 \in [0,1]$, and the collection $\mathscr{L}$ consisting of the bounded on $\mathbb{R}$ solutions of

(5) corresponding to different values of ζ_0 is a Li-Yorke chaotic set, which admits infinitely many periodic solutions with periods 2ρ for each natural number ρ.

Figure 3 shows the solution $x(t) = \{x_{ij}(t)\}$ of the SICNN (5) with $\zeta_0 = 0.192$ corresponding to the initial data $x_{11}(t_0) = 0.1407$, $x_{12}(t_0) = 0.1548$, $x_{13}(t_0) = 0.1092$, $x_{21}(t_0) = 0.9168$, $x_{22}(t_0) = 0.1451$, $x_{23}(t_0) = 0.3276$, $x_{31}(t_0) = 0.1046$, $x_{32}(t_0) = 0.0992$, $x_{33}(t_0) = 0.2518$, where $t_0 = 0.192$. It is seen in Figure 3 that each cell of the SICNN (5) possesses chaos.

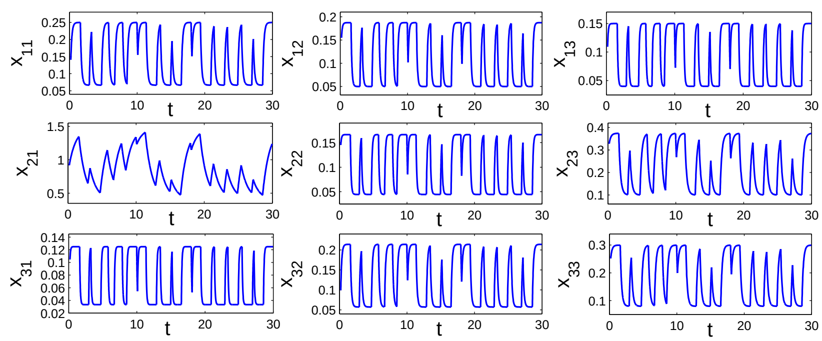

FIGURE 3. The chaotic behavior of SICNN (5).

Next, we consider the impulsive SICNN

$$\begin{aligned} \frac{dy_{ij}(t)}{dt} &= -\bar{a}_{ij}y_{ij}(t) - \sum_{\overline{C}_{hl}\in N_1(i,j)} \overline{C}_{ij}^{hl} f_1(y_{hl}(t))y_{ij}(t) + \overline{L}_{ij}(t),\ t \neq \theta_k, \\ \Delta y_{ij}|_{t=\theta_k} &= \bar{b}_{ij}y_{ij}(\theta_k) + \sum_{\overline{C}_{hl}\in N_1(i,j)} \overline{D}_{ij}^{hl} g_1(y_{hl}(\theta_k))y_{ij}(\theta_k) + \overline{I}_{ij}^{k}, \end{aligned} \tag{6}$$

where $i,j = 1,2,3$, $f_1(s) = 0.4s^{7/2}$, $g_1(s) = 0.2s^2$, $\theta_k = 2k$, $k \in \mathbb{Z}$, $\overline{I}_{11}^k = 0.005$, $\overline{I}_{12}^k = 0.012$, $\overline{I}_{13}^k = 0.005$, $\overline{I}_{21}^k = -0.01$, $\overline{I}_{22}^k = 0.013$, $\overline{I}_{23}^k = -0.004$, $\overline{I}_{31}^k = 0.01$, $\overline{I}_{32}^k = -0.007$, $\overline{I}_{33}^k = -0.006$ for each k,

$$\begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} & \bar{a}_{13} \\ \bar{a}_{21} & \bar{a}_{22} & \bar{a}_{23} \\ \bar{a}_{31} & \bar{a}_{32} & \bar{a}_{33} \end{pmatrix} = \begin{pmatrix} 7 & 4 & 8 \\ 5 & 9 & 6 \\ 10 & 7 & 5 \end{pmatrix},$$

$$\begin{pmatrix} \bar{b}_{11} & \bar{b}_{12} & \bar{b}_{13} \\ \bar{b}_{21} & \bar{b}_{22} & \bar{b}_{23} \\ \bar{b}_{31} & \bar{b}_{32} & \bar{b}_{33} \end{pmatrix} = \begin{pmatrix} -0.5 & 0.6 & -0.4 \\ 0.4 & -0.5 & -0.3 \\ 0.7 & 0.5 & -0.2 \end{pmatrix},$$

$$\begin{pmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} \\ \overline{C}_{21} & \overline{C}_{22} & \overline{C}_{23} \\ \overline{C}_{31} & \overline{C}_{32} & \overline{C}_{33} \end{pmatrix} = \begin{pmatrix} 0.006 & 0.001 & 0.004 \\ 0 & 0.003 & 0.009 \\ 0.012 & 0.005 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \overline{D}_{11} & \overline{D}_{12} & \overline{D}_{13} \\ \overline{D}_{21} & \overline{D}_{22} & \overline{D}_{23} \\ \overline{D}_{31} & \overline{D}_{32} & \overline{D}_{33} \end{pmatrix} = \begin{pmatrix} 0.001 & 0.005 & 0.007 \\ 0.009 & 0 & 0.008 \\ 0.006 & 0.003 & 0.002 \end{pmatrix}.$$

Define the function $\varphi(v) = \{\varphi_{ij}(v)\}$, where $v = \{v_{ij}\}$, $i,j = 1,2,3$, through the equations $\varphi_{11}(v) = 3v_{33}^2$, $\varphi_{12}(v) = \arctan(v_{32})$, $\varphi_{13}(v) = 4(0.3 + 2v_{31})^3$, $\varphi_{21}(v) =$

v_{11}, $\varphi_{22}(v) = \tanh(10v_{12})$, $\varphi_{23}(v) = v_{13}^{3/2} + 0.08$, $\varphi_{31}(v) = 0.3v_{21} + 0.1\sin(v_{21})$, $\varphi_{32}(v) = 5v_{22} + 0.2v_{22}^3$, $\varphi_{33}(v) = 0.15 + \dfrac{5v_{23}^2}{1+v_{23}}$. In network (6), we set $\overline{L}_{ij}(t) = \varphi_{ij}(x(t))$. That is, we make use of the outputs of (5) as external inputs for the impulsive SICNN (6).

It is worth noting that the nonlinear function φ satisfies the inequality (3) on the compact region in which the chaotic attractor of system (5) takes place. Accordingly, the set $\mathscr{L}_\varphi$ whose elements are of the form $\varphi(x(t))$, $x(t) \in \mathscr{L}$, is Li-Yorke chaotic.

One can verify that the conditions $(C1)-(C8)$ hold for the network (6) with $p = 1$, $T = 2$, $K = 4$, $\overline{c} = 0.04$, $\overline{d} = 0.041$, $M = 0.0095$, $L_0 = 0.1051$, $b_0 \approx 0.499613$, $H_0 \approx 4.333645$, $\lambda \approx 3.764998$ and $\delta \approx 0.045279$, where the approximations for b_0, H_0, λ and δ are given with accuracy of six digits in the decimal part. Therefore, the dynamics of the network (6) is Li-Yorke chaotic according to Theorem 3.3.

In the impulsive SICNN (6), let us use the solution of (5) that is represented in Figure 3. Figure 4 depicts the output of (6) with the initial data $y_{11}(t_0) = 0.0119$, $y_{12}(t_0) = 0.0306$, $y_{13}(t_0) = 0.0541$, $y_{21}(t_0) = 0.0491$, $y_{22}(t_0) = 0.0635$, $y_{23}(t_0) = 0.0103$, $y_{31}(t_0) = 0.0339$, $y_{32}(t_0) = 0.0346$, $y_{33}(t_0) = 0.0419$, where $t_0 = 0.192$. Figure 4 supports our theoretical results such that the SICNN (6) exhibits chaotic motions. The 3−dimensional projection of the same solution on the $y_{13} - y_{21} - y_{33}$ space is shown in Figure 5, which confirms one more time the presence of chaos in the dynamics of the network (6).

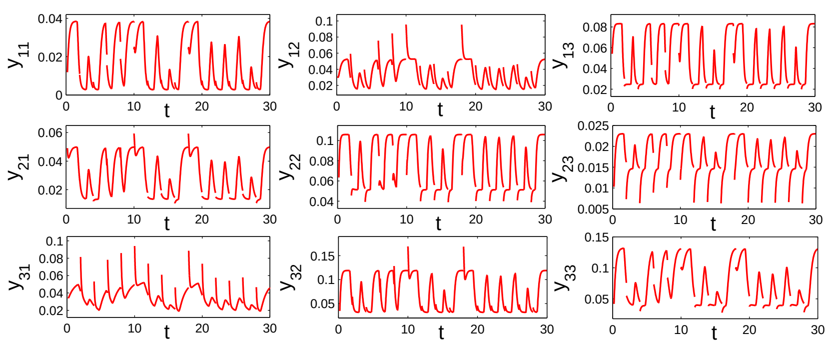

FIGURE 4. The chaotic behavior of SICNN (6).

Now, let us take into account the impulsive SICNN

$$\begin{aligned}
\frac{dz_{ij}(t)}{dt} &= -\widetilde{a}_{ij} z_{ij}(t) - \sum_{\widetilde{C}_{hl}\in N_1(i,j)} \widetilde{C}_{ij}^{hl} f_2(z_{hl}(t)) z_{ij}(t) + \widetilde{L}_{ij}(t),\ t \neq \eta_k,\\
\Delta z_{ij}|_{t=\eta_k} &= \widetilde{b}_{ij} z_{ij}(\eta_k) + \sum_{\widetilde{C}_{hl}\in N_1(i,j)} \widetilde{D}_{ij}^{hl} g_2(z_{hl}(\eta_k)) z_{ij}(\eta_k) + \widetilde{I}_{ij}^k,
\end{aligned} \tag{7}$$

where $i,j = 1,2,3$, $f_2(s) = 0.6s^3$, $g_2(s) = 0.1s^4$, $\eta_k = 4k$, $k \in \mathbb{Z}$, $\widetilde{I}_{ij}^k = 0.02(-1)^k$ for each i,j and k,

$$\begin{pmatrix} \widetilde{a}_{11} & \widetilde{a}_{12} & \widetilde{a}_{13} \\ \widetilde{a}_{21} & \widetilde{a}_{22} & \widetilde{a}_{23} \\ \widetilde{a}_{31} & \widetilde{a}_{32} & \widetilde{a}_{33} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.5 & 0.2 \\ 0.4 & 0.9 & 0.2 \\ 0.6 & 0.7 & 0.4 \end{pmatrix},$$

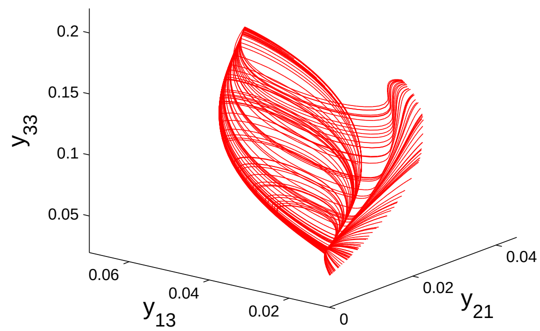

FIGURE 5. The 3-dimensional projection of the chaotic trajectory of the impulsive SICNN (6) on the $y_{13} - y_{21} - y_{33}$ space.

$$\begin{pmatrix} \widetilde{b}_{11} & \widetilde{b}_{12} & \widetilde{b}_{13} \\ \widetilde{b}_{21} & \widetilde{b}_{22} & \widetilde{b}_{23} \\ \widetilde{b}_{31} & \widetilde{b}_{32} & \widetilde{b}_{33} \end{pmatrix} = \begin{pmatrix} 0.12 & 0.21 & 0.24 \\ 0.16 & 0.18 & 0.01 \\ 0.32 & 0.15 & 0.35 \end{pmatrix},$$

$$\begin{pmatrix} \widetilde{C}_{11} & \widetilde{C}_{12} & \widetilde{C}_{13} \\ \widetilde{C}_{21} & \widetilde{C}_{22} & \widetilde{C}_{23} \\ \widetilde{C}_{31} & \widetilde{C}_{32} & \widetilde{C}_{33} \end{pmatrix} = \begin{pmatrix} 0.002 & 0 & 0.007 \\ 0.001 & 0.004 & 0.003 \\ 0.005 & 0.009 & 0.008 \end{pmatrix},$$

$$\begin{pmatrix} \widetilde{D}_{11} & \widetilde{D}_{12} & \widetilde{D}_{13} \\ \widetilde{D}_{21} & \widetilde{D}_{22} & \widetilde{D}_{23} \\ \widetilde{D}_{31} & \widetilde{D}_{32} & \widetilde{D}_{33} \end{pmatrix} = \begin{pmatrix} 0.005 & 0.008 & 0.003 \\ 0.006 & 0.004 & 0.003 \\ 0.009 & 0.002 & 0.007 \end{pmatrix}.$$

We will demonstrate numerically the appearance of near-periodic discontinuous chaos in the dynamics of the SICNN (7) when the networks (5) and (7) are weakly connected.

First of all, consider the network (7) with the external inputs $L_{11}(t) = 0.008 + 0.004\sin(\pi t/2)$, $L_{12}(t) = 0.014 + 0.006\cos(\pi t/4)$, $L_{13}(t) = 0.005 + 0.005\cos(\pi t/2)$, $L_{21}(t) = 0.013 + 0.015\sin(\pi t/4)$, $L_{22}(t) = 0.032 + 0.017\sin(\pi t/4)$, $L_{23}(t) = 0.018 + 0.009\cos(\pi t/2)$, $L_{31}(t) = 0.019 + 0.004\cos(\pi t/2)$, $L_{32}(t) = 0.012 + 0.009\cos(\pi t/2)$ and $L_{33}(t) = 0.007 + 0.005\sin(\pi t/4)$, which are periodic functions, so that the network admits a unique periodic solution. The output of SICNN (7) corresponding to the initial data $z_{11}(0.5) = 0.0431$, $z_{12}(0.5) = 0.0573$, $z_{13}(0.5) = 0.0512$, $z_{21}(0.5) = 0.0396$, $z_{22}(0.5) = 0.0495$, $z_{23}(0.5) = 0.1091$, $z_{31}(0.5) = 0.0602$, $z_{32}(0.5) = 0.0417$, $z_{33}(0.5) = 0.0361$ is shown in Figure 6, where it is seen that the represented output approaches to the periodic solution of the network (7).

In order to obtain motions that behave chaotically around the discontinuous periodic solution shown in Figure 6, we utilize the external inputs $L_{11}(t) = 0.008 + 0.004\sin(\pi t/2)+0.015x_{21}(t)$, $L_{12}(t) = 0.014+0.006\cos(\pi t/4)+0.091x_{22}(t)$, $L_{13}(t) = 0.005+0.005\cos(\pi t/2)+0.018x_{23}(t)$, $L_{21}(t) = 0.013+0.015\sin(\pi t/4)+0.039x_{11}(t)$, $L_{22}(t) = 0.032 + 0.017\sin(\pi t/4) + 0.084x_{12}(t)$, $L_{23}(t) = 0.018 + 0.009\cos(\pi t/2) + 0.048x_{13}(t)$, $L_{31}(t) = 0.019 + 0.004\cos(\pi t/2) + 0.083x_{31}(t)$, $L_{32}(t) = 0.012 + 0.009\cos(\pi t/2) + 0.075x_{32}(t)$, $L_{33}(t) = 0.007 + 0.005\sin(\pi t/4) + 0.035x_{33}(t)$ in (7), where $x(t) = \{x_{ij}(t)\}$ are the outputs of SICNN (5).

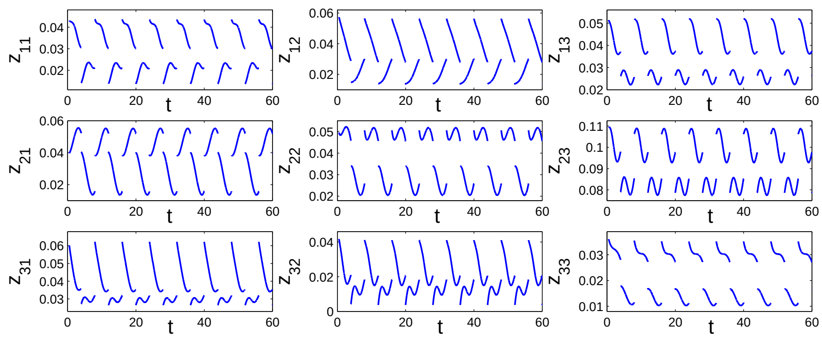

FIGURE 6. The periodic solution of SICNN (7) with the external inputs $L_{11}(t) = 0.008 + 0.004\sin(\pi t/2)$, $L_{12}(t) = 0.014 + 0.006\cos(\pi t/4)$, $L_{13}(t) = 0.005 + 0.005\cos(\pi t/2)$, $L_{21}(t) = 0.013 + 0.015\sin(\pi t/4)$, $L_{22}(t) = 0.032 + 0.017\sin(\pi t/4)$, $L_{23}(t) = 0.018 + 0.009\cos(\pi t/2)$, $L_{31}(t) = 0.019 + 0.004\cos(\pi t/2)$, $L_{32}(t) = 0.012 + 0.009\cos(\pi t/2)$, $L_{33}(t) = 0.007 + 0.005\sin(\pi t/4)$.

The conditions $(C1) - (C8)$ are valid for the SICNN (7) with $p = 2$, $T = 8$, $K = 1.8225$, $\overline{c} = 0.039$, $\overline{d} = 0.047$, $M = 0.0021$, $L_0 = 0.04185$, $b_0 \approx 1.009943$, $H_0 \approx 0.756194$, $\lambda \approx 0.146222$ and $\delta \approx 0.224958$, where the approximations for b_0, H_0, λ and δ are given with accuracy of six digits in the decimal part. According to Theorem 3.3, the network (7) possesses chaos in the sense of Li-Yorke.

Making use of the solution of (5) that is depicted in Figure 3, we represent in Figure 7 the output of the impulsive SICNN (7) corresponding to the initial data $z_{11}(t_0) = 0.0921$, $z_{12}(t_0) = 0.0353$, $z_{13}(t_0) = 0.0852$, $z_{21}(t_0) = 0.0416$, $z_{22}(t_0) = 0.0551$, $z_{23}(t_0) = 0.1241$, $z_{31}(t_0) = 0.0682$, $z_{32}(t_0) = 0.0367$, $z_{33}(t_0) = 0.0479$, where $t_0 = 0.192$. One can observe that the represented motion behaves chaotically near the periodic solution shown in Figure 6.

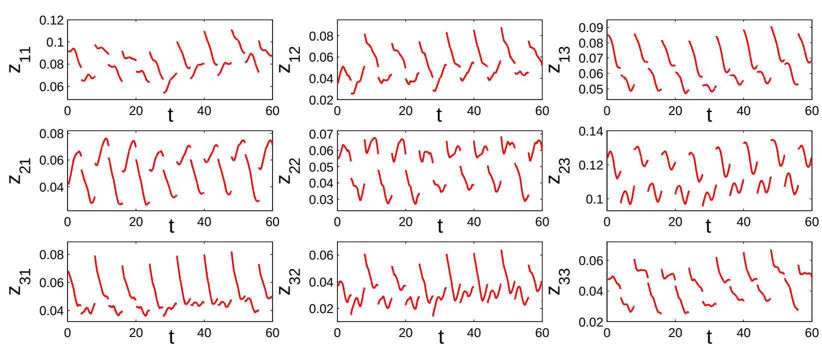

FIGURE 7. The appearance of near-periodic discontinuous chaos in the SICNN (7).

It is worth noting that the neural system (5)-(6)-(7) is designed according to the core mechanism shown in Figure 1. The connection topology of the neural system

(5)-(6)-(7) is represented in Figure 8, where one can see that the outputs of SICNN (5) are used as external inputs to the cells $\overline{C}_{ij}$ and $\widetilde{C}_{ij}$, $i, j = 1, 2, 3$, of both (6) and (7).

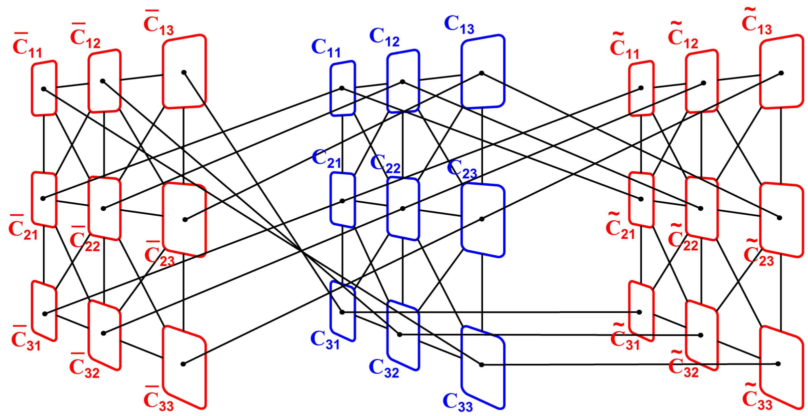

FIGURE 8. The connection topology of the neural system (5)-(6)-(7). The cells $\overline{C}_{ij}$ and $\widetilde{C}_{ij}$, $i, j = 1, 2, 3$, of (6) and (7) are influenced by the outputs of (5).

5. **Conclusions.**

In this study, SICNNs with fixed moments of impulses under the influence of chaotic external inputs are considered. The description of Li-Yorke chaos for the multidimensional dynamics of impulsive SICNNs is given. This is the first time in the literature that discontinuous Li-Yorke chaos is rigorously approved not only for SICNNs, but also in neuroscience. The presence of the ingredients of Li-Yorke chaos, proximality and frequent separation, are mathematically verified. The presented technique is appropriate for impulsive SICNNs with arbitrary number of cells.

Another novelty in the present paper is the consideration of the impacts with the cell and shunting principles. The advantage of the novelty is grounded to the arguments of the studies [26, 36, 37].

By the presented results, it is possible to obtain arbitrarily high dimensional neural systems by means of the core or chain mechanisms (see Figures 1 and 2) as well as their combinations. We illustrated the usefulness of our results by taking into account a neural system consisting of three SICNNs in Section 4. It is worth noting that the obtained chaos in the neural system (5)-(6)-(7) is controllable [11, 13, 16], and a way to control the chaos of the neural system is to stabilize an unstable periodic solution of the SICNN (5). For instance, the Ott-Grebogi-Yorke (OGY) [76] and Pyragas [83] control methods applied to the logistic map can be used for that purpose. The problem of period-doubling route to chaos [39, 86] and extension of intermittency [81] by impulsive SICNNs can also be considered through the presented method. Moreover, our approach can be useful for modeling secure communication systems [21, 54, 71, 72, 104].

The appearance of cyclic irregular behavior in neural systems was observed by Freeman and his collaborators [92, 108]. We numerically demonstrated the presence of near-periodic discontinuous chaotic motions of SICNNs. The obtained result can be useful for investigations of weakly coupled impulsive neural networks.

Acknowledgments. The authors wish to express their sincere gratitude to the referees for the helpful criticism and valuable suggestions, which helped to improve the paper significantly.

Appendix A. **Verification of the proximality and frequent separation features.** In this part of the paper, the proofs of Lemma 3.1 and Lemma 3.2 will be provided. We start with the proof of Lemma 3.1.

Proof of Lemma 3.1. Set $\alpha = \lambda - K\overline{c}(M + L_0H_0) - (p/T)\ln(1 + K\overline{d}(M + L_0H_0))$ and $R = 2K \max\limits_{(i,j)} \Big[\dfrac{MH_0 \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} + M_{ij}}{\lambda_{ij}} + \dfrac{pMH_0 \sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl}}{1 - e^{-\lambda_{ij}T}}\Big]$. The number α is positive by condition $(C6)$. Fix an arbitrary small positive number ϵ and an arbitrary large positive number E which satisfies the inequality $E \geq \dfrac{2}{\alpha}\ln\left(\dfrac{\gamma R[1 + K\overline{d}(M + L_0H_0)]^p}{\epsilon}\right)$, where γ is a number such that

$$\gamma \geq 1 + \frac{K}{\lambda} + \frac{K^2\overline{c}}{\lambda\alpha}(M + L_0H_0)[1 + K\overline{d}(M + L_0H_0)]^p$$
$$+\frac{K^2\overline{d}p}{\lambda(1 - e^{-\alpha T})}(M + L_0H_0)[1 + K\overline{d}(M + L_0H_0)]^p.$$

We assume without loss of generality that $\epsilon < \dfrac{R\gamma\lambda}{K}$. Since the pair $\big(L(t), \overline{L}(t)\big) \in \mathscr{L}\times\mathscr{L}$ is proximal, there exists an interval $J = [\sigma, \sigma + E_1]$ with $E_1 \geq E$ such that $\big\|L(t) - \overline{L}(t)\big\| < \epsilon/\gamma$ for all $t \in J$.

By means of the relations

$$\phi_L^{ij}(t) = -\int_{-\infty}^{t} u_{ij}(t,s)\Big[\sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} f(\phi_L^{hl}(s))\phi_L^{ij}(s) - L_{ij}(s)\Big]ds$$
$$+\sum_{-\infty<\theta_k<t} u_{ij}(t,\theta_k+)\Big[\sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl} g(\phi_L^{hl}(\theta_k))\phi_L^{ij}(\theta_k) + I_{ij}^k\Big]$$

and

$$\phi_{\overline{L}}^{ij}(t) = -\int_{-\infty}^{t} u_{ij}(t,s)\Big[\sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} f(\phi_{\overline{L}}^{hl}(s))\phi_{\overline{L}}^{ij}(s) - \overline{L}_{ij}(s)\Big]ds$$
$$+\sum_{-\infty<\theta_k<t} u_{ij}(t,\theta_k+)\Big[\sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl} g(\phi_{\overline{L}}^{hl}(\theta_k))\phi_{\overline{L}}^{ij}(\theta_k) + I_{ij}^k\Big],$$

one can obtain for $t \geq \sigma$ that

$$\phi_L^{ij}(t) - \phi_{\overline{L}}^{ij}(t) = -\int_{-\infty}^{\sigma} u_{ij}(t,s)\Big[\sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} f(\phi_L^{hl}(s))\phi_L^{ij}(s) - L_{ij}(s)$$
$$-\sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} f(\phi_{\overline{L}}^{hl}(s))\phi_{\overline{L}}^{ij}(s) + \overline{L}_{ij}(s)\Big]ds$$
$$-\int_{\sigma}^{t} u_{ij}(t,s)\Big[\sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} f(\phi_L^{hl}(s))\phi_L^{ij}(s) - L_{ij}(s)$$

$$
\begin{aligned}
&- \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} f(\phi_{\overline{L}}^{hl}(s))\phi_{\overline{L}}^{ij}(s) + \overline{L}_{ij}(s)\Big] ds \\
&+ \sum_{-\infty<\theta_k\le\sigma} u_{ij}(t,\theta_k+) \sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl}\Big[g(\phi_L^{hl}(\theta_k))\phi_L^{ij}(\theta_k) - g(\phi_{\overline{L}}^{hl}(\theta_k))\phi_{\overline{L}}^{ij}(\theta_k)\Big] \\
&+ \sum_{\sigma<\theta_k<t} u_{ij}(t,\theta_k+) \sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl}\Big[g(\phi_L^{hl}(\theta_k))\phi_L^{ij}(\theta_k) - g(\phi_{\overline{L}}^{hl}(\theta_k))\phi_{\overline{L}}^{ij}(\theta_k)\Big].
\end{aligned}
$$

If t belongs to the interval J, then making use of the inequalities

$$
\begin{aligned}
&\Big| -\int_{-\infty}^{\sigma} u_{ij}(t,s)\Big[\sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} f(\phi_L^{hl}(s))\phi_L^{ij}(s) - L_{ij}(s) \\
&- \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} f(\phi_{\overline{L}}^{hl}(s))\phi_{\overline{L}}^{ij}(s) + \overline{L}_{ij}(s)\Big] ds\Big| \\
&\le \int_{-\infty}^{\sigma} 2Ke^{-\lambda_{ij}(t-s)}\Big(MH_0 \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} + M_{ij}\Big) ds \\
&= \frac{2K(MH_0 \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} + M_{ij})}{\lambda_{ij}} e^{-\lambda_{ij}(t-\sigma)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big| \sum_{-\infty<\theta_k\le\sigma} u_{ij}(t,\theta_k+) \sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl}\Big[g(\phi_L^{hl}(\theta_k))\phi_L^{ij}(\theta_k) - g(\phi_{\overline{L}}^{hl}(\theta_k))\phi_{\overline{L}}^{ij}(\theta_k)\Big]\Big| \\
&\le \sum_{-\infty<\theta_k\le\sigma} 2KMH_0 e^{-\lambda_{ij}(t-\theta_k)} \sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl} \\
&\le 2pKMH_0 \frac{\sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl}}{1-e^{-\lambda_{ij}T}} e^{-\lambda_{ij}(t-\sigma)},
\end{aligned}
$$

we attain that

$$
\begin{aligned}
&\Big|\phi_L^{ij}(t) - \phi_{\overline{L}}^{ij}(t)\Big| \le 2K\Big[\frac{MH_0 \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl} + M_{ij}}{\lambda_{ij}} \\
&+ \frac{pMH_0 \sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl}}{1-e^{-\lambda_{ij}T}}\Big] e^{-\lambda_{ij}(t-\sigma)} \\
&+ \int_{\sigma}^{t} Ke^{-\lambda_{ij}(t-s)} \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl}\Big(\big|f(\phi_L^{hl}(s))\big|\Big|\phi_L^{ij}(s) - \phi_{\overline{L}}^{ij}(s)\Big| \\
&+ \Big|\phi_{\overline{L}}^{ij}(s)\Big|\big|f(\phi_L^{hl}(s)) - f(\phi_{\overline{L}}^{hl}(s))\big|\Big) ds \\
&+ \sum_{\sigma<\theta_k<t} Ke^{-\lambda_{ij}(t-\theta_k)} \sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl}\Big(\big|g(\phi_L^{hl}(\theta_k))\big|\Big|\phi_L^{ij}(\theta_k) - \phi_{\overline{L}}^{ij}(\theta_k)\Big| \\
&+ \Big|\phi_{\overline{L}}^{ij}(\theta_k)\Big|\big|g(\phi_L^{hl}(\theta_k)) - g(\phi_{\overline{L}}^{hl}(\theta_k))\big|\Big) \\
&+ \int_{\sigma}^{t} Ke^{-\lambda_{ij}(t-s)}\big|L_{ij}(s) - \overline{L}_{ij}(s)\big| ds.
\end{aligned}
$$

The last inequality implies for $t \in J$ that

$$
\|\phi_L(t) - \phi_{\overline{L}}(t)\| \le Re^{-\lambda(t-\sigma)} + \frac{K\epsilon}{\gamma\lambda}(1 - e^{-\lambda(t-\sigma)})
$$

$$+\int_{\sigma}^{t} K\overline{c}(M+L_0H_0)e^{-\lambda(t-s)}\left\|\phi_L(s)-\phi_{\overline{L}}(s)\right\|ds$$
$$+\sum_{\sigma<\theta_k<t} K\overline{d}(M+L_0H_0)e^{-\lambda(t-\theta_k)}\left\|\phi_L(\theta_k)-\phi_{\overline{L}}(\theta_k)\right\|.$$

Define the functions $u(t)=e^{\lambda t}\left\|\phi_L(t)-\phi_{\overline{L}}(t)\right\|$ and $\psi(t)=\left(R-\dfrac{K\epsilon}{\gamma\lambda}\right)e^{\lambda\sigma}+\dfrac{K\epsilon}{\gamma\lambda}e^{\lambda t}$. In this case, we have that

$$u(t)\le\psi(t)+\int_{\sigma}^{t} K\overline{c}(M+L_0H_0)u(s)ds+\sum_{\sigma<\theta_k<t} K\overline{d}(M+L_0H_0)u(\theta_k),\ t\in J.$$

The application of the Gronwall's Lemma for piecewise continuous functions to the last inequality yields

$$u(t)\le\psi(t)+\int_{\sigma}^{t} K\overline{c}(M+L_0H_0)[1+K\overline{d}(M+L_0H_0)]^{i((s,t))}$$
$$\times\psi(s)e^{K\overline{c}(M+L_0H_0)(t-s)}ds$$
$$+\sum_{\sigma<\theta_k<t} K\overline{d}(M+L_0H_0)[1+K\overline{d}(M+L_0H_0)]^{i((\theta_k,t))}\psi(\theta_k)e^{K\overline{c}(M+L_0H_0)(t-\theta_k)}.$$

By virtue of the equation

$$1+\int_{\sigma}^{t} K\overline{c}(M+L_0H_0)[1+K\overline{d}(M+L_0H_0)]^{i((s,t))}e^{K\overline{c}(M+L_0H_0)(t-s)}ds$$
$$+\sum_{\sigma<\theta_k<t} K\overline{d}(M+L_0H_0)[1+K\overline{d}(M+L_0H_0)]^{i((\theta_k,t))}e^{K\overline{c}(M+L_0H_0)(t-\theta_k)}$$
$$=[1+K\overline{d}(M+L_0H_0)]^{i((\sigma,t))}e^{K\overline{c}(M+L_0H_0)(t-\sigma)},$$

one can obtain that

$$u(t)\le[1+K\overline{d}(M+L_0H_0)]^{i((\sigma,t))}e^{K\overline{c}(M+L_0H_0)(t-\sigma)}\left(R-\frac{K\epsilon}{\gamma\lambda}\right)e^{\lambda\sigma}+\frac{K\epsilon}{\gamma\lambda}e^{\lambda t}$$
$$+K\overline{c}(M+L_0H_0)\frac{K\epsilon}{\gamma\lambda}\int_{\sigma}^{t} e^{\lambda s}[1+K\overline{d}(M+L_0H_0)]^{i((s,t))}e^{K\overline{c}(M+L_0H_0)(t-s)}ds$$
$$+K\overline{d}(M+L_0H_0)\frac{K\epsilon}{\gamma\lambda}\sum_{\sigma<\theta_k<t} e^{\lambda\theta_k}[1+K\overline{d}(M+L_0H_0)]^{i((\theta_k,t))}$$
$$\times e^{K\overline{c}(M+L_0H_0)(t-\theta_k)}.$$

Since the inequality

$$[1+K\overline{d}(M+L_0H_0)]^{i((t_1,t_2))}e^{K\overline{c}(M+L_0H_0)(t_2-t_1)}\le[1+K\overline{d}(M+L_0H_0)]^{p}e^{(\lambda-\alpha)(t_2-t_1)}$$

holds for all real numbers t_1 and t_2 with $t_1<t_2$, we have for $t\in J$ that

$$u(t)\le R[1+K\overline{d}(M+L_0H_0)]^{p}e^{\lambda t}e^{-\alpha(t-\sigma)}$$
$$+\frac{K\epsilon}{\gamma\lambda}e^{\lambda t}\Big[1-(1+K\overline{d}(M+L_0H_0))^{p}e^{-\alpha(t-\sigma)}\Big]$$
$$+\frac{K^2\overline{c}\epsilon}{\gamma\lambda\alpha}(M+L_0H_0)[1+K\overline{d}(M+L_0H_0)]^{p}\Big(1-e^{-\alpha(t-\sigma)}\Big)e^{\lambda t}$$

$$+\frac{K^2\overline{d}p\epsilon}{\gamma\lambda(1-e^{-\alpha T})}(M+L_0H_0)[1+K\overline{d}(M+L_0H_0)]^p\Big(1-e^{-\alpha(t-\sigma+T)}\Big)e^{\lambda t}.$$

If we multiply both sides of the last inequality by $e^{-\lambda t}$, then we get

$$\begin{aligned}&\|\phi_L(t)-\phi_{\overline{L}}(t)\|\leq R[1+K\overline{d}(M+L_0H_0)]^p e^{-\alpha(t-\sigma)}\\&+\frac{K\epsilon}{\gamma\lambda}\Big[1-(1+K\overline{d}(M+L_0H_0))^p e^{-\alpha(t-\sigma)}\Big]\\&+\frac{K^2\overline{c}\epsilon}{\gamma\lambda\alpha}(M+L_0H_0)[1+K\overline{d}(M+L_0H_0)]^p\Big(1-e^{-\alpha(t-\sigma)}\Big)\\&+\frac{K^2\overline{d}p\epsilon}{\gamma\lambda(1-e^{-\alpha T})}(M+L_0H_0)[1+K\overline{d}(M+L_0H_0)]^p\Big(1-e^{-\alpha(t-\sigma+T)}\Big).\end{aligned}$$

Thus, one can obtain for $t\in J$ that

$$\begin{aligned}&\|\phi_L(t)-\phi_{\overline{L}}(t)\|< R[1+K\overline{d}(M+L_0H_0)]^p e^{-\alpha(t-\sigma)}\\&+\frac{K\epsilon}{\gamma\lambda}\Big[1+\frac{K\overline{c}}{\alpha}(M+L_0H_0)(1+K\overline{d}(M+L_0H_0))^p\\&+\frac{K\overline{d}p}{1-e^{-\alpha T}}(M+L_0H_0)(1+K\overline{d}(M+L_0H_0))^p\Big].\end{aligned}$$

Since the number E is sufficiently large such that

$$E\geq\frac{2}{\alpha}\ln\left(\frac{\gamma R[1+K\overline{d}(M+L_0H_0)]^p}{\epsilon}\right),$$

we have that

$$R[1+K\overline{d}(M+L_0H_0)]^p e^{-\alpha(t-\sigma)}\leq\frac{\epsilon}{\gamma},\ \ t\in[\sigma+E/2,\sigma+E_1].$$

Hence, the inequality

$$\begin{aligned}&\|\phi_L(t)-\phi_{\overline{L}}(t)\|<\frac{\epsilon}{\gamma}\Big[1+\frac{K}{\lambda}+\frac{K^2\overline{c}}{\lambda\alpha}(M+L_0H_0)(1+K\overline{d}(M+L_0H_0))^p\\&+\frac{K^2\overline{d}p}{\lambda(1-e^{-\alpha T})}(M+L_0H_0)(1+K\overline{d}(M+L_0H_0))^p\Big]\\&\leq\epsilon\end{aligned}$$

holds for all $t\in J^1$, where $J^1=[\sigma+E/2,\sigma+E_1]$. We note that the interval J^1 has a length no less than $E/2$. Consequently, the couple $(\phi_L(t),\phi_{\overline{L}}(t))\in\mathscr{A}\times\mathscr{A}$ is proximal. □

The proof of Lemma 3.2 is as follows.

Proof of Lemma 3.2. Because the couple $\big(L(t),\overline{L}(t)\big)\in\mathscr{L}\times\mathscr{L}$ is frequently (ϵ_0,Δ) separated for some $\epsilon_0>0$ and $\Delta>0$, there exist infinitely many disjoint intervals $J_q, q\in\mathbb{N}$, each with a length no less than Δ, such that $\big\|L(t)-\overline{L}(t)\big\|>\epsilon_0$ for each t from these intervals. In the proof, we will verify the existence of numbers $\epsilon_1>0$, $\overline{\Delta}>0$ and infinitely many disjoint intervals $J_q^1\subset J_q, q\in\mathbb{N}$, each with length $\overline{\Delta}$, such that the inequality $\|\phi_L(t)-\phi_{\overline{L}}(t)\|>\epsilon_1$ holds for each t from the intervals $J_q^1, q\in\mathbb{N}$.

According to the equicontinuity of $\mathscr{L}$, one can find a positive number $\tau<\Delta$, such that for any $t_1,t_2\in\mathbb{R}$ with $|t_1-t_2|<\tau$, the inequality

$$\big|\big(L_{ij}(t_1)-\overline{L}_{ij}(t_1)\big)-\big(L_{ij}(t_2)-\overline{L}_{ij}(t_2)\big)\big|<\frac{\epsilon_0}{2}\tag{8}$$

holds for all $1 \leq i \leq m, 1 \leq j \leq n$.

Suppose that for each $q \in \mathbb{N}$, the number s_q denotes the midpoint of the interval J_q. Let us define a sequence $\{\kappa_q\}$ through the equation $\kappa_q = s_q - \tau/2$.

Let us fix an arbitrary $q \in \mathbb{N}$. One can find integers i_0, j_0, such that

$$\left|L_{i_0j_0}(s_q) - \overline{L}_{i_0j_0}(s_q)\right| = \left\|L(s_q) - \overline{L}(s_q)\right\| > \epsilon_0. \tag{9}$$

Making use of the inequality (8), for all $t \in [\kappa_q, \kappa_q + \tau]$ we have

$$\begin{aligned}
&\left|L_{i_0j_0}(s_q) - \overline{L}_{i_0j_0}(s_q)\right| - \left|L_{i_0j_0}(t) - \overline{L}_{i_0j_0}(t)\right| \\
&\leq \left|\left(L_{i_0j_0}(t) - \overline{L}_{i_0j_0}(t)\right) - \left(L_{i_0j_0}(s_q) - \overline{L}_{i_0j_0}(s_q)\right)\right| \\
&< \frac{\epsilon_0}{2},
\end{aligned}$$

and therefore, by means of (9), we attain that the inequality

$$\left|L_{i_0j_0}(t) - \overline{L}_{i_0j_0}(t)\right| > \left|L_{i_0j_0}(s_q) - \overline{L}_{i_0j_0}(s_q)\right| - \frac{\epsilon_0}{2} > \frac{\epsilon_0}{2} \tag{10}$$

is valid for all $t \in [\kappa_q, \kappa_q + \tau]$.

For each i and j, one can find numbers $\zeta^q_{ij} \in [\kappa_q, \kappa_q + \tau]$ such that

$$\int_{\kappa_q}^{\kappa_q+\tau} \left(L(s) - \overline{L}(s)\right) ds = \tau \left(L_{11}(\zeta^q_{11}) - \overline{L}_{11}(\zeta^q_{11}), \ldots, L_{mn}(\zeta^q_{mn}) - \overline{L}_{mn}(\zeta^q_{mn})\right).$$

Thus, according to the inequality (10), we have that

$$\left\|\int_{\kappa_q}^{\kappa_q+\tau} \left(L(s) - \overline{L}(s)\right) ds\right\| \geq \tau \left|L_{i_0j_0}(\zeta^q_{i_0j_0}) - \overline{L}_{i_0j_0}(\zeta^q_{i_0j_0})\right| > \frac{\tau\epsilon_0}{2}. \tag{11}$$

Making use of the relations

$$\begin{aligned}
\phi^{ij}_L(t) = \phi^{ij}_L(\kappa_q) &- \int_{\kappa_q}^{t} \Big(a_{ij} + \sum_{C_{hl}\in N_r(i,j)} C^{hl}_{ij} f(\phi^{hl}_L(s))\Big)\phi^{ij}_L(s)ds + \int_{\kappa_q}^{t} L_{ij}(s)ds \\
&+ \sum_{\kappa_q \leq \theta_k < t} \Big(b_{ij} + \sum_{C_{hl}\in N_r(i,j)} D^{hl}_{ij} g(\phi^{hl}_L(\theta_k))\Big)\phi^{ij}_L(\theta_k) + \sum_{\kappa_q \leq \theta_k < t} I^k_{ij}
\end{aligned}$$

and

$$\begin{aligned}
\phi^{ij}_{\overline{L}}(t) = \phi^{ij}_{\overline{L}}(\kappa_q) &- \int_{\kappa_q}^{t} \Big(a_{ij} + \sum_{C_{hl}\in N_r(i,j)} C^{hl}_{ij} f(\phi^{hl}_{\overline{L}}(s))\Big)\phi^{ij}_{\overline{L}}(s)ds + \int_{\kappa_q}^{t} \overline{L}_{ij}(s)ds \\
&+ \sum_{\kappa_q \leq \theta_k < t} \Big(b_{ij} + \sum_{C_{hl}\in N_r(i,j)} D^{hl}_{ij} g(\phi^{hl}_{\overline{L}}(\theta_k))\Big)\phi^{ij}_{\overline{L}}(\theta_k) + \sum_{\kappa_q \leq \theta_k < t} I^k_{ij}
\end{aligned}$$

we obtain that

$$\begin{aligned}
\left|\phi^{ij}_L(\kappa_q + \tau) - \phi^{ij}_{\overline{L}}(\kappa_q + \tau)\right| &\geq \left|\int_{\kappa_q}^{\kappa_q+\tau} \left(L_{ij}(s) - \overline{L}_{ij}(s)\right) ds\right| \\
&- \left|\phi^{ij}_L(\kappa_q) - \phi^{ij}_{\overline{L}}(\kappa_q)\right| - \int_{\kappa_q}^{\kappa_q+\tau} a_{ij}\left|\phi^{ij}_L(s) - \phi^{ij}_{\overline{L}}(s)\right| ds
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{\kappa_q\le\theta_k<\kappa_q+\tau}|b_{ij}|\left|\phi_L^{ij}(\theta_k)-\phi_{\overline{L}}^{ij}(\theta_k)\right|\\
&-\int_{\kappa_q}^{\kappa_q+\tau}\sum_{C_{hl}\in N_r(i,j)}C_{ij}^{hl}\left|f(\phi_L^{hl}(s))\phi_L^{ij}(s)-f(\phi_{\overline{L}}^{hl}(s))\phi_{\overline{L}}^{ij}(s)\right|ds\\
&-\sum_{\kappa_q\le\theta_k<\kappa_q+\tau}\sum_{C_{hl}\in N_r(i,j)}D_{ij}^{hl}\left|g(\phi_L^{hl}(\theta_k))\phi_L^{ij}(\theta_k)-g(\phi_{\overline{L}}^{hl}(\theta_k))\phi_{\overline{L}}^{ij}(\theta_k)\right|\\
&\ge\left|\int_{\kappa_q}^{\kappa_q+\tau}\left(L_{ij}(s)-\overline{L}_{ij}(s)\right)ds\right|-\sup_{t\in[\kappa_q,\kappa_q+\tau]}\|\phi_L(t)-\phi_{\overline{L}}(t)\|\\
&-\tau a_{ij}\sup_{t\in[\kappa_q,\kappa_q+\tau]}\|\phi_L(t)-\phi_{\overline{L}}(t)\|-\frac{p}{T}(T+\tau)|b_{ij}|\sup_{t\in[\kappa_q,\kappa_q+\tau]}\|\phi_L(t)-\phi_{\overline{L}}(t)\|\\
&-\tau(M+L_0H_0)\sum_{C_{hl}\in N_r(i,j)}C_{ij}^{hl}\sup_{t\in[\kappa_q,\kappa_q+\tau]}\|\phi_L(t)-\phi_{\overline{L}}(t)\|\\
&-\frac{p}{T}(T+\tau)(M+L_0H_0)\sum_{C_{hl}\in N_r(i,j)}D_{ij}^{hl}\sup_{t\in[\kappa_q,\kappa_q+\tau]}\|\phi_L(t)-\phi_{\overline{L}}(t)\|\,.
\end{aligned}
$$

By means of the inequality (11), one can show that

$$
\begin{aligned}
&\sup_{t\in[\kappa_q,\kappa_q+\tau]}\|\phi_L(t)-\phi_{\overline{L}}(t)\|\ge\|\phi_L(\kappa_q+\tau)-\phi_{\overline{L}}(\kappa_q+\tau)\|\\
&>\frac{\tau\epsilon_0}{2}-(1+P_0)\sup_{t\in[\kappa_q,\kappa_q+\tau]}\|\phi_L(t)-\phi_{\overline{L}}(t)\|\,,
\end{aligned}
$$

where

$$
\begin{aligned}
P_0=\max_{(i,j)}\Big[&\tau a_{ij}+\tau(M+L_0H_0)\sum_{C_{hl}\in N_r(i,j)}C_{ij}^{hl}+\frac{p}{T}(T+\tau)|b_{ij}|\\
&+(M+L_0H_0)\frac{p}{T}(T+\tau)\sum_{C_{hl}\in N_r(i,j)}D_{ij}^{hl}\Big].
\end{aligned}
$$

Hence, we have that $\sup_{t\in[\kappa_q,\kappa_q+\tau]}\|\phi_L(t)-\phi_{\overline{L}}(t)\|>M_0$, where $M_0=\dfrac{\tau\epsilon_0}{2\,(2+P_0)}$.

Set $M_1=2\max_{(i,j)}\Big(H_0a_{ij}+MH_0\sum_{C_{hl}\in N_r(i,j)}C_{ij}^{hl}+M_{ij}\Big)$, $b_1=\max_{(i,j)}\Big(|b_{ij}|+(M+H_0L_0)\sum_{C_{hl}\in N_r(i,j)}D_{ij}^{hl}\Big)$ and $\underline{\theta}=\min_{1\le k\le p}(\theta_{k+1}-\theta_k)$. Define the numbers

$$
\epsilon_1=\frac{M_0}{2}\min\Big\{b_0-L_0H_0\overline{d},\frac{1}{1+b_1}\Big\}
$$

and

$$
\overline{\Delta}=\min\Big\{\underline{\theta},\frac{M_0}{2M_1(2+b_1)},\frac{M_0\left(b_0-L_0H_0\overline{d}\right)}{2M_1(1+b_0-L_0H_0\overline{d})}\Big\}.
$$

Suppose that there exists $\xi_q\in[\kappa_q,\kappa_q+\tau]$ such that $\sup_{t\in[\kappa_q,\kappa_q+\tau]}\|\phi_L(t)-\phi_{\overline{L}}(t)\|=\|\phi_L(\xi_q)-\phi_{\overline{L}}(\xi_q)\|\,.$

Let $\kappa_q^1=\begin{cases}\xi_q, & \text{if } \xi_q\le\kappa_q+\tau/2\\ \xi_q-\overline{\Delta}, & \text{if } \xi_q>\kappa_q+\tau/2\end{cases}$. Since $\overline{\Delta}\le\underline{\theta}$, there exists at most one impulsive moment on the interval $(\kappa_q^1,\kappa_q^1+\overline{\Delta})$.

We shall start by considering the case $\xi_q > \kappa_q + \dfrac{\tau}{2}$. Assume that there exists an impulsive moment $\theta_{k_0} \in (\kappa_q^1, \kappa_q^1 + \overline{\Delta})$. For $t \in (\theta_{k_0}, \kappa_q^1 + \overline{\Delta})$, making use of the equation

$$\phi_L^{ij}(t) - \phi_{\overline{L}}^{ij}(t) = \Big(\phi_L^{ij}(\xi_q) - \phi_{\overline{L}}^{ij}(\xi_q)\Big) - \int_{\xi_q}^t a_{ij}\left(\phi_L^{ij}(s) - \phi_{\overline{L}}^{ij}(s)\right) ds$$
$$- \int_{\xi_q}^t \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl}\left(f(\phi_L^{hl}(s))\phi_L^{ij}(s) - f(\phi_{\overline{L}}^{hl}(s))\phi_{\overline{L}}^{ij}(s)\right) ds$$
$$+ \int_{\xi_q}^t \left(L_{ij}(s) - \overline{L}_{ij}(s)\right) ds,$$

one can verify that $\|\phi_L(t) - \phi_{\overline{L}}(t)\| > M_0 - \overline{\Delta} M_1 > \dfrac{M_0}{2} > \epsilon_1$. In particular, we have that

$$\|\phi_L(\theta_{k_0}+) - \phi_{\overline{L}}(\theta_{k_0}+)\| > M_0 - \overline{\Delta} M_1.$$

Because the inequality

$$\left|\phi_L^{ij}(\theta_{k_0}+) - \phi_{\overline{L}}^{ij}(\theta_{k_0}+)\right| \le |1+b_{ij}| \left|\phi_L^{ij}(\theta_{k_0}) - \phi_{\overline{L}}^{ij}(\theta_{k_0})\right|$$
$$+M \sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl}\left|\phi_L^{ij}(\theta_{k_0}) - \phi_{\overline{L}}^{ij}(\theta_{k_0})\right|$$
$$+H_0 L_0 \sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl}\left|\phi_L^{hl}(\theta_{k_0}) - \phi_{\overline{L}}^{hl}(\theta_{k_0})\right|$$

is valid for each i and j, it is easy to obtain that

$$\|\phi_L(\theta_{k_0}) - \phi_{\overline{L}}(\theta_{k_0})\| \ge \frac{\|\phi_L(\theta_{k_0}+) - \phi_{\overline{L}}(\theta_{k_0}+)\|}{1+b_1} > \frac{M_0 - \overline{\Delta} M_1}{1+b_1}.$$

For $t \in (\kappa_q^1, \theta_{k_0})$, the relation

$$\phi_L^{ij}(t) - \phi_{\overline{L}}^{ij}(t) = \Big(\phi_L^{ij}(\theta_{k_0}) - \phi_{\overline{L}}^{ij}(\theta_{k_0})\Big) - \int_{\theta_{k_0}}^t a_{ij}\left(\phi_L^{ij}(s) - \phi_{\overline{L}}^{ij}(s)\right) ds$$
$$- \int_{\theta_{k_0}}^t \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl}\left(f(\phi_L^{hl}(s))\phi_L^{ij}(s) - f(\phi_{\overline{L}}^{hl}(s))\phi_{\overline{L}}^{ij}(s)\right) ds$$
$$+ \int_{\theta_{k_0}}^t \left(L_{ij}(s) - \overline{L}_{ij}(s)\right) ds$$

implies that

$$\|\phi_L(t) - \phi_{\overline{L}}(t)\| > \frac{M_0 - \overline{\Delta} M_1}{1+b_1} - \overline{\Delta} M_1 \ge \frac{M_0}{2(1+b_1)} \ge \epsilon_1.$$

On the other hand, if none of the impulsive moments belong to $(\kappa_q^1, \kappa_q^1 + \overline{\Delta})$, then for each t from this interval we have that $\|\phi_L(t) - \phi_{\overline{L}}(t)\| > M_0 - \overline{\Delta} M_1 > \epsilon_1$. Therefore, in the case of $\xi_q > \kappa_q + \tau/2$, the inequality $\|\phi_L(t) - \phi_{\overline{L}}(t)\| > \epsilon_1$ holds for all $t \in (\kappa_q^1, \kappa_q^1 + \overline{\Delta})$, regardless of the existence of an impulsive moment in this interval.

Next, we consider the case $\xi_q \le \kappa_q + \dfrac{\tau}{2}$. If there exists an impulsive moment $\theta_{k_0} \in (\kappa_q^1, \kappa_q^1 + \overline{\Delta})$, then one can use a similar evaluation as in the case discussed above to show for $t \in (\kappa_q^1, \theta_{k_0}]$ that $\|\phi_L(t) - \phi_{\overline{L}}(t)\| > M_0 - \overline{\Delta} M_1 > \epsilon_1$. Moreover, the inequality

$$\left|\phi_L^{ij}(\theta_{k_0}+) - \phi_{\overline{L}}^{ij}(\theta_{k_0}+)\right| \geq \Big(|1+b_{ij}| - M \sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl}\Big) \left|\phi_L^{ij}(\theta_{k_0}) - \phi_{\overline{L}}^{ij}(\theta_{k_0})\right|$$
$$- \sum_{C_{hl}\in N_r(i,j)} D_{ij}^{hl} L_0 H_0 \left|\phi_L^{hl}(\theta_{k_0}) - \phi_{\overline{L}}^{hl}(\theta_{k_0})\right|$$

yields

$$\|\phi_L(\theta_{k_0}+) - \phi_{\overline{L}}(\theta_{k_0}+)\| \geq \big(b_0 - L_0 H_0 \overline{d}\big) \|\phi_L(\theta_{k_0}) - \phi_{\overline{L}}(\theta_{k_0})\|$$
$$> \big(b_0 - L_0 H_0 \overline{d}\big)(M_0 - \overline{\Delta} M_1).$$

Thus, for $t \in (\theta_{k_0}, \kappa_q^1 + \overline{\Delta})$, the relation

$$\phi_L^{ij}(t) - \phi_{\overline{L}}^{ij}(t) = \Big(\phi_L^{ij}(\theta_{k_0}+) - \phi_{\overline{L}}^{ij}(\theta_{k_0}+)\Big) - \int_{\theta_{k_0}}^t a_{ij}\Big(\phi_L^{ij}(s) - \phi_{\overline{L}}^{ij}(s)\Big)\,ds$$
$$- \int_{\theta_{k_0}}^t \sum_{C_{hl}\in N_r(i,j)} C_{ij}^{hl}\Big(f(\phi_L^{hl}(s))\phi_L^{ij}(s) - f(\phi_{\overline{L}}^{kl}(s))\phi_{\overline{L}}^{ij}(s)\Big)\,ds$$
$$+ \int_{\theta_{k_0}}^t \big(L_{ij}(s) - \overline{L}_{ij}(s)\big)\,ds$$

implies that

$$\|\phi_L(t) - \phi_{\overline{L}}(t)\| > \big(b_0 - L_0 H_0 \overline{d}\big)(M_0 - \overline{\Delta} M_1) - \overline{\Delta} M_1 \geq \frac{\big(b_0 - L_0 H_0 \overline{d}\big) M_0}{2} \geq \epsilon_1.$$

Therefore, for all $t \in (\kappa_q^1, \kappa_q^1 + \overline{\Delta})$ we have that $\|\phi_L(t) - \phi_{\overline{L}}(t)\| > \epsilon_1$. One can also show that the same inequality holds even if no impulsive moments exist inside the interval $(\kappa_q^1, \kappa_q^1 + \overline{\Delta})$.

Now, suppose that there exists an impulsive moment $\theta_{l_0} \in [\kappa_q, \kappa_q + \tau]$ such that

$$\sup_{t\in[\kappa_q, \kappa_q+\tau]} \|\phi_L(t) - \phi_{\overline{L}}(t)\| = \|\phi_L(\theta_{l_0}+) - \phi_{\overline{L}}(\theta_{l_0}+)\|.$$

Let us define $\kappa_q^1 = \begin{cases} \theta_{l_0}, & \text{if } \theta_{l_0} \leq \kappa_q + \tau/2 \\ \theta_{l_0} - \overline{\Delta}, & \text{if } \theta_{l_0} > \kappa_q + \tau/2 \end{cases}$. It is worth noting that the interval $(\kappa_q^1, \kappa_q^1 + \overline{\Delta})$ does not contain any impulsive moments. If $\theta_{l_0} > \kappa_q + \dfrac{\tau}{2}$, then using the inequality

$$\|\phi_L(\theta_{l_0}) - \phi_{\overline{L}}(\theta_{l_0})\| \geq \frac{\|\phi_L(\theta_{l_0}+) - \phi_{\overline{L}}(\theta_{l_0}+)\|}{1+b_1} > \frac{M_0}{1+b_1},$$

one can verify for $t \in (\kappa_q^1, \kappa_q^1 + \overline{\Delta})$ that

$$\|\phi_L(t) - \phi_{\overline{L}}(t)\| \geq \|\phi_L(\theta_{l_0}) - \phi_{\overline{L}}(\theta_{l_0})\| - \overline{\Delta} M_1 > \frac{M_0}{1+b_1} - \overline{\Delta} M_1 > \frac{M_0}{2(1+b_1)} \geq \epsilon_1.$$

In a similar way, if $\theta_{l_0} \leq \kappa_q + \dfrac{\tau}{2}$, then we have for $t \in (\kappa_q^1, \kappa_q^1 + \overline{\Delta})$ that

$$\|\phi_L(t) - \phi_{\overline{L}}(t)\| \geq \|\phi_L(\theta_{l_0}+) - \phi_{\overline{L}}(\theta_{l_0}+)\| - \overline{\Delta} M_1 > M_0 - \overline{\Delta} M_1 > \frac{M_0}{2} > \epsilon_1.$$

Hence, on each of the intervals $J_q^1 = (\kappa_q^1, \kappa_q^1 + \overline{\Delta})$, $q \in \mathbb{N}$, the inequality

$$\|\phi_L(t) - \phi_{\overline{L}}(t)\| > \epsilon_1$$

holds. Consequently, the couple of functions $(\phi_L(t), \phi_{\overline{L}}(t)) \in \mathscr{A} \times \mathscr{A}$ is frequently $(\epsilon_1, \overline{\Delta})-$separated. □

REFERENCES

[1] M. Abeles, *Neural Codes for Higher Brain Functions*, in Information Processing by the Brain (ed. H.J. Markowitsch), Toronto, Hans Huber Publishers, 1988.

[2] S. Ahmad and I. M. Stamova, Global exponential stability for impulsive cellular neural networks with time-varying delays, *Nonlinear Anal.: Theory Methods Appl.,* **69** (2008), 786–795.

[3] K. Aihara and G. Matsumoto, Chaotic oscillations and bifurcations in squid giant axons, in *Chaos* (ed. A.V. Holden), Manchester, UK, Manchester University Press, (1986), 257–269.

[4] K. Aihara, T. Takebe and M. Toyoda, Chaotic neural networks, *Phys. Lett. A,* **144** (1990), 333–340.

[5] M. U. Akhmet, Creating a chaos in a system with relay, *Int. J. Qual. Theory Differ. Equat. Appl.,* **3** (2009), 3–7.

[6] M. U. Akhmet, Devaney's chaos of a relay system, *Commun. Nonlinear Sci. Numer. Simulat.,* **14** (2009), 1486–1493.

[7] M. U. Akhmet, Li-Yorke chaos in the system with impacts, *J. Math. Anal. Appl.,* **351** (2009), 804–810.

[8] M. U. Akhmet, Homoclinical structure of the chaotic attractor, *Commun. Nonlinear Sci. Numer. Simulat.,* **15** (2010), 819–822.

[9] M. Akhmet, *Principles of Discontinuous Dynamical Systems*, New York, Springer, 2010.

[10] M. Akhmet, *Nonlinear Hybrid Continuous/Discrete-Time Models*, Atlantis Press, Paris, 2011.

[11] M. U. Akhmet and M. O. Fen, Chaotic period-doubling and OGY control for the forced Duffing equation, *Commun. Nonlinear. Sci. Numer. Simulat.,* **17** (2012), 1929–1946.

[12] M. U. Akhmet and M. O. Fen, Shunting inhibitory cellular neural networks with chaotic external inputs, *Chaos,* **23** (2013), 023112, 9pp.

[13] M. U. Akhmet and M. O. Fen, Replication of chaos, *Commun. Nonlinear Sci. Numer. Simulat.,* **18** (2013), 2626–2666.

[14] M. Akhmet and E. Yılmaz, *Neural Networks with Discontinuous/Impact Activations*, Springer, New York, 2014.

[15] M. U. Akhmet and M. O. Fen, Entrainment by chaos, *J. Nonlinear Sci.,* **24** (2014), 411–439.

[16] M. Akhmet and M. O. Fen, Chaotification of impulsive systems by perturbations, *Int. J. Bifurcation and Chaos,* **24** (2014), 1450078, 16pp.

[17] M. Akhmet and M. O. Fen, Generation of cyclic/toroidal chaos by Hopfield neural networks, *Neurocomputing,* **145** (2014), 230–239.

[18] M. Akhmet and M. O. Fen, *Replication of Chaos in Neural Networks, Economics and Physics*, Springer-Verlag, Berlin, Heidelberg, 2016.

[19] E. Akin and S. Kolyada, Li-Yorke sensitivity, *Nonlinearity,* **16** (2003), 1421–1433.

[20] P. Balasubramaniam, M. Kalpana and R. Rakkiyappan, Stationary oscillation of interval fuzzy cellular neural networks with mixed delays under impulsive perturbations, *Neural Comput. & Applic.,* **22** (2013), 1645–1654.

[21] P. Balasubramaniam and P. Muthukumar, Synchronization of chaotic systems using feedback controller: An application to Diffie-Hellman key exchange protocol and ElGamal public key cryptosystem, *J. Egyptian Math. Soc.,* **22** (2014), 365–372.

[22] R. Barrio, M. A. Martinez, S. Serrano and A. Shilnikov, Macro- and micro-chaotic structures in the Hindmarsh-Rose model of bursting neurons, *Chaos,* **24** (2014), 023128, 11pp.

[23] A. Bouzerdoum and R. B. Pinter, A shunting inhibitory motion detector that can account for the functional characteristics of fly motion-sensitive interneurons, in *Proceedings of IJCNN International Joint Conference on Neural Networks,* (1990), 149–153.

[24] A. Bouzerdoum, B. Nabet and R. B. Pinter, Analysis and analog implementation of directionally sensitive shunting inhibitory cellular neural networks, in *Visual Information Processing: From Neurons to Chips,* Proc. SPIE, **1473** (1991), 29–38.

[25] A. Bouzerdoum and R. B. Pinter, Nonlinear lateral inhibition applied to motion detection in the fly visual system, *Nonlinear Vision* (ed. R.B. Pinter, B. Nabet), Boca Raton, CRC Press, (1992), 423–450.

[26] A. Bouzerdoum and R. B. Pinter, Shunting inhibitory cellular neural networks: Derivation and stability analysis, *IEEE Trans. Circuits Systems-I: Fund. Theory and Appl.,* **40** (1993), 215–221.
[27] M. Cai and W. Xiong, Almost periodic solutions for shunting inhibitory cellular neural networks without global Lipschitz and bounded activation functions, *Phys. Lett. A,* **362** (2007), 417–423.
[28] J. Cao, Global stability conditions for delayed CNNs, *IEEE Trans. Circuits Syst.-I: Fund. Theory Appl.,* **48** (2001), 1330–1333.
[29] J. Cao and J. Lu, Adaptive synchronization of neural networks with or without time-varying delay, *Chaos,* **16** (2006), 013133, 6pp.
[30] J. Cao, D. W. C. Ho and X. Huang, LMI-based criteria for global robust stability of bidirectional associative memory networks with time delay, *Nonlinear Anal.,* **66** (2007), 1558–1572.
[31] R. Caponetto, M. Lavorgna and L. Occhipinti, Cellular neural networks in secure transmission applications, in *Proceedings of CNNA96: Fourth IEEE International Workshop on Cellular Neural Networks and Their Applications,* Seville, Spain, (1996), 411–416.
[32] G. A. Carpenter and S. Grossberg, The ART of adaptive pattern recognition by a self-organizing neural network, *Computer,* **21** (1988), 77–88.
[33] A. Chen and J. Cao, Almost periodic solution of shunting inhibitory CNNs with delays, *Phys. Lett. A,* **298** (2002), 161–170.
[34] L. Chen and H. Zhao, Global stability of almost periodic solution of shunting inhibitory cellular neural networks with variable coefficients, *Chaos Soliton. Fract.,* **35** (2008), 351–357.
[35] H. N. Cheung, A. Bouzerdoum and W. Newland, Properties of shunting inhibitory cellular neural networks for colour image enhancement, in *Proceedings of 6th International Conference on Neural Information Processing,* Perth, Western Australia, (1999), 1219–1223.
[36] L. O. Chua and L. Yang, Cellular neural networks: Theory, *IEEE Trans. Circuits Syst.,* **35** (1988), 1257–1272.
[37] L. O. Chua and L. Yang, Cellular neural networks: Applications, *IEEE Trans. Circuits Syst.,* **35** (1988), 1273–1290.
[38] H.-S. Ding, J. Liang and T.-J. Xiao, Existence of almost periodic solutions for SICNNs with time-varying delays, *Phys. Lett. A,* **372** (2008), 5411–5416.
[39] M. J. Feigenbaum, Universal behavior in nonlinear systems, *Los Alamos Sci.,* **1** (1980), 4–27.
[40] W. J. Freeman, Tutorial on neurobiology: From single neurons to brain chaos, *Int. J. Bifurcation and Chaos,* **2** (1992), 451–482.
[41] K. Fukushima, Analysis of the process of visual pattern recognition by the neocognitron, *Neural Netw.*, **2** (1989), 413–420.
[42] W. Gerstner and W. M. Kistler, *Spiking Neuron Models: Single Neurons, Populations, Plasticity*, Cambridge University Press, United Kingdom, 2002.
[43] J. Guckenheimer and R. A. Oliva, Chaos in the Hodgkin-Huxley Model, *SIAM J. Appl. Dyn. Syst.,* **1** (2002), 105–114.
[44] Z. Gui and W. Ge, Periodic solution and chaotic strange attractor for shunting inhibitory cellular neural networks with impulses, *Chaos,* **16** (2006), 033116, 10pp.
[45] J. Hale and H. Koçak, *Dynamics and Bifurcations*, Springer-Verlag, New York, 1991.
[46] W. Hu, C. Li and S. Wu, Stochastic robust stability for neutral-type impulsive interval neural networks with distributed time-varying delays, *Neural Comput. Appl.,* **21** (2012), 1947–1960.
[47] X. Huang and J. Cao, Almost periodic solution of shunting inhibitory cellular neural networks with time-varying delay, *Phys. Lett. A,* **314** (2003), 222–231.
[48] E. M. Izhikevich, Weakly connected quasi-periodic oscillators, FM interactions, and multiplexing in the brain, *SIAM J. Appl. Math.,* **59** (1999), 2193–2223.
[49] S. Jankowski, A. Londei, C. Mazur and A. Lozowski, Synchronization phenomena in 2D chaotic CNN, in Proceedings of CNNA-94 Third IEEE International Workshop on Cellular Neural Networks and Their Applications, *Rome, Italy,* (1994), 339–344.
[50] M. E. Jernigan and G. F. McLean, Lateral inhibition and image processing, in *Nonlinear Vision (ed. R.B. Pinter, B. Nabet),* Boca Raton, FL, CRC Press, (1992), 451–462.
[51] M. Kalpana and P. Balasubramaniam, Stochastic asymptotical synchronization of chaotic Markovian jumping fuzzy cellular neural networks with mixed delays and Wiener process based on sampled-data control, *Chin. Phys. B,* **22** (2013), 078401.
[52] E. Kaslik and S. Balint, Complex and chaotic dynamics in a discrete-time-delayed Hopfield neural network with ring architecture, *Neural Netw.,* **22** (2009), 1411–1418.

[53] Q. Ke and B. J. Oommen, Logistic neural networks: Their chaotic and pattern recognition properties, *Neurocomputing,* **125** (2014), 184–194.

[54] A. Khadra, X. Liu and X. Shen, Application of impulsive synchronization to communication security, *IEEE Trans. Circuits Syst.-I: Fund. Theory Appl.,* **50** (2003), 341–351.

[55] P. Kloeden and Z. Li, Li-Yorke chaos in higher dimensions: A review, *J. Differ. Equ. Appl.,* **12** (2006), 247–269.

[56] J. Kuroiwa, N. Masutani, S. Nara and K. Aihara, Chaotic wandering and its sensitivity to external input in a chaotic neural network, in *Proceedings of the 9th International Conference on Neural Information Processing* (ed. L. Wang, J.C. Rajapakse, K. Fukushima, S.Y. Lee, X. Yao), Singapore, Orchid Country Club, **1** (2002), 353–357.

[57] J. Lei and Z. Lei, The chaotic cipher based on CNNs and its application in network, in *Proceedings of International Symposium on Intelligence Information Processing and Trusted Computing,* (2011), 184–187.

[58] K. Li, X. Zhang and Z. Li, Global exponential stability of impulsive cellular neural networks with time-varying and distributed delay, *Chaos Soliton. Fract.,* **41** (2009), 1427–1434.

[59] P. Li, Z. Li, W. A. Halang and G. Chen, Li-Yorke chaos in a spatiotemporal chaotic system, *Chaos Soliton. Fract.,* **33** (2007), 335–341.

[60] Y. Li, C. Liu and L. Zhu, Global exponential stability of periodic solution for shunting inhibitory CNNs with delays, *Phys. Lett. A,* **337** (2005), 46–54.

[61] Y. Li and J. Shu, Anti-periodic solutions to impulsive shunting inhibitory cellular neural networks with distributed delays on time scales, *Commun. Nonlinear Sci. Numer. Simulat.,* **16** (2011), 3326–3336.

[62] T. Y. Li and J. A. Yorke, Period three implies chaos, *The American Mathematical Monthly,* **82** (1975), 985–992.

[63] W. Lin and J. Ruan, Chaotic dynamics of an integrate-and-fire circuit with periodic pulse-train input, *IEEE Trans. Circuits Syst.-I: Fund. Theory Appl.,* **50** (2003), 686–693.

[64] B. Liu and L. Huang, Existence and stability of almost periodic solutions for shunting inhibitory cellular neural networks with time-varying delays, *Chaos Soliton. Fract.,* **31** (2007), 211–217.

[65] Q. Liu and S. Zhang, Adaptive lag synchronization of chaotic Cohen-Grossberg neural networks with discrete delays, *Chaos,* **22** (2012), 033123, 5pp.

[66] W. Liu and L. Wang, Variable thresholds in the chaotic cellular neural network, in *Proceedings of International Joint Conference on Neural Networks,* Orlando, Florida, USA, (2007), 711–714.

[67] J. Lu, D. W. C. Ho, J. Cao and J. Kurths, Exponential synchronization of linearly coupled neural networks with impulsive disturbances, *IEEE Trans. Neural Netw.,* **22** (2011), 329–335.

[68] W. Lu and T. Chen, Synchronization of coupled connected neural networks with delays, *IEEE Trans. Circuits Syst.-I Regul. Pap.,* **51** (2004), 2491–2503.

[69] F. R. Marotto, Snap-back repellers imply chaos in $\mathbf{R}^n$, *J. Math. Anal. Appl.,* **63** (1978), 199–223.

[70] B. L. McNaughton, C. A. Barnes and P. Andersen, Synaptic efficacy and EPSP summation in granule cells of rat fascia dentata studied in vitro, *J. Neurophysiol,* **46** (1981), 952–966.

[71] P. Muthukumar and P. Balasubramaniam, Feedback synchronization of the fractional order reverse butterfly-shaped chaotic system and its application to digital cryptography, *Nonlinear Dynamics,* **74** (2013), 1169–1181.

[72] P. Muthukumar, P. Balasubramaniam and K. Ratnavelu, Synchronization of a novel fractional order stretch-twist-fold (STF) flow chaotic system and its application to a new authenticated encryption scheme (AES), *Nonlinear Dynamics,* **77** (2014), 1547–1559.

[73] S. Nara and P. Davis, Chaotic wandering and search in a cycle-memory neural network, *Prog. Theor. Phys.,* **88** (1992), 845–855.

[74] S. Nara, P. Davis, M. Kawachi and H. Totsuji, Chaotic memory dynamics in a recurrent neural network with cycle memories embedded by pseudoinverse method, *Int. J. Bifurcation and Chaos,* **5** (1995), 1205–1212.

[75] M. Ohta and K. Yamashita, A chaotic neural network for reducing the peak-to-average power ratio of multicarrier modulation, in *Proceedings of International Joint Conference on Neural Networks,* **2** (2003), 864–868.

[76] E. Ott, C. Grebogi and J. A. Yorke, Controlling chaos, *Phys. Rev. Lett.,* **64** (1990), 1196–1199.

[77] C. Ou, Almost periodic solutions for shunting inhibitory cellular neural networks, *Nonlinear Anal.: Real World Appl.,* **10** (2009), 2652–2658.

[78] L. Pan and J. Cao, Anti-periodic solution for delayed cellular neural networks with impulsive effects, *Nonlinear Anal.: Real World Appl.,* **12** (2011), 3014–3027.

[79] G. Peng and L. Huang, Anti-periodic solutions for shunting inhibitory cellular neural networks with continuously distributed delays, *Nonlinear Anal.: Real World Appl.,* **10** (2009), 2434–2440.

[80] R. B. Pinter, R. M. Olberg and E. Warrant, Luminance adaptation of preferred object size in identified dragonfly movement detectors, in *Proceedings of IEEE International Conference on Systems, Man and Cybernetics*, **2** (1989), 682–686.

[81] Y. Pomeau and P. Manneville, Intermittent transition to turbulence in dissipative dynamical systems, *Commun. Math. Phys.,* **74** (1980), 189–197.

[82] A. Potapov and M. K. Ali, Robust chaos in neural networks, *Phys. Lett. A,* **277** (2000), 310–322.

[83] K. Pyragas, Continuous control of chaos by self-controlling feedback, *Phys. Rev. A,* **170** (1992), 421–428.

[84] D. J. Rijlaarsdam and V. M. Mladenov, Synchronization of chaotic cellular neural networks based on Rössler cells, in *Proceedings of 8th Seminar on Neural Network Applications in Electrical Engineering,* NEUREL (ed. B. Reljin, S. Stankovié), Serbia, University of Belgrade (2006), pp. 41–43.

[85] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[86] E. Sander and J. A. Yorke, Period-doubling cascades galore, *Ergod. Th.* & *Dynam. Sys.,* **31** (2011), 1249–1267.

[87] J. Shao, Anti-periodic solutions for shunting inhibitory cellular neural networks with time-varying delays, *Phys. Lett. A,* **372** (2008), 5011–5016.

[88] Y. Shi and G. Chen, Chaos of discrete dynamical systems in complete metric spaces, *Chaos Soliton. Fract.,* **22** (2004), 555–571.

[89] Y. Shi and G. Chen, Discrete chaos in Banach spaces, *Sci. China Ser. A Math.,* **48** (2005), 222–238.

[90] Y. Shi, P. Zhu and K. Qin, Projective synchronization of different chaotic neural networks with mixed time delays based on an integral sliding mode controller, *Neurocomputing,* **123** (2014), 443–449.

[91] M. Shibasaki and M. Adachi, Response to external input of chaotic neural networks based on Newman-Watts model, in *Proceedings of WCCI 2012 IEEE World Congress on Computational Intelligence* (ed. J. Liu, C. Alippi, B. Bouchon-Meunier, G. W. Greenwood, H. A. Abbass), Brisbane, Australia, (2012), 1–7.

[92] C. A. Skarda and W. J. Freeman, How brains make chaos in order to make sense of the world?, *Behav. Brain Sci.,* **10** (1987), 161–173.

[93] X. Song, X. Xin and W. Huang, Exponential stability of delayed and impulsive cellular neural networks with partially Lipschitz continuous activation functions, *Neural Netw.,* **29-30** (2012), 80–90.

[94] I. M. Stamova and R. Ilarionov, On global exponential stability for impulsive cellular neural networks with time-varying delays, *Comput. Math. Appl.,* **59** (2010), 3508–3515.

[95] J. Sun, Stationary oscillation for chaotic shunting inhibitory cellular neural networks with impulses, *Chaos,* **17** (2007), 043123, 5pp.

[96] J. A. K. Suykens, M. E. Yalcin and J. Vandewalle, Coupled chaotic simulated annealing processes, in *Proceedings of the 2003 IEEE International Symposium on Circuits and Systems,* Bangkok, Thailand, **3** (2003), 582–585.

[97] I. Tsuda, Chaotic itinerancy as a dynamical basis of hermeneutics in brain and mind, *World Futures,* **32** (1991), 167–184.

[98] X. Wang, Period-doublings to chaos in a simple neural network: An analytical proof, *Complex Systems,* **5** (1991), 425–441.

[99] Q. Wang and X. Liu, Exponential stability of impulsive cellular neural networks with time delay via Lyapunov functionals, *Appl. Math. Comput.,* **194** (2007), 186–198.

[100] Z. Wang, H. Zhang and B. Jiang, LMI-based approach for global asymptotic stability analysis of recurrent neural networks with various delays and structures, *IEEE Trans. Neural Netw.,* **22** (2011), 1032–1045.

[101] B. Wu, Y. Liu and J. Lu, New results on global exponential stability for impulsive cellular neural networks with any bounded time-varying delays, *Math. Comput. Model.,* **55** (2012), 837–843.

[102] Y. Xia, J. Cao and Z. Huang, Existence and exponential stability of almost periodic solution for shunting inhibitory cellular neural networks with impulses, *Chaos Soliton. Fract.,* **34** (2007), 1599–1607.

[103] L. Yan, H. He and P. Xiong, Algebraic condition of control for multiple time-delayed chaotic cellular neural networks, in *Proceedings of Fourth International Workshop on Advanced Computational Intelligence,* Wuhan, Hubei, China, (2011), 596–600.

[104] T. Yang and L. O. Chua, Impulsive control and synchronization of nonlinear dynamical systems and application to secure communication, *Int. J. Bifurcation and Chaos,* **7** (1997), 645–664.

[105] T. Yang, *Impulsive Systems and Control: Theory and Applications*, Nova Science, New York, 2001.

[106] Y. Yang and J. Cao, Stability and periodicity in delayed cellular neural networks with impulsive effects, *Nonlinear Anal.: Real World Appl.,* **8** (2007), 362–374.

[107] Z. Yang and D. Xu, Stability analysis of delay neural networks with impulsive effects, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.,* **13** (2006), 563–573.

[108] Y. Yao and W. J. Freeman, Model of biological pattern recognition with spatially chaotic dynamics, *Neural Netw.,* **3** (1990), 153–170.

[109] Z. Yifeng and H. Zhengya, A secure communication scheme based on cellular neural network, in *Proceedings of IEEE International Conference on Intelligent Processing Systems,* **1** (1997), 521–524.

[110] W. Yu, J. Cao and W. Lu, Synchronization control of switched linearly coupled neural networks with delay, *Neurocomputing,* **73** (2010), 858–866.

[111] W. Zhao and H. Zhang, On almost periodic solution of shunting inhibitory cellular neural networks with variable coefficients and time-varying delays, *Nonlinear Anal.: Real World Appl.,* **9** (2008), 2326–2336.

[112] Q. Zhou, B. Xiao, Y. Yu and L. Peng, Existence and exponential stability of almost periodic solutions for shunting inhibitory cellular neural networks with continuously distributed delays, *Chaos Soliton. Fract.,* **34** (2007), 860–866.

[113] F. Zou and J. A. Nossek, A chaotic attractor with cellular neural networks, *IEEE Trans. Circuits Syst.,* **38** (1991), 811–812.

[114] F. Zou and J. A. Nossek, Bifurcation and chaos in cellular neural networks, *IEEE Trans. Circuits Syst.-I: Fund. Theory Appl.,* **40** (1993), 166–173.

Received December 2014; revised December 2015.

E-mail address: monur.fen@gmail.com
E-mail address: marat@metu.edu.tr