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Exogenous Versus Endogenous for Chaotic Business Cycles

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Abstract

We propose a novel approach to generate chaotic business cycles in a deterministic setting. Rather than producing chaos endogenously, we consider aggregate economic models with limit cycles and equilibriums, subject them to chaotic exogenous shocks and obtain chaotic cyclical motions. Thus, we emphasize that chaotic cycles, which are inevitable in economics, are not only interior properties of economic models, but also can be considered as a result of interaction of several economical systems. This provides a comprehension of chaos (unpredictability, lack of forecasting) and control of chaos as a global economic phenomenon from the deterministic point of view.

We suppose that the results of our paper are contribution to the mixed exogenous-endogenous theories of business cycles in classification by P.A. Samuelson [1]. Moreover, they demonstrate that the irregularity of the extended chaos can be structured, and this distinguishes them from the generalized synchronization. The advantage of the knowledge of the structure is that by applying instruments, which already have been developed for deterministic chaos, one can control the chaos, emphasizing a parameter or a type of motion. For the globalization of cyclic chaos phenomenon we utilize new mechanisms such as entrainment by chaos, attraction of chaotic cycles by equilibriums and bifurcation of chaotic cycles developed in our earlier papers.

1 Introduction

Business cycles are a commonly accepted phenomenon in economics. However, we do not actually observe perfectly periodic motions in economic variables. Instead, economic data is highly irregular. One way to reflect this in economic models is to allow for stochastic processes. Deterministic differential equations can also be turned into a better picture of economic reality by introducing chaos.[a]

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[a]There exists a third approach, which is somewhere in-between the two, where iterated function systems generated by the optimal policy functions for a class of stochastic growth models converge to invariant distributions with support over fractal sets [2].

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Chaotic economic systems can be viewed as unpredictable due to their sensitivity to initial values, which makes forecasting extremely difficult [3–6]. This is known also as the *butterfly effect* [7]. Devaney [8] proposed that *sensitivity* in conjunction with other properties, namely *transitivity* and *density of periodic solutions*, be considered as ingredients of chaos. An alternative way to prove the presence of chaos is by observing the *period-doubling cascade* [9]. This chaos is also sensitive, since there are infinitely many solutions with different periods and they are *unstable*. We utilize these ways of observing chaos in our paper. Importantly, irregularity based on theoretical *deterministic chaos* can be visualized in simulations.

One should remark that it is not only sensitivity that can be considered as a mathematical representation of unpredictability, but also the existence of infinitely many *unstable* periodic solutions. Indeed, while the presence of a single periodic solution can be accepted as a strong indicator of predictability (if one knows the values of the process during the period, then one knows all its future values), with infinitely many *unstable* periodic solutions all the cycles are unstable, and the trajectory of the dynamics wanders around, visiting neighborhoods of the cycles in an unpredictable way. That is the reason why in the literature the proof of the existence of a periodic-doubling cascade is accepted as evidence of chaos. Stabilizing periodic solutions is named in chaos theory as control of chaos.

Chaos theory could provide a new approach to economic policy-making. Economists believed initially that chaotic dynamics is not only unpredictable, but also un-controllable. The results of Ott et al. [10] showed that control of a chaos can be made by a very small corrections of parameters [11, 12]. This and related methods have been widely applied to economic models, as exemplified by Holyst et al. [13], Kaas [14], Mendes and Mendes [15], Chen and Chen [16] and many others.

In the classic book [1] it is observed that while forced oscillator systems naturally emerge in theoretical investigations of several technical and physical devices, economic examples for this special family of functions have only rarely been provided. The main reason for this deficiency may lie in the fact that the necessary periodicity of the dynamic forcing may not be obvious in most economic applications. Our proposals are to apply *deterministic and chaotic* exogenous shocks to economic models and make them more realistic.

One may view chaos (the lack of forecasting) as undesirable in economics, but unavoidable. Hence a deterministic economic model is realistic if it exhibits chaotic motions. We suggest considering the presence of chaos in a model not only as an indication of its adequacy, but also as a measure of its power. Indeed, the presence of chaos implies that the model generates infinitely many aperiodic motions and motions with different periods, which are unstable, and consequently easily affected by control and sustained in a desirable mode. In other words, deterministic chaos is essential for the flexibility and high-speed adjustment of economic models, an indispensable feature in the modern world.

The principal novelty of our investigation is that we create a chaotic perturbation, plug it in a regular dynamic system, and find that similar chaos is inherited by the solutions of the new system. We call this as *the input-output mechanism of chaos generation*. This approach has been widely applied to differential equations before, but for regular inputs. In the studies [17–21], the mechanisms for generating chaos in systems with asymptotically stable equilibria are provided. In contrast, in [22–25] unpredictability in the solutions of differential equations was considered a result of random perturbations with small probability.

P.A. Samuelson [1] accepts purely endogenous theory as "self-generating" cycle. Following this opinion we understand chaos as endogenous if it is self-generated by an economic model. One can find detailed analysis of the endogenous chaos in the books [6, 26, 27] and paper [3], which are very seminal sources on the subject. The dynamics arise in duopoly models [28] and in simple ad hoc macroeconomic models [29, 30]. By applying the Li-Yorke theorem it is shown in [31, 32] that an overlapping generations model of the Gale type could generate endogenous chaotic cycles. Discrete equations have been applied to investigate the presence of chaos in papers [33,34], where models representing a capital stock with a maximum capital-labor ratio and a Malthusian agrarian economy are investigated. In [34–36] endogenous chaotic cycles are demonstrated in growth cycle models. The multiplier-accelerator model of Samuelson [1] has been modified for generation of chaotic endogenous cycles and investigated in [37–39]. Investigations in Kaldor's type models, which are originated from [37, 40] and finalized in [41], showed that they could generate endogenous chaos.

Economists of the first half of the last century already felt a strong need for a theory of irregularities, particularly of irregular business cycles. In his classic book, Samuelson [1] observes that while forced oscillator systems naturally emerge in theoretical investigations of several technical and physical devices and phenomena, economic examples for this special family of functions have only rarely been provided. The main reason for this dearth of evidence may lie in the fact that the necessary periodicity of the dynamic forcing may not be obvious in most economic applications. That is, economic phenomena do not display the kind of regularity that physical phenomena do. Samuelson [1] states that "... in a physical system there are grand conservation laws of nature, which guarantee that the system must fall on the thin line between stability and instability. But there is nothing in the economic world corresponding to these laws ...". In a passage Samuelson [1] suggests that "It is to be stressed that the exogenous impulses which keep the cycle alive need not themselves be even quasi-oscillatory in character." Thus, he was already talking about irregular business cycles that emerge as a result of *irregular exogenous shocks*. Moreover, he recognized that "most economists are eclectic and prefer a combination of endogenous and exogenous theories." Accordingly, in the present paper we consider economic models that admit *endogenous* business cycles and are perturbed by *exogenous* chaotic disturbances. Examples of models possessing limit cycles are Kaldor-Kalecki models and Lienard type equations with relaxation oscillations which are popular in economics. Next, the systems are subject to *exogenous* chaotic disturbances, sensitive and with infinitely many unstable periodic solutions.

We propose two techniques of obtaining exogenous chaotic cycles as solutions of differential equations. In the first approach, an economic model with a limit cycle is perturbed chaotically to produce a chaotic business cycle. In the second one, we consider a system with an equilibrium, perturb it by cyclic chaos and observe that a chaotic business cycle emerges as a result. While the first method is theoretically verified in [42], the second method of cyclic chaos generation is new and is demonstrated in the present study through simulations. Currently, we study cases where the shocks enter the system additively, but future investigations may involve more complex scenarios, where the disturbance enters the main functions of the economic model.

Goodwin [5] argues that the apparent unpredictability of economic systems is due to deterministic chaos as much as to exogenous shocks. In this sense, our results can be interpreted as the *transmission of unpredictability* from one economic system to another, and even models that do not admit irregularity in isolation can eventually be contaminated with chaos. Thus, we provide support to the idea that unpredictability is a *global phenomenon* in economics, and demonstrate one of the mechanisms for this contagion. Considering the current extensive globalisation process, this is a good depiction of reality.

Our results demonstrate that the control may become not a local (applied to an isolated model) but a global phenomenon with strong effectiveness such that control applied to a model, which is realizable easily (for example, the logistic map or Feichtinger's generic model), can be sufficient to rule the process in all models joined with the controlled one. Another benefit of our studies is that in the literature controls are applied to those systems which are simple and low-dimensional. Control of chaos becomes difficult as the dimensions of the systems increase and the construction of Poincaré sections becomes complicated. Chaos control cannot be achieved if we do not know the period of unstable motion to be controlled. In our case, the control is applicable to models of arbitrary dimensions as long as the basic period of the generator is known. For these reasons, the possibility to control generated chaos by controlling the exogenous shocks that produce the said chaos is appealing. It is especially appealing from a policy-maker's point of view, as it offers a cost-effective way to regulate an economic system.

Control of chaos is nowadays a synonym to the suppression of chaos. Thus, our results give another way of suppression of chaos. If we find the controllable link (member) in a chain (collection) of connected chaotic systems, then we can suppress chaos in the whole chain. This is the effective consequence of our studies.

1.1 Organization of the paper

The paper is organized in the following way. In the next section we describe the input-output mechanism that serves as the basis of chaos extension and formulate two theorems that provide theoretical support to the subsequent discussion. In Section 3 economic models with regular motions - stable equilibrium and orbitally stable cycle - are introduced. These models are chaotically perturbed in the following section to obtain the main economic dynamics of the paper. More precisely, Section 4 considers a constellation of five economic models connected unilaterally. The extension of chaos near an equilibrium attractor, the entrainment of limit business cycles by chaos, the bifurcation of a chaotic cycle, and the attraction of a chaotic cycle are the scenarios of the appearance of chaos, and in some cases of chaotic business cycles, in economic models that we demonstrate. The effects of applying the Ott-Grebogi-Yorke (OGY) control [10] to the models will also be presented. Section 5 provides simulation results for the entrainment of limit business cycles by chaos of economic models with time delay. We compare in detail our method of chaos generation with that based on the synchronization of chaos [43–45] in Section 6. In particular, we argue that chaotic business cycles in the paper cannot be obtained by the synchronization of chaotic systems. In Section 7 we discuss our results from the point of view of self-organization, and particularly synergetics of Haken [46]. We summarize the obtained results in Section 8.

2 The input-output mechanism and its applications

To explain the input-output mechanism of chaos generation, let us introduce systems, which we call *the base-system, the replicator and the generator*. They are intensively used in the manuscript. Consider the following system of differential equations,

$$z' = B(z), \tag{1}$$

where $B : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function. The system (1) is called *the base-system*.

Next, we subdue the base-system to a perturbation, $I(t)$, which will be called an *input* and obtain the following system,

$$y' = B(y) + I(t), \tag{2}$$

which will be called as *the replicator*.

Suppose that the input $I(t)$ admits a certain property, say, it is a bounded function. Assume that there exists a unique solution, $y(t)$, of the replicator system (2), with the same property. This solution is called an *output*. The process of obtaining the solution $y(t)$ by applying the perturbation $I(t)$ to the base-system (1) is called *the input-output mechanism*. It is known that for certain base-systems, if the input is a periodic, almost periodic, bounded function, then there exists an output that is also a periodic, almost periodic, bounded function. In our paper, we consider inputs of a different nature: chaotic functions and set of cyclic chaotic functions. The motions that are in the chaotic attractor of the Lorenz system [7], considered altogether, give us an example of a chaotic set of functions. Each element of this set is considered as a chaotic function. Both a set of functions and a single function can serve as an input (as well as an output), and we will use both types of inputs and outputs in this study.

We consider base-systems of two kinds: (i) systems with asymptotically stable equilibria, (ii) systems with limit cycles. In the former case, we will talk about attraction of chaos by equilibria, and in particular, attraction of cyclic chaos by equilibria. If the base-system admits a limit cycle, then we talk about the entrainment of limit cycles by chaos or just about entrainment by chaos [42]. If the limit cycle in a base-system is the result of a Hopf bifurcation [47], we will also talk about the bifurcation of the cyclic chaos.

In our previous papers [17,20,21,48,49] we analyzed the extension of chaos when the base system possesses an asymptotically stable equilibrium. The present paper focuses mostly on the generation of cyclic chaos through unilateral coupling of multiple systems.

The main source of chaos in theory are difference and differential equations. For this reason we consider in our manuscript, inputs, which are solutions of some systems of differential or discrete equations equations. These systems will be called *generators*.[b]

Thus, we consider the following system of differential equations,

$$x' = G(t,x), \tag{3}$$

where the function $G : [0,\infty) \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous in all of its arguments. We assume that system (3) possesses a chaotic attractor, and we call this system a *generator*. If $x(t)$ is a solution of the system from the chaotic attractor, then we take

$$I(t) = \varepsilon\psi(x(t)),$$

and use the function $I(t)$ in equation (2). Here, ε is a non-zero real number and the function $\psi : \mathbb{R}^m \to \mathbb{R}^n$ is continuous. Since we use $x(t)$ as a perturbation in system (2), we call it a *chaotic solution*. The chaotic solutions may be irregular as well as regular (periodic and unstable) [8, 54–56]. In this study we will utilize also the logistic map [8] as a generator.

System (3) is called sensitive if there exist positive numbers ε_0 and Δ such that for an arbitrary positive number δ_0 and for each chaotic solution $x(t)$ of (3), there exist a chaotic solution $\bar{x}(t)$ of the same system and an interval $J \subset [0,\infty)$, with a length no less than Δ, such that $\|x(0)-\bar{x}(0)\| < \delta_0$ and $\|x(t)-\bar{x}(t)\| > \varepsilon_0$ for all $t \in J$.

For a given chaotic solution $x(t)$ of (3), let us denote by $\eta_{x(t)}(t,y_0)$, $y_0 \in \mathbb{R}^n$, the solution of (2) with $\eta_{x(t)}(0,y_0) = y_0$. System (2) replicates the sensitivity of (3) if there exist positive numbers ε_1 and $\overline{\Delta}$ such that for an arbitrary positive number δ_1 and for each solution $\eta_{x(t)}(t,y_0)$, there exist an interval $J^1 \subset [0,\infty)$, with a length no less than $\overline{\Delta}$, and a solution $\eta_{\bar{x}(t)}(t,y_1)$ such that $\|y_0 - y_1\| < \delta_1$ and $\left\|\eta_{x(t)}(t,y_0) - \eta_{\bar{x}(t)}(t,y_1)\right\| > \varepsilon_1$ for all $t \in J^1$. Moreover, we say that system (2) is chaotic if it replicates the sensitivity of (3) and the coupled system $(3)+(2)$ possesses infinitely many unstable periodic solutions in a bounded region.

Next, we will formulate a theorem that forms the mathematical basis of the paper.

The following conditions are required:

(C1) System (1) admits a non-constant and *orbitally stable* periodic solution;

(C2) System (3) possesses sensitivity and is chaotic through period-doubling cascade;

(C3) The functions B and G are bounded;

(C4) There exists a positive number L_B such that

$$\|B(z_1) - B(z_2)\| \le L_B \|z_1 - z_2\|,$$

for all $z_1,\, z_2 \in \mathbb{R}^n$;

(C5) There exists a positive number L_ψ such that

$$\|\psi(x_1) - \psi(x_2)\| \ge L_\psi \|x_1 - x_2\|,$$

for all $x_1, x_2 \in \mathbb{R}^m$.

The following assertion is based on the results in [42].

Theorem 1. *If conditions $(C1)-(C5)$ hold and $|\varepsilon|$ is sufficiently small, then there exists a neighborhood $\mathscr{U}$ of the orbitally stable limit cycle of (1) such that solutions of (2) which start inside $\mathscr{U}$ behave chaotically around the limit cycle. That is, the solutions are sensitive and there are infinitely many unstable periodic solutions.*

[b] In future work, economic time series that have been tested for the presence of deterministic chaos may be considered (see [4,50–53].)

3 Economic models: the base systems

In what follows, we will require regular systems, that is, models with asymptotically stable equilibria or limit cycles, that can be perturbed to generate chaotic business cycles. In this part of the paper we propose three economic models to be used as base systems.

3.1 Kaldor-Kalecki model with a steady equilibrium

Consider the following model of an aggregate economy:

$$\begin{aligned} Y' &= \alpha[I(Y,K) - S(Y,K)], \\ K' &= I(Y,K) - \delta K, \end{aligned} \tag{4}$$

where Y is income, K is capital stock, I is gross investment, and S is savings. Income changes proportionally to the excess demand in the goods market, and the second equation is a standard capital accumulation equation. The constant depreciation rate δ and the adjustment coefficient α are positive. This model was studied in detail in [26] and [27]. It admits a stable equilibrium under certain conditions on the functions involved.

Let us consider the following specification of system (4) with $I(Y,K) = Y - aY^3 + bK$, $S(Y,K) = sY$,

$$\begin{aligned} Y' &= \alpha[(1-s)Y - aY^3 + bK], \\ K' &= Y - aY^3 + bK - \delta K, \end{aligned} \tag{5}$$

where the constant parameters satisfy $a > 0$, $b < 0$, $0 < s < 1$ and $0 < \delta < 1$.

One can see that a steady state of (5) with positive coordinates

$$Y^* = \sqrt{\frac{\delta(1-s)+bs}{a\delta}}, \qquad K^* = \frac{s}{\delta}\sqrt{\frac{\delta(1-s)+bs}{a\delta}},$$

exists only if $\delta s < \delta + bs$.

The transformations $Y = y + Y^*$, $K = k + K^*$, applied to (5), give us the system

$$\begin{aligned} y' &= \alpha[(2(s-1) - \frac{3bs}{\delta})y + bk - ay^3 - 3\sqrt{\frac{a\delta(1-s)+abs}{\delta}}y^2], \\ k' &= (3s-2-\frac{3bs}{\delta})y + (b-\delta)k - ay^3 - 3\sqrt{\frac{a\delta(1-s)+abs}{\delta}}y^2. \end{aligned} \tag{6}$$

3.2 A model with a business cycle

We also investigate the idealized macroeconomic model with foreign capital investment,

$$\begin{aligned} S' &= \alpha Y + pS(k - Y^2), \\ Y' &= v(S+F), \\ F' &= mS - rY, \end{aligned} \tag{7}$$

where $S(t)$ are savings of households, $Y(t)$ is Gross Domestic Product (GDP), $F(t)$ is foreign capital inflow, k is potential GDP, and t is time. If k is set to 1, then Y, S, F are measured as multiples of potential output. The parameters represent corresponding ratios: α is the variation of the marginal propensity to save, p is the ratio of capitalised profit, $\frac{1}{v}$ is the capital-output ratio, m is the capital inflow-savings ratio and r is the debt refund-output ratio. The model in (7) was introduced by Bouali [57], and later studied by Bouali et al. [58] and Pribylova [59].

Consider system (7) with specified coefficients,

$$\begin{aligned} S' &= \alpha Y + 0.1S(1 - Y^2), \\ Y' &= 0.5(S+F), \\ F' &= 0.19S - 0.25Y. \end{aligned} \tag{8}$$

According to [59], the system (8) admits Hopf bifurcation at $\alpha = \alpha_0 \equiv 0.25$ and an orbitally stable cycle appears as α decreases.

3.3 A Kaldor-Kalecki model

Let us consider the system,

$$\begin{aligned} Y' &= \alpha[I(Y,K) - S(Y,k)], \\ K' &= I(Y(t-\tau),K) - \delta K. \end{aligned} \tag{9}$$

System (9) is a Kaldor model with time delay. Kalecki [60] introduced the idea that there may be a time lag between the time an investment decision is made and the time investment is realised. The Kaldor-Kalecki model (9) was formalised by Krawiec and Szydlowski [61], where investment depends on income at the time investment decisions are taken and on capital stock at the time investment is finished. One can find additional information on the models with delay in the papers [62, 63].

We will study the specification

$$\begin{aligned} Y' &= 1.5[\tanh(Y) - 0.25K - (4/3)Y], \\ K' &= \tanh(Y(t-\tau)) - 0.5K. \end{aligned} \tag{10}$$

According to Zhang and Wei [64], the model admits an orbitally stable limit cycle for $\tau > 5.4$. More precisely, the periodic solution appearance follows a Hopf bifurcation so that the origin is asymptotically stable if $\tau < 5.4$, and the origin loses its stability and the cycle bifurcates from the origin for $\tau > 5.4$.

4 Extension of chaos in a constellation of economical models

To provide a comprehensive illustration for the discussion in the previous sections, we will consider a constellation of five unilaterally connected economic models denoted by A_k, $k = 1,\dots,5$. The topology of the connection is presented in Figure 1, and the models are formulated in system (11). We will show that the chaos that appears in A_1 spreads to all the other models. A_2 serves as a replicator of the chaos of A_1 and as a generator of chaos in A_3 and A_4. Model A_4 is a replicator of the chaos of A_2 and a generator of chaos in A_5.

The following is a system of five unidirectionally coupled models $A_1 - A_5$.

$$\begin{aligned}
&\kappa_{j+1} = \mu\kappa_j(1-\kappa_j), \big\} A_1 \\
&\left.\begin{aligned} y_1' &= (1/8)y_1 - (5/16)k_1 - a_1 y_1^3 - \frac{3\sqrt{a_1}}{2} y_1^2 + \nu_1(t,\theta), \\ k_1' &= (1/4)y_1 - (3/8)k_1 - a_1 y_1^3 - \frac{3\sqrt{a_1}}{2} y_1^2, \end{aligned}\right\} A_2 \\
&\left.\begin{aligned} y_2' &= (1/3)y_2 - k_2 - a_2 y_2^3 - \frac{\sqrt{6a_2}}{2} y_2^2 + 0.6 y_1(t) + \nu_2(t,\zeta), \\ k_2' &= (1/2)y_2 - (5/4)k_2 - a_2 y_2^3 - \frac{\sqrt{6a_2}}{2} y_2^2, \end{aligned}\right\} A_3 \\
&\left.\begin{aligned} S' &= 0.23Y + 0.1S(1-Y^2), \\ Y' &= 0.5(S+F) + 2(y_1(t)+0.5), \\ F' &= 0.19S - 0.25Y, \end{aligned}\right\} A_4 \\
&\left.\begin{aligned} y_3' &= (3/5)y_3 - (4/5)k_3 - a_3 y_3^3 - \frac{3\sqrt{a_3}}{\sqrt{10}} y_3^2 + 0.01Y(t), \\ k_3' &= (7/10)y_3 - (9/10)k_3 - a_3 y_3^3 - \frac{3\sqrt{a_3}}{\sqrt{10}} y_3^2, \end{aligned}\right\} A_5
\end{aligned} \tag{11}$$

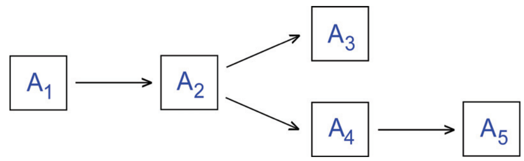

Fig. 1 The connection topology of the systems $A_1 - A_5$.

where a_1, a_2, a_3 are constants and the piecewise constant functions $v_1(t,\theta)$ and $v_2(t,\zeta)$ are defined as follows:

$$v_1(t,\theta) = \begin{cases} 0.019, & \text{if } \theta_{2j} < t \le \theta_{2j+1}, \\ 0.002, & \text{if } \theta_{2j-1} < t \le \theta_{2j}, \end{cases} \tag{12}$$

and

$$v_2(t,\zeta) = \begin{cases} 0.0006, & \text{if } \zeta_{2j} < t \le \zeta_{2j+1}, \\ 0.0017, & \text{if } \zeta_{2j-1} < t \le \zeta_{2j}, \end{cases}. \tag{13}$$

The sequence $\theta = \{\theta_j\}$, $j \in \mathbb{Z}$, of the discontinuity instants of the function (12) satisfies the relation $\theta_j = j + \kappa_j$, where the sequence $\{\kappa_j\}$ is a solution of the logistic map A_1 with $\kappa_0 \in [0,1]$. The sequence $\zeta = \{\zeta_j\}$, $j \in \mathbb{Z}$, of the discontinuity instants of (13) satisfies the relation $\zeta_j = 2\sqrt{2}j$ for each j.

Examples of shocks of the form (12) and (13) are natural disasters and extreme events in general, such as market crashes. They take a finite number of values (an earthquake either happens or not), but their timing is irregular or regular.

4.1 Description of the models A_1 to A_5

Equation A_1 is the logistic map, which will be used as the main source of chaos in system (11). The interval $[0,1]$ is invariant under the iterations of the map for the parameter values $\mu \in (0,4]$ [65], and for $\mu = 3.8$ it is chaotic through period-doubling cascade [66]. The logistic map plays a very important role in many fields of science, particularly in economics. It can be used to describe economic variables. In [67] the logistic map emerges as the law of motion of the price of the non-numeraire good in a simple discrete-time model of an exchange economy with two goods under Walrasian tatonnement. Benhabib and Day [32] showed that a logistic map describes optimal consumption in a simple overlapping generations model with a quadratic utility function, and Mitra and Sorger [68] proved that the logistic map can be the optimal policy function of a regular dynamic optimisation problem, if and only if the discount factor does not exceed $1/16$.

The logistic map A_1 is the generator of chaos for the global system (11) and as we mentioned above, a generator can be not only with continuous dynamics, but also with discrete, and even hybrid, i.e., combining both continuous and discrete. In fact the whole model (11) is an example of a hybrid system.

System A_2 describes the aggregate economy of Country 1. It is a perturbed Kaldor model (6), obtained by setting $\alpha = 1$, $s = 1/8$, $\delta = 1/16$ and $b = -5/16$. In the absence of the perturbation function $v_1(t,\theta)$, the model possesses an asymptotically stable equilibrium provided that the number a_1 is sufficiently small. One can verify that the associated linear system admits complex conjugate eigenvalues $(-1 \pm i)/8$. The function (12) describes a rainfall shock that impacts on the agricultural sector and through it on the total output. The higher value of v_1 implies normal rainfall, while the lower value is drought, which leads to lower agricultural production and slower output growth.

Using the results of [17, 20, 48], one can state that the chaoticity of the logistic map A_1 with $\mu = 3.8$ makes the function $v_1(t,\theta)$ behave chaotically, and system A_2 is chaotic through period-doubling cascade for the same value of the parameter μ. That is, it admits infinitely many unstable periodic solutions and exhibits sensitivity. For each natural number p, the system possesses an unstable periodic solution with period $2p$. Next, in its own turn system A_2 is the generator for the systems A_3 and A_4.

System A_3 reflects the dynamics of Country 2. It is obtained by using the coefficients $\alpha = 1$, $s = 1/6$, $\delta = 1/4$, $b = -1$ in the Kaldor model (6) and by perturbing it with the solutions of A_2 as well as with the periodic function (13). The associated linear system has the eigenvalues $(-11 \pm \sqrt{73})/24$. In the absence of the perturbation terms $0.6y_1(t)$ and $v_2(t,\zeta)$, and if the number a_2 is sufficiently small, the system admits an asymptotically stable equilibrium. The term $0.6y_1(t)$ describes the effect exports from Country 2 to Country 1, modelled as a function of the income of Country 1, $y_1(t)$, have on the rate of change in the income of Country 2. The function $v_2(t,\zeta)$ reflects productivity shocks in Country 2, which is a binary variable. The higher value of v_2 stands for faster productivity growth, and the lower value for slower productivity growth, which leads to slower output growth.

Since the periodic motions that are embedded in the chaotic attractor of system A_2 with $\mu = 3.8$ and the function (13) have incommensurate periods, one can confirm using the results of [21] that system A_3 is chaotic with infinitely many quasi-periodic solutions in the basis. This will be shown through simulations in Figure 9.

System A_4 describes the aggregate economy of Country 3. It is obtained by perturbing system (8) with the solutions of A_2. It is a replicator with respect to system A_2, while the term $2(y_1(t)+0.5)$ is the input. This term represents the effect of exports from Country 3 to Country 1, modelled as a function of income in Country 1, $y_1(t)$, on the rate of growth of income in Country 3.

In the absence of perturbations, A_4 possesses an orbitally stable limit cycle [59]. Theorem 1 implies that system A_4 admits chaotic business cycles, provided that the value of the parameter $\mu = 3.8$ is used in system A_2. Since the orbitally stable cycle of system (8) occurs through a Hopf bifurcation, one can talk about the *bifurcation of the cyclic chaos*.

System A_5 models the dynamics of Country 4. It is constructed by perturbing the Kaldor model (6) with the solutions of A_4, or in economic terms, by perturbing the aggregate economy with exports from Country 4 to Country 3, which are a fraction of the income of Country 3, $Y(t)$. The eigenvalues of the associated linear system are $-1/5$ and $-1/10$. In the absence of the perturbation term $0.01Y(t)$, the system possesses an asymptotically stable equilibrium, for sufficiently small values of a_3. We will make use of system A_5 to demonstrate the *attraction of chaotic business cycles*.

4.2 Simulations

In this part of the paper, we will demonstrate numerically the chaotic behavior in system (11). In what follows, we will use $a_1 = 3\times 10^{-6}$, $a_2 = 10^{-6}$, $a_3 = 5\times 10^{-6}$, $\mu = 3.8$ and $\kappa_0 = 0.63$.

Let us start with system A_2. Setting the initial data $y_1(t_0) = 0.12$, $k_1(t_0) = 0.08$, where $t_0 = 0.63$, we graph in Figure 2 the y_1 coordinate of system A_2. It is seen in the figure that system A_2 behaves chaotically.

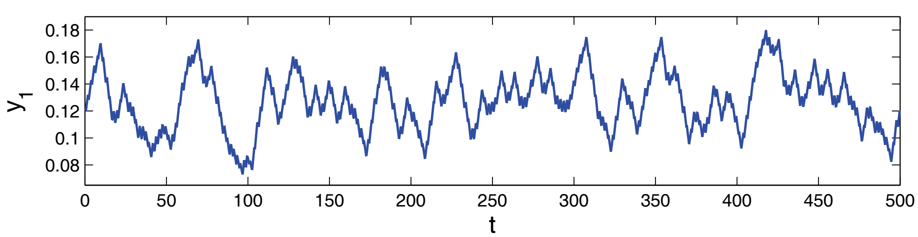

Fig. 2 The graph of the y_1 coordinate of system A_2.

To show the extension of chaos by system A_3, we use the solution in Figure 2 as perturbation in system A_3 and present in Figure 3 the time series of the y_2 coordinate of A_3. The initial data $y_2(t_0) = 0.95$, $k_2(t_0) = 0.38$, where $t_0 = 0.63$, is used in the simulation. Figure 3 reveals that the chaos of system A_2 is extended such that the system A_3 also possesses chaos. In order to confirm the extension of chaos once more, we depict in Figure 4 the projection of the trajectory of the coupled Kaldor system $A_2 - A_3$, corresponding to the same initial data, on the $y_1 - k_1 - y_2$ space.

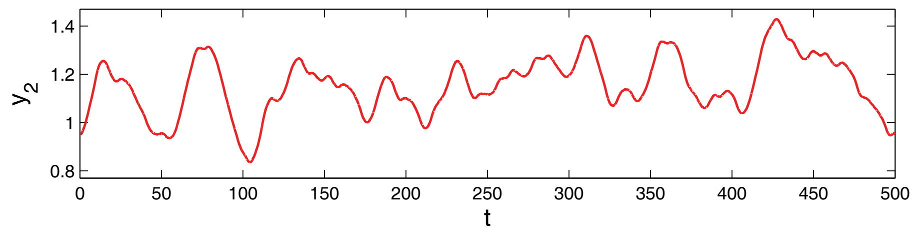

Fig. 3 Extension of chaos by system A_3.

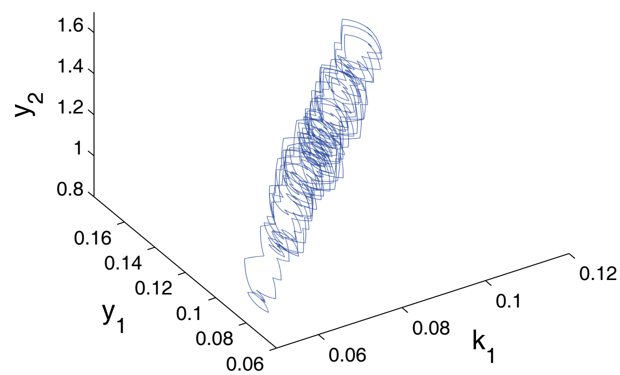

Fig. 4 The projection of the chaotic trajectory of the coupled Kaldor-Kalecki system $A_2 - A_3$ on the $y_1 - k_1 - y_2$ space.

Next, we continue with system A_4. We take into account system A_4 with the solution of A_2 that is represented in Figure 2, and show the trajectory of A_4 with $S(t_0) = 1.67$, $Y(t_0) = 0.94$, $F(t_0) = -5.15$, where $t_0 = 0.63$, in Figure 5. One can observe in Figure 5 that the system A_4 admits a chaotic business cycle.

In order to observe the attraction of the cyclic chaos of system A_4, we again use the solution of A_4 with $S(t_0) = 1.67$, $Y(t_0) = 0.94$, $F(t_0) = -5.15$, where $t_0 = 0.63$, in A_5 and depict in Figure 6 the trajectory of system A_5 with $y_3(t_0) = 0.72$, $k_3(t_0) = 0.56$. It is seen in Figure 6 that the chaotic business cycle of A_4 is attracted by A_5, and the cyclic irregular behavior is extended.

4.3 Control of extended chaos

The source of the chaotic motions in system (11) is the logistic map A_1. Therefore, to control the chaos of the entire system, one has to stabilize an unstable periodic solution of the logistic map. The OGY control method [10, 69] is one of the possible ways to do this. We proceed by briefly explaining the method.

Suppose that the parameter μ in the logistic map A_1 is allowed to vary in the range $[3.8 - \varepsilon, 3.8 + \varepsilon]$, where ε is a given small number. That is, it is not possible (say, it is prohibitively costly or practically infeasible) to simply shift the value of μ to a level that generates non-chaotic dynamics. Let us consider an arbitrary solution $\{\kappa_j\}$, $\kappa_0 \in [0,1]$, of the map and denote by $\kappa^{(q)}$, $q = 1, 2, \ldots, p$, the target unstable $p-$periodic orbit to be stabilized. In the OGY control method [69], at each iteration step j after the control mechanism is switched on, we consider the logistic map with the parameter value $\mu = \bar{\mu}_j$, where

$$\bar{\mu}_j = 3.8[1 + \frac{(2\kappa^{(q)} - 1)(\kappa_j - \kappa^{(q)})}{\kappa^{(q)}(1 - \kappa^{(q)})}], \tag{14}$$

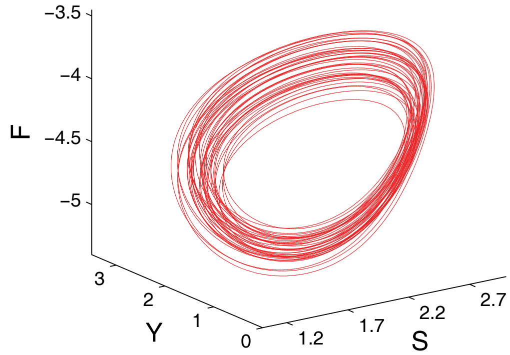

Fig. 5 Chaotic business cycle of system A_4.

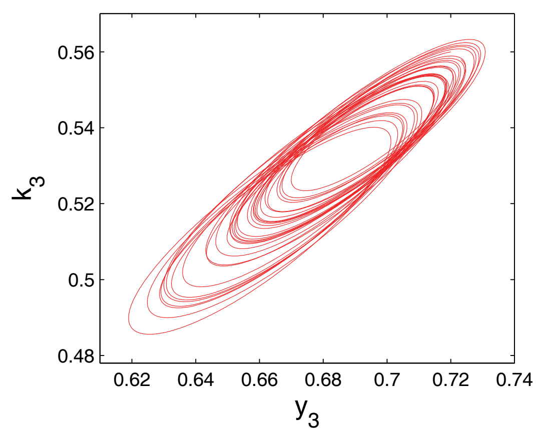

Fig. 6 Attraction of cyclic chaos by system A_5.

provided that the number on the right-hand side of the formula (14) belongs to the interval $[3.8-\varepsilon, 3.8+\varepsilon]$. In other words, we apply a perturbation in the amount of $\frac{3.8(2\kappa^{(q)}-1)(\kappa_j-\kappa^{(q)})}{\kappa^{(q)}(1-\kappa^{(q)})}$ to the parameter $\mu = 3.8$ of the logistic map, if the trajectory $\{\kappa_j\}$ is sufficiently close to the target periodic orbit. This perturbation makes the map behave regularly so that at each iteration step the orbit κ_j is forced to be located in a small neighborhood of a previously chosen periodic orbit $\kappa^{(q)}$. Unless the parameter perturbation is applied, the orbit κ_j moves away from $\kappa^{(q)}$ due to the instability. If $\left|\frac{3.8(2\kappa^{(jq)}-1)(\kappa_j-\kappa^{(q)})}{\kappa^{(q)}(1-\kappa^{(q)})}\right| > \varepsilon$, we set $\bar{\mu}_j = 3.8$, so that the system evolves at its original parameter value, and wait until the trajectory $\{\kappa_j\}$ enters a sufficiently small neighborhood of the periodic orbit $\kappa^{(q)}$, $q = 1,2,\ldots,p$, such that the inequality $-\varepsilon \leq \frac{3.8(2\kappa^{(q)}-1)(\kappa_j-\kappa^{(q)})}{\kappa^{(q)}(1-\kappa^{(q)})} \leq \varepsilon$ holds. If this is the case, the control of chaos is not achieved immediately after switching on the control mechanism. Instead, there is a transition time before the desired periodic orbit is stabilized. The transition time increases if the number ε decreases [11].

The chaos of system A_2 can be stabilized by controlling an unstable periodic orbit of the logistic map A_1, since the map gives rise to the presence of chaos in the system. By applying the OGY control method around the fixed point 2.8/3.8 of the logistic map, we stabilize the corresponding unstable 2−periodic solution of system A_2. The simulation result is seen in Figure 7. We used the same initial data as in Figure 2. It is seen in Figure 7 that the OGY control method successfully controls the chaos of system A_2. The control is switched on at $t = \theta_{50}$

and switched off at $t = \theta_{280}$. The values $\kappa_0 = 0.63$ and $\varepsilon = 0.08$ are utilized in the simulation. The control becomes dominant approximately at $t = 150$ and its effect lasts approximately until $t = 340$, after which the instability becomes dominant and irregular behavior develops again.

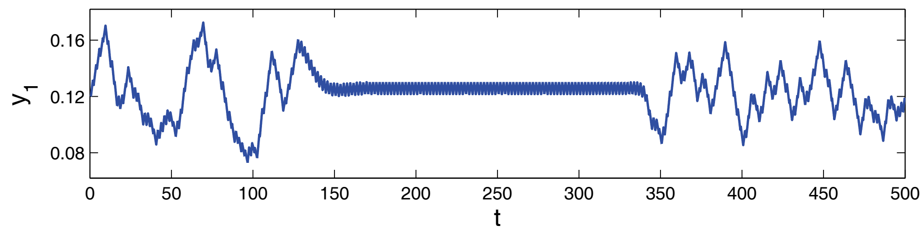

Fig. 7 The chaos control of system A_2. The OGY control method is applied around the fixed point $2.8/3.8$ of the logistic map. The value $\varepsilon = 0.08$ is used.

Next, we will demonstrate the stabilization of an unstable quasi-periodic solution of system A_3. We suppose that an unstable quasi-periodic solution of A_3 can be stabilized by controlling the chaos of system A_2. We use the solution shown in Figure 7 as the perturbation in system A_3, and represent in Figure 8 the solution of A_3 with $y_2(t_0) = 0.95$, $k_2(t_0) = 0.38$, where $t_0 = 0.63$. Similarly to system A_2, it seen in the figure that the chaos of A_3 is controlled approximately for $150 \leq t \leq 340$.

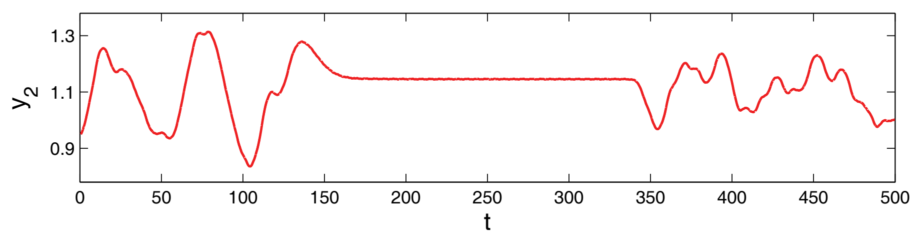

Fig. 8 The chaos control of system A_3. It is observable in the figure that controlling the chaos of system A_3 makes the chaos of system A_2 to be also controlled.

To reveal that the stabilized solution is indeed quasi-periodic, we depict in Figure 9 the graph of the same solution for $200 \leq t \leq 300$. Figure 9 manifests that application of the OGY control method to system A_2 makes an unstable quasi-periodic solution of A_3 to be stabilized. On the other hand, the stabilized torus of the coupled system $A_2 - A_3$ is shown in Figure 9.

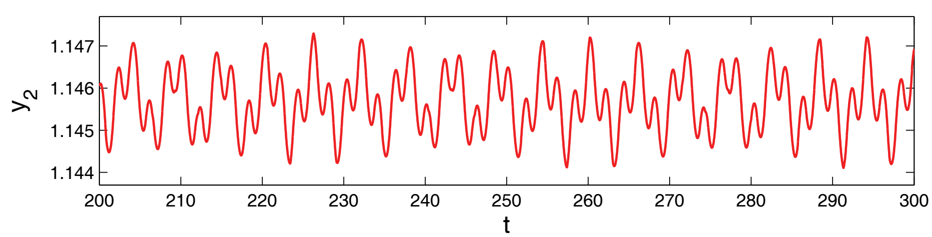

Fig. 9 The stabilized quasi-periodic solution of system A_3.

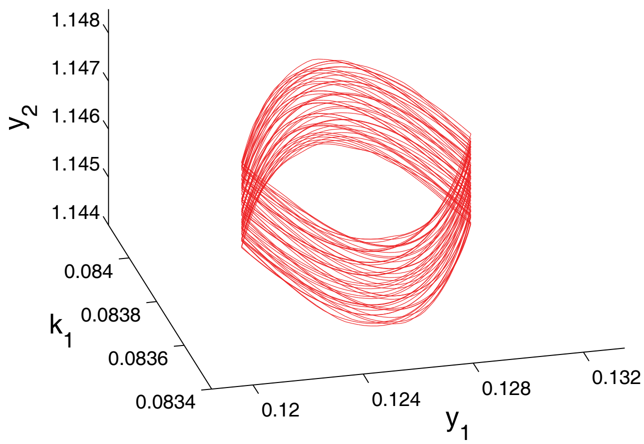

Fig. 10 The stabilized torus of the coupled Kaldor-Kalecki system $A_2 - A_3$

5 Chaotic business cycles in Kaldor-Kalecki model with time delay

This section considers the phenomenon of chaos extension by utilizing an economical model with time lag (15). We are devoting a separate discussion to this model, since the result for this case does not have theoretical support at the moment. The extension of chaos can be only observed numerically in our example, but in the future one can prove the entrainment of the limit cycle by chaos for functional differential equations using the results of the paper [42]. In this section, we will demonstrate numerically the formation of chaotic business cycles in the Kaldor-Kalecki model with time delay.

Let us take into account the system,

$$\begin{aligned} & x'' + 5(x^2-1)x' + x = 5\cos(2.467t), \} B_1 \\ & \left.\begin{aligned} Y' &= 1.5\,[\tanh(Y) - 0.25K - (4/3)Y] + 0.0045x(t), \\ K' &= \tanh(Y(t-\tau)) - 0.5K. \end{aligned}\right\} B_2 \end{aligned} \tag{15}$$

Equation B_1 is the chaotic Van der Pol oscillator, which is used as the generator system in (15). Van der Pol type equations have played a role in economic modelling [5, 26, 70]. It is shown by Parlitz and Lauterborn [71] that equation B_1 is chaotic through period-doubling cascade. The process of period-doubling is described by Thompson and Stewart [72]. This implies that there are infinitely many *unstable* periodic solutions of B_1, all with different periods. Due to the absence of stability, any solution that starts near the periodic motions behaves *irregularly*. We will interpret the solution $x(t)$ as an irregular productivity shock.

System B_2 is the Kaldor-Kalecki model and it is the result of the perturbation of the model (10) of an aggregate economy with a productivity shock. We will observe numerically the appearance of a chaotic business cycle, and in particular, the entrainment of the limit cycle of system (10) by chaos, in the next simulations.

Let us take $\tau = 5.5$ in B_2 so that the system possesses an orbitally stable limit cycle in the absence of perturbation [64]. We make use of the solution $x(t)$ of B_1 with $x(0) = 1.1008$, $x'(0) = -1.5546$, and present in Figure 11 the solution of B_2 with the initial condition $Y(t) = u(t)$ and $K(t) = v(t)$ for $t \in [-\tau, 0]$, where $u(t) = -0.057$ and $v(t) = 0.063$ are constant functions. Figure 11 reveals that the dynamics of B_2 exhibits chaotic business cycles. This result shows that our theory of chaotic business cycles can be extended to systems with time delay.

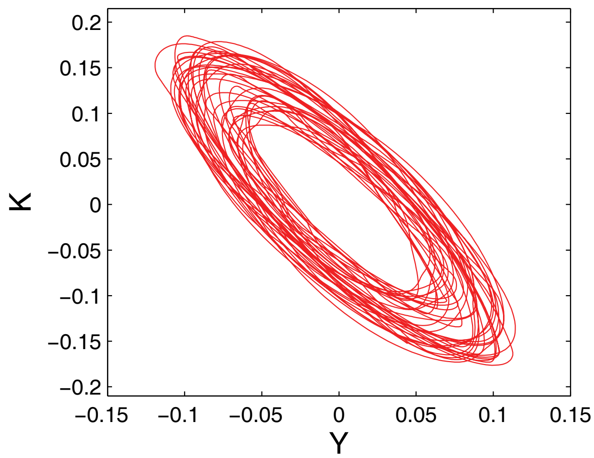

Fig. 11 The appearance of chaotic business cycle in the Kaldor-Kalecki model B_2.

6 Chaos extension versus synchronization

Generalized synchronization characterizes the dynamics of a response system that is driven by the output of a chaotic driving system [11, 43, 45, 73, 74]. Suppose that the dynamics of the drive and response are governed by the following systems with a skew product structure

$$x' = D(x) \tag{16}$$

and

$$y' = R(y, K(x)), \tag{17}$$

respectively, where $x \in \mathbb{R}^p, y \in \mathbb{R}^q$. Synchronization [45] is said to occur if there exist sets I_x, I_y of initial conditions and a transformation ϕ, defined on the chaotic attractor of (16), such that for all $x(0) \in I_x$, $y(0) \in I_y$ the relation $\lim_{t\to\infty} \|y(t) - \phi(x(t))\| = 0$ holds. In this case, a motion that starts on $I_x \times I_y$ collapses onto a manifold $M \subset I_x \times I_y$ of synchronized motions. The transformation ϕ is not required to exist for the transient trajectories. When ϕ is the identity, the identical synchronization takes place [11, 44].

It is formulated in paper [43] that generalized synchronization occurs if and only if for all $x_0 \in I_x$, $y_{10}, y_{20} \in I_y$, the following asymptotic stability criterion holds:

$$\lim_{t\to\infty} \|y(t, x_0, y_{10}) - y(t, x_0, y_{20})\| = 0,$$

where $y(t, x_0, y_{10})$, $y(t, x_0, y_{20})$ denote the solutions of (17) with $y(0, x_0, y_{10}) = y_{10}$, $y(0, x_0, y_{20}) = y_{20}$ and the same $x(t)$, $x(0) = x_0$.

A numerical method that can be used to investigate coupled systems for generalized synchronization is the auxiliary system approach [11, 73]. Let us investigate the coupled economic model $A_2 - A_4$ for generalized synchronization by means of the auxiliary system approach.

Consider the auxiliary system

$$\begin{aligned} S_0' &= 0.23Y_0 + 0.1S_0(1 - Y_0^2), \\ Y_0' &= 0.5(S_0 + F_0) + 2y_3(t), \\ F_0' &= 0.19S_0 - 0.25Y_0. \end{aligned} \tag{18}$$

System (18) is an identical copy of system A_4.

By marking the trajectory of system $A_2 - A_4 - (18)$ with initial data $y_1(t_0) = 0.12$, $k_1(t_0) = 0.08$, $S(t_0) = 1.67$, $Y(t_0) = 0.94$, $F(t_0) = -5.15$, $S_0(t_0) = 2.63$, $Y_0(t_0) = 0.84$, $F_0(t_0) = -2.89$ at times $t = \theta_j$, $j = 1,2,\ldots,9000$, and omitting the first 500 iterations, we obtain the stroboscopic plot whose projection on the $Y - Y_0$ plane is shown in Figure 12. Since the plot is not placed on the line $Y_0 = Y$, we conclude that generalized synchronization does not occur in the couple $A_2 - A_4$.

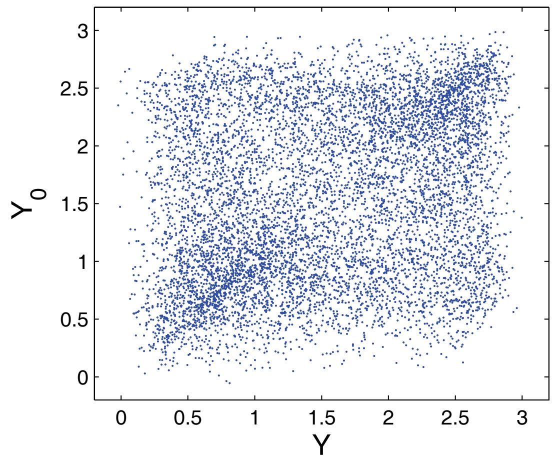

Fig. 12 The auxiliary system approach shows that the systems A_2 and A_4 are not synchronized in the generalized sense.

It is worth noting that generalized synchronization does not take place also in the dynamics of the unidirectionally coupled subsystems B_1 and B_2, which are mentioned in Section 5, and this can be verified by means of the auxiliary system approach [11, 73] as well.

7 The global unpredictability, synergetics and self-organization

The idea of the transition of chaos from one system to another, as well as the arrangement of chaos in an ordered way, can be viewed through the lens of self-organization [46, 75]. Durrenmatt [76] explained that "... a system is self-organizing if it acquires a spatial, temporal or functional structure without specific interference from the outside. By 'specific' we mean that the structure of functioning is not impressed on the system, but the system is acted upon from the outside in a nonspecific fashion." There are three approaches to self-organization, namely thermodynamic (dissipative structures), synergetic and the autowaves approach. For the theory of dynamical systems (e.g. differential equations) the phenomenon means that an autonomous system of equations admits a regular and stable motion (periodic, quasiperiodic, almost periodic). These are called autowaves processes [77] or self-excited oscillations [78] in the literature. We are inclined to add to the list one more phenomenon-chaos extension. For example, consider the collection of systems $A_1, A_2, \ldots, A_5$ once again, where A_1 is the original generator of chaos. Because of the connections and the conditions discovered in our analysis, all the other subsystems, $A_2, \ldots, A_5$, are also chaotic. We believe this is a self-organization phenomenon, that is, a coherent behavior of a large number of systems [46].

Haken [46], a German theoretical physicist, introduced a new interdisciplinary field of science, synergetics, which deals with the origins and the evolution of spatiotemporal structures. Synergetics is based in large part on the dynamical systems theory. One of the crucial features of systems in synergetics is self-organization, which was discussed above. According to Haken [46], the central question in synergetics is whether there are general principles which govern the self-organized formation of structures and/or functions. The main concepts of the theory are instability, order parameters, and slaving [46].

Instability is understood as the formation or collapse of structures (patterns) [75]. This is very common in fluid dynamics, lasers, chemistry and biology [46, 75, 79, 80]. A number of examples of instability can be found in the literature on morphogenesis [81], and pattern formation examples can be found in fluid dynamics. The phenomenon is called instability because a former state of fluid transforms into a new one, loses its ability to persist, and becomes unstable. One can view the formation of chaos in systems $A_2,\ldots,A_5$ in our results as instability. Even though processes in finite dimensional spaces are considered, chaotic attractors are assumed to be not single trajectories, but collections of infinitely many trajectories with complex topologies. One may say that they are somehow in-between objects of ordinary differential equations and partial differential equations. This allows us to also talk about dissipative structures [75], due to the "density" of the chaotic trajectories in the space.

Order parameters, when applied to differential equations theory, are those phase variables whose behavior produces the main properties of a macroscopic structure and which dominate all other variables in the formation, so that the latter can even depend on the order parameters functionally. The dependence that is proved (discovered) mathematically is what is called slaving. It is not difficult to see that the variables of system A_1 are order parameters, and they determine the chaotic behavior of the joined systems' variables.

8 Conclusion

We provide examples of models of aggregate economy where the main variables exhibit cycle-like motion with chaotic elements. Thus, we obtain an irregular business cycle in a deterministic setting. This provides a modelling alternative to the business cycle literature relying on stochastic variation in the economy. Additionally, our investigation highlights the variety of ways of generating chaos in an economic model. Previous work has focused on generating chaos and, in particular, chaotic business cycles *endogenously* (see [6, 26, 27, 57, 58]). Our method of creating chaos has its own relevance for economics, since we show the role of *exogenous* shocks in the appearance of chaos in models that otherwise do not exhibit irregular behavior. It can also be said that our work provides a missing link in the research on the origins of irregularities in economic time series. While the literature on endogenous chaos was a response to the view that exogenous stochastic shocks are the source of fluctuations in the economy (see [3]), this paper is a response to the former, in that it provides a role for exogenous chaotic disturbances in producing these fluctuations, and thus completes the circle.

Baumol and Benhabib [3] summarized the significance of chaos research for economics: "Chaos theory has at least equal power in providing caveats for both the economic analysis and the policy designer. For example, it warns us that apparently random behavior may not be random at all. It demonstrates dramatically the dangers of extrapolation and the difficulties that can beset economic forecasting generally. It provides the basis for the construction of simple models of the behavior of rational agents, showing how even these can yield extremely complex developments. It has served as the basis for models of learning behavior and has been shown to arise naturally in a number of standard equilibrium models. It offers additional insights about the economic source of oscillations in a number of economic models."

Indeed, applications of chaos theory have illustrated the possibility of producing complex dynamics in deterministic settings [5, 26, 27, 57–59, 82], with some papers specifically focusing on building "chaotic business cycles" [83]. Chaos is generated endogenously, and its appearance hinges on the values of some crucial parameters of the model. The main novelty of the present study is that we start with a model that is not endogenously complex. In one case (model A_4 in the main body), we assume that the system has a limit cycle, where the limit cycle is understood to be a closed orbit that is also an attractor [84]. We then subject the model to chaotic exogenous shocks and obtain a perturbed system that admits chaotic motions. The chaos emerging around the original limit cycle is cycle-like, and therefore can be called a chaotic business cycle. This approach is based on rigorous mathematical theory [17, 20], and we provide numerical simulations. In another case (model A_5 in the main body), we subject a system with an asymptotically stable equilibrium to chaotic cyclic shocks, which produces a chaotic business cycle in the original model, as well. We demonstrate this scenario with simulations, as this approach does not have theoretical underpinnings as yet.

In this paper we show that it is possible to produce a chaotic business cycle in a very natural way - take a system of differential equations with a limit cycle as a point of departure, and introduce a chaotic exogenous disturbance. An example of an exogenous disturbance is a technology shock to the economy which affects output, holding all other variables constant. We describe it using solutions of chaos generator models. We use them to demonstrate the proposed approach, and other formulations can be studied in future work. For example, one can use actual economic time series, such as commodity prices, that have been tested for deterministic chaos [4,53,85,86]. Moreover, shocks other than technology shocks can be considered, in view of the on-going debate between two literatures supporting and rejecting the importance of technology shocks for generating business cycles [22,87,88].

Our results give more theoretical lights on the processes, as we suggest a mathematical apparatus, which describe rigorously *extension of chaos*, increases its complexity, and provides new structures of *effective control* for clusters of economic models.

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Non-Abelian Bell Polynomials and Their some Applications for Integrable Systems

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Abstract

The noncommutative Bell polynomials and their dual Bell polynomials are presented, respectively, which are extensively applied to mathematics and physics. We make use of them to exhibit a method for generating integrable hierarchies of evolution equations. As applications, we obtain the Burgers hierarchy and a convection-diffusion equation which can be applied to fluid mechanics, specially, be used to represent mass transformations in fluid systems under some constrained conditions. As reduced cases, the Burgers equation which has extensive applications in physics is followed to produce. Furthermore, we obtain a set of nonlinear evolution equations with four potential functions which reduces to a new nonlinear equation similar to the Calogero-Degasperis-Fokas equation. Finally, we discrete the convection-diffusion equation and obtain its three kinds of finite-difference schemes, that is,the weighted implicit difference scheme and the Lax difference scheme. Their some properties including truncation errors, compatibilities and stabilities based on the Von Neumann condition are discussed in detail.

1 Introduction

Since the notation on the Bell polynomials was proposed [1], some various forms of the Bell polynomials were introduced. For example, the simplest Bell polynomials are given by exponential powers as follows

$$Y_{nt}(y) = Y_n(y_t, y_{2t}, \ldots, y_{nt}) = e^{-y}\partial_t^n e^y, \tag{1}$$

where $y = e^{\alpha t} - \alpha_0 \equiv \alpha_1 t + \frac{\alpha_2}{2!}t^2 + \cdots$.

The first three members are showed as follows

$$Y_0 = 1,\ Y_1 = y_t,\ Y_2 = y_t^2 + y_{2t},\ Y_3 = y_t^3 + 3y_t y_{2t} + y_{3t}. \tag{2}$$

Later, Gilson, Lambert, et al. [2,3] proposed a general generalization of the Bell polynomials. That is, for $n_k \geq 0, k = 1,2,\ldots,l$ being arbitrary integers. Set $f = f(x_1,\ldots,x_l)$ to be a C^∞ multi-variable function, the following polynomials

$$Y_{n_1x_1,\ldots,n_lx_l}(f) \equiv \exp(-f)\partial_{x_1}^{n_1}\ldots\partial_{x_l}^{n_l}\exp(f) \tag{3}$$

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are called the multi-dimensional Bell polynomials.

A simple case, we choose $f = f(x,t)$ and employ (3) to get the following reductions

$$Y_x(f) = f_x, Y_{2x}(f) = f_{2x} + f_x^2, Y_{3x}(f) = f_{3x} + 3f_x f_{2x} + f_x^3, \ldots$$

In terms of the Bell polynomials (3), the multi-dimensional binary Bell polynomials can be defined by

$$\mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(v,w) = \mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(f)|_{f_{r_1x_1,\ldots,r_lx_l}} = \begin{cases} v_{r_1x_1,\ldots,r_lx_l}, & r_1+\cdots+r_l \text{is odd}, \\ w_{r_1x_1,\ldots,r_lx_l}, & r_1+\cdots+r_l \text{is even}. \end{cases} \tag{4}$$

In [2], a link between the binary Bell polynomials and the standard Hirota bilinear operators was given by

$$\mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(v = \ln\frac{F}{G}, w = \ln FG) = (FG)^{-1}D_{x_1}^{n_1}\ldots D_{x_l}^{n_l}F \bullet G, \tag{5}$$

where

$$D_{x_1}^{n_1}\ldots D_{x_l}^{n_l} f \bullet G = (\partial_{x_1} - \partial_{x_1'})^{n_1}\ldots(\partial_{x_l} - \partial_{x_l'})^{n_l} F(x_1,\ldots,x_l)G(x_1',\ldots,x_{x_l}')|_{x_1'=x_1,\ldots,x_l'=x_l}.$$

A special case of (5) is given by when $F = G$:

$$G^{-2}D_{x_1}^{n_1}\ldots D_{x_l}^{n_l} G \bullet G = \mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(0, q = 2\ln G) = \begin{cases} 0, & n_1+\cdots+n_l \text{is odd}, \\ P_{n_1x_1,\ldots,n_lx_l}(q), & n_1+\cdots+n_l \text{is even}. \end{cases} \tag{6}$$

Based on the above known theory, the integrability of some nonlinear differential equations were discussed, including bilinear representations, Lax pairs, Bäcklund transformations and infinite conservation laws, and so on[4-8]. Specially, Fan, Liu, et al[4,9,10] extended the Bell polynomials to the super-integrability, and obtained Lax pairs, bilinear representations of some super-integrable equations. All the works mentioned as above are based on the commutative Bell polynomials. However, as for the investigation of noncommutative Bell polynomials and their applications, few results were exhibited up to now, to our knowledge. The papers[11,12] once introduced some properties of noncommutative Bell polynomials and obtained Burgers hierarchy and the KP hierarchy as well as the KdV hierarchy with the help of the Sato theory and the Hirota formulism. Based on the facts, we shall choose suitable Lax pairs with the aid of powers of first-order differential operator to generate integrable hierarchies of evolution type, which are expressed by noncommutative Bell polynomials. A few examples are showed, which produce the Burgers hierarchy and a convection-diffusion equation under some constrained conditions. Specially, under without any condition, we generate a set of equations with 4-potential functions which reduce to a new nonlinear equation similar to the Calogero-Degasperis-Fokas equation. It is remarkable that the way to generate the Burgers hierarchy is different from that in [11], where it was given directly by a recurrence relation given by Ma Wen-Xiu. But, in the paper we derive it starting from a Lax pair, which is expressed by dual Bell polynomials. As reduced cases, we get the Burgers equation and the convection-diffusion equation, which have important applications in physics. Finally, we produce two various weighted implicit difference schemes and a Lax-difference scheme of the convection-diffusion equation. Furthermore, we discuss some properties of the difference schemes including the truncation errors, compatibilities and stabilities by making use of the Von Neumann condition.

2 Noncommutative Bell polynomials

Definition 1 [11]: Let A be an associative (generally noncommutative) algebra over the field of real numbers R with a unit element 1, and let $y_1, y_2, \ldots,$ denote variable elements of A. The unique solution $B_n = B_n(y_1,\ldots,y_n)(n = 0,1,2,\ldots)$ of the following initial value problem

$$B_{n+1} =: \sum_{k=0}^{n} \binom{n}{k} B_{n-k} y_{k+1}, \tag{7}$$

for $n = 0,1,2,\ldots$ together with the initial condition

$$B_0 = 1 \tag{8}$$

are called the (noncommutative) Bell polynomials. The homogeneous parts are defined by

$$B_n = \sum_{p=1}^{n} B_{n,p}, B_{n,p}(\lambda y_1,\ldots,\lambda y_n) = \lambda^p B_{n,p}(y_1,\ldots,y_n). \tag{9}$$

The first four members in (7) present that

$$\begin{cases} B_1 = y_1, B_2 = y_2 + y_1^2 = B_{2,1} + B_{2,2}, \\ B_3 = y_3 + y_2 y_1 + 2y_1 y_2 + y_1^3, \\ B_4 = y_4 + y_3 y_1 + 3y_2^2 + 3y_1 y_3 + y_2 y_1^2 + 2y_1 y_2 y_1 + 3y_1^2 y_2 + y_1^4. \end{cases} \tag{10}$$

When the Bell polynomials are commutative, they are just the previous Eq.(3). For example, B_3 is the same as $Y_{3x}(f)$ in Eq.(3) when B_3 is commutative.
The dual Bell polynomials are given by [11]:

$$B^*_{n+1} =: \sum_{k=0}^{n} \binom{n}{k} y_{k+1} B^*_{n-k}, B^*_0 = 1, \tag{11}$$

that is, the unique solutions $B^*_n = B^*_n(y_1,\ldots,y_n)(n = 0,1,2,\ldots)$ of the initial problem (11) are called the dual (noncommutative) Bell polynomials. Similar to the previous statement, we have

$$\begin{cases} B^*_n = \sum\limits_{d=1}^{n} B^*_{n,d}, \\ B^*_{n,d} = \sum\limits_{n_2,\ldots,n_d=1} \binom{n-1}{n_2}\binom{n_2-1}{n_3}\cdots\binom{n_{d-1}-1}{n_d} y_{n-n_2} y_{n_2-n_3}\cdots y_{n_{d-1}-n_d} y_{n_d}. \end{cases} \tag{12}$$

In what follows, we state the link between the Bell polynomials and the powers of a first-order differential operator.

Theorem 1. *([11]) Assume $L = D + u$, where $u \in A$, the derivative operator $D : A \to A$ is an endomorphism which possesses the property*

$$D(y_1 y_2) = (Dy_1)y_2 + y_1(Dy_2), D1 = 0. \tag{13}$$

Then we have

$$L^n = \sum_{k=0}^{n} \binom{n}{k} B_k(u,u',\ldots,u^{(n-1)}) D^{n-k}. \tag{14}$$

Proof It is easy to see that

$$L(y_1 y_2) = (D+u)(y_1 y_2) = (Dy_1)y_2 + y_1(Dy_2) + uy_1 y_2 = (Ly_1)y_2 + y_1(Dy_2),$$

by induction we have

$$L^n(y_1 y_2) = \sum_{k=0}^{n} \binom{n}{k} (L^k y_1)(D^{n-k} y_2).$$

Take $y_1 = 1, y_2 = y$, we have

$$L^n y = \sum_{k=0}^{n} \binom{n}{k} (L^k 1)(D^{n-k} y). \tag{15}$$

Set $y = u$, one infers that

$$L^{n+1}1 = L^n L^1 1 = L^n u = \sum_{k=0}^{n} \binom{n}{k} (L^k 1) D^{n-k} u,$$

$$L^0 = 1.$$

Thus, we arrive at

$$L^n 1 = B_n(u, u', \ldots, u^{(n-1)}).$$

That is,

$$(D+u)^n 1 = B_n(u, u', \ldots, u^{(n-1)}), \tag{16}$$

which gives a way for generating the Bell polynomials by the first-order differential operator. From (16), we get a corollary:

$$B_{n+1}(u, u', \ldots, u^{(n)}) = (D+u) B_n(u, u', \ldots, u^{(n-1)}). \tag{17}$$

In addition, suppose Ψ denotes a smooth $N \times N$ matrix-valued function, and U is the same size with Ψ. The first-order linear initial value problem

$$\frac{d\Psi}{dx} = \Psi U, \Psi(0) = I, \tag{18}$$

where I stands for a unit matrix with the same size with Ψ, has the exponetial integral. By induction, we see that

$$\Psi^{(n)} = \Psi B_n(U, U', \ldots, U^{(n-1)}). \tag{19}$$

Similarly, as for the following linear initial value problem

$$\frac{d\Phi}{dx} = U\Phi, \Phi(0) = I, \tag{20}$$

where Φ and U are all $N \times N$ matrix-valued functions, we have

$$\Phi^{(n)} = B_n^*(U, U', \ldots, U^{(n-1)})\Phi. \tag{21}$$

3 Integrable hierarchies of equations and some reductions

The methods for generating integrable hierarchies of evolution equations are much richer, here we only want to employ the Tu scheme [13] to discuss producing integrable hierarchies combined with the noncommutative Bell polynomials. Now we first simply recall the Tu scheme.

Let G denote a finite-dimensional Lie algebra over C, and $\tilde{G}$ be the corresponding loop algebra

$$\tilde{G} = G \otimes C[\lambda, \lambda^{-1}],$$

where $C[\lambda, \lambda^{-1}]$ is the set of Laurent polynomials in λ. Assume $\{E_1, \ldots, E_d\}$ is a basis of G. Then $\{E_1(n), \ldots, E_d(n) | n \in \mathbf{Z}\}$ provides a basis of $\tilde{G}$, where $E_i(n) = E_i \otimes \lambda^i = E_i \lambda^n$. If set

$$\text{ker ad R} = \{x | x \in \tilde{G}, [x, R] = 0\},$$

$$\text{im ad R} = \{x | \exists y \in \tilde{G}, x = [y, R]\}$$

then it holds that

$$\tilde{G} = \text{ker ad R} \oplus \text{im ad R}, \text{ker ad R} \quad \text{is commutative}.$$

Different gradations of $\tilde{G}$ can be available, such as deg $(X \otimes \lambda^n) = n, X \in G$. Based on above notations, a spectral problem is introduced by

$$\varphi_x = U\varphi, \tag{22}$$

where $U = R + \sum_{i=1}^{p} u_i e_i, R, e_1, \ldots, e_p \tilde{G}$ satisfy that
(i)$R, e_1, \ldots, e_p$ are linear independent,
(ii) R is pseudo-regular,
(iii) $\alpha > 0, \alpha > \varepsilon_i (i = 1, 2, \ldots, p)$,
here $\alpha = \deg(R), \varepsilon_i = \deg(e_i)$.

Step 1: Solving the stationary 0-curvature equation

$$V_x = [U, V], \tag{23}$$

where U, V are the Lax matrices in spetral probelms

$$\varphi_x = U\varphi, \varphi_t = V\varphi. \tag{24}$$

Step 2: Taking a solution V of Eq.(24) and searching for a modified term $\Delta_n \in \tilde{G}$ such that for

$$V^{(n)} = (\lambda^n V)_+ + \Delta_n, \tag{25}$$

it holds that

$$-V_x^{(n)} + [U, V^{(n)}] \in Ce_1 + \ldots + Ce_p,$$

where $g_+ = \sum_{m=0}^{n} g_m \lambda^{n-m}$.

Step 3: With the aid of 0-curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0, \tag{26}$$

we could get an integrable hierarchies of evolution equations.

Step 4: Employing the trace identity proposed by Tu[13] deduces the Hamiltonian structure of Eq.(26). In what follows, we make use of the Tu scheme and the Bell polynomials to deduce two integrable hierarchies. Consider the following spectral problems

$$\frac{d\Psi}{dx} = \Psi U, \tag{27}$$

$$\frac{d\Psi}{dt} = \Psi^{(n)} + \Psi\Delta_n, \tag{28}$$

where $\Psi^{(n)}$ denotes $n-$th order derivative with respect to the variable x, Δ_n is a matrix with the same size as the matrix U. The compatibility condition of Eqs.(27) and (28) combined with Eq.(19) gives rise to

$$U_t = B_{n+1}(U, U', \ldots, U^{(n)}) - B_n(U, U', \ldots, U^{(n-1)})U + \Delta_{n,x} + [U, \Delta_n] \tag{29}$$

which denotes an integrable hierarchy expressed by the Bell polynomials.

If $N \times N$ matrices Φ and U satisfy the following linear differential equations

$$\frac{d\Phi}{dx} = U\Phi, \tag{30}$$

$$\frac{d\Psi}{dt} = \Psi^{(n)} + \Delta_n \phi. \tag{31}$$

Similarly, we can obtain the following evolution system combined with Eq.(21):

$$U_t = B^*_{n+1}(U,U',\ldots,U^{(n)}) - UB^*_n(U,U',\ldots,U^{(n-1)}) + \Delta_{n,x} + [\Delta_n,U]. \tag{32}$$

Thus, with the help of the Tu scheme and Eqs.(18)-(21), we obained two integrable hierarchies (29) and (32). In the following, we give their applications by examples. If U represents a scalar function smooth enough, then from Eq.(11) we have

$$B^*_1 = U, B^*_2 = U^2 + U_x, B^*_3 = U^3 + UU_x + 2U_xU + U_{xx}, \tag{33}$$

which is a special case of Eq.(10). Take $\Delta_n = 0$, Eq.(32) admits when $n = 2$:

$$U_t = B^*_3 - UB^*_2 = 2U_xU + U_{xx}, \tag{34}$$

which is the exact Burgers equation. Therefore, when take $\Delta_n = 0$, Eq.(32) is just right the Burgers hierarchy

$$U_t = B^*_{n+1}(U,U',\ldots,U^{(n)}) - UB^*_n(U,U',\ldots,U^{(n-1)}) = DB^*_n(U,U',\ldots,U^{(n-1)}), \tag{35}$$

which is consistent with the reslut in [11].

Remark 1: In paper[11,14],the authors directly gave a Bell recurrence relation. When the variable U is a scalar function, the Bell polynomials are the same with their dual Bell polynomials. Hence, Eq.(35) is completely same as that in [11].

If U denotes a $N\times N$ matrix, then Eq.(34) is a formal matrix Burgers equation which can derive some integrable equations. As an illustrated example, we consider the 2×2 matrix U, and set $U = \sigma h + qe + rf$, where $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, q and r are all smooth derivative functions, σ stands for a parameter, by using Eq.(34), we have that

$$U_{xx} + 2U_xU = \begin{pmatrix} 2q_xr & -2\sigma q_x + q_{xx} \\ 2\sigma r_x + r_{xx} & -2r_xq \end{pmatrix}. \tag{36}$$

Take $\Delta_2 = \begin{pmatrix} \delta_1 & \delta_2 \\ \delta_3 & \delta_4 \end{pmatrix}$, and substitute it into Eq.(34), we get that

$$\begin{cases} q_t = -2\sigma q_x + q_{xx} + q(\delta_1 - \delta_4) - 2\sigma\delta_2 + \delta_{2x}, \\ r_t = 2\sigma r_x + r_{xx} + 2\sigma\delta_3 + r(\delta_4 - \delta_1) + \delta_{3x}, \end{cases} \tag{37}$$

$$\delta_{1x} + r\delta_2 - q\delta_3 + 2q_xr = 0, \tag{38}$$

$$\delta_{4x} + q\delta_3 - r\delta_2 + 2r_xq = 0. \tag{39}$$

Case 1: When $\delta_1 = \delta_4 = \sigma = 0, qr = \alpha(t)$, here $\alpha(t)$ is an arbitrary function in t, $\delta_2 = \delta_3 = 0$, Eq.(37) reduces to the well-known heat equation

$$u_t = u_{xx}.$$

Case 2: When $\delta_1 = \delta_4 = \delta = -qr, \delta_2 = e^{2\sigma x}, \delta_3 = e^{-2\sigma x}$, Eq.(37) reduces to convection-diffusion equations, respectively:

$$q_t = -2\sigma q_x + q_{xx}, \tag{40}$$

$$r_t = 2\sigma r_x + r_{xx}. \tag{41}$$

Equations(38) and (39) become

$$re^{2\sigma x} - qe^{-2\sigma x} + q_xr - qr_x = 0. \tag{42}$$

That is, under the constraine condition (42), Eq.(37) can be reduced to the convection-diffusion equation which describes physical phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes: diffusion and convection. Hencefore, it is important to further investigate its properties.

Case 3: Multiply r,q in the first-equation and the second equation in Eq.(37), then add them together, we can get

$$(qr)_t = 2\sigma(qr_x - rq_x) + rq_{xx} + qr_{xx} + 2\sigma q\delta_3 - 2\sigma r\delta_2 + r\delta_{2x} + q\delta_{3x}. \tag{43}$$

Set $r = \sigma = 1$ Eq.(43) reduces to

$$q_t = -2q_x + q_{xx} + 2q\delta_3 - 2\delta_2 + \delta_{2x} + q\delta_{3x}, \tag{44}$$

which is a generalized convection-diffusion equation containing the free terms δ_2 and δ_3. Oviously, Eq.(40) is a special case of Eq.(44).

Remark 2: If set $\Delta_n = 0$ in Eq.(29), we can get an integrable hierarchy of evolution type:

$$U_{t_n} = B_{n+1}(U,U',\ldots,U^{(n)}) - B_n(U,U',\ldots,U^{(n-1)})U. \tag{45}$$

With the help of Eq.(17), Eq.(45) can be written as

$$U_{t_n} = DB_n(U,U',\ldots,U^{(n-1)}) + [U,B_n(U,U',\ldots,U^{(n-1)})]. \tag{46}$$

When U is a scalar smooth function in x and t, Eq.(45) is the same as Eq.(35), they are all the Burgers hierarchies. When U is a $N\times N$ matrix, obviously, Eqs.(35) and (45) are all the generalized forms of the Burgers hierarchy[11]. A special case of Eq.(46) when $n = 3$ presents that

$$U_t = D(U^3 + U_{xx}) + 2U^2U_x - UU_xU - U_xU^2 + D(2UU_x + U_xU) + [U,U_{xx}]. \tag{47}$$

If set

$$U = \begin{pmatrix} s & q \\ r & w \end{pmatrix},$$

we have

$$D(U^3 + U_{xx}) = \begin{pmatrix} \partial(s^3 + 2qr + qrw + s_{xx}) & \partial(s^2q + q^2r + qsw + qw^2 + q_{xx}) \\ \partial(rs^2 + qr^2 + rw^2 + rsw + r_{xx}) & \partial(w^3 + 2qrw + qrs + w_{xx}) \end{pmatrix},$$

$$[U,U_{xx}] = \begin{pmatrix} qr - xx - q_{xx}r & sq_{xx} + qw_{xx} - qs_{xx} - wq_{xx} \\ rs_{xx} + wr_{xx} - sr_{xx} - rw_{xx} & rq_{xx} - qr_{xx} \end{pmatrix},$$

$$D(2UU_x + U_xU) = \begin{pmatrix} \partial(3ss_x + 2qr_x + q_xr) & \partial(2sq_x + 2qw_x + qs_x + wq_x) \\ \partial(2rs_x + 2wr_x + sr_x + w_xr) & \partial(2rq_x + qr_x + 3ww_x) \end{pmatrix},$$

$$2U^2U_x - UU_xU - U_xU^2 = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix},$$

where

$m_1 = 2qrs_x + qsr_x + 2qwr_x - r^2q_x - qrw_x - rsq_x - q_xrw,$

$m_2 = 2s^2q_x + qrq_x + 2qsw_x + qww_x - 2qss_x - q^2r_x - wsq_x - qws_x - q_xw^2,$

$m_3 = rss_x + 2rws_x + qrr_x + 2w^2r_x - wsr_x - r^2q_x - r_xs^2 - w_xrs - 2rww_x,$

$m_4 = 2rsq_x + rwq_x + qrw_x - qrs_x - 2qwr_x - qsr_x.$

Substituting the above matrix values into Eq.(47) yields that

$$\begin{cases} s_t = \partial(s^3+2qr+qrw+s_{xx})+qr_{xx}-q_{xx}r+\partial(ss_x+2qr_x+q_xr)+2qrs_x+qsr_x+2qwr_x \\ \quad -r^2q_x-qrw_x-rsq_x-q_xrw, \\ q_t = \partial(s^2q+q^2r+qsw+qw^2+q_{xx})+sq_{xx}+qw_{xx}-qs_{xx}-wq_{xx}+\partial(2sq_x+2qw_x+qs_x+wq_x) \\ \quad +2s^2q_x+qrq_x+2qsw_x+qww_x-2qss_x-a^2r_x-wsq_x-qws_x-q_xw^2, \\ r_t = \partial(rs^2+qr^2+rw^2+rsw+r_{xx})+rs_{xx}+wr_{xx}-sr_{xx}-rw_{xx}+\partial(2rs_x+2wr_x+sr_x+w_xr) \\ \quad +rss_x+2rws_x+qrr_x+2w^2r_x-wsr_x-r^2q_x-r_xs^2-w_xrs2rww_x, \\ w_t = \partial(w^3+2qrw+qrs+w_{xx})+rq_{xx}-qr_{xx}+\partial(2rq_{xx}+qr_{xx}+3ww_x)+2rsq_x+rwq_x \\ \quad +qrw_x-qrs_x-2qwr_x-qsr_x. \end{cases} \tag{48}$$

When $s=w=0, q=r$, Eq.(48) reduces to

$$\begin{cases} q_t = q_{xxx}+3q^2q_x, \\ qq_{xxx}+q_xq_{xx}=0, \end{cases} \tag{49}$$

which can be written as

$$q_t = -\frac{q_xq_{xx}}{q}+3q^2q_x. \tag{50}$$

Set $\frac{q_x}{q}=u_x$, then we have $q=e^u, q_x=e^uu_x, q_{xx}=e^u(u_x^2+u_{xx})$. Inserting these into Eq.(50) yields

$$u_t = -u_x^3-u_xu_{xx}+3e^{2u}u_x, \tag{51}$$

which is similar to the Calogero-Degasperis-Fokas equation.

4 Different discrete equations of the convection-diffusion equation

In the section we shall discuss some discrete equations and their stabilities of the convection-diffusion equation obtained as above in the paper so that these results could provide the potential approaches for generating some approximate solutions and properties of the convection-diffusion equation. Actually,in [15], the central difference scheme, the modified central difference scheme, the upwind difference scheme, Samarskii scheme and Crank-Nicolson difference scheme of the standard convection-diffusion equation were once presented. In what follows, we shall discuss its other difference schemes and some properties, such as the truncation errors, compatibility and stability. First of all, we introduce a few notations.

Denote an approximate value $u(x_j,t_n)$ by u_j^n on the mesh node (x_j,t_n). Set h and τ to denote the mesh spacings of the space variable x ad the time variable t. A first-order derivative of a smooth function $u(x,t)$ is given by

$$\frac{\partial u}{\partial x}(x_j,t_n) = \frac{u_{j+1}^n-u_j^n}{h}+O(h),$$
$$\frac{\partial u}{\partial t}(x_j,t_n) = \frac{u_j^{n+1}-u_j^n}{\tau}+O(\tau).$$

The second-order derivative is given by

$$\frac{\partial^2 u}{\partial x^2}(x_j,t_n) = \frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}+O(h^2),\ldots$$

In the section, we shall consider the various difference schemes of Eq.(40). A forward difference scheme of Eq.(40) presents

$$\frac{q_j^{n+1}-q_j^n}{\tau}+2\lambda\frac{q_{j+1}^n-q_j^n}{h} = \frac{q_{j+1}^n-2q_j^n+q_{j-1}^n}{h^2}, \tag{52}$$

which can be written as

$$q_j^{n+1} = q_j^n + \mu(q_{j+1}^n - q_j^n) + \nu(q_{j+1}^n - 2q_j^n + q_{j-1}^n), \tag{53}$$

where $\mu = \frac{\tau}{h}, \nu = \frac{\tau}{h^2}$.

A backward difference scheme of Eq.(40) is given by

$$\frac{q_j^n - q_j^{n-1}}{\tau} + 2\lambda \frac{q_{j+1}^n - q_j^n}{h} = \frac{q_{j+1}^n - 2q_j^n + q_{j-1}^n}{h^2}, \tag{54}$$

or

$$\frac{q_j^n - q_{j^{n-1}}}{\tau} + 2\lambda \frac{q_{j+1}^n - q_j^n}{h} = \frac{q_{j+1}^n - 2q_j^n + q_{j-1}^n}{h^2}. \tag{55}$$

That is, we have

$$q_j^n = q_j^{n-1} + \mu(q_{j+1}^n - q_j^n) + \nu(q_{j+1}^n - 2q_j^n + q_{j-1}^n), \tag{56}$$

or

$$q_j^n = q_j^{n-1} + \mu(q_{j+1}^{n-1} - q_j^{n-1}) + \nu(q_{j+1}^n - 2q_j^n + q_{j-1}^n). \tag{57}$$

Rewrite Eq.(52) as the following form

$$\frac{q_j^n - q_j^{n-1}}{\tau} + 2\lambda \frac{q_{j+1}^{n-1} - q_j^{n-1}}{h} = \frac{q_{j+1}^{n-1} - 2q_j^{n-1} + q_{j-1}^{n-1}}{h^2}. \tag{58}$$

Multiply Eq.(54) by θ, Eq.(58) by $(1-\theta)$, then add them together, we can obtain that

$$-\theta(\mu+\nu)q_{j+1}^n + [1+\theta(2\nu+\mu)]q_j^n - \theta\nu q_{j-1}^n = (1-\theta)(\nu+\mu)q_{j+1}^{n-1} - (1-\theta)(\mu+2\nu)q_j^{n-1} + (1-\theta)\nu q_{j-1}^{n-1}, \tag{59}$$

where $0 \le \theta \le 1$.

Similarly, by employing (55) and (58), we can get that

$$-\theta\nu q_{j+1}^n + (1+2\theta\nu)q_j^n - \theta\nu q_{j-1}^n = [\mu+(1-\theta)\nu]q_{j+1}^{n-1} - [\mu+2(1-\theta)\nu]q_j^{n-1} + (1-\theta)\nu q_{j-1}^{n-1}, \tag{60}$$

where $0 \le \theta \le 1$.

We call Eqs.(59) and (60) the weighted implicit difference schemes, they are all the two-level difference schemes.

In what follows, we only discuss some properties of Eq.(60), the properties of Eq.(59) can be discussed similarly.

Definition 2 ([15]) (Truncation error): Suppose $F(u) = 0$ is a partial differential equation, the corresponding finite difference scheme is assumed to be $g_h(u_j^n) = 0$. If employ the solutions $u(x_j, t_n)$ on the mesh spacing node (x_j, t_n) replacing all the solutions u_j^n of the difference equation, then the difference value of two sides of the equation is called the truncation error, denoted by $T(x_j, t_n)$.

Definition 3 If the mesh spacings τ and $h \to 0$, the truncation error $T(x_j, t_n) \to 0$, then we call the difference scheme $g_h(u_j^n) = 0$ and the differential $F(u) = 0$ compatible.

Theorem 2 (Von Numann condition)**.** *A necessary condition for the finite-difference scheme*

$$u_j^{n+1} = C(x_j, \tau)u_j^n \tag{61}$$

to be stable presents that when $\tau \le \tau_0, n\tau \le T$, it holds that

$$|\lambda_j(G(\tau,k)| \le 1 + M\tau, j = 1, 2, \ldots, p,$$

where $C(x_j, \tau)$ is a difference operator,$G(\tau,k)$ is a growth factor of the difference scheme,$\lambda_j(G(\tau,k))$ represent the eigenvalues of $G(\tau,k)$.

With the aid of the above theorem, we can obtain the following results:

Theorem 3. *([14]) If the growth matrix $G(\tau,k)$ of a difference scheme is regular, then the Von Neumann condition is necessary and sufficient for the difference scheme to be stable.*

Now we consider the truncation errors of the finite-difference equation (60) by making use of Taylor expansion:

$$\begin{aligned} & T(x_j,t_n) \\ = & \frac{q(x_j,t_n)-q(x_j,t_{n-1})}{\tau}+2\lambda\frac{q(x_j+h,t_{n-1})-q(x_j,t_{n-1})}{h} \\ & -\theta\frac{q(x_j+h,t_n)-2q(x_j,t_n)+q(x_j-h,t_n)}{h^2} \\ & -(1-\theta)\frac{q(x_j+h,t_{n-1})-2q(x_j,t_{n-1})+q(x_j-h,t_{n-1})}{h^2} \\ = & \frac{\partial q}{\partial t}(x_j,t_{n-1})+2\lambda\frac{\partial q}{\partial x}(x_j,t_{n-1})-\frac{\partial^2 q}{\partial x^2}(x_j,t_{n-1})+\frac{1}{2}\frac{\partial^2 q}{\partial t^2}(x_i,t_{n-1})\tau+\lambda\frac{\partial^2 q}{\partial x^2}(x_i,t_{n-1})h \\ & +\theta(\frac{\partial^2 q}{\partial x^2}(x_j,t_{n-1})-\frac{\partial^2 q}{\partial x^2}(x_j,t_n))-\frac{\theta}{12}\frac{\partial^4 q}{\partial x^4}(x_j,t_n)h^2-\frac{1-\theta}{12}\frac{\partial^4 q}{\partial x^4}(x_j,t_{n-1})h^2+O(\tau^2+h^3) \\ = & \frac{1}{2}\frac{\partial^2 q}{\partial t^2}(x_j,t_n)\tau+\lambda\frac{\partial^2 q}{\partial x^2}(x_j,t_{n-1})h-\theta\frac{\partial^3 q}{\partial t\partial x^2}(x_j,\eta)\tau-\frac{\theta}{12}\frac{\partial^4 q}{\partial x^4}(x_j,t_n)h^2 \\ & -\frac{1-\theta}{12}\frac{\partial^4 q}{\partial x^4}(x_j,t_{n-1})h^2+O(\tau^2+h^3), \end{aligned} \tag{62}$$

where $\eta\in(t_n-\tau,t_n+\tau)$. Hence, the truncation error of Eq.(60) reads $O(\tau+h^2)$. When $\tau,h\to 0$, we have $T(x_j,t_n)\to 0$. Therefore, the difference scheme (60) is compatible with the difference equation (40).
Next, we shall discuss the stability of the difference scheme (60) by employing the Fourier method. Set $q_j^n=v^n e^{ikjh}, i^2=-1$ and insert it into Eq.(60), we obtain that

$$(-\theta\nu e^{ikh}-\theta\nu e^{-ikh}+2\theta\nu+1)v^n=[(\mu+(1-\theta)\nu)e^{ikh}+(1-\theta)\nu e^{-ikh}-2(1-\theta)\nu-\mu]v^{n-1}. \tag{63}$$

Hence, we get that again

$$v^n=\frac{[\mu+2(1-\theta)\nu](\cos kh-1)+i\mu\sin kh}{1+2\theta\nu(1-\cos kh)}v^{n-1}.$$

Thus, we obtain the growth factor

$$G(\tau,k)=\frac{[\mu+2(1-\theta)\nu](\cos kh-1)+i\mu\sin kh}{1+2\theta\nu(1-\cos kh)}, \tag{64}$$

the modulus of the complex-valued function (64) read that

$$|G(\tau,k)|^2=\frac{\mu^2\sin^2 kh+[\mu+2(1-\theta)\nu]^2(\cos kh-1)^2}{[1+2\theta\nu(1-\cos kh)]^2}. \tag{65}$$

A direct verification indicates that when

$$(15\theta^2+2\theta-\frac{5}{4}h^2)\nu^2+9\theta\nu+1\geq 0, \tag{66}$$

the following relation holds

$$|G(\tau,k)|\leq 1.$$

According to theorems, that is the Von Neumann condition, we conclude that the finite-difference scheme (60) is stable under the condition (66).

In what follows, we want to generate a Lax-difference scheme of Eq.(40). In terms of the known difference operators in [15], we discrete the differential equation (40) as follows

$$\frac{u_j^{n+1}-\frac{1}{2}(u_{j+1}^n+u_{j-1}^n)}{\tau}+2\lambda\frac{u_{j+1}^n-u_{j-1}^n}{2h}=\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}, \tag{67}$$

which can be rewritten as

$$u_j^{n+1}=(\frac{1}{2}-2\lambda\mu+\nu)u_{j+1}^n+(\frac{1}{2}+2\lambda\mu+\nu)u_{j-1}^n-2\nu u_j^n, \tag{68}$$

where $\mu=\frac{\tau}{2h},\nu=\frac{\tau}{h^2}$.

The difference scheme (68) is an explicit form whose stability can be discussed as above. Set $u_j^n=v^n e^{ikjh}$ and insert it into Eq.(68), we get that

$$v^{n+1}=[(1+2\nu)\cos kh-2\nu-4i\lambda\mu\sin kh]v^n.$$

Hence, we obtain the growth factor

$$G(\tau,k)=(1+2\nu)\cos kh-2\nu-4i\lambda\mu\sin kh,$$

$$\begin{aligned}|G(\tau,k)|^2= &\ 16\lambda^2\mu^2 sin^2 kh+[(1+2\nu)\cos kh-2\nu]^2=[(1+2\nu)^2-16\lambda^2\mu^2]\cos^2 kh\\ &-4\nu(1+2\nu)\cos kh+16\lambda^2\mu^2+4\nu^2.\end{aligned}$$

When $\nu=-\frac{1}{2},\sin kh\neq 0,|G(\tau,k)|^2=1+16\lambda^2\mu^2\sin^2 kh>1$, the difference scheme (68) is not stable according to the Von Neumann condition.

When $\sin kh=0$, i.e., $kh=K\pi,K\in\mathbf{Z},|G(\tau,k)|=1$, the difference scheme (68) is stable.

When $\cos kh=0$, i.e. $kh=2K\pi+\frac{\pi}{2},K=0,\pm1,\pm2,\ldots,16\lambda^2\mu^2+4\nu^2\leq 1$, the difference scheme (68) is stable.

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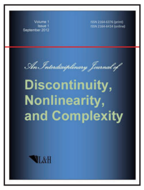
Discontinuity, Nonlinearity, and Complexity 5(2) (2016) 133–144



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Review on Finite Difference Method for Reaction-Diffusion Equation Defined on a Circular Domain

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Abstract

In this paper, a finite difference method for a non-linear reaction diffusion equation defined on a circular domain is presented. A simple second-order finite difference treatment of polar coordinate singularity for Laplacian operator, the centered difference approximations, the treatments for Neumann boundary problems are used to discretize this equation. By using this method, numerical solutions can be computed. In the end, we give two applications of reaction diffusion predator-prey models with modified Leslie-Gower and Holling type II functional responses.

1 Introduction

A reaction-diffusion equation is a partial differential equation which comprises reaction and diffusion terms:

$$\frac{\partial u}{\partial t} = D\Delta u + f(u), \tag{1}$$

where $u = u(t,x)$ is a state variable and describes density/concentration of a substance or a population at position $x \in \Omega \subset \mathbb{R}^n$ and at time t, Ω is an open domain, Δ is the Laplacian operator and D is a diagonal matrix of diffusion coefficients.

This type of equations was introduced by Fisher [2] and Kolmogorov, Petrovsky and Piskunov [3] to describe the spreading of biological populations. Some of these equations can be solved analytically and numerically but for a large number of equations, the analytical solution is unknown and can be approximated by using numerical methods. In the literature, there exist many numerical methods for the resolution of different problems, here are some used to discretize a system of equations: the finite element method, finite difference method, finite volume method. In this review, we are interested in finite difference method.

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For the finite difference method (Thomée [15]), the domain is represented by a finite number of points $x_i = \Omega_h$ called nodes of the mesh and the solution is represented by a set of values u_i approaching $u(x_i)$. The method replaces the partial derivatives by differences or combinations of these punctual values of the function using truncated Taylor developments. The advantages of this method are its simplicity of writing and low computational cost.

The finite difference method (Richtmyer and Morton [6]; Hildebrand [7]) is widely used by many authors for approximating numerical solution of reaction-diffusion equations. Many authors (for example Ascher et al. [8]; Pao [9]; Jerome [16]; Li et al. [14]) also used this method to study stability and convergence result.

In [1], the author considered the following reaction-diffusion model defined:

$$\begin{cases} \dfrac{\partial u(t,x,y)}{\partial t} = D_1 \Delta u(t,x,y) + f(u(t,x,y), v(t,x,y)) & (x,y) \in \Omega, t > 0, \\ \dfrac{\partial v(t,x,y)}{\partial t} = D_2 \Delta v(t,x,y) + g(u(t,x,y), v(t,x,y)) & (x,y) \in \Omega, t > 0, \\ \dfrac{\partial u(t,x,y)}{\partial n} = \dfrac{\partial v(t,x,y)}{\partial n} = 0, & (x,y) \in \partial\Omega, \end{cases} \tag{2}$$

$u(t,x,y)$ and $v(t,x,y)$ represents the densities of populations, $\Omega = [0,L] \times [0,L]$ and Δu is the Laplacian operator

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \tag{3}$$

D_1 and D_2 are the diffusion coefficients, $f(u,v)$ and $g(u,v)$ model the local activity (absence of diffusion). Authors in [13] are numerically solved this system with appropriate functions f and g by using finite difference method on square domain and with Neumann boundary conditions.

In this review, we extend this method to a 2-D reaction diffusion system defined on a circular domain ($\Omega = \{(x,y) \in \mathbf{R}^2 / x^2 + y^2 < R^2\}$) and with Neumann boundary conditions. To do that, we strive to linearize the reaction-diffusion system using the finite difference method [4] in polar coordinates. To apply this method to two dimensions, a simple division in reaction-diffusion equation defines the node at any point of the mesh of the circular domain.

The organization of the remaining part of the paper is as follows: In Section 2, we present a finite difference discretization for equation (2) given in polar coordinates. In Section 3, we apply this method to two component reaction diffusion predator-prey model defined on a disk. Then, we extend this result to three component reaction diffusion predator-prey model.

2 Discretization of reaction-diffusion equation defined on a disk domain

In this section, through the finite difference method and the principle of the numerical method used in [5], we solve numerically equation (2) defined on a disk domain $\Omega = \{(x,y) \in \mathbf{R}^2 / x^2 + y^2 < R^2\}$.

As $(x,y) \in \Omega$, we can make the following change of variables:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \quad \text{where} \quad \begin{cases} r = \sqrt{x^2 + y^2}, \\ \theta = \tan^{-1}\left(\frac{y}{x}\right). \end{cases}$$

Without loss of generalities we also denote

$$\begin{cases} u(t,x,y) = u(t, r\cos(\theta), r\sin(\theta)) = u(t,r,\theta), \\ v(t,x,y) = v(t, r\cos(\theta), r\sin(\theta)) = v(t,r,\theta). \end{cases}$$

Therefore the Laplacian operator in polar coordinates is given by :

$$\Delta_{r\theta} u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{4}$$

and

$$\begin{cases} f(u(t,x,y),v(t,x,y)) = f(u(t,r,\theta),v(t,r,\theta)), \\ g(u(t,x,y),v(t,x,y)) = g(u(t,r,\theta),v(t,r,\theta)). \end{cases}$$

The Neumann boundary conditions in polar coordinates becomes:

$$\begin{cases} \dfrac{\partial u(t,x,y)}{\partial n}|_{\forall (x,y)\in\partial\Omega} = \partial_r u(t,r,\theta)|_{\text{ for } r=R \text{ Radial derivative}}, \\ \dfrac{\partial v(t,x,y)}{\partial n}|_{\forall (x,y)\in\partial\Omega}) = \partial_r v(t,r,\theta)|_{\text{ for } r=R \text{ Radial derivative}}. \end{cases}$$

Then system (2) can be written as

$$\begin{cases} \dfrac{\partial u(t,r,\theta)}{\partial t} = D_1\Delta_{r\theta}u(t,r,\theta) + f(u(t,r,\theta),v(t,r,\theta)) \text{ for } (r,\theta)\in\mathscr{D} \text{ and } t>0, \\ \dfrac{\partial v(t,r,\theta)}{\partial t} = D_2\Delta_{r\theta}v(t,r,\theta) + g(u(t,r,\theta),v(t,r,\theta)) \text{ for } (r,\theta)\in\mathscr{D} \text{ and } t>0, \\ \partial_r u(.,r,\theta) = \partial_r v(.,r,\theta) = 0, \quad \text{for } r=R \text{ Radial derivative}, \end{cases} \tag{5}$$

where $\mathscr{D} = \{(r,\theta) : 0 < r < R, 0 \le \theta < 2\pi\}$.

Equation (2) is written

$$\frac{\partial u(t,r,\theta)}{\partial t} = D_1\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right) + f(u(t,r,\theta),v(t,r,\theta)). \tag{6}$$

We see that, equation (6) has a singularity at the origin $r = 0$, (see [11]). This singularity is due to the representation of the equation in polar coordinates. If f is regular enough, the solution itself is nonsingular at the origin. In order to have the desired regularity and accuracy, the classical finite difference scheme uses a uniformly integers grid with some conditions at the origin. This pole conditions act as a numerical boundary condition at the origin which is needed in finite difference scheme.

Considering the Neumann boundary conditions [12] and by discretization, we obtain the following approximation of equation (15). For $n = 1,\dots,N$, with $N = \frac{T}{\Delta t}$, $i = 1,\dots,P+1$, and $j = 1,\dots,M+1$ we find $\{u^n_{i,j}, v^n_{i,j}\}$ such that

$$\begin{cases} \partial_n u^n_{i,j} = \Delta_{r_i\theta_j}u^n_{i,j} + f(\overrightarrow{u^n_{i,j}}, \overrightarrow{u^{n-1}_{i,j}}), \\ \partial_n v^n_{i,j} = \delta\Delta_{r_i\theta_j}v^n_{i,j} + g(\overrightarrow{u^n_{i,j}}, \overrightarrow{u^{n-1}_{i,j}}), \end{cases} \tag{7}$$

where $\overrightarrow{u^n_{i,j}} = (u^n_{i,j}, v^n_{i,j})^T$ denotes the two-dimensional approximation at the point (r_i, θ_j, t_n) with $t_n = n\Delta t$. The approximations of the initial conditions are given as:

$$u^0_{i,j} = u_0(r_i,\theta_j),\ v^0_{i,j} = v_0(r_i,\theta_j).$$

We choose a grid such that the points are integers in azimuthal direction and half-integer in radial direction (see Fig. 1):

$$r_i = (i-\frac{1}{2})\Delta r,\ \theta_j = (j-1)\Delta\theta, \tag{8}$$

where

$$\Delta r = \frac{2}{2P+1},\ \Delta\theta = \frac{2\pi}{M}.$$

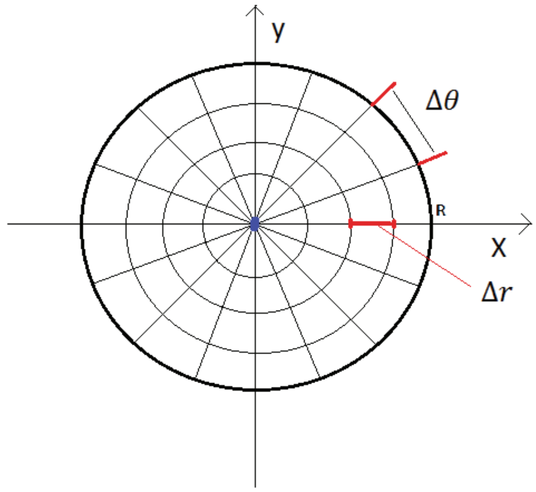

Fig. 1 The circular mesh.

For $i = 2,\ldots,P$ and $j = 1,\ldots,M$ and using the centered difference method to discretize the Laplacian operator, we have

$$\Delta_{r_i\theta_j}u^n_{i,j} \approx \frac{u^n_{i+1,j}+u^n_{i-1,j}-2u^n_{i,j}}{\Delta r^2}+\frac{u^n_{i+1,j}-u^n_{i-1,j}}{2r_i\Delta r}+\frac{u^n_{i,j+1}+u^n_{i,j-1}-2u^n_{i,j}}{r_i^2\Delta\theta^2}. \tag{9}$$

From the Neumann boundary conditions (the flow is zero on the edge)

$$\frac{u^n_{P+1,j}-u^n_{P,j}}{\Delta r}=0, \tag{10}$$

so the numerical boundary values at $r = 1$, $u^n_{P+1,j}$ can be approximated by $u^n_{P,j}$, and $u^n_{i,0} = u^n_{i,M}$, $u^n_{i,1} = u^n_{i,M+1}$ since u is 2π periodic in θ. At $i = 1$, equation (9) becomes

$$\Delta_{r_1\theta_j}u^n_{1,j} \approx \frac{u^n_{2,j}+u^n_{0,j}-2u^n_{1,j}}{\Delta r^2}+\frac{u^n_{2,j}-u^n_{0,j}}{2r_1\Delta r}+\frac{u^n_{1,j+1}+u^n_{1,j-1}-2u^n_{1,j}}{r_1^2\Delta\theta^2} \tag{11}$$

since $r_1 = \frac{\Delta r}{2}$, the term $u^n_{0,j}$ is simplified and the equation (11) is written by

$$\Delta_{r_1\theta_j}u^n_{1,j} \approx \frac{2(u^n_{2,j}-u^n_{1,j})}{\Delta r^2}+\frac{u^n_{1,j+1}+u^n_{1,j-1}-2u^n_{1,j}}{r_1^2\Delta\theta^2}. \tag{12}$$

In order to approach $\partial_n u^n_{i,j}$, we use the implicit Euler method,

$$\partial_n u^n_{i,j} = \frac{u^n_{i,j}-u^{n-1}_{i,j}}{\Delta t}.$$

Finally we obtain the following equation

$$\begin{cases} \dfrac{u^n_{i,j}-u^{n-1}_{i,j}}{\Delta t} = D_1\left(\dfrac{u^n_{i+1,j}+u^n_{i-1,j}-2u^n_{i,j}}{\Delta r^2}+\dfrac{u^n_{i+1,j}-u^n_{i-1,j}}{2r_i\Delta r}+\dfrac{u^n_{i,j+1}+u^n_{i,j-1}-2u^n_{i,j}}{r_i^2\Delta\theta^2}\right)+f(\overrightarrow{u^n_{i,j}},\overrightarrow{u^{n-1}_{i,j}}), \\ \dfrac{v^n_{i,j}-v^{n-1}_{i,j}}{\Delta t} = D_2\left(\dfrac{v^n_{i+1,j}+v^n_{i-1,j}-2v^n_{i,j}}{\Delta r^2}+\dfrac{v^n_{i+1,j}-v^n_{i-1,j}}{2r_i\Delta r}+\dfrac{v^n_{i,j+1}+v^n_{i,j-1}-2v^n_{i,j}}{r_i^2\Delta\theta^2}\right)+f(\overrightarrow{v^n_{i,j}},\overrightarrow{v^{n-1}_{i,j}}). \end{cases} \tag{13}$$

3 Application

In this section, we apply the above method to 2-D two component reaction-diffusion predator-prey model with modified Leslie-Gower and Beddington-DeAngelis functional response and then we extend this method to 2-D three reaction diffusion component predator-prey model.

3.1 Example of a predator-prey model of two species

Let us now consider the model with two component:

$$\begin{cases} \dfrac{\partial u(t,x,y)}{\partial t} = D_1 \Delta u(t,x,y) + (a_1 - b_1 u(t,x,y) - \dfrac{c_1 v(t,x,y)}{d_1 u(t,x,y) + d_2 v(t,x,y) + k_1}) u(t,x,y) \\ \dfrac{\partial v(t,x,y)}{\partial t} = D_2 \Delta v(t,x,y) + (a_2 - \dfrac{c_2 v(t,x,y)}{u(t,x,y) + k_2}) v(t,x,y). \end{cases} \tag{14}$$

This two species food chain model describes a prey population u which serves as food for a predator v. $u(t,x,y)$ and $v(t,x,y)$ represent population densities at time t and the position (x,y) defined on a circular domain Ω with radius R (i.e. $\Omega = \{(x,y) \in \mathbb{R}^2 / x^2 + y^2 < R^2\}$), $r_1, a_1, b_1, k_1, r_2, a_2$, and k_2 are positive parameters, a_1 is the growth rate of prey u, a_2 describes the growth rate of predator v, b_1 measures the strength of competition among individuals of species u, c_1 is the maximum value of the per capita reduction of u due to v, c_2 has a similar meaning to c_1, k_1 measures the extent protection to which environment provides to prey u, k_2 has a similar meaning to k_1 relatively to the predator v, d_1 and d_2 are two positive constants, D_1 and D_2 are the diffusions coefficients of the preys and the predators.

In polar coordinates model (14) is written as follows:

$$\begin{cases} \dfrac{\partial u(t,r,\theta)}{\partial t} = D_1 \Delta_{r\theta} u(t,r,\theta) + f(u(t,r,\theta), v(t,r,\theta)) & \text{for } (r,\theta) \in \mathscr{D} \text{ and } t > 0, \\ \dfrac{\partial v(t,r,\theta)}{\partial t} = D_2 \Delta_{r\theta} v(t,r,\theta) + g(u(t,r,\theta), v(t,r,\theta)) & \text{for } (r,\theta) \in \mathscr{D} \text{ and } t > 0, \\ \partial_r u(.,r,\theta) = \partial_r v(.,r,\theta) = 0 & \text{for } r = R \text{ Radial derivative}, \end{cases} \tag{15}$$

where

$$\begin{cases} f(u(t,r,\theta), v(t,r,\theta)) = (a_1 - b_1 u(t,r,\theta) - \dfrac{c_1 v(t,r,\theta)}{d_1 u(t,r,\theta) + d_2 v(t,r,\theta) + k_1}) u(t,r,\theta), \\ g(u(t,r,\theta), v(t,r,\theta)) = (a_2 - \dfrac{c_2 v(t,r,\theta)}{u(t,r,\theta) + k_2}) v(t,r,\theta). \end{cases} \tag{16}$$

$u(t,r,\theta)$ and $v(t,r,\theta)$ represent the population densities at time t and the position (r,θ).
By computation, one can show that system (15) has four equilibrium points:

$$E_0 = (0,0), E_1 = (1,0), E_2 = (0,e_2), E^* = (u^*, v^*),$$

where

$$u^* = \frac{1 - a - e_1 + \sqrt{(a + e_1 - 1)^2 + 4(e_1 - a e_2)}}{2}, \tag{17}$$

and

$$v^* = u^* + e_2. \tag{18}$$

Next, we consider the disc domain $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 400\}$ and the boundary conditions are of the Neumann type. In order to avoid numerical artifacts, the values of the time (Δt) and space steps $(\Delta r$ and $\Delta\theta)$

have been chosen sufficiently small and satisfying the CFL (Courant-Friedrichs-Levy) stability criterion for reaction diffusion equation.

Therefore, we obtain the following system

$$\begin{cases} \partial_n u_{i,j}^n = D_1 \Delta_{r_i \theta_j} u_{i,j}^n + f(\overrightarrow{u_{i,j}^n}, \overrightarrow{u_{i,j}^{n-1}}) \\ \partial_n v_{i,j}^n = D_2 \Delta_{r_i \theta_j} v_{i,j}^n + g(\overrightarrow{u_{i,j}^n}, \overrightarrow{u_{i,j}^{n-1}}) \end{cases} \tag{19}$$

with

$$\begin{cases} f(\overrightarrow{u_{i,j}^n}, \overrightarrow{u_{i,j}^{n-1}}) = a_1 u_{i,j}^{n-1} - b_1 u_{i,j}^{n-1} |u_{i,j}^{n-1}| - \dfrac{c_1 v_{i,j}^{n-1}}{d_1 |u_{i,j}^{n-1}| + d_2 |v_{i,j}^{n-1}| + k_1} u_{i,j}^{n-1} \\ g(\overrightarrow{u_{i,j}^n}, \overrightarrow{u_{i,j}^{n-1}}) = a_2 v_{i,j}^{n-1} - \dfrac{c_2 v_{i,j}^{n-1}}{|u_{i,j}^{n-1}| + k_2} v_{i,j}^{n-1}. \end{cases} \tag{20}$$

The linear system associated to system (15) is

$$AZ = B$$

and the unknown vector $Z = \begin{pmatrix} \overrightarrow{u}^n \\ \overrightarrow{v}^n \end{pmatrix}$ is defined by

$$\overrightarrow{u}^n = \begin{pmatrix} u_1^n \\ u_2^n \\ \vdots \\ \vdots \\ u_{M-1}^n \\ u_M^n \end{pmatrix}, \overrightarrow{v}^n = \begin{pmatrix} v_1^n \\ v_2^n \\ \vdots \\ \vdots \\ v_{M-1}^n \\ v_M^n \end{pmatrix}, \text{ with } u_j^n = \begin{pmatrix} u_{1,j}^n \\ u_{2,j}^n \\ \vdots \\ \vdots \\ u_{P-1,j}^n \\ u_{P,j}^n \end{pmatrix} \text{ and } v_j^n = \begin{pmatrix} v_{1,j}^n \\ v_{2,j}^n \\ \vdots \\ \vdots \\ v_{P-1,j}^n \\ v_{P,j}^n \end{pmatrix}$$

and the matrix A can be written as

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where

$$\begin{pmatrix} A_1 = I + D_1 \Delta t L \\ A_2 = I + D_2 \Delta t L \end{pmatrix}$$

in which I is the identity matrix and the size of the matrix L is $((P+1) \times (P+1))$ and written as follows:

$$\mathbf{L} = \begin{pmatrix} Q-2S & S & 0 & \dots & 0 & S \\ S & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & S \\ S & 0 & \dots & 0 & S & Q-2S \end{pmatrix},$$

where

$$\mathbf{Q} = \begin{pmatrix} -2 & 1+\lambda_1 & 0 & \dots & \dots & 0 \\ 1-\lambda_2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 1+\lambda_i & \ddots & \vdots \\ \vdots & \ddots & 1-\lambda_i & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -2 & 1+\lambda_{P-1} \\ 0 & \dots & \dots & 0 & 1-\lambda_P & 1+\lambda_P \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} \beta_1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \beta_i & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & 0 & \beta_P \end{pmatrix}$$

with

$$\beta_i = \frac{1}{(i-0.5)^2\Delta\theta^2}, \lambda_i = \frac{1}{(i-0.5)}, i = 1, ..., P.$$

The known vector $B = \begin{pmatrix} \overrightarrow{u}^{n-1} + \Delta t \overrightarrow{f} \\ \overrightarrow{v}^{n-1} + \Delta t \overrightarrow{g} \end{pmatrix}$ is defined by

$$B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ \vdots \\ B_{M-1} \\ B_M \end{pmatrix}, \; B_j = \begin{pmatrix} \Delta r^2(u_{1,j}^{n-1} + \Delta t f_{1,j}^{n-1}) \\ \vdots \\ \Delta r^2(u_{P,j}^{n-1} + \Delta t f_{P,j}^{n-1}) \\ \Delta r^2(v_{1,j}^{n-1} + \Delta t g_{1,j}^{n-1}) \\ \vdots \\ \Delta r^2(v_{P,j}^{n-1} + \Delta t g_{P,j}^{n-1}) \end{pmatrix}.$$

The initial conditions are small perturbation in the vicinity of equilibrium point (u^*, v^*) and are chosen as:

$$u_0(r_i, \theta_j) = u^*((r_i \cos\theta_j)^2 + (r_i \sin\theta_j)^2) = u^* r_i^2 < 400, \tag{21}$$

$$v_0(r_i, \theta_j) = v^*((r_i \cos\theta_j)^2 + (r_i \sin\theta_j)^2) = v^* r_i^2 < 400. \tag{22}$$

The values of the used parameters are given by

$$a_1 = 1, \; a_2 = 0.02, \; b_1 = 1, \; k_1 = 0.2, \; k_2 = 0.1, \; d_1 = 0.9, \; d_2 = 0.1, \; c_1 = 1.1,$$

$$c_2 = 0.02, \; D_1 = 1, \; D_2 = 1. \tag{23}$$

We suppose that the two species diffuse in the same way, (i.e. $D_1 = D_2$). In Fig. (2), the numerical solutions $u_{i,j}^n$ and $v_{i,j}^n$ of the predator-prey system are plotted. The left figures are the spatial distribution of prey population and on the right ones are of the predator population. We observe that the spatial distribution is a spiral wave type for system (15).

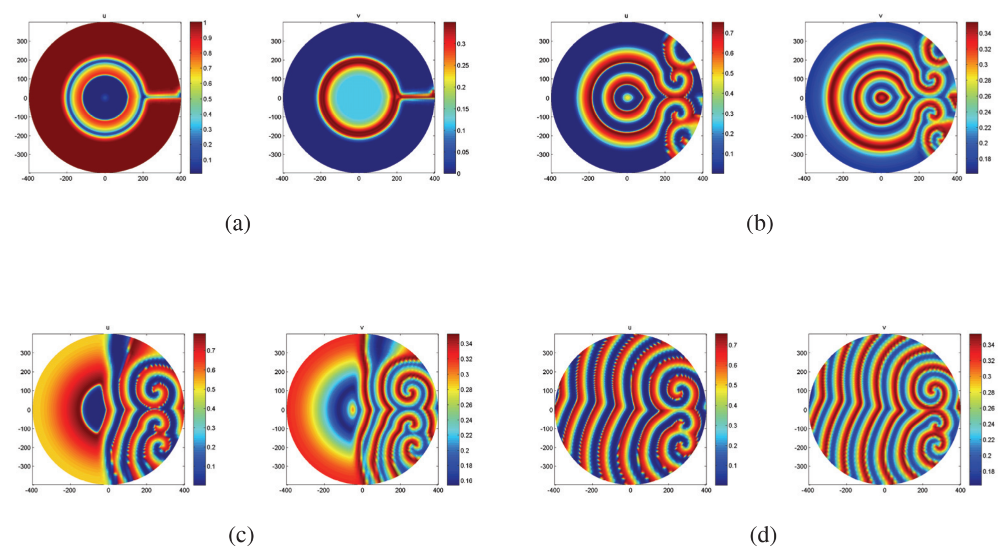

Fig. 2 Spatial distribution of species of system (15), for different values of time t, (a) t=100, (b) t=1000, (c) t=1800, (d) t=6000

3.2 Example of a three species predator-prey

In this example, we consider a three-species food chain model consisting of prey, intermediate predator and top-predator, modeled by a system of three reaction-diffusion equations defined on a circular spatial domain and incorporates the Holling type II and a modified Leslie-Gower functional response. The first species denoted U is the only food source of the second V. As well, intermediate predator V is the only prey of a top-predator W. Local interactions between species U and V are modeled by Lotka-Volterra type scheme and the interactions between species W and V has been modeled by Leslie-Gower scheme [17]. The spatio-temporal system can be written as follows (see [10]):

$$\begin{cases} \dfrac{\partial U(T,x,y)}{\partial T} = D_1 \Delta U(T,x,y) + (a_0 - b_0 U(T,x,y) - \dfrac{v_0 V(T,x,y)}{U(T,x,y)+d_0}) U(T,x,y), \\ \dfrac{\partial V(T,x,y)}{\partial T} = D_2 \Delta V(T,x,y) + (-a_1 + \dfrac{v_1 U(T,x,y)}{U(T,x,y)+d_0} - \dfrac{v_2 W(T,x,y)}{V(T,x,y)+d_2}) V(T,x,y), \\ \dfrac{\partial W(T,x,y)}{\partial T} = D_3 \Delta W(T,x,y) + (c_3 - \dfrac{v_3 W(T,x,y)}{V(T,x,y)+d_3}) W(T,x,y), \\ \dfrac{\partial U}{\partial n} = \dfrac{\partial V}{\partial n} = \dfrac{\partial W}{\partial n} = 0, \\ U(0,x,y) = U_0(x,y) \geq 0,\ V(0,x,y) = V_0(x,y) \geq 0,\ W(0,x,y) = W_0(x,y) \geq 0, \end{cases} \tag{24}$$

where $U(T,x,y)$, $V(T,x,y)$ and $W(T,x,y)$ are the densities of prey, intermediate predator and top-predator, respectively, at time T and position (x,y) defined on a circular domain Ω with radius R (i.e. $\Omega = \{(x,y) \in \mathbf{R}^2 / x^2 + y^2 < R^2\}$. The three species are assumed to diffuse at rates D_i (i = 1, 2, 3). The parameters a_0, b_0, v_0, d_0, a_1, v_1, v_2, d_2, c_3, v_3 and d_3 are assumed to be positive constants and are defined as follows: a_0 is the growth rate of the prey U, b_0 measures the mortality due to the competition between individuals of the species U, v_0 is the maximum extent that the rate of reduction by individual U can reach, d_0 measures the protection that the species U and V benefit through the environment, a_1 represents the death rate of V in the absence of U, v_1, v_2 and v_3 are the the maximum value that the rate of reduction by the individual of U, V and W can reach respectively, d_2 is the value of V for which the rate of elimination by individual V becomes $\frac{v_2}{2}$, c_3 describes the growth rate of W, assuming that there is the same number of males and females and d_3 represents the residual loss caused by high scarcity of prey V of the specie W.

Using the following transformations and by change of variables to polar coordinates:

$$U = \frac{a_0}{b_0} u,\ V = \frac{a_0^2}{b_0 v_0} v,\ W = \frac{a_0^3}{b_0 v_0 v_2} w,\ T = \frac{t}{a_0},$$

and

$$a = \frac{b_0 d_0}{a_0},\ b = \frac{a_1}{a_0},\ c = \frac{v_1}{a_0},\ d = \frac{d_2 v_0 b_0}{a_0^2},\ p = \frac{c_3 a_0^2}{v_0 b_0 v_2},\ q = \frac{v_3}{v_2},\ s = \frac{d_3 v_0 b_0}{a_0^2},\ \delta_1 = \frac{D_1}{a_0},\ \delta_2 = \frac{D_2}{a_0},\ \delta_3 = \frac{D_3}{a_0}$$

the spatio-temporal system (24) becomes

$$\begin{cases} \dfrac{\partial u(t,r,\theta)}{\partial t} = \delta_1 \Delta_{r\theta} u(t,r,\theta) + (1 - u(t,r,\theta) - \dfrac{v(t,r,\theta)}{u(t,r,\theta)+a}) u(t,r,\theta), & \forall (r,\theta) \in \Gamma,\ t > 0 \\ \dfrac{\partial v(t,r,\theta)}{\partial t} = \delta_2 \Delta_{r\theta} v(t,r,\theta) + (-b + \dfrac{c u(t,r,\theta)}{u(t,r,\theta)+a} - \dfrac{w(t,r,\theta)}{v(t,r,\theta)+d}) v(t,r,\theta), & \forall (r,\theta) \in \Gamma,\ t > 0 \\ \dfrac{\partial w(t,r,\theta)}{\partial t} = \delta_3 \Delta_{r\theta} w(t,r,\theta) + (p - \dfrac{q w(t,r,\theta)}{v(t,r,\theta)+s}) w(t,r,\theta) & \forall (r,\theta) \in \Gamma,\ t > 0 \\ \partial_r u(.,r,\theta) = \partial_r v(.,r,\theta) = \partial_r w(.,r,\theta) = 0 \text{ for } r = R \text{ (radial derivative)}, \\ u(0,r,\theta) = u_0(r,\theta) \geq 0,\ v(0,r,\theta) = v_0(r,\theta) \geq 0,\ w(0,r,\theta) = w_0(r,\theta) \geq 0. \end{cases} \tag{25}$$

$u(t,r,\theta)$, $v(t,r,\theta)$ and $w(t,r,\theta)$ represent the population densities at time t and the position $(r,\theta)\in\Gamma$, $\Gamma=\{(r,\theta): 0<r<R, 0\le\theta<2\pi\}$.

By computation, system (25) has four trivial equilibrium points $E_0=(0,0,0)$, $E_1=(1,0,0)$, $E_2=(0,0,\frac{sp}{q})$, $E_3=(1,0,\frac{sp}{q})$ and a positive nontrivial one $E^*=(u^*,v^*,w^*)$ which exists if and only if the following inequalities hold

$$qc > bq+p \text{ and } qc-bq-p > a(bq+p), \tag{26}$$

such that

$$u^*=\frac{a(bq+p)}{qc-bq-p},\ v^*=(1-u^*)(u^*+a) \text{ and } w^*=\frac{p(v^*+s)}{q}. \tag{27}$$

Therefore, we obtain the following system

$$\begin{cases}\partial_n u^n_{i,j}=\delta_1\Delta_{r_i\theta_j}u^n_{i,j}+f(\overrightarrow{u^n_{i,j}},\overrightarrow{u^{n-1}_{i,j}}),\\ \partial_n v^n_{i,j}=\delta_2\Delta_{r_i\theta_j}v^n_{i,j}+g(\overrightarrow{u^n_{i,j}},\overrightarrow{u^{n-1}_{i,j}}),\\ \partial_n w^n_{i,j}=\delta_3\Delta_{r_i\theta_j}w^n_{i,j}+h(\overrightarrow{u^n_{i,j}},\overrightarrow{u^{n-1}_{i,j}}),\end{cases} \tag{28}$$

with

$$\begin{cases}f(\overrightarrow{u^n_{i,j}},\overrightarrow{u^{n-1}_{i,j}})=u^{n-1}_{i,j}-u^{n-1}_{i,j}|u^{n-1}_{i,j}|-\dfrac{v^{n-1}_{i,j}}{|u^{n-1}_{i,j}|+a}u^{n-1}_{i,j},\\ g(\overrightarrow{u^n_{i,j}},\overrightarrow{u^{n-1}_{i,j}})=-bv^{n-1}_{i,j}+\dfrac{cu^{n-1}_{i,j}}{|u^{n-1}_{i,j}|+a}v^{n-1}_{i,j}-\dfrac{w^{n-1}_{i,j}}{|v^{n-1}_{i,j}|+d}v^{n-1}_{i,j},\\ h(\overrightarrow{u^n_{i,j}},\overrightarrow{u^{n-1}_{i,j}})=pw^{n-1}_{i,j}-\dfrac{qw^{n-1}_{i,j}}{|v^{n-1}_{i,j}|+s}w^{n-1}_{i,j}.\end{cases} \tag{29}$$

The linear system associated with system (15) is given by

$$DH=C.$$

The unknown vector $H=\begin{pmatrix}\overrightarrow{u}^n\\ \overrightarrow{v}^n\\ \overrightarrow{w}^n\end{pmatrix}$ is defined by

$$\overrightarrow{u}^n=\begin{pmatrix}u^n_1\\ u^n_2\\ \vdots\\ \vdots\\ u^n_{M-1}\\ u^n_M\end{pmatrix},\ \overrightarrow{v}^n=\begin{pmatrix}v^n_1\\ v^n_2\\ \vdots\\ \vdots\\ v^n_{M-1}\\ v^n_M\end{pmatrix},\ \overrightarrow{w}^n=\begin{pmatrix}w^n_1\\ w^n_2\\ \vdots\\ \vdots\\ w^n_{M-1}\\ w^n_M\end{pmatrix},\ \text{with } u^n_j=\begin{pmatrix}u^n_{1,j}\\ u^n_{2,j}\\ \vdots\\ \vdots\\ u^n_{P-1,j}\\ u^n_{P,j}\end{pmatrix},\ v^n_j=\begin{pmatrix}v^n_{1,j}\\ v^n_{2,j}\\ \vdots\\ \vdots\\ v^n_{P-1,j}\\ v^n_{P,j}\end{pmatrix},\ w^n_j=\begin{pmatrix}w^n_{1,j}\\ w^n_{2,j}\\ \vdots\\ \vdots\\ w^n_{P-1,j}\\ w^n_{P,j}\end{pmatrix}$$

and the matrix A can be written as

$$D=\begin{pmatrix}D_1 & 0 & 0\\ 0 & D_2 & 0\\ 0 & 0 & D_3\end{pmatrix},$$

$$\begin{pmatrix}D_1=I+\delta_1\Delta tL\\ D_2=I+\delta_2\Delta tL\\ D_3=I+\delta_2\Delta tL\end{pmatrix}.$$

The known vector $C = \begin{pmatrix} \overrightarrow{u}^{n-1} + \Delta t\, \overrightarrow{f} \\ \overrightarrow{v}^{n-1} + \Delta t\, \overrightarrow{g} \\ \overrightarrow{w}^{n-1} + \Delta t\, \overrightarrow{h} \end{pmatrix}$ is defined by

$$C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ \vdots \\ C_{M-1} \\ C_M \end{pmatrix}, \; C_j = \begin{pmatrix} \Delta r^2 (u_{1,j}^{n-1} + \Delta t f_{1,j}^{n-1}) \\ \vdots \\ \Delta r^2 (u_{P,j}^{n-1} + \Delta t f_{P,j}^{n-1}) \\ \Delta r^2 (v_{1,j}^{n-1} + \Delta t g_{1,j}^{n-1}) \\ \vdots \\ \Delta r^2 (v_{P,j}^{n-1} + \Delta t g_{P,j}^{n-1}) \\ \Delta r^2 (w_{1,j}^{n-1} + \Delta t h_{1,j}^{n-1}) \\ \vdots \\ \Delta r^2 (w_{P,j}^{n-1} + \Delta t h_{P,j}^{n-1}) \end{pmatrix}.$$

We simulate the spatial distributions of the three populations in the limited field $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 50\}$. The boundary conditions are of Neumann type, (i.e. there is no emigration or immigration of populations). The initial conditions are a small perturbation in the vicinity of equilibrium point (u^*, v^*, w^*) and are chosen as

$$u_0(r_i, \theta_j) = u^*((r_i \cos\theta_j)^2 + (r_i \sin\theta_j)^2) = u^* r_i^2 < 50, \tag{30}$$

$$v_0(r_i, \theta_j) = v^*((r_i \cos\theta_j)^2 + (r_i \sin\theta_j)^2) = v^* r_i^2 < 50,$$

$$w_0(r_i, \theta_j) = w^*((r_i \cos\theta_j)^2 + (r_i \sin\theta_j)^2) = w^* r_i^2 < 50$$

and parameters values are:

$$a_0 = 0.5,\ a_1 = 0.4,\ b_0 = 0.36,\ d_0 = 0.3,\ d_2 = 0.4,\ d_3 = 0.4,\ v_0 = 0.4,\ v_1 = 0.8,\ v_2 = 0.4,\ v_3 = 0.6. \tag{31}$$

From Fig. (3), different types of dynamics are observed when the bifurcation parameter c_3 varies.

4 Conclusions

In this paper, we have considered a nonlinear reaction-diffusion equation defined on a circular domain with the Neumann boundary conditions. We used the implicit Euler scheme to approach the derivative in time and the finite difference method to approximate the Laplacian operator in polar-coordinates. So, we extract a linear system in the form $AX = B$ which is necessary for the numerical solution of such equation.

To provide efficiency of this method, we have presented two applications arising from mathematical ecology.

A MATLAB code was also developed with the assumptions that the values of the time step (Δt) and space steps $(\Delta r$ and $\Delta\theta)$ have been chosen sufficiently small (number of nodes on the radius and the perimeter is very large) and satisfying the CFL (Courant-Friedrichs-Levy) stability criterion for reaction diffusion equation.

We chose a set of fixed parameters and the initial conditions depends on the points of the grid on the radius and the numbers of nodes of the mesh on the radius and perimeter are set. Figure (2) represents the evolution of the spatial distribution of two species for different values of time. We observe from this figure that when increasing the value of time, the number of iterations in time increases (to calculate the solution), so the solution of the system converges to a stable state. Figure (3) represents the evolution of the spatial distribution when the control parameter varies.

Therefore, the advantages of this method are: the simplicity of implementation, effectiveness, ability to construct approximations to high orders. Other methods such as finite element and finite volume can often be interpreted as finite difference schemes in the case of regular mesh.

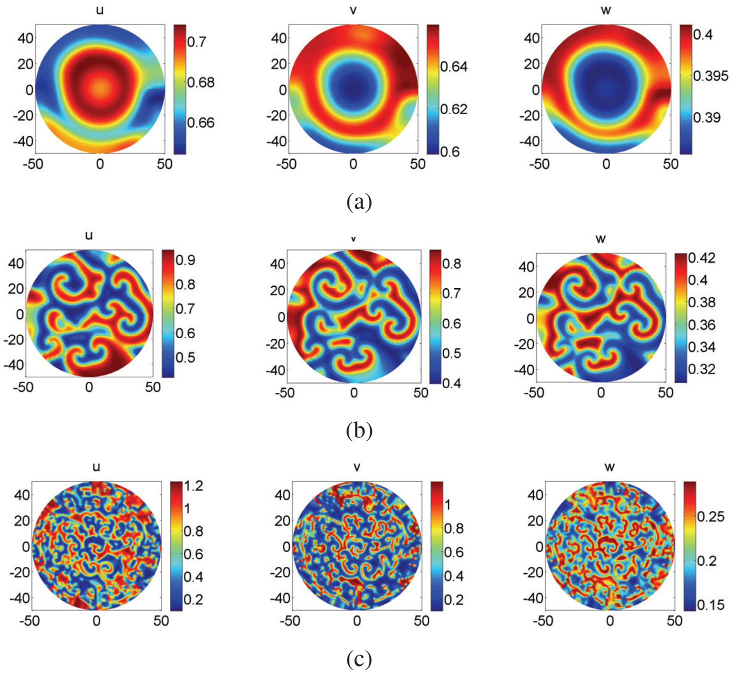

Fig. 3 Spatial distribution of prey (first column), predator (second column) and top predator (third column) are population densities of system (25). Spatial patterns are obtained with diffusivity coefficients $\delta_1 = 0.02$, $\delta_2 = 0.01$ and $\delta_3 = 0.05$, for fixed time $t = 12000$ at different bifurcation parameter $c_3 = 0.23$ (a), c_3=0.22 (b), $c_3 = 0.15$ (c)

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Asymptotic Behavior of Solutions of Singular Integro-differential Equations

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Abstract

We study the asymptotic behavior of the two-point integral boundary value problem for third order integro-differential equations with the small parameter at two highest derivatives. The asymptotic estimations of the solution of the integral boundary value problem is obtained. The obtained results shown that the solution of integral boundary value problem on both sides of given segment has the initial jumps with different orders.

1 Introduction

Mathematical models of many processes in physics, chemistry, biology, mechanics and technics often consists of differential and integro-differential equations containing small parameters in the highest derivatives. Such equations are now called singularly perturbed. Theory of asymptotic integration of singularly perturbed equations has become purposefully developed starting with the works of L. Schlesinger [1], G.D. Birkhoff [2], P. Noaillon [3]. In a further development of the main trends of the theory W. Wasow [4], A.H. Nayfeh [5], M. Nagumo [6], A. N. Tikhonov [7,8], M.I. Vishik, L.A. Lusternik [9,10], N.N. Bogolyubov, U.A Mitropolsky [11], A.B. Vasilieva and V.F. Butuzov [12,13], R.E. O'Malley [14,15], D.R. Smith [16], W. Eckhaus [17], K. W. Chang and F. A. Howes [18], J. Kevorkian and J.D. Cole [19], Sanders and F. Verhulst [20], E.F. Mischenko and N.X. Rozov [21], S.A. Lomov [22], K.A. Kasymov [23-26] and others have made a significant contribution. For a broad class of singularly perturbed problems effective asymptotic methods to build a uniform approximation with any degree of accuracy in the small parameter were developed. In the works of M. I. Vishik, L.A. Lyusternik [10] and K.A. Kasymov [24] first studied initial problems for singularly perturbed nonlinear equations of the second order with unbounded initial conditions when the small parameter tends to zero. These problems are called the Cauchy problems with an initial jump. A characteristic feature of these problems is that the solution of singularly perturbed problems tends to the solutions of the degenerate equations with the

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changed initial conditions. In this case we say that there is a phenomenon of the initial jump of the solution. The most common cases of the Cauchy problem with initial jumps for singularly perturbed nonlinear systems of ordinary and integro-differential equations, and partial differential equations of hyperbolic type was studied by K.A. Kasymov [23,25,26].

This paper considers the asymptotic behavior of the solution of a two-point boundary value problems for linear integro-differential equations of the third order with the small parameter at the two highest derivatives when the roots of the additional characteristic equation have opposite signs. Derivatives of the solution on the ends of the interval become infinitely large for sufficiently small values of the parameter. In this case we say that there is a phenomenon of the boundary jump.

2 Statement of the problem and Preliminary materials

Consider the following singularly perturbed linear integro-differential equation

$$L_\varepsilon y \equiv \varepsilon^2 y''' + \varepsilon A_0(t)y'' + A_1(t)y' + A_2(t)y = F(t) + \int_0^1 \sum_{i=0}^{1} H_i(t,x)y^{(i)}(x,\varepsilon)dx \tag{1}$$

with boundary conditions

$$y(0,\varepsilon) = \alpha, \quad y'(0,\varepsilon) = \beta, \quad y(1,\varepsilon) = \gamma + \int_0^1 \sum_{i=0}^{1} a_i(x)y^{(i)}(x,\varepsilon)dx, \tag{2}$$

where $\varepsilon > 0$ is a small parameter, α, β, γ are known constants independent of ε.

Assume that following conditions hold:

1. Functions $A_i(t), i = \overline{0,2}$, $F(t)$, $a_i(t), i = 0,1$ are sufficiently smooth and defined on the interval $0 \le t \le 1$, $H_0(t,x), H_1(t,x)$ – are defined in the domain $D = \{0 \le t \le 1, 0 \le x \le 1\}$ and sufficiently smooth.

II. The roots $\mu_i(t),\ i = 1,2$ of "additional characteristic equation" $\mu^2 + A_0(t)\mu + A_1(t) = 0$ satisfies the following inequalities $\mu_1(t) < -\gamma_1 < 0,\ \ \mu_2(t) > \gamma_2 > 0$.

III. 1 is not an eigenvalue of the kernel

$$H(t,s) = \frac{H_1(t,s)}{A_1(s)} + \int_s^1 \frac{1}{A_1(s)}\left(H_0(t,x) - H_1(t,x)\frac{A_2(x)}{A_1(x)}\right)\exp\left(-\int_s^x \frac{A_2(p)}{A_1(p)}dp\right)dx.$$

IV. $a_1(1) \ne 1$.

For the fundamental system of solutions of singularly perturbed homogeneous differential equation

$$L_\varepsilon y \equiv \varepsilon^2 y''' + \varepsilon A_0(t)y'' + A_1(t)y' + A_2(t)y = 0 \tag{3}$$

the following asymptotic representation holds as $\varepsilon \to 0$

$$\begin{aligned}
y_1^{(q)}(t,\varepsilon) &= \frac{1}{\varepsilon^q}\exp\left(\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx\right)\cdot\left(\mu_1^q(t)y_{10}(t) + O(\varepsilon)\right), \quad q = \overline{0,2},\\
y_2^{(q)}(t,\varepsilon) &= \frac{1}{\varepsilon^q}\exp\left(-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx\right)\cdot\left(\mu_2^q(t)y_{20}(t) + O(\varepsilon)\right), \quad q = \overline{0,2},\\
y_3^{(q)}(t,\varepsilon) &= y_{30}^{(q)}(t) + O(\varepsilon), \quad q = \overline{0,2},
\end{aligned} \tag{4}$$

where $y_{30}(t) = \exp(-\int_0^t \frac{A_2(x)}{A_1(x)}dx)$, functions $y_{i0}(t), i = 1,2$ are solutions of the problem: $p_i(t)\cdot y'_{i0}(t) + q_i(t)\cdot y_{i0}(t) = 0$, $y_{i0}(0) = 1, i = 1,2$, where $p_i(t) = (A_0(t)+2\mu_i(t))\mu_i(t) \neq 0$; $q_i(t) = A_2(t)+A_0(t)\mu_i'(t)+3\mu_i(t)\mu_i'(t)$.

The proof of the formulas (4) are readily obtained from the known theorems of L. Schlesinger [1] and G.D.Birkhoff [2] and P.Noaillon [3].

For the Wronskian $W(t,\varepsilon)$ in view (4) the following asymptotic representation holds as $\varepsilon \to 0$:

$$\begin{aligned} W(t,\varepsilon) =& \frac{1}{\varepsilon^3}\exp\Big(\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx - \frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx\Big)(y_{10}(t)y_{20}(t)y_{30}(t)\mu_1(t)\mu_2(t) \\ &\times(\mu_2(t)-\mu_1(t)) + O(\varepsilon)). \end{aligned} \tag{5}$$

We introduce the functions

$$K_0(t,s,\varepsilon) = \frac{P_0(t,s,\varepsilon)}{W(s,\varepsilon)}; \quad K_1(t,s,\varepsilon) = \frac{P_1(t,s,\varepsilon)}{W(s,\varepsilon)}, \tag{6}$$

where $P_0(t,s,\varepsilon)$, $P_1(t,s,\varepsilon)$ are the third order determinant obtained from the Wronskian $W(s,\varepsilon)$ by replacing the third row with $y_1(t,\varepsilon), 0, y_3(t,\varepsilon)$ and $0, y_2(t,\varepsilon), 0$ respectively. Functions $K_0(t,s,\varepsilon), K_1(t,s,\varepsilon)$ satisfies the homogeneous equation (3) with respect to variable t. Function $K(t,s,\varepsilon) = K_0(t,s,\varepsilon)+K_1(t,s,\varepsilon)$ is called the Cauchy function and it is a solution of the problem:

$$L_\varepsilon K(t,s,\varepsilon) = 0, \quad K(s,s,\varepsilon) = 0, \quad K'(s,s,\varepsilon) = 0, \quad K''(s,s,\varepsilon) = 1.$$

For the functions $K_0(t,s,\varepsilon)$, $K_1(t,s,\varepsilon)$ in view (4), (5), (6) the following asymptotic representation holds as $\varepsilon \to 0$:

$$\begin{aligned} K_0^{(q)}(t,s,\varepsilon) =& \varepsilon^2\Big(\frac{y_{30}^{(q)}(t)}{y_{30}(s)\mu_1(s)\mu_2(s)} - \frac{\mu_1^q(t)y_{10}(t)}{\varepsilon^q y_{10}(s)\mu_1(s)(\mu_2(s)-\mu_1(s))}\exp\Big(\frac{1}{\varepsilon}\int_s^t \mu_1(x)dx\Big) \\ &+O(\varepsilon)\Big), \quad t \geq s \\ K_1^{(q)}(t,s,\varepsilon) =& \varepsilon^2\Big(\frac{\mu_2^q(t)y_{20}(t)}{\varepsilon^q y_{20}(s)\mu_2(s)(\mu_2(s)-\mu_1(s))}\exp\Big(-\frac{1}{\varepsilon}\int_t^s \mu_2(x)dx\Big)+O(\varepsilon)\Big), t \leq s, q = \overline{0,2}. \end{aligned} \tag{7}$$

From (7) we obtain the following asymptotic estimations:

$$\begin{aligned} \left|K_0^{(q)}(t,s,\varepsilon)\right| \leq& C\varepsilon^2 + \frac{C}{\varepsilon^{q-2}}e^{-\gamma_1\frac{t-s}{\varepsilon}}, \quad q = 0,1,2,\ t \geq s \\ \left|K_1^{(q)}(t,s,\varepsilon)\right| \leq& \frac{C}{\varepsilon^{q-2}}e^{-\gamma_2\frac{s-t}{\varepsilon}},\ q = 0,1,2,\ t \leq s, \end{aligned} \tag{8}$$

where $C > 0$ is a constant independent of ε.

Let functions $\Phi_i(t,\varepsilon), i = 1,2,3$ are solutions for the following problem:

$$\begin{aligned} &L_\varepsilon \Phi_i(t,\varepsilon) = 0, \quad i = 1,2,3, \\ &\Phi_1(0,\varepsilon) = 1, \quad \Phi_1'(0,\varepsilon) = 0, \quad \Phi_1(1,\varepsilon) - \int_0^1 \sum_{i=0}^1 a_i(x)\Phi_1^{(i)}(x,\varepsilon)dx = 0 \\ &\Phi_2(0,\varepsilon) = 0, \quad \Phi_2'(0,\varepsilon) = 1, \quad \Phi_2(1,\varepsilon) - \int_0^1 \sum_{i=0}^1 a_i(x)\Phi_2^{(i)}(x,\varepsilon)dx = 0 \\ &\Phi_3(0,\varepsilon) = 0, \quad \Phi_3'(0,\varepsilon) = 0, \quad \Phi_3(1,\varepsilon) - \int_0^1 \sum_{i=0}^1 a_i(x)\Phi_3^{(i)}(x,\varepsilon)dx = 1. \end{aligned} \tag{9}$$

Functions $\Phi_i(t,\varepsilon)$, $i = 1,2,3$ are called boundary functions and can be represented in the form:

$$\Phi_i(t,\varepsilon) = \frac{I_i(t,\varepsilon)}{I(\varepsilon)}, \tag{10}$$

where

$$I(\varepsilon) = \begin{vmatrix} y_1(0,\varepsilon) & y_2(0,\varepsilon) & y_3(0,\varepsilon) \\ y_1'(0,\varepsilon) & y_2'(0,\varepsilon) & y_3'(0,\varepsilon) \\ y_1(1,\varepsilon) - \int_0^1 \sum_{i=0}^1 a_i(x) y_1^{(i)}(x,\varepsilon)dx & y_2(1,\varepsilon) - \int_0^1 \sum_{i=0}^1 a_i(x) y_2^{(i)}(x,\varepsilon)dx & y_3(1,\varepsilon) - \int_0^1 \sum_{i=0}^1 a_i(x) y_3^{(i)}(x,\varepsilon)dx \end{vmatrix}$$
$$= \frac{1}{\varepsilon}((1-a_1(1))\mu_1(0)y_{20}(1) + O(\varepsilon)) \neq 0, \tag{11}$$

$I_i(t,\varepsilon)$ is the determinant obtained from $I(\varepsilon)$ by replacing the i-th row by the fundamental set of solutions $y_1(t,\varepsilon), y_2(t,\varepsilon), y_3(t,\varepsilon)$ of the equation $L_\varepsilon y = 0$.

For boundary functions $\Phi_i(t,\varepsilon)$, $i = 1,2,3$ from (10) in view (4), (11) we obtain asymptotic representation as $\varepsilon \to 0$:

$$\begin{aligned}
\Phi_1^{(q)}(t,\varepsilon) =& y_{30}^{(q)}(t) - \frac{\mu_1(t)y_{10}(t)y_{30}'(0)}{\varepsilon^{q-1}\cdot\mu_1(0)} \exp(\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx) \\
&+ \frac{\mu_2^q(t)y_{20}(t)(y_{30}(1) - \int_0^1 \sum_{i=0}^1 a_i(x)y_{30}^{(i)}(x)dx)}{\varepsilon^q\cdot(1-a_1(1))y_{20}(1)} \exp(-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx) \\
&+ O(\varepsilon + \frac{1}{\varepsilon^{q-2}}\exp(\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx) + \frac{1}{\varepsilon^{q-1}}\exp(-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx)),\ q = \overline{0,2}, \\
\Phi_2^{(q)}(t,\varepsilon) =& -\varepsilon\frac{y_{30}^{(q)}(t)}{\mu_1(0)} + \frac{\mu_1^q(t)y_{10}(t)}{\varepsilon^{q-1}\cdot\mu_1(0)}\exp(\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx) \\
&+ \frac{\mu_2^q(t)y_{20}(t)(y_{30}(1) - \int_0^1 \sum_{i=0}^1 a_i(x)y_{30}^{(i)}(x)dx)}{\varepsilon^{q-1}\cdot\mu_1(0)(1-a_1(1))y_{20}(1)} \exp(-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx) \\
&+ O(\varepsilon^2 + \varepsilon^{2-q}\exp(\frac{1}{\varepsilon}\int_0^t \mu_1(x)dx) + \varepsilon^{2-q}\exp(-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx)),\ q = \overline{0,2}, \\
\Phi_3^{(q)}(t,\varepsilon) =& \frac{1}{\varepsilon^q}\cdot\frac{\mu_2^q(t)y_{20}(t)}{(1-a_1(1))y_{20}(1)}\cdot\exp(-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx) + O(\frac{1}{\varepsilon^{q-1}}\exp(-\frac{1}{\varepsilon}\int_t^1 \mu_2(x)dx)),\ q = \overline{0,2}.
\end{aligned} \tag{12}$$

From (12) we obtain the following asymptotic estimations as $\varepsilon \to 0$:

$$\begin{aligned}
\left|\Phi_1^{(q)}(t,\varepsilon)\right| &\leq C + \frac{C}{\varepsilon^{q-1}}e^{-\gamma_1\frac{t}{\varepsilon}} + \frac{C}{\varepsilon^q}e^{-\gamma_2\frac{1-t}{\varepsilon}},\ q = 0,1,2, \\
\left|\Phi_2^{(q)}(t,\varepsilon)\right| &\leq C\varepsilon + \frac{C}{\varepsilon^{q-1}}e^{-\gamma_1\frac{t}{\varepsilon}} + \frac{C}{\varepsilon^{q-1}}e^{-\gamma_2\frac{1-t}{\varepsilon}},\ q = 0,1,2, \\
\left|\Phi_3^{(q)}(t,\varepsilon)\right| &\leq \frac{C}{\varepsilon^q}e^{-\gamma_2\frac{1-t}{\varepsilon}},\ q = 0,1,2,
\end{aligned} \tag{13}$$

where $C > 0$ is a constant independent of ε.

3 Main results

We seek the solution of the boundary value problem (1) and (2) in the form:

$$\begin{aligned}
y(t,\varepsilon) =& C_1\Phi_1(t,\varepsilon) + C_2\Phi_2(t,\varepsilon) + C_3\Phi_3(t,\varepsilon) \\
&+ \frac{1}{\varepsilon^2}\int_0^t K_0(t,s,\varepsilon)z(s,\varepsilon)ds - \frac{1}{\varepsilon^2}\int_t^1 K_1(t,s,\varepsilon)z(s,\varepsilon)ds,
\end{aligned} \tag{14}$$

where $\Phi_i(t,\varepsilon), i = 1,2,3$ are boundary functions and expressed by the formula (10), functions $K_0(t,s,\varepsilon), K_1(t,s,\varepsilon)$ can be represented by the formula (6), $C_i, i = 1,2,3$ are unknown constants, $z(t,\varepsilon)$ is an unknown function. Substituting (14) into equation (1) we obtain that $z(t,\varepsilon)$ satisfies the following Fredholm integral equation of the second kind:

$$z(t,\varepsilon) = f(t,\varepsilon) + \int_0^1 H(t,s,\varepsilon) z(s,\varepsilon) ds, \tag{15}$$

where

$$f(t,\varepsilon) = F(t) + C_1 \int_0^1 \sum_{i=0}^1 H_i(t,x) \Phi_1^{(i)}(x,\varepsilon) dx + C_2 \int_0^1 \sum_{i=0}^1 H_i(t,x) \Phi_2^{(i)}(x,\varepsilon) dx + C_3 \int_0^1 \sum_{i=0}^1 H_i(t,x) \Phi_3^{(i)}(x,\varepsilon) dx,$$

$$H(t,s,\varepsilon) = \frac{1}{\varepsilon^2} \int_s^1 \sum_{i=0}^1 H_i(t,x) K_0^{(i)}(x,s,\varepsilon) dx - \frac{1}{\varepsilon^2} \int_0^s \sum_{i=0}^1 H_i(t,x) K_1^{(i)}(x,s,\varepsilon) dx.$$

In view of condition III integral equation (15) has an unique solution, that can be represented in the form

$$z(t,\varepsilon) = f(t,\varepsilon) + \int_0^1 R(t,s,\varepsilon) f(s,\varepsilon) ds, \tag{16}$$

where $R(t,s,\varepsilon)$ is a resolvent of the kernel $H(t,s,\varepsilon)$. Substituting (16) into equation (14) we obtain solution of the boundary value problem (1) and (2) in the form

$$y(t,\varepsilon) = \sum_{i=1}^3 C_i Q_i(t,\varepsilon) + P(t,\varepsilon), \tag{17}$$

where

$$\begin{aligned} Q_i(t,\varepsilon) &= \Phi_i(t,\varepsilon) + \frac{1}{\varepsilon^2} \int_0^t K_0(t,s,\varepsilon) \overline{\varphi_i}(s,\varepsilon) ds - \frac{1}{\varepsilon^2} \int_t^1 K_1(t,s,\varepsilon) \overline{\varphi_i}(s,\varepsilon) ds, \\ P(t,\varepsilon) &= \frac{1}{\varepsilon^2} \int_0^t K_0(t,s,\varepsilon) \overline{F}(s,\varepsilon) ds - \frac{1}{\varepsilon^2} \int_t^1 K_1(t,s,\varepsilon) \overline{F}(s,\varepsilon) ds, \\ \overline{\varphi_i}(s,\varepsilon) &= \int_0^1 \sum_{j=0}^1 \overline{H}_j(s,x,\varepsilon) \Phi_i^{(j)}(x,\varepsilon) dx, \overline{F}(s,\varepsilon) = F(s) + \int_0^1 R(s,p,\varepsilon) F(p) dp, \\ \overline{H}_j(s,x,\varepsilon) &\equiv H_j(s,x) + \int_0^1 R(s,p,\varepsilon) H_j(p,x) dp = \overline{H}_j(s,x) + O(\varepsilon). \end{aligned} \tag{18}$$

Now, we determine the unknown constants C_i, $i = 1,2,3$ in (17). For determining these constants we substitute (17) into (2). Thus, we need to solve the system of algebraic equation

$$\begin{cases} C_1 Q_1(0,\varepsilon) + C_2 Q_2(0,\varepsilon) + C_3 Q_3(0,\varepsilon) = \alpha - P(0,\varepsilon), \\ C_1 Q_1'(0,\varepsilon) + C_2 Q_2'(0,\varepsilon) + C_3 Q_3'(0,\varepsilon) = \beta - P'(0,\varepsilon), \\ C_1 \overline{Q_1}(1,\varepsilon) + C_2 \overline{Q_2}(1,\varepsilon) + C_3 \overline{Q_3}(1,\varepsilon) = \gamma - \overline{P}(1,\varepsilon) \end{cases} \tag{19}$$

where

$$\overline{Q_i}(1,\varepsilon) = Q_i(1,\varepsilon) - \int_0^1 \sum_{j=0}^1 a_j(x) Q_i^{(j)}(x,\varepsilon) dx, i = \overline{1,3}$$

$$\overline{P}(1,\varepsilon) = P(1,\varepsilon) - \int_0^1 \sum_{j=0}^1 a_j(x) P^{(j)}(x,\varepsilon) dx.$$

For the main determinant $\delta(\varepsilon)$ of the system (19) in view (7), (12), (18) we have asymptotic representation as $\varepsilon \to 0$: $\delta(\varepsilon) = \overline{\delta} + O(\varepsilon)$, where $\overline{\delta} = \int_0^1 \frac{\overline{H}_1(s,1)}{(1-a_1(1))A_1(s)y_{30}(s)} (y_{30}(1) - a_1(s) y_{30}(s) - \int\limits_0^1 \sum_{i=0}^1 a_i(x) y_{30}^{(i)}(x) dx) ds$.

Assume that the following condition is valid:

V. $\overline{\delta} \neq 0$.

The obtained results formulate the following theorem.

Theorem 1. *Let the conditions I-V are valid. Then boundary value problem (1) and (2) on the interval* $[0,1]$ *has an unique solution, expressed by the formula (17), where* $Q_i(t,\varepsilon), P(t,\varepsilon)$ *are defined by the formula (18),* $C_i,\ i=1,2,3$ *-are solutions of the system (19).*

Theorem 2. *If the conditions I-V are valid, then for the solution* $y(t,\varepsilon)$ *boundary value problem (1) and (2) and its derivatives the following asymptotic estimation holds as* $\varepsilon \to 0$ *:*

$$\begin{aligned}
|y(t,\varepsilon)| \leq & C(|\alpha|+\varepsilon|\beta|+|\gamma|\max_{0\leq t\leq 1}|H_1(t,1)|+\max_{0\leq t\leq 1}|F(t)|) \\
& + C\varepsilon\, e^{-\gamma_1 \frac{t}{\varepsilon}}(|\alpha|+|\beta|+|\gamma|\max_{0\leq t\leq 1}|H_1(t,1)|+\max_{0\leq t\leq 1}|F(t)|) \\
& + Ce^{-\gamma_2 \frac{1-t}{\varepsilon}}(|\alpha|+\varepsilon|\beta|+|\gamma|\max_{0\leq t\leq 1}|H_1(t,1)|+\max_{0\leq t\leq 1}|F(t)|), \\
\left|y'(t,\varepsilon)\right| \leq & C(|\alpha|+\varepsilon|\beta|+|\gamma|\max_{0\leq t\leq 1}|H_1(t,1)|+\max_{0\leq t\leq 1}|F(t)|) \\
& + C\, e^{-\gamma_1 \frac{t}{\varepsilon}}(|\alpha|+|\beta|+|\gamma|\max_{0\leq t\leq 1}|H_1(t,1)|+\max_{0\leq t\leq 1}|F(t)|) \\
& + \frac{C}{\varepsilon}e^{-\gamma_2 \frac{1-t}{\varepsilon}}(|\alpha|+\varepsilon|\beta|+|\gamma|\max_{0\leq t\leq 1}|H_1(t,1)|+\max_{0\leq t\leq 1}|F(t)|), \\
\left|y''(t,\varepsilon)\right| \leq & C(|\alpha|+\varepsilon|\beta|+|\gamma|\max_{0\leq t\leq 1}|H_1(t,1)|+\max_{0\leq t\leq 1}|F(t)|) \\
& + \frac{C}{\varepsilon}\, e^{-\gamma_1 \frac{t}{\varepsilon}}(|\alpha|+|\beta|+|\gamma|\max_{0\leq t\leq 1}|H_1(t,1)|+\max_{0\leq t\leq 1}|F(t)|) \\
& + \frac{C}{\varepsilon^2}e^{-\gamma_2 \frac{1-t}{\varepsilon}}(|\alpha|+\varepsilon|\beta|+|\gamma|\max_{0\leq t\leq 1}|H_1(t,1)|+\max_{0\leq t\leq 1}|F(t)|),
\end{aligned} \tag{20}$$

where $C>0$ *is a constant independent of* ε.

Proof. From (18) in view (8), (13) for the functions $Q_i^{(q)}(t,\varepsilon),\ i=1,2,3$ and $P^{(q)}(t,\varepsilon), q=0,1,2$ we obtain the following asymptotic estimations:

$$\begin{aligned}
\left|Q_1^{(q)}(t,\varepsilon)\right| &\leq C(1+\frac{1}{\varepsilon^{q-1}}e^{-\gamma_1\frac{t}{\varepsilon}}+\frac{1}{\varepsilon^{q}}e^{-\gamma_2\frac{1-t}{\varepsilon}}), \\
\left|Q_2^{(q)}(t,\varepsilon)\right| &\leq C(\varepsilon+\frac{1}{\varepsilon^{q-1}}e^{-\gamma_1\frac{t}{\varepsilon}}+\frac{1}{\varepsilon^{q-1}}e^{-\gamma_2\frac{1-t}{\varepsilon}}), \\
\left|Q_3^{(q)}(t,\varepsilon)\right| &\leq C(1+\frac{1}{\varepsilon^{q}}e^{-\gamma_2\frac{1-t}{\varepsilon}}), \\
\left|P^{(q)}(t,\varepsilon)\right| &\leq C\max_{0\leq t\leq 1}|F(t)|\,(1+\frac{1}{\varepsilon^{q-1}}e^{-\gamma_1\frac{t}{\varepsilon}}+\frac{1}{\varepsilon^{q-1}}e^{-\gamma_2\frac{1-t}{\varepsilon}}).
\end{aligned} \tag{21}$$

Then using estimations (21) from (17) we obtain asymptotic estimation (20). Theorem 2 is proved.

The theorem implies that the solution of the problem (1) and (2) at point $t=0$ has the phenomenon of the first order initial jump and at point $t=1$ has the phenomenon of the zero order initial jump, i.e. $y(0,\varepsilon)=O(1),\ \ y'(0,\varepsilon)=O(1),\ \ y''(0,\varepsilon)=O(\frac{1}{\varepsilon})$ and $y(1,\varepsilon)=O(1),\ \ y'(1,\varepsilon)=O(\frac{1}{\varepsilon}), y''(1,\varepsilon)=O(\frac{1}{\varepsilon^2})$. In this case we say that the solution of the boundary value problem (1) and (2) has the the phenomena of the boundary jumps.

4 Conclusions

We investigate how integral terms can effect asymptotic behavior of solutions of singularly perturbed integro-differential equations. Scientific novelties of the problem are as follows. Firstly, a solution to this problem has the boundary layer on both ends of the segment, and secondly, at the both ends of the segment has an initial jumps different orders in view infinitely large growth of the derivatives. Namely, at the left point of the segment the solution of the considered problem has an initial jump of the first order, and the right point is the point of the zero order initial jumps. The order of the initial jump depends on the order of derivatives in the integrand of the equation (1).

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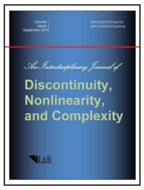
Discontinuity, Nonlinearity, and Complexity 5(2) (2016) 153–165



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On the Solvability of Nonlocal Boundary Value Problem for the Systems of Impulsive Hyperbolic Equations with Mixed Derivatives

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Abstract

A nonlocal boundary value problem for a system of impulsive hyperbolic equations at the fixed times is considered. The questions of existence, uniqueness, and construction of algorithms for finding the solutions to this problem are studied. By introducing the additional parameters as values of solutions on specific lines the considered problem is reduced to the problem consisting of the Goursat problem for a system of hyperbolic equations and the Cauchy problem for ordinary differential equations. The algorithms for finding the approximate solutions of latter problem are obtained and their convergence to the solution of original problem is proved. Conditions for existence of a unique solution to the nonlocal boundary value problem with impulse effects are set in the terms of initial data.

1 Introduction

Nonlocal boundary value problems for systems of hyperbolic equations have been investigated by numerous authors [1]– [2]. In the theory of nonlocal boundary value problems, many works were devoted to the study of periodic boundary value problems for hyperbolic partial differential equations (see [3]– [6] and references therein). Different problems of the theories of nuclear reactors, automatic controls, electrical and mechanical engineering, earthquake monitoring, and dynamical systems lead to the boundary value problems for impulsive differential equations. Periodic and almost periodic solutions to the systems of impulsive differential equations were comprehensively investigated in [7]– [15]. Periodic and some other types of nonlocal boundary value problems for impulsive hyperbolic equations were studied in [16]– [19].

Nonlocal boundary value problem for the systems of impulsive hyperbolic equations with data given on the characteristics was considered in [20]. Problems of existence and uniqueness of solution and continuous dependence of this solution on initial data were investigated by the method of introduction of functional parameters [21]– [26]. Sufficient conditions of unique solvability for the considering problem were established in the terms of coefficients of hyperbolic system, boundary and impulsive matrices.

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In this paper, a new class of nonlocal boundary value problems for the system of impulsive hyperbolic equations is investigated by the method of introduction of functional parameters. We construct the algorithms for finding the approximate solutions to the investigated problems. Conditions for existence of unique solution to the nonlocal boundary value problem for the system of impulsive hyperbolic equations are established in the terms of initial data.

We consider the following nonlocal boundary value problem for the system of impulsive hyperbolic equations of second-order at the fixed times in the domain $\bar{\Omega} = [0,T] \times [0,\omega]$:

$$\frac{\partial^2 u}{\partial t \partial x} = A(t,x)\frac{\partial u}{\partial x} + B(t,x)\frac{\partial u}{\partial t} + C(t,x)u + f(t,x), \qquad t \neq t_i, \tag{1}$$

$$u(t,0) = \psi(t), \qquad t \in [0,T], \tag{2}$$

$$P_2(x)\frac{\partial u(0,x)}{\partial x} + P_1(x)\frac{\partial u(t,x)}{\partial t}|_{t=0} + P_0(x)u(0,x) + S_2(x)\frac{\partial u(T,x)}{\partial x}$$
$$+S_1(x)\frac{\partial u(t,x)}{\partial t}|_{t=T} + S_0(x)u(T,x) = \varphi(x), \quad x \in [0,\omega], \tag{3}$$

$$M_i(x)\lim_{t\to t_i+0}\frac{\partial u(t,x)}{\partial x} - L_i(x)\lim_{t\to t_i-0}\frac{\partial u(t,x)}{\partial x} = \sum_{0<t_j\le t_i} V_{ij}(x)\lim_{t\to t_j+0}\frac{\partial u(t,x)}{\partial x} + \varphi_i(x),\ i = \overline{1,k}, \tag{4}$$

where $u = col(u_1,u_2,\ldots,u_n)$, the $n\times n$ matrices $A(t,x)$, $B(t,x)$, and $C(t,x)$, and the n vector-function $f(t,x)$ are piecewise continuous on $\bar{\Omega}$ with possible discontinuities at lines $t = t_i$, the $n\times n$ matrices $P_i(x)$ and $S_i(x)$, $i = \overline{0,2}$, $M_j(x)$, $L_j(x)$ and $V_{mj}(x)$, and the n-vector functions $\varphi(x)$, $\varphi_j(x)$, $j = \overline{1,k}$, $m = \overline{1,k}$, are continuous on $[0,\omega]$, the n-vector function $\psi(t)$ is continuously piecewise differentiable on $[0,T]$ with possible discontinuities at the lines $t = t_i$, $0 < t_1 < t_2 < \cdots < t_k < T$, and $||u(t,x)|| = \max_{i=\overline{1,n}}|u_i(t,x)|$, $||A(t,x)|| = \max_{i=\overline{1,n}}\sum_{j=1}^{n}|a_{ij}(t,x)|$.

Let $PC(\bar{\Omega},R^n)$ be a set of functions $u : \bar{\Omega} \to R^n$ continuous in $\bar{\Omega}$ at $t \neq t_i$, having the left-handed limits $\lim_{t\to t_i-0} u(t,x)$, for which the continuous right-handed limits at $t = t_i$, $i = \overline{1,k}$, exist.

Function $u(t,x) \in PC(\bar{\Omega},R^n)$, which has partial derivatives

$$\frac{\partial u(t,x)}{\partial x} \in PC(\bar{\Omega},R^n), \frac{\partial u(t,x)}{\partial t} \in PC(\bar{\Omega},R^n), \frac{\partial^2 u(t,x)}{\partial t\partial x} \in PC(\bar{\Omega},R^n),$$

is called a solution to problem (1)–(4) if it satisfies the system (1) for all $(t,x) \in \Omega$, except the lines $t = t_i$, $i = \overline{1,k}$, as well the boundary conditions (2), (3), and conditions of impulse effects at the fixed times (4).

To solve problem (1)–(4), we use the method of introduction of functional parameters [21]– [26]. This method is based on the introduction of additional parameters as the values of desired solution at the variable t on the certain lines of domain Ω. Nonlocal boundary value problem for a system of hyperbolic equations is reduced to an equivalent problem containing 1) the Goursat problem for system of hyperbolic equations with functional parameters depending on x and 2) the functional relations with respect to parameters. Properties of solution and its partial derivatives are transferred to the functional parameters as well. In [20], problem (1)–(4) was considered under $M_i(x) = I - U_i(x)$, $L_i(x) = I$, and $V_{ij}(x) = 0$, where I is the identity matrix of dimension n, and the $n\times n$ matrices $U_i(x)$ are continuous on $[0,\omega]$, $i, j = \overline{1,k}$. Based on the mentioned method the conditions of unique solvability for considered problem have been obtained and the algorithms for finding the solutions in the terms of initial data have been constructed. In addition, the coefficients of system of hyperbolic equations (1) are assumed continuous on $\bar{\Omega}$, and boundary function $\psi(t)$ is continuously differentiable on $[0,T]$.

In the present paper, we investigate the problem (1)–(4) for the initial data which are piecewise continuous on $\bar{\Omega}$ and $[0,T]$. We determine the sufficient conditions for unique solvability of problem (1)–(4) in the terms of initial data by using the method of introduction of functional parameters. We also offer an algorithm for finding of approximate solution to problem (1)–(4). The additional parameters are introduced as the values of desired function on the characteristics $t = t_i$, $i = \overline{0,k}$. Here $t_0 = 0$ and $t_{k+1} = T$.

Note that the algorithms offered for construction of solutions can be successfully implemented in mathematical software packages MathCad and MatLab. Results obtained for linear problem (1)–(4) can be used while solving the corresponding nonlocal nonlinear boundary value problems for systems of hyperbolic equations with impulse effects.

2 Scheme of method of introduction of functional parameters and the main result

Using the straight lines $t = t_i$, $i = \overline{1,k}$, we split the domain Ω into subdomains $\Omega_r = [t_{r-1},t_r)\times[0,\omega]$, $r = \overline{1,k+1}$. Let by $u_r(t,x)$ denote the restriction of function $u(t,x)$ to Ω_r, $r = \overline{1,k+1}$. Introducing the parameters $\mu_r(x) = u_r(t_{r-1},x)$, $r = \overline{1,k+1}$, and making the substitution $u(t,x) = \widetilde{u}_r(t,x)+\mu_r(x)$, $(t,x)\in\Omega_r$, $r = \overline{1,k+1}$, we reduce problem (1)-(4) to the following equivalent problem with functional parameters

$$\begin{aligned}\frac{\partial^2\widetilde{u}_r}{\partial t\partial x} = {} & A(t,x)\frac{\partial\widetilde{u}_r}{\partial x}+B(t,x)\frac{\partial\widetilde{u}_r}{\partial t}+C(t,x)\widetilde{u}_r+A(t,x)\dot{\mu}_r(x)\\ & +C(t,x)\mu_r(x)+f(t,x), \qquad (t,x)\in\Omega_r, \qquad r=\overline{1,k+1},\end{aligned} \tag{5}$$

$$\widetilde{u}_r(t_{r-1},x) = 0, \qquad x\in[0,\omega], \qquad r = \overline{1,k+1}, \tag{6}$$

$$\widetilde{u}_r(t,0) = \psi(t)-\psi(t_{r-1}), \qquad t\in[t_{r-1},t_r), \qquad r=\overline{1,k+1}, \tag{7}$$

$$\begin{aligned}P_2(x)\dot{\mu}_1(x)+P_1(x)\frac{\partial\widetilde{u}_1(t,x)}{\partial t}\Big|_{t=0} & +P_0(x)\mu_1(x)+S_2(x)\dot{\mu}_{k+1}(x)+S_2(x)\lim_{t\to T-0}\frac{\partial\widetilde{u}_{k+1}(t,x)}{\partial x}\\ & +S_1(x)\lim_{t\to T-0}\frac{\partial\widetilde{u}_{k+1}(t,x)}{\partial t}+S_0(x)\mu_{k+1}(x)+S_0(x)\lim_{t\to T-0}\widetilde{u}_{k+1}(t,x) = \varphi(x), \quad x\in[0,\omega],\end{aligned} \tag{8}$$

$$M_i(x)\dot{\mu}_{i+1}(x)-L_i(x)\dot{\mu}_i(x)-L_i(x)\lim_{t\to t_i-0}\frac{\partial\widetilde{u}_i(t,x)}{\partial x} = \sum_{j=1}^{i}V_{ij}(x)\dot{\mu}_{j+1}(x)+\varphi_i(x), \quad i=\overline{1,k}. \tag{9}$$

Solution to problem (5)-(9) is a system of pairs $(\mu(x),\widetilde{u}([t],x))$ with elements $\mu(x) = (\mu_1(x),\mu_2(x),\ldots,\mu_{k+1}(x))'$, $\widetilde{u}([t],x) = (\widetilde{u}_1(t,x),\widetilde{u}_2(t,x),\ldots,\ \widetilde{u}_{k+1}(t,x))'$, where the functions $\widetilde{u}_r(t,x)$ and their partial derivatives $\frac{\partial\widetilde{u}_r(t,x)}{\partial x}$, $\frac{\partial\widetilde{u}_r(t,x)}{\partial t}$, and $\frac{\partial^2\widetilde{u}_r(t,x)}{\partial t\partial x}$ are continuous on Ω_r, $r=\overline{1,k+1}$, and they have the finite left-hand side limits $\lim\limits_{t\to t_r-0}\frac{\partial\widetilde{u}_r(t,x)}{\partial x}$, $r=\overline{1,k+1}$, the functions $\mu_r(x)$ are continuously differentiable by x on $[0,\omega]$ and they satisfy the system of hyperbolic equations (5) and conditions (6)–(9).

Problems (1)–(4) and (5)–(9) are equivalent in the following sense: if the function $u(t,x)$ is a solution to problem (1)–(4), then the system of pairs $(\mu(x),\widetilde{u}([t],x))$, where $\mu(x) = (\mu_1(x),\mu_2(x),\ldots,\ \mu_{k+1}(x))'$, $\widetilde{u}([t],x) = (\widetilde{u}_1(t,x),\widetilde{u}_2(t,x),\ldots,\ \widetilde{u}_{k+1}(t,x))'$, $u_r(t,x) = u(t,x)$, $(t,x)\in\Omega_r$, $r=\overline{1,k+1}$, $\lim\limits_{t\to T-0}u_{k+1}(t,x) = u(T,x)$, $\mu_r(x) = u_r(t_{r-1},x)$, $\widetilde{u}_r(t,x) = u_r(t,x)-u_r(t_{r-1},x)$, $r=\overline{1,k+1}$, is a solution to problem (5)–(9); conversely, if $(\mu_r(x),\ \widetilde{u}_r(t,x))$, $r=\overline{1,k+1}$, is a solution to problem (5)–(9), then function $u(t,x)$, defined by the equalities

$$\begin{aligned}u(t,x) = {} & \mu_r(x)+\widetilde{u}_r(t,x), \qquad (t,x)\in\Omega_r, \quad r=\overline{1,k+1}\\ u(T,x) = {} & \mu_{k+1}(x)+\lim_{t\to T-0}\widetilde{u}_{k+1}(t,x),\end{aligned}$$

is a solution to problem (1)–(4). Unlike the problem (1)–(4), in this case, we have the initial conditions (6) specified as values of unknown function on the characteristics $t = t_{r-1}$, $r=\overline{1,k+1}$. For fixed $\mu_r(x)$ and $\dot{\mu}_r(x)$, $r=\overline{1,k+1}$, the functions $\widetilde{u}_r(t,x)$, $r=\overline{1,k+1}$, are the solutions to the Goursat problem on Ω_r with conditions

(6) and (7). Introducing the notations $\widetilde{v}_r(t,x) = \frac{\partial \widetilde{u}_r(t,x)}{\partial x}$ and $\widetilde{w}_r(t,x) = \frac{\partial \widetilde{u}_r(t,x)}{\partial t}$, in view of relations (6) and (7), we obtain $\widetilde{v}_r(t_{r-1},x) = 0$, $\widetilde{w}_r(t,0) = \dot{\psi}(t)$, and reduce the Goursat problem to the following system of three integral equations

$$\begin{aligned}
\widetilde{w}_r(t,x) = {} & \dot{\psi}(t) + \int_0^x [A(t,\xi)\widetilde{v}_r(t,\xi) + B(t,\xi)\widetilde{w}_r(t,\xi) + C(t,\xi)\widetilde{u}_r(t,\xi) \\
& + f(t,\xi) + A(t,\xi)\dot{\mu}_r(\xi) + C(t,\xi)\mu_r(\xi)]d\xi, \quad (10) \\
\widetilde{v}_r(t,x) = {} & \int_{t_{r-1}}^t [A(\tau,x)\widetilde{v}_r(\tau,x) + B(\tau,x)\widetilde{w}_r(\tau,x) + C(\tau,x)\widetilde{u}_r(\tau,x) \\
& + f(\tau,x) + A(\tau,x)\dot{\mu}_r(x) + C(\tau,x)\mu_r(x)]d\tau, \quad (11) \\
\widetilde{u}_r(t,x) = {} & \psi(t) - \psi(t_{r-1}) + \int_{t_{r-1}}^t d\tau \int_0^x [A(\tau,\xi)\widetilde{v}_r(\tau,\xi) + B(\tau,\xi)\widetilde{w}_r(\tau,\xi) \\
& + C(\tau,\xi)\widetilde{u}_r(\tau,\xi) + f(\tau,\xi) + A(\tau,\xi)\dot{\mu}_r(\xi) + C(\tau,\xi)\mu_r(\xi)]d\xi. \quad (12)
\end{aligned}$$

Substitute the corresponding right-hand side of (11) instead of $\widetilde{v}_r(\tau,x)$ and repeat this process ν $(\nu = 1,2,\ldots)$ times. This yields the following representation for the function $\widetilde{v}_r(t,x)$:

$$\widetilde{v}_r(t,x) = G_{\nu r}(t,x,\widetilde{v}_r) + H_{\nu r}(t,x,\widetilde{u}_r,\widetilde{w}_r) + F_{\nu r}(t,x) + D_{\nu r}(t,x)\dot{\mu}_r(x) + E_{\nu r}(t,x)\mu_r(x), \quad (13)$$

where

$$\begin{aligned}
G_{\nu r}(t,x,\widetilde{v}_r) = {} & \int_{t_{r-1}}^t A(\tau_1,x)\cdots\int_{t_{r-1}}^{\tau_{\nu-2}} A(\tau_{\nu-1},x)\int_{t_{r-1}}^{\tau_{\nu-1}} A(\tau_\nu,x)\widetilde{v}_r(\tau_\nu,x)d\tau_\nu\ldots d\tau_1, \\
H_{\nu r}(t,x,\widetilde{u}_r,\widetilde{w}_r) = {} & \int_{t_{r-1}}^t [B(\tau_1,x)\widetilde{w}_r(\tau_1,x) + C(\tau_1,x)\widetilde{u}_r(\tau_1,x)]d\tau_1 + \cdots \\
& + \int_{t_{r-1}}^t A(\tau_1,x)\ldots\int_{t_{r-1}}^{\tau_{\nu-2}} A(\tau_{\nu-1},x)\int_{t_{r-1}}^{\tau_{\nu-1}} [B(\tau_\nu,x)\widetilde{w}_r(\tau_\nu,x) + C(\tau_\nu,x)\widetilde{u}_r(\tau_\nu,x)]d\tau_\nu\ldots d\tau_1, \\
F_{\nu r}(t,x) = {} & \int_{t_{r-1}}^t f(\tau_1,x)d\tau_1 + \cdots + \int_{t_{r-1}}^t A(\tau_1,x)\cdots\int_{t_{r-1}}^{\tau_{\nu-2}} A(\tau_{\nu-1},x)\int_{t_{r-1}}^{\tau_{\nu-1}} f(\tau_\nu,x)d\tau_\nu\ldots d\tau_1, \\
D_{\nu r}(t,x) = {} & \int_{t_{r-1}}^t A(\tau_1,x)d\tau_1 + \cdots + \int_{t_{r-1}}^t A(\tau_1,x)\cdots\int_{t_{r-1}}^{\tau_{\nu-1}} A(\tau_\nu,x)d\tau_\nu\ldots d\tau_1, \\
E_{\nu r}(t,x) = {} & \int_{t_{r-1}}^t C(\tau_1,x)d\tau_1 + \cdots + \int_{t_{r-1}}^t A(\tau_1,x)\cdots\int_{t_{r-1}}^{\tau_{\nu-2}} A(\tau_{\nu-1},x)\int_{t_{r-1}}^{\tau_{\nu-1}} C(\tau_\nu,x)d\tau_\nu\ldots d\tau_1,
\end{aligned}$$

$(t,x) \in \Omega_r$, $r = \overline{1,k+1}$.

Passing at right-hand side of (13) to the limit at $t \to t_r - 0$, we find the limits $\lim_{t \to t_r - 0} \widetilde{v}_r(t,x)$, $r = \overline{1,k+1}$, $x \in [0,\omega]$. For the unknown functions $\mu_r(x)$, $r = \overline{1,k+1}$, substituting these functions into (8) and (9), we get the following system of $k+1$ ordinary differential equations of the first order unsolved with respect to the derivatives:

$$Q_\nu(x)\dot{\mu}(x) = E_\nu(x)\mu(x) + F_\nu(x) + H_\nu(x,\widetilde{u},\widetilde{w}) + G_\nu(x,\widetilde{v}), \quad (14)$$

where

$$Q_\nu(x)=\begin{vmatrix} -P_2(x) & 0 & 0 & .. & 0 & -S_2(x)\tilde{D}_{k+1}(x)\\ L_1(x)\tilde{D}_1(x) & V_{11}(x)-M_1(x) & 0 & .. & 0 & 0\\ 0 & L_2(x)\tilde{D}_2(x)+V_{21}(x) & V_{22}(x)-M_2(x) & .. & 0 & 0\\ .. & .. & .. & .. & .. & ..\\ 0 & V_{k1}(x) & V_{k2}(x) & .. & L_k(x)\tilde{D}_k(x)+V_{k(k-1)}(x) & V_{kk}(x)-M_k(x)\end{vmatrix},$$

$$\tilde{D}_i(x)= I+D_{\nu i}(t_i,x), \qquad i=\overline{1,k+1},$$

$$E_\nu(x)=\begin{vmatrix} P_0(x) & 0 & 0 & \dots & 0 & S_0(x)+S_2(x)E_{\nu(k+1)}(T,x)\\ L_1(x)E_{\nu 1}(t_1,x) & 0 & 0 & \dots & 0 & 0\\ 0 & L_2(x)E_{\nu 2}(t_2,x) & 0 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & L_k(x)E_{\nu k}(t_k,x) & 0\end{vmatrix},$$

$$F_\nu(x)= \big(S_2(x)F_{\nu(k+1)}(T,x)-\varphi(x),-L_1(x)F_{\nu 1}(t_1,x)-\varphi_1(x),\dots,-L_k(x)F_{\nu k}(t_k,x)-\varphi_k(x)\big)',$$

$$H_\nu(x,\widetilde{u},\widetilde{w})= \big(S_2(x)H_{\nu(k+1)}(T,x,\widetilde{u}_{k+1},\widetilde{w}_{k+1})+P_1(x)\widetilde{w}_1(0,x)+S_1(x)\widetilde{w}_{k+1}(T,x)$$
$$+S_0(x)\widetilde{u}_{k+1}(T,x),-L_1(x)H_{\nu 1}(t_1,x,\widetilde{u}_1,\widetilde{w}_1),\dots,-L_k(x)H_{\nu k}(t_k,x,\widetilde{u}_k,\widetilde{w}_k)\big)',$$

$$G_\nu(x,\widetilde{v})= \big(S_2(x)G_{\nu(k+1)}(T,x,\widetilde{v}_{k+1}),-L_1(x)G_{\nu 1}(t_1,x,\widetilde{v}_1),\dots,-L_k(x)G_{\nu k}(t_k,x,\widetilde{v}_k)\big)'.$$

Using the matching conditions at the points $(t_{r-1},0)$, $r=\overline{1,k+1}$, we get

$$\lambda_r(0)=\psi(t_{r-1}), \qquad r=\overline{1,k+1}. \tag{15}$$

If the functions $\mu_r(x)$ and $\dot{\mu}_r(x)$, $r=\overline{1,k+1}$ are known, we solve the system of integral equations (10)–(12) and obtain the functions $\widetilde{u}_r(t,x)$, $\widetilde{w}_r(t,x)$ and $\widetilde{v}_r(t,x)$. Using the system of functions $(\mu_r(x)+\widetilde{u}_r(t,x))$, we find the solution to the original problem. If the functions $\widetilde{u}_r(t,x)$, $\widetilde{w}_r(t,x)$ and $\widetilde{v}_r(t,x)$ are known, we solve equation (14) under condition (15) and determine $\dot{\mu}_r(x)$ and $\mu_r(x)$. Further, we get the solution to problem (1)–(4) from the system of functions $(\mu_r(x)+\widetilde{u}_r(t,x))$.

Here, the functions $\mu_r(x)$, $\dot{\mu}_r(x)$, $\widetilde{u}_r(t,x)$, $\widetilde{w}_r(t,x)$, and $\widetilde{v}_r(t,x)$ are unknown. Thus, by the iterative method, the solution to integral equations (10)–(12) and (14) with condition (15) is obtained as the limits of sequences $\{\mu_r^{(m)}(x)\}$, $\{\dot{\mu}_r^{(m)}(x)\}$, $\{\widetilde{u}_r^{(m)}(t,x)\}$, $\{\widetilde{w}_r^{(m)}(t,x)\}$, and $\{\widetilde{v}_r^{(m)}(t,x)\}$ according to the following algorithm.

Step 1. On right-hand side of (14) assume that $\mu_r(x)=\psi(t_{r-1})$, $\widetilde{u}_r(t,x)=\psi(t)-\psi(t_{r-1})$, $\widetilde{w}_r(t,x)=\dot{\psi}(t)$ and $\widetilde{v}_r(t,x)=0$, and that the matrix $Q_\nu(x)$ is invertible for all $x\in[0,\omega]$. Thus, equation (14) yields that $\dot{\mu}_r^{(0)}(x)$, $r=\overline{1,k+1}$ exist. By using conditions (15), we get the functions $\mu_r^{(0)}(x)$: $\mu_r^{(0)}(x)=\psi(t_{r-1})+\int_0^x\dot{\mu}_r^{(0)}(\xi)d\xi$. Determine the functions $\widetilde{u}_r^{(0)}(t,x)$, $\widetilde{w}_r^{(0)}(t,x)$, and $\widetilde{v}_r^{(0)}(t,x)$, $r=\overline{1,k+1}$ from the system of integral equations (10)–(12), where $\mu_r(x)=\mu_r^{(0)}(x)$ and $\dot{\mu}_r(x)=\dot{\mu}_r^{(0)}(x)$.

Step 2. Using system (14) with $\mu_r(x)=\mu_r^{(0)}(x)$, $\widetilde{u}_r(t,x)=\widetilde{u}_r^{(0)}(t,x)$, $\widetilde{w}_r(t,x)=\widetilde{w}_r^{(0)}(t,x)$, and $\widetilde{v}_r(t,x)=\widetilde{v}_r^{(0)}(t,x)$, $r=\overline{1,k+1}$ in view of invertibility of $Q_\nu(x)$ for $x\in[0,\omega]$, we find $\dot{\mu}_r^{(1)}(x)$, $r=\overline{1,k+1}$. By using conditions (15) once again, we obtain that $\mu_r^{(1)}(x)$: $\mu_r^{(1)}(x)=\psi(t_{r-1})+\int_0^x\dot{\mu}_r^{(1)}(\xi)d\xi$.

Determine the functions $\widetilde{u}_r^{(1)}(t,x)$, $\widetilde{w}_r^{(1)}(t,x)$ and $\widetilde{v}_r^{(1)}(t,x)$, $r=\overline{1,k+1}$ from the system of integral equations (10)–(12), where $\mu_r(x)=\mu_r^{(1)}(x)$ and $\dot{\mu}_r(x)=\dot{\mu}_r^{(1)}(x)$. And so on.

Method of introduction of functional parameters splits the process of determination of unknown functions into two stages:

(i) determination of introduced functional parameters $\mu_r(x)$ and $\dot{\mu}_r(x)$ from relation (14) with condition (15);

(ii) determination of unknown functions $\widetilde{u}_r(t,x)$, $\widetilde{w}_r(t,x)$ and $\widetilde{v}_r(t,x)$ from the system of integral equations (10)–(12).

Implementation of offered algorithm and unique solvability of problem (1)–(4) is guaranteed by the conditions of next assertion.

Theorem 1. *For some $\nu,\ \nu \in \mathbb{N}$, assume that the $[n(k+1)\times n(k+1)]$ matrix $Q_\nu(x)$ is invertible for all $x\in[0,\omega]$ and the following inequalities are true:*

(a) $||[Q_\nu(x)]^{-1}|| \le \gamma_\nu(x)$,

(b) $q_\nu(x) = \gamma_\nu(x)\cdot \max(||S_2(x)||, \max\limits_{i=\overline{1,k}} ||L_i(x)||)\cdot [e^{\alpha(x)h} - 1 - \sum\limits_{j=1}^{\nu} \frac{[\alpha(x)h]^j}{j!}] \le \chi < 1,$

where $\gamma_\nu(x)$ is a positive function continuous by $x\in[0,\omega]$, $\alpha(x) = \max\limits_{r=\overline{1,k+1}} \sup\limits_{t\in[t_{r-1},t_r)} ||A(t,x)||$, $h = \max\limits_{i=\overline{1,k+1}}(t_i - t_{i-1})$, and χ is a constant.

Then nonlocal boundary value problem for the system of impulsive hyperbolic equations (1) – (4) has a unique solution.

Proof. Under the assumption imposed on the data of problem, we get the following inequalities

$$||E_\nu(x)|| \le \ ||P_0(x)|| + ||S_0(x)|| + \max\{||S_2(x)||, \max_{i=\overline{1,k}} ||L_i(x)||\} h \sum_{j=0}^{\nu-1} \frac{[\alpha(x)h]^j}{j!} \max_{r=\overline{1,k+1}} \sup_{t\in[t_{r-1},t_r)} ||C(t,x)||,$$

$$||F_\nu(x)|| \le \ ||\varphi(x)|| + \max_{i=\overline{1,k}} ||\varphi_i(x)|| + \max\{||S_2(x)||, \max_{i=\overline{1,k}} ||L_i(x)||\} h \sum_{j=0}^{\nu-1} \frac{[\alpha(x)h]^j}{j!} \max_{r=\overline{1,k+1}} \sup_{t\in[t_{r-1},t_r)} ||f(t,x)||,$$

$$||H_\nu(x,\widetilde{u},\widetilde{w})|| \le \ a_0(x) \max_{r=\overline{1,k+1}} \sup_{t\in[t_{r-1},t_r)} [||\widetilde{w}_r(t,x)|| + ||\widetilde{u}_r(t,x)||], \tag{16}$$

where

$$a_0(x) = \ ||P_1(x)|| + ||S_1(x)|| + ||S_0(x)|| + \max\{||S_2(x)||, \max_{i=\overline{1,k}} ||L_i(x)||\} h \sum_{j=0}^{\nu-1} \frac{[\alpha(x)h]^j}{j!}$$
$$\times \max\{ \max_{r=\overline{1,k+1}} \sup_{t\in[t_{r-1},t_r)} ||B(t,x)||, \max_{r=\overline{1,k+1}} \sup_{t\in[t_{r-1},t_r)} ||C(t,x)||\}.$$

Let $\tilde{C}(\Omega_r, R^n)$ be a set of functions $\widetilde{u}_r : \Omega_r \longrightarrow R^n$ continuous and bounded on Ω_r.

By virtue of condition (a), for fixed $\mu_r(x)$, $\widetilde{u}_r(t,x)$, $\widetilde{w}_r(t,x)$ and $\widetilde{v}_r(t,x)$, $r=\overline{1,k+1}$, the system of functions $\dot{\mu}_r(x)$, $r=\overline{1,k+1}$, is uniquely determined from the system of equations (14) and

$$\dot{\mu}(x) = [Q_\nu(x)]^{-1}\{E_\nu(x)\mu(x) + F_\nu(x) + H_\nu(x,\widetilde{u},\widetilde{w}) + G_\nu(x,\widetilde{v})\}, \quad x\in[0,\omega],\ \mu\in R^{n(k+1)}.$$

For any r, $r=\overline{1,k+1}$, and fixed $\mu_r(x)\in C([0,\omega],R^n)$ and $\dot{\mu}_r(x)\in C([0,\omega],R^n)$, the system of integral equations (10)–(12) possesses a unique solution $\{\widetilde{u}_r(t,x), \widetilde{w}_r(t,x), \widetilde{v}_r(t,x)\}$, where $\widetilde{u}_r$, $\widetilde{w}_r$, and $\widetilde{v}_r$ belong to $\tilde{C}(\Omega_r,R^n)$, and the following estimates are true:

$$\sup_{t\in[t_{r-1},t_r)} ||\widetilde{v}_r(t,x)|| \le \ [e^{\alpha(x)(t_r-t_{r-1})} - 1]||\dot{\mu}_r(x)|| + (t_r - t_{r-1})e^{\alpha(x)(t_r-t_{r-1})}$$
$$\{ \sup_{t\in[t_{r-1},t_r)} ||C(t,x)||\cdot||\mu_r(x)|| + \sup_{t\in[t_{r-1},t_r)} ||f(t,x)||$$
$$+ \max(\sup_{t\in[t_{r-1},t_r)} ||B(t,x)||, \sup_{t\in[t_{r-1},t_r)} ||C(t,x)||)\} \sup_{t\in[t_{r-1},t_r)} [||\widetilde{u}_r(t,x)|| + ||\widetilde{w}_r(t,x)||], \tag{17}$$

$$\sup_{t\in[t_{r-1},t_r)}[||\widetilde{u}_r(t,x)||+||\widetilde{w}_r(t,x)||]$$
$$\leq \{\sup_{t\in[t_{r-1},t_r)}||\psi(t)-\psi(t_{r-1})||+\sup_{t\in[t_{r-1},t_r)}||\dot{\psi}(t)||$$
$$+(1+t_r-t_{r-1})\int_0^x[1+\alpha(\xi)(t_r-t_{r-1})e^{\alpha(\xi)(t_r-t_{r-1})}]\sup_{t\in[t_{r-1},t_r)}||f(t,\xi)||d\xi$$
$$+(1+t_r-t_{r-1})\int_0^x\alpha(\xi)(t_r-t_{r-1})e^{\alpha(\xi)(t_r-t_{r-1})}||\dot{\mu}_r(\xi)||d\xi$$
$$+(1+t_r-t_{r-1})\int_0^x[1+\alpha(\xi)(t_r-t_{r-1})e^{\alpha(\xi)(t_r-t_{r-1})}]\sup_{t\in[t_{r-1},t_r)}||C(t,\xi)||\cdot||\mu_r(\xi)||d\xi\}$$
$$\cdot\exp\{(1+t_r-t_{r-1})\int_0^x[1+\alpha(\xi)(t_r-t_{r-1})e^{\alpha(\xi)(t_r-t_{r-1})}]$$
$$\max\{\sup_{t\in[t_{r-1},t_r)}||B(t,\xi)||,\sup_{t\in[t_{r-1},t_r)}||C(t,\xi)||\}d\xi\}. \tag{18}$$

Following estimates are obtained from the first and second steps of offered algorithm

$$\max_{r=\overline{1,k+1}}||\dot{\mu}_r^{(0)}(x)||\leq \gamma_\nu(x)(||E_\nu(x)||\max_{r=\overline{1,k+1}}||\psi(t_{r-1})||+||F_\nu(x)||$$
$$+a_0(x)\max_{r=\overline{1,k+1}}\sup_{t\in[t_{r-1},t_r)}[||\dot{\psi}(t)||+||\psi(t)-\psi(t_{r-1})||])=d_1(x),$$
$$\max_{r=\overline{1,k+1}}||\mu_r^{(0)}(x)-\psi(t_{r-1})||\leq \int_0^x d_1(\xi)d\xi=d_2(x),$$
$$\max_{r=\overline{1,k+1}}||\dot{\mu}_r^{(1)}(x)-\dot{\mu}_r^{(0)}(x)||\leq \gamma_\nu(x)||E_\nu(x)||\cdot d_2(x)+\chi\cdot d_2(x)$$
$$+\gamma_\nu(x)[e^{b_1(x)}a_0(x)+a_1(x)]\max_{r=\overline{1,k+1}}\sup_{t\in[t_{r-1},t_r)}[||\dot{\psi}(t)||+||\psi(t)-\psi(t_{r-1})||]$$
$$+\gamma_\nu(x)[a_2(x)e^{b_1(x)}(1+h)\int_0^x(1+\alpha(\xi)he^{\alpha(\xi)h})$$
$$\max_{r=\overline{1,k+1}}\sup_{t\in[t_{r-1},t_r)}||f(t,\xi)||d\xi+a_1(x)\max_{t\in[0,T]}||f(t,x)||\Big]$$
$$+\gamma_\nu(x)e^{b_1(x)}[(1+h)a_2(x)b_1(x)+a_1(x)\max_{r=\overline{1,k+1}}\sup_{t\in[t_{r-1},t_r)}||C(t,x)||\Big]$$
$$\times(\max_{r=\overline{1,k+1}}||\psi(t_{r-1})||+\int_0^x||d_2(\xi)||d\xi)=d(x),$$

where

$$a_1(x)=h\max\{||S_2(x)||,\max_{i=\overline{1,k}}||L_i(x)||\}[e^{\alpha(x)h}-1-...-\frac{(\alpha(x)h)^{\nu-1}}{(\nu-1)!}],$$
$$a_2(x)=||P_1(x)||+||S_1(x)||+||S_0(x)||+\max\{||S_2(x)||,\max_{i=\overline{1,k}}||L_i(x)||\}he^{\alpha(x)h}a_0(x)$$
$$\times\max[\max_{r=\overline{1,k+1}}\sup_{t\in[t_{r-1},t_r)}||B(t,x)||,\max_{r=\overline{1,k+1}}\sup_{t\in[t_{r-1},t_r)}||C(t,x)||],$$
$$b_1(x)=\int_0^x\big[1+\alpha(\xi)[t_r-t_{r-1}]e^{\alpha(\xi)[t_r-t_{r-1}]}\big]\max[\max_{r=\overline{1,k+1}}\sup_{t\in[t_{r-1},t_r)}||B(t,\xi)||,\max_{r=\overline{1,k+1}}\sup_{t\in[t_{r-1},t_r)}||C(t,\xi)||]d\xi.$$

In the view of integral equation (11) and the Bellman - Gronwall inequality for the differences of successive approximations $\widetilde{v}_r^{(m)}(t,x)-\widetilde{v}_r^{(m-1)}(t,x)$, we obtain the estimate

$$\begin{aligned}||\widetilde{v}_r^{(m)}(t,x)-\widetilde{v}_r^{(m-1)}(t,x)|| \leq\ & [e^{\alpha(x)(t-t_{r-1})}-1]||\dot{\mu}_r^{(m)}(x)-\dot{\mu}_r^{(m-1)}(x)|| \\ & +(t-t_{r-1})e^{\alpha(x)(t-t_{r-1})}\Big(\sup_{t\in[t_{r-1},t_r)}\max\{||B(t,x)||,||C(t,x)||\} \\ & \times\sup_{t\in[t_{r-1},t_r)}[||\widetilde{w}_r^{(m)}(t,x)-\widetilde{w}_r^{(m-1)}(t,x)||+||\widetilde{u}_r^{(m)}(t,x)-\widetilde{u}_r^{(m-1)}(t,x)||] \\ & +\sup_{t\in[t_{r-1},t_r)}||C(t,x)||\cdot||\mu_r^{(m)}(x)-\mu_r^{(m-1)}(x)||\Big), \quad r=\overline{1,k+1}. \qquad (19)\end{aligned}$$

For the differences of successive approximations $\mu_r^{(m)}(x)-\mu_r^{(m-1)}(x)$, $\widetilde{u}_r^{(m)}(t,x)-\widetilde{u}_r^{(m-1)}(t,x)$, and $\widetilde{w}_r^{(m)}(t,x)-\widetilde{w}_r^{(m-1)}(t,x)$, $r=\overline{1,k+1}$, $m=1,2,...$, in view of inequalities (17)–(19), we get

$$||\mu_r^{(m)}(x)-\mu_r^{(m-1)}(x)|| \leq \int_0^x ||\dot{\mu}_r^{(m)}(\xi)-\dot{\mu}_r^{(m-1)}(\xi)||d\xi, \qquad (20)$$

$$\begin{aligned}&\max_{r=\overline{1,k+1}}\sup_{t\in[t_{r-1},t_r)}[||\widetilde{w}_r^{(m)}(t,x)-\widetilde{w}_r^{(m-1)}(t,x)||+||\widetilde{u}_r^{(m)}(t,x)-\widetilde{u}_r^{(m-1)}(t,x)||] \\ &\quad\leq \int_0^x b_2(\xi,x)\max_{r=\overline{1,k+1}}||\dot{\mu}_r^{(m)}(\xi)-\dot{\mu}_r^{(m-1)}(\xi)||d\xi, \qquad (21)\end{aligned}$$

where

$$\begin{aligned}b_2(\xi,x) =\ & e^{b_1(x)}(1+t_r-t_{r-1})[\alpha(\xi)(t_r-t_{r-1})e^{\alpha(\xi)(t_r-t_{r-1})}+b_3(x)], \\ b_3(x) =\ & \int_0^x[1+\alpha(\xi)(t_r-t_{r-1})e^{\alpha(\xi)(t_r-t_{r-1})}]\max_{r=\overline{1,k+1}}\sup_{t\in[t_{r-1},t_r)}||C(t,\xi)||d\xi.\end{aligned}$$

Therefore, we have

$$\begin{aligned}& ||\widetilde{v}_r^{(m)}(t,x)-\widetilde{v}_r^{(m-1)}(t,x)|| \\ \leq\ & [e^{\alpha(x)(t-t_{r-1})}-1]||\dot{\mu}_r^{(m)}(x)-\dot{\mu}_r^{(m-1)}(x)||+[t-t_{r-1}]e^{\alpha(x)(t-t_{r-1})} \\ & \int_0^x[\max\{\max_{r=\overline{1,k+1}}\sup_{t\in[t_{r-1},t_r)}||B(t,x)||,\max_{r=\overline{1,k+1}}\sup_{t\in[t_{r-1},t_r)}||C(t,x)||\}b_2(\xi,x) \\ & +\max_{r=\overline{1,k+1}}\sup_{t\in[t_{r-1},t_r)}||C(t,x)||]\cdot||\dot{\mu}_r^{(m)}(\xi)-\dot{\mu}_r^{(m-1)}(\xi)||d\xi.\end{aligned}$$

By using estimate (16), we arrive to the following relation for the difference $\dot{\mu}_r^{(m+1)}(x)-\dot{\mu}_r^{(m)}(x)$:

$$\begin{aligned}& \max_{r=\overline{1,k+1}}||\dot{\mu}_r^{(m+1)}(x)-\dot{\mu}_r^{(m)}(x)|| \\ \leq\ & ||[Q_\nu(x)]^{-1}||[||E_\nu(x)||\cdot\max_{r=\overline{1,k+1}}||\mu_r^{(m)}(x)-\mu_r^{(m-1)}(x)|| \\ & +a_0(x)\max_{r=\overline{1,k+1}}\sup_{t\in[t_{r-1},t_r)}[||\widetilde{w}_r^{(m)}(t,x)-\widetilde{w}_r^{(m-1)}(t,x)||+||\widetilde{u}_r^{(m)}(t,x)-\widetilde{u}_r^{(m-1)}(t,x)||] \\ & +\max(||S_2(x)||,\max_{i=\overline{1,k}}||L_i(x)||) \\ & \times\max_{r=\overline{1,k+1}}\{\int_{t_{r-1}}^{t_r}\alpha(x)...\int_{t_{r-1}}^{\tau_{\nu-2}}\alpha(x)\int_{t_{r-1}}^{\tau_{\nu-1}}\alpha(x)||\widetilde{v}_r^{(m)}(\tau_\nu,x)-\widetilde{v}_r^{(m-1)}(\tau_\nu,x)||d\tau_\nu d\tau_{\nu-1}...d\tau_1\}].\end{aligned}$$

Substituting inequality (18) into this relation, taking the repeated integrals, and applying the estimates (20) and (21), we obtain

$$\begin{aligned} &\max_{r=\overline{1,k+1}} ||\dot{\mu}_r^{(m+1)}(x) - \dot{\mu}_r^{(m)}(x)|| \\ \leq\ &\chi \max_{r=\overline{1,k+1}} ||\dot{\mu}_r^{(m)}(x) - \dot{\mu}_r^{(m-1)}(x)|| \\ &+ \int_0^x a_3(\xi,x) \max_{r=\overline{1,k+1}} ||\dot{\mu}_r^{(m)}(\xi) - \dot{\mu}_r^{(m-1)}(\xi)||d\xi, \end{aligned} \tag{22}$$

where

$$\begin{aligned} a_3(\xi,x) = \ &\gamma_\nu(x)\cdot\{\max\{||S_2(x)||, \max_{i=\overline{1,k}}||L_i(x)||\}h \\ &\max_{r=\overline{1,k+1}} \sup_{t\in[t_{r-1},t_r)} ||C(t,x)|| \sum_{j=0}^{\nu-1} \frac{[\alpha(x)h]^j}{j!} + a_0(x)b_0(\xi,x) \\ &+ \max\{||S_2(x)||, \max_{i=\overline{1,k}}||L_i(x)||\}h(e^{\alpha(x)h} - \sum_{j=0}^{\nu-1}\frac{[\alpha(x)h]^j}{j!}) \\ &[\max_{r=\overline{1,k+1}} \sup_{t\in[t_{r-1},t_r)} ||C(t,x)|| \\ &+ \max\{\max_{r=\overline{1,k+1}} \sup_{t\in[t_{r-1},t_r)} ||B(t,x)||, \max_{r=\overline{1,k+1}} \sup_{t\in[t_{r-1},t_r)} ||C(t,x)||\}a_0(\xi,x)]\}. \end{aligned}$$

In view of (22), we come to the following inequality for the function $\Lambda_m(x) = \max_{r=\overline{1,k+1}} ||\dot{\mu}_r^{(m+1)}(x) - \dot{\mu}_r^{(m)}(x)||$:

$$\begin{aligned} \Lambda_m(x) \leq\ &\sum_{j=0}^{m} \frac{m!}{(m-j)!\cdot j!}\cdot\chi^{m-j}\cdot\frac{1}{j!}(\int_0^x a_3(\xi,x)d\xi)^j \cdot \max_{x\in[0,\omega]} d(x) \\ &\leq \chi^m \sum_{j=0}^{m} \frac{m!}{(m-j)!\cdot j!}\cdot\frac{1}{j!}\left(\frac{\tilde{a}_3}{\chi}\right)^j\cdot\tilde{d}, \end{aligned} \tag{23}$$

where $\tilde{a}_3 = \max_{x\in[0,\omega]} \int_0^x a_3(\xi,x)d\xi$ and $\tilde{d} = \max_{x\in[0,\omega]} d(x)$. Since $\chi\in(0,1)$, we choose a number $\theta\in(0,(1-\chi)/\chi)$ and consider the sequence $z_m = \frac{1}{m!}(\frac{\tilde{b}_1}{\theta\chi})^m$. It is easily seen that $\lim_{m\to\infty} z_m = 0$, i.e. $z^* = 0$.

According to the corollary of Toeplitz theorem from the theory of limits, this implies that

$$\tilde{z}_m = \frac{1}{(1+\theta)^m}\sum_{j=0}^{m}\frac{m!}{(m-j)!\cdot j!}\cdot\theta^j\cdot z_j \to 0 \qquad \text{as} \qquad m\to\infty.$$

Then a number $d_3 > 0$ bounding the sequence $\tilde{z}_k$ exists and inequality (23) holds. This yields the following main estimate

$$\Lambda_m(x) \leq \chi^m(1+\theta)^m\cdot\tilde{z}_m\cdot\tilde{d} \leq \chi_1^m\cdot\tilde{d}\cdot d_3,$$

where $\chi_1 = \chi(1+\theta) < 1$, i.e. the sequence $\{\Lambda_m(x)\}$ is majorized by a geometric progression. This implies the uniform convergence of series $\sum_{m=1}^{\infty}\Lambda_m(x)$ as $x\in[0,\omega]$ guaranteeing the uniform convergence of sequence $\{\dot{\mu}_r^{(m)}(x)\}$ to the function $\dot{\mu}_r^*(x)$ continuous on $x\in[0,\omega]$ for all $r=\overline{1,k+1}$. Uniform convergence of sequence $\{\mu_r^{(m)}(x)\}$ to the function $\mu_r^*(x)\in C([0,\omega],R^n)$ follows from (20). Based on estimates (21) and (19), we conclude that the sequences $\{\widetilde{u}_r^{(m)}(t,x)\}$, $\{\widetilde{w}_r^{(m)}(t,x)\}$ and $\{\widetilde{v}_r^{(m)}(t,x)\}$, $r=\overline{1,k+1}$, converge uniformly by $(t,x)\in\Omega_r$ to the functions $\widetilde{u}_r^*(t,x)$, $\widetilde{w}_r^*(t,x)$ and $\widetilde{v}_r^*(t,x)$, respectively, which belong to $\tilde{C}(\Omega_r,R^n)$. Obviously, the function $u^*(t,x)$ obtained from the system of functions $(\mu_r^*(x)+\widetilde{u}_r^*(t,x))$ is the solution to problem (1)–(4). Prove the uniqueness of solution to problem (1)–(4) from contrary. Assume that two solutions $u^*(t,x)$ and $u^{**}(t,x)$ exist. Then the corresponding systems of pairs $(\mu_r^*(x),\widetilde{u}_r^*(t,x))$ and $(\mu_r^{**}(x),\ \widetilde{u}_r^{**}(t,x))$, $r=\overline{1,k+1}$, are the solutions to equivalent problem (5)–(9). Functions $\mu_r^*(x)$ and $\mu_r^{**}(x)$, $r=\overline{1,k+1}$ satisfy the systems

$$\dot{\mu}^*(x)=[Q_\nu(x)]^{-1}\{E_\nu(x)\mu^*(x)+F_\nu(x)+H_\nu(x,\widetilde{u}^*,\widetilde{w}^*)+G_\nu(x,\widetilde{v}^*)\}, \tag{24}$$

$$\dot{\mu}^{**}(x)=[Q_\nu(x)]^{-1}\{E_\nu(x)\dot{\mu}^{**}(x)+F_\nu(x)+H_\nu(x,\widetilde{u}^{**},\widetilde{w}^{**})+G_\nu(x,\widetilde{v}^{**})\}. \tag{25}$$

By analogy with (17) and (18), it follows from the system of integral equations (10)–(12), that

$$\begin{aligned}
&\sup_{t\in[_{r-1},t_r)}||\widetilde{v}_r^*(t,x)-\widetilde{v}_r^{**}(t,x)||\\
\leq\ &[e^{\alpha(x)(t_r-t_{r-1})}-1]||\dot{\mu}_r^*(x)-\dot{\mu}_r^{**}(x)||\\
&+(t_r-t_{r-1})e^{\alpha(x)(t_r-t_{r-1})}\sup_{t\in[t_{r-1},t_r)}||C(t,x)||\cdot||\mu_r^*(x)-\mu_r^{**}(x)||\\
&+(t_r-t_{r-1})e^{\alpha(x)(t_r-t_{r-1})}\max\{\sup_{t\in[t_{r-1},t_r)}||B(t,x)||,\ \sup_{t\in[t_{r-1},t_r)}||C(t,x)||\}\\
&\times\sup_{t\in[t_{r-1},t_r)}[||\widetilde{u}_r^*(t,x)-\widetilde{u}_r^{**}(t,x)||+||\widetilde{w}_r^*(t,x)-\widetilde{w}_r^{**}(t,x)||],
\end{aligned}$$

$$\begin{aligned}
&\sup_{t\in[t_{r-1},t_r)}[||\widetilde{u}_r^*(t,x)-\widetilde{u}_r^{**}(t,x)||+||\widetilde{w}_r^*(t,x)-\widetilde{w}_r^{**}(t,x)||]\\
\leq\ &\{(1+t_r-t_{r-1})\int_0^x\alpha(\xi)(t_r-t_{r-1})e^{\alpha(\xi)(t_r-t_{r-1})}||\dot{\mu}_r^*(\xi)-\dot{\mu}_r^{**}(\xi)||d\xi\\
&+(1+t_r-t_{r-1})\int_0^x[1+\alpha(\xi)(t_r-t_{r-1})e^{\alpha(\xi)(t_r-t_{r-1})}]\sup_{t\in[t_{r-1},t_r)}||C(t,\xi)||\cdot||\mu_r^*(\xi)-\mu_r^{**}(\xi)||d\xi\}\\
&\times\exp\{\int_0^x(1+\alpha(\xi)(t_r-t_{r-1})e^{\alpha(\xi)(t_r-t_{r-1})})\max\{\sup_{t\in[t_{r-1},t_r)}||B(t,\xi)||,\ \sup_{t\in[t_{r-1},t_r)}||C(t,\xi)||\}d\xi\}.
\end{aligned}$$

Further, by analogy with (20) and (21), we obtain

$$\begin{aligned}
&||\mu_r^*(x)-\mu_r^{**}(x)||\leq\int_0^x||\dot{\mu}_r^*(\xi)-\dot{\mu}_r^{**}(\xi)||d\xi,\\
&\max_{r=\overline{1,k+1}}\sup_{t\in[t_{r-1},t_r)}[||\widetilde{u}_r^*(t,x)-\widetilde{u}_r^{**}(t,x)||+||\widetilde{w}_r^*(t,x)-\widetilde{w}_r^{**}(t,x)||]\\
&\leq\int_0^x b_2(\xi,x)\max_{r=\overline{1,k+1}}||\dot{\mu}_r^*(\xi)-\dot{\mu}_r^{**}(\xi)||d\xi.
\end{aligned} \tag{26}$$

Then, in view of systems (24) and (25), we have the following estimate for the differences $\dot{\mu}_r^*(x)-\dot{\mu}_r^{**}(x)$

$$\max_{r=\overline{1,k+1}}||\dot{\mu}_r^*(x)-\dot{\mu}_r^{**}(x)||\leq\chi\max_{r=\overline{1,k+1}}||\dot{\mu}_r^*(x)-\dot{\mu}_r^{**}(x)||+\int_0^x a_3(\xi,x)\max_{r=\overline{1,k+1}}||\dot{\mu}_r^*(\xi)-\dot{\mu}_r^{**}(\xi)||d\xi.$$

This yields

$$\max_{r=\overline{1,k+1}}||\dot{\mu}_r^*(x)-\dot{\mu}_r^{**}(x)||\leq\frac{1}{1-\chi}\int_0^x\overline{a}_3(\xi)\max_{r=\overline{1,k+1}}||\dot{\mu}_r^*(\xi)-\dot{\mu}_r^{**}(\xi)||d\xi, \tag{27}$$

where $\overline{a}_3(\xi) = \max_{x\in[0,\omega]} a_3(\xi,x)$. By using the Bellman-Gronwall inequality and inequality (27), we find $\max_{r=\overline{1,k+1}} ||\dot{\mu}_r^*(x) - \dot{\mu}_r^{**}(x)|| = 0$. From the relations

$$\mu_r^*(x) = \psi(t_{r-1}) + \int_0^x \dot{\mu}_r^*(\xi)d\xi, \quad \mu_r^{**}(x) = \psi(t_{r-1}) + \int_0^x \dot{\mu}_r^{**}(\xi)d\xi,$$

we obtain $\mu_r^*(x) = \mu_r^{**}(x)$, $r = \overline{1,k+1}$. Then, it follows from inequality (26) that $\widetilde{u}_r^*(t,x) = \widetilde{u}_r^{**}(t,x)$ for all $(t,x) \in \Omega_r$, $r = \overline{1,k+1}$, and $u^*(t,x) = u^{**}(t,x)$. The proof of Theorem is complete.

Example. On $[0,1]\times[0,1]$ we consider the nonlocal boundary value problem with impulse effect for the system of equations

$$\frac{\partial^2 u}{\partial t \partial x} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \frac{\partial u}{\partial x}, \qquad t \neq \frac{1}{2}, \tag{28}$$

$$u(t,0) = 0, \qquad t \in [0,1], \tag{29}$$

$$\frac{\partial u(0,x)}{\partial x} - \frac{\partial u(1,x)}{\partial x} = 0, \quad x \in [0,1], \tag{30}$$

$$\lim_{t\to\frac{1}{2}+0} \frac{\partial u(t,x)}{\partial x} - e^{\frac{1}{2}} \lim_{t\to\frac{1}{2}-0} \frac{\partial u(t,x)}{\partial x} = e \lim_{t\to\frac{1}{2}+0} \frac{\partial u(t,x)}{\partial x} - 1, \tag{31}$$

where $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\alpha(x) = 2$, $h = \frac{1}{2}$.

The conditions of Theorem are fulfilled for $\nu = 1$: the (4×4) matrix

$$Q_1(x) = \begin{pmatrix} -1 & 0 & \frac{3}{2} & 0 \\ 0 & -1 & 0 & 2 \\ \frac{3}{2}e^{\frac{1}{2}} & 0 & e-1 & 0 \\ 0 & 2e^{\frac{1}{2}} & 0 & e-1 \end{pmatrix}$$

is invertible for all $x \in [0,1]$

$$[Q_1(x)]^{-1} = \begin{pmatrix} -4\cdot\frac{e-1}{9e^{\frac{1}{2}}-4+4e} & 0 & \frac{6}{9e^{\frac{1}{2}}-4+4e} & 0 \\ 0 & -\frac{e-1}{4e^{\frac{1}{2}}-1+e} & 0 & \frac{2}{4e^{\frac{1}{2}}-1+e} \\ 6\cdot\frac{e^{\frac{1}{2}}}{9e^{\frac{1}{2}}-4+4e} & 0 & \frac{4}{9e^{\frac{1}{2}}-4+4e} & 0 \\ 0 & 2\cdot\frac{e^{\frac{1}{2}}}{4e^{\frac{1}{2}}-1+e} & 0 & \frac{1}{4e^{\frac{1}{2}}-1+e} \end{pmatrix},$$

$||[Q_1(x)]^{-1}|| \le 0.64$, and $\quad q_1(x) = 0.64\cdot e^{\frac{1}{2}}\cdot[e-1-1] \le 0.76 < 1$.

Problem (28)-(31) has a unique solution of the following form

$$u(t,x) = \begin{pmatrix} \frac{e}{e^2-1}e^t x \\ \frac{e^2}{e^{\frac{7}{2}}-1}e^{2t} x \end{pmatrix} \quad \text{for } (t,x) \in [0,\frac{1}{2})\times[0,1]; \qquad u(t,x) = \frac{1}{1-e}\begin{pmatrix} \frac{1}{e^2-1}e^t x \\ \frac{1}{e^{\frac{7}{2}}-1}e^{2t} x \end{pmatrix} \quad \text{for } (t,x) \in [\frac{1}{2},1]\times[0,1].$$

3 Conclusion

Thus, Theorem 1 provides the sufficient conditions for the existence of unique solution to problem (1)–(4) in terms of initial data, i.e in the terms of coefficient matrix $A(t,x)$, bounding matrices $S_2(x)$, $P_2(x)$ and the matrices of impulse effects $M_i(x)$, $L_i(x)$, $V_{ij}(x)$, $i = \overline{1,k}$, $j = \overline{1,k}$.

Invertibility of matrix $Q_\nu(x)$ of dimension $n(k+1)\times n(k+1)$, which is composed block by block from the integrals of matrix $A(t,x)$, matrices of boundary conditions and impulse effects, is the main condition for existence of unique solution to problem (1)–(4). Assuming the invertibility of $n\times n$ matrices $V_{ii}(x)-M_i(x)$ or $L_i(x)[I+D_{\nu i}(t_i,x)]$, $i=\overline{1,k}$, we can show that for all $x\in[0,\omega]$ the invertibility of $n(k+1)\times n(k+1)$ matrix $Q_\nu(x)$ is equivalent to the invertibility of $n\times n$ matrices $\tilde{M}_\nu(x)$ or $\tilde{L}_\nu(x)$ of special form. Similar to the formulas of Lemma 1 and 2 from [20], we can obtain the explicit form of the matrices $\tilde{M}_\nu(x)$ or $\tilde{L}_\nu(x)$ and the recurrence formulas as well.

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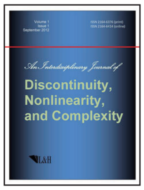

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Nonlinear Dissipation for Some Systems of Critical NLS Equations in Two Dimensions

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Abstract

We prove the global well-posedness in $H^1(\mathbb{R}^2,\mathbb{C}^N)$ for certain systems of the critical Nonlinear Schrödinger equations coupled linearly or nonlinearly with nonlinear supercritical dissipation terms, generalizing the previous result of [1] obtained for a single equation of this kind.

1 Introduction

There has been a substantial amount of work on nonlinear partial differential equations involving dissipative terms accomplished in recent years. Article [1] is devoted to the studies of the focusing, critical in two dimensions (with the cubic nonlinearity) Nonlinear Schrödinger (NLS) equation. It was established that when linear or nonlinear dissipative terms are incorporated in such an equation, it becomes globally well -posed in $H^1(\mathbb{R}^2)$. Work [2] is a numerical approach to the studies of singular solutions of the critical and supercritical NLS equation with the nonlinear dissipation. In paper [3] the authors show the global well-posedness for the cubic NLS with nonlinear damping when the external quadratic confining potential is present. The present article is the generalization of the ideas of [1] to the case of certain systems of NLS equations with the cubic nonlinearities, which are critical in two dimensions. We show the arrest of collapse occurring when the supercritical nonlinear dissipation is involved in such systems of equations. Apparently, such an effect has similarities with the enhanced binding in nonrelativistic Quantum Electrodynamics (QED) (see e.g. [4], [5], [6]). It is well known that in $\mathbb{R}^3$ the Schrödinger operator with a negative, shallow, short-range potential does not possess square integrable bound states. It was established that when the quantized radiation field in the Pauli-Fierz model is turned on, the particle absorbs the energy from it, its mass is getting increased and the negative eigenvalues with corresponding eigenfunctions belonging to $L^2(\mathbb{R}^3)$ appear. Certain systems of coupled NLS equations without dissipation were studied in [7] and [8] from the point of view of understanding the spectral stability of solitary

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waves. In the present work we will be using the $H^1(\mathbb{R}^2,\mathbb{C}^N)$ Sobolev space equipped with the norm

$$\|\psi\|^2_{H^1(\mathbb{R}^2,\mathbb{C}^N)} := \|\psi\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} + \|\nabla\psi\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} = \sum_{k=1}^{N}\{\|\psi_k\|^2_{L^2(\mathbb{R}^2)} + \|\nabla\psi_k\|^2_{L^2(\mathbb{R}^2)}\}$$

for a vector-function $\psi = (\psi_1,\psi_2,...,\psi_N)^T$. The inner product of two square integrable functions $f,g \in L^2(\mathbb{R}^2)$ will be designated as

$$(f,g)_{L^2(\mathbb{R}^2)} := \int_{\mathbb{R}^2} f(x)\bar{g}(x)dx.$$

The first part of the article deals with the system of N focusing critical NLS equations in two dimensions with the supercritical nonlinear dissipation, coupled linearly, such that

$$i\frac{\partial\psi_k}{\partial t} = -\Delta\psi_k - i\delta_k|\psi_k|^p\psi_k - |\psi_k|^2\psi_k + \sum_{s=1}^{N} a_{ks}\psi_s, \quad 1 \le k \le N. \tag{1}$$

The coupling matrix here and in system (2) is assumed to be arbitrary, constant in space and time and Hermitian, such that $a_{ks} = \overline{a_{sk}}$ for all $1 \le s,k \le N$. Here and in (2) $\delta_k > 0,\ 1 \le k \le N$ are meant to be arbitrary constants as well. Let us denote $\delta_{min} := min\{\delta_k\}_{k=1}^N > 0$. For the supercritical power involved in the nonlinear dissipative term both in (1) and (2) we have $p := 2(1+\alpha)$ with some constant $\alpha > 0$. The initial condition for systems (1) and for (2) analogously would be $\psi(x,0) = \psi_0(x) \in H^1(\mathbb{R}^2,\mathbb{C}^N)$. In the absence of nonlinear dissipation in two dimensions, for system (1) we have the blow up, which can be studied by virtue of the fairly standard scaling argument for NLS type equations (see e.g. Section 6 of [10]). The situation in (2) will depend on the choice of coefficients a_{ks}. Our first main result is as follows.

Theorem 1. *For every initial condition $\psi(x,0) \in H^1(\mathbb{R}^2,\mathbb{C}^N)$, there is a unique mild solution $\psi(x,t),\ t \in [0,\infty)$ of (1) with $\psi(x,t) \in H^1(\mathbb{R}^2,\mathbb{C}^N)$.*

The second part of the article is devoted to the studies of the system of critical NLS equations in two dimensions with the nonlinear supercritical dissipation coupled nonlinearly, namely

$$i\frac{\partial\psi_k}{\partial t} = -\Delta\psi_k - i\delta_k|\psi_k|^p\psi_k - \frac{1}{N}\sum_{s=1}^{N} a_{ks}|\psi_s|^2\psi_k, \quad 1 \le k \le N. \tag{2}$$

We also assume for the system above that for all $k,s = 1,...,N$ we have a_{ks} real valued and symmetric, namely $a_{ks} = a_{sk}$ and $|a_{ks}| \le a$ with some constant $a > 0$. Our second main statement is as follows.

Theorem 2. *For an arbitrary initial condition $\psi(x,0) \in H^1(\mathbb{R}^2,\mathbb{C}^N)$, there exists a unique mild solution $\psi(x,t),\ t \in [0,\infty)$ of system (2) with $\psi(x,t) \in H^1(\mathbb{R}^2,\mathbb{C}^N)$.*

First we turn our attention to the case of the supercritical nonlinear dissipation in a system of critical NLS equations coupled linearly.

2 The system of focusing, critical NLS with the nonlinear dissipation coupled linearly

Proof of Theorem 1. We rewrite the system of equations (1) as

$$\frac{\partial\psi_k}{\partial t} = i\Delta\psi_k + (F[\psi])_k, \quad 1 \le k \le N \tag{3}$$

with $(F[\psi])_k := -\delta_k|\psi_k|^p\psi_k + i|\psi_k|^2\psi_k - i\sum_{s=1}^{N} a_{ks}\psi_s$. The mild solution of our system satisfies the Duhamel's principle

$$\psi_k(x,t) = e^{it\Delta}\psi_{0,k}(x) + e^{it\Delta}\int_0^t e^{-is\Delta}(F[\psi(s)])_k ds, \quad 1 \le k \le N$$

in $H^1(\mathbb{R}^2,\mathbb{C}^N)$ for $t\in[0,T)$. The local well-posedness for our problem can be established using the Strichartz estimates (see e.g. Section 4 of [10] for the standard argument for the NLS type equations). Moreover,

$$\lim_{t\to T^-}\|\psi(t)\|_{H^1(\mathbb{R}^2,\mathbb{C}^N)}=\infty$$

if T is finite. With a slight abuse of notations we will be using the same letter T in such context in the proof of the consecutive theorem as well. Our goal is to establish that this solution is in fact global in time. Using the system of equations (3), we easily compute for $t\in[0,T)$:

$$\frac{d}{dt}\|\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)}=-2\sum_{k=1}^N\delta_k\int_{\mathbb{R}^2}|\psi_k(t)|^{p+2}dx-2Im\sum_{k=1}^N(\psi_k(t),\sum_{s=1}^N a_{ks}\psi_s(t))_{L^2(\mathbb{R}^2)}.$$

Note that the last expression in the right side of the identity above vanishes since the coupling matrix in our system is Hermitian as assumed. Thus, we arrive at

$$\frac{d}{dt}\|\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)}=-2\sum_{k=1}^N\delta_k\int_{\mathbb{R}^2}|\psi_k(t)|^{p+2}dx\le 0,\quad t\in[0,T), \tag{4}$$

which is analogous to formula (3.13) of [1] proven for the single NLS equation. Hence

$$\|\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)}\le\|\psi_0\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)},\quad t\in[0,T) \tag{5}$$

and for $t\in[0,T)$ we have

$$\|\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)}+2\sum_{k=1}^N\delta_k\int_0^t\int_{\mathbb{R}^2}|\psi_k(x,s)|^{p+2}dxds=\|\psi_0\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)}. \tag{6}$$

Therefore, the $L^2(\mathbb{R}^2,\mathbb{C}^N)$ norm of our mild solution is well under control. Using the system of equations (3) and taking the sufficiently regular solutions, we evaluate

$$\begin{aligned}\frac{d}{dt}\|\nabla\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)}=\;&2\sum_{k=1}^N\delta_k Re(\bar{\psi}_k(t)\Delta\psi_k(t),|\psi_k(t)|^p)_{L^2(\mathbb{R}^2)}\\&-2\sum_{k=1}^N Im(\bar{\psi}_k(t)\Delta\psi_k(t),|\psi_k(t)|^2)_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{7}$$

In the computation above we used that

$$Im\sum_{k=1}^N(-\Delta\psi_k(t),\sum_{s=1}^N a_{ks}\psi_s(t))_{L^2(\mathbb{R}^2)}=0.$$

Indeed, since the coupling matrix for our system is constant and Hermitian as assumed, its product with the Laplacian operator is self-adjoint and therefore, the term above vanishes. We will make use of the trivial identity

$$\bar{\psi}_k\Delta\psi_k-\psi_k\Delta\bar{\psi}_k=div(\bar{\psi}_k\nabla\psi_k-\psi_k\nabla\bar{\psi}_k),\quad 1\le k\le N \tag{8}$$

to obtain

$$Im(\bar{\psi}_k\Delta\psi_k,|\psi_k|^2)_{L^2(\mathbb{R}^2)}=Im\int_{\mathbb{R}^2}\psi_k^2(\nabla\bar{\psi}_k)^2dx. \tag{9}$$

A straightforward computation yields that the first term in the right side of (7) equals to

$$-2\sum_{k=1}^N\delta_k Re\int_{\mathbb{R}^2}\bar{\psi}_k(t)\nabla\psi_k(t).\nabla|\psi_k(t)|^p dx-2\sum_{k=1}^N\delta_k\int_{\mathbb{R}^2}|\nabla\psi_k(t)|^2|\psi_k(t)|^p dx. \tag{10}$$

Thus we arrive at

$$\begin{aligned}\frac{d}{dt}\|\nabla\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} = & -2\mathrm{Im}\sum_{k=1}^{N}\int_{\mathbb{R}^2}\psi_k^2(t)(\nabla\bar{\psi}_k(t))^2dx\\ & -2\sum_{k=1}^{N}\delta_k Re\int_{\mathbb{R}^2}\psi_k(t)\nabla\bar{\psi}_k(t).\nabla|\psi_k(t)|^p dx\\ & -2\sum_{k=1}^{N}\delta_k\int_{\mathbb{R}^2}|\nabla\psi_k(t)|^2|\psi_k(t)|^p dx, \quad t\in[0,T).\end{aligned} \tag{11}$$

A trivial calculation gives us

$$Re\int_{\mathbb{R}^2}\psi_k\nabla|\psi_k|^p.\nabla\bar{\psi}_k dx = \frac{p}{4}\int_{\mathbb{R}^2}[\nabla|\psi_k|^2]^2|\psi_k|^{p-2}dx \geq 0, \quad 1\leq k\leq N. \tag{12}$$

Let us introduce the following auxiliary quantity

$$P_k(t) := \int_{\mathbb{R}^2}|\psi_k|^p|\nabla\psi_k|^2dx, \quad 1\leq k\leq N. \tag{13}$$

We use the Hölder's inequality to obtain the following estimate from above

$$\int_{\mathbb{R}^2}|\psi_k|^2|\nabla\psi_k|^{\frac{4}{p}}|\nabla\psi_k|^{2-\frac{4}{p}}dx \leq \Big(P_k(t)\Big)^{\frac{2}{p}}\|\nabla\psi_k\|_{L^2(\mathbb{R}^2)}^{\frac{2(p-2)}{p}}. \tag{14}$$

Note that $p>2$ as assumed. We recall the Young's inequality

$$ab \leq \frac{a^p}{p}+\frac{b^q}{q} \tag{15}$$

for $a,b\geq 0$ and $\frac{1}{p}+\frac{1}{q}=1$. In the context of our work the conjugate exponents are $\frac{p}{2}$ and $\frac{p}{p-2}$. This gives us the following upper bound for the right side of inequality (14)

$$\delta_k P_k(t)+\frac{p-2}{p}\Big(\frac{2}{\delta_k p}\Big)^{\frac{2}{p-2}}\|\nabla\psi_k\|^2_{L^2(\mathbb{R}^2)}. \tag{16}$$

Let us introduce the constant

$$C(\delta_{min},p) := 2\frac{p-2}{p}\Big(\frac{2}{\delta_{min}p}\Big)^{\frac{2}{p-2}} > 0. \tag{17}$$

Hence, due to the estimates above, we arrive at the differential inequality

$$\frac{d}{dt}\|\nabla\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} \leq C(\delta_{min},p)\|\nabla\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)}, \quad t\in[0,T). \tag{18}$$

Apparently, differential inequality (18) yields the bound

$$\|\nabla\psi(x,t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} \leq \|\nabla\psi_0\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)}e^{C(\delta_{min},p)t}, \quad t\in[0,T). \tag{19}$$

Note that similarly to single NLS equations studied in [1], the supercritical nonlinear dissipation gives us the factor exponentially growing in time. Via the blow-up alternative, (19) implies that the system of NLS equations (1) is globally well-posed in $H^1(\mathbb{R}^2,\mathbb{C}^N)$.

3 The system of critical NLS with nonlinear dissipation coupled nonlinearly

Proof of Theorem 2. Clearly, the system of equations (2) can be easily written as

$$\frac{\partial \psi_k}{\partial t} = i\Delta\psi_k + (G[\psi])_k, \quad 1 \le k \le N \tag{20}$$

with $(G[\psi])_k := -\delta_k|\psi_k|^p\psi_k + \frac{i}{N}\sum_{s=1}^{N} a_{ks}|\psi_s|^2\psi_k$. The mild solution of system (20) satisfies the Duhamel's principle

$$\psi_k(x,t) = e^{it\Delta}\psi_{0,k}(x) + e^{it\Delta}\int_0^t e^{-is\Delta}(G[\psi(s)])_k ds, \quad 1 \le k \le N$$

in $H^1(\mathbb{R}^2, \mathbb{C}^N)$ for $t \in [0,T)$ and the local well- posedness can be established via the standard argument for NLS type equations by applying the Strichartz estimates (see e.g. Section 4 of [10]). Furthermore,

$$\lim_{t\to T^-}\|\psi(t)\|_{H^1(\mathbb{R}^2,\mathbb{C}^N)} = \infty$$

if T is finite. We are going to prove that such solution is global in time. A straightforward computation yields that estimates (4), (5) and (6) hold here as well. Using system (20) and considering sufficiently regular solutions, we obtain for $t \in [0,T)$:

$$\begin{aligned}\frac{d}{dt}\|\nabla\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} = & \ 2\sum_{k=1}^{N}\delta_k Re(\bar{\psi}_k(t)\Delta\psi_k(t), |\psi_k(t)|^p)_{L^2(\mathbb{R}^2)} \\ & -2\sum_{k=1}^{N} Im\frac{1}{N}\sum_{s=1}^{N} a_{ks}(\bar{\psi}_k(t)\Delta\psi_k(t), |\psi_s(t)|^2)_{L^2(\mathbb{R}^2)}.\end{aligned}$$

For the first term in the right side of the identity above we will use formula (10). A straightforward computation yields

$$Im(\bar{\psi}_k\Delta\psi_k, |\psi_s|^2)_{L^2(\mathbb{R}^2)} = -Im(\bar{\psi}_k\nabla\psi_k, \nabla|\psi_s|^2)_{L^2(\mathbb{R}^2,\mathbb{C}^2)}.$$

Clearly, via the Schwarz inequality we obtain

$$\left|Im(\bar{\psi}_k\nabla\psi_k, \bar{\psi}_s\nabla\psi_s)_{L^2(\mathbb{R}^2,\mathbb{C}^2)}\right| \le \|\bar{\psi}_k\nabla\psi_k\|_{L^2(\mathbb{R}^2)}\|\bar{\psi}_s\nabla\psi_s\|_{L^2(\mathbb{R}^2)},$$

$$\left|Im(\bar{\psi}_k\nabla\psi_k, \psi_s\nabla\bar{\psi}_s)_{L^2(\mathbb{R}^2,\mathbb{C}^2)}\right| \le \|\bar{\psi}_k\nabla\psi_k\|_{L^2(\mathbb{R}^2)}\|\psi_s\nabla\bar{\psi}_s\|_{L^2(\mathbb{R}^2)}.$$

Therefore, we arrive at

$$-2\sum_{k=1}^{N}\mathrm{Im}\frac{1}{N}\sum_{s=1}^{N} a_{ks}(\bar{\psi}_k\Delta\psi_k, |\psi_s|^2)_{L^2(\mathbb{R}^2)} \le 4a\sum_{k=1}^{N}\|\bar{\psi}_k\nabla\psi_k\|^2_{L^2(\mathbb{R}^2)}, \tag{21}$$

such that

$$\begin{aligned}& \frac{d}{dt}\|\nabla\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} \\ & +2\sum_{k=1}^{N}\delta_k Re\int_{\mathbb{R}^2}\bar{\psi}_k(t)\nabla\psi_k(t).\nabla|\psi_k(t)|^p dx \\ & +2\sum_{k=1}^{N}\delta_k\int_{\mathbb{R}^2}|\nabla\psi_k(t)|^2|\psi_k(t)|^p dx \\ \le & \ 4a\sum_{k=1}^{N}\|\bar{\psi}_k(t)\nabla\psi_k(t)\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}$$

We use identity (12) for the second term in the left side of the inequality above. The argument analogous to (14) and (16) yields

$$4a\sum_{k=1}^{N}\int_{\mathbb{R}^2}|\psi_k|^2|\nabla\psi_k|^2dx \le \sum_{k=1}^{N}2\delta_k P_k(t)+\tilde{C}(\delta_{min},p)\|\nabla\psi\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)}$$

with $P_k(t)$ given by (13) and

$$\tilde{C}(\delta_{min},p):=4a\frac{p-2}{p}\Big(\frac{4a}{\delta_{min}p}\Big)^{\frac{2}{p-2}}>0.$$

Hence we obtain the differential inequality

$$\frac{d}{dt}\|\nabla\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} \le \tilde{C}(\delta_{min},p)\|\nabla\psi(t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)}, \quad t\in[0,T)$$

Finally, we arrive at the bound

$$\|\nabla\psi(x,t)\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)} \le \|\nabla\psi_0\|^2_{L^2(\mathbb{R}^2,\mathbb{C}^N)}e^{\tilde{C}(\delta_{min},p)t}, \quad t\in[0,T). \tag{22}$$

Analogously to (19), the supercritical nonlinear dissipation gives us the factor exponentially growing in time. By means of the blow-up alternative, this implies that the system of NLS equations (2) is globally well-posed in $H^1(\mathbb{R}^2,\mathbb{C}^N)$.

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Dynamical Systems Generated by a Gonosomal Evolution Operator

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Abstract

In this paper we consider discrete-time dynamical systems generated by gonosomal evolution operators of sex linked inheritance. Mainly we study dynamical systems of a hemophilia, which biologically is a group of hereditary genetic disorders that impair the body's ability to control blood clotting or coagulation, which is used to stop bleeding when a blood vessel is broken. We give an algebraic model of the biological system corresponding to the hemophilia. The evolution of such system is studied by a nonlinear (quadratic) gonosomal operator. In a general setting, this operator is considered as a mapping from $\mathbb{R}^n$, $n \geq 2$ to itself. In particular, for a gonosomal operator at $n = 4$ we explicitly give all (two) fixed points. Then limit points of the trajectories of the corresponding dynamical system are studied. Moreover we consider a normalized version of the gonosomal operator. In the case $n = 4$, for the normalized gonosomal operator we show uniqueness of fixed point and study limit points of the dynamical system.

1 Introduction

In biology it is important a proper understanding of living populations at all levels. The relevant mathematics undoubtedly requires of nonlinear analysis, in particular a nonlinear dynamical system, compounded stochastic processes modeling, and the creative implementation of the computer methodology.

The action of genes is manifested statistically in sufficiently large communities of matching individuals (belonging to the same species). These communities are called *populations* [1]. The population exists not only in space but also in time, i.e. it has its own life cycle. The basis for this phenomenon is reproduction by mating. Mating in a population can be free or subject to certain restrictions.

The whole population in space and time comprises discrete generations $F_0, F_1, \ldots$ The generation F_{n+1} is the set of individuals whose parents belong to the F_n generation. A *state* of a population is a distribution of probabilities of the different types of organisms in every generation.

A type partition is called differentiation. The simplest example is sex differentiation. In bisexual population any kind of differentiation must agree with the sex differentiation, i.e. all the organisms of one type must

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belong to the same sex. Thus, it is possible to speak of male and female types (see for example [2], [3], [4] for mathematical models of bisexual population).

In many biological systems, sex is determined genetically: males and females have different alleles or even different genes that specify their sexual morphology. In animals, this is often accompanied by *chromosomal* differences. Determination genetically is generally through chromosome combinations of XY (for example: humans, mammals), ZW (birds), $X0$ (in this variant of the XY system, females have two copies of the sex chromosome (XX) but males have only one ($X0$). The 0 denotes the absence of a second sex chromosome. There are some sex linked systems which depends on temperature and even some of systems have sex change phenomenon (see [5] for a detailed review.) We note that the behavior of sex-linked system can be investigated by studying of nonlinear dynamical systems, such systems are not fully understood yet. A search in MathSciNet gives about 15 mathematical papers which are related to sex linked models (see for example, [6], [7], [8], [9]). In papers [10], [11] we attempted to introduce thermodynamic methods in biology. In [4] an algebra associated to a sex change is constructed.

In this paper we consider evolution (dynamical system) of a hemophilia. Recall that *hemophilia* is a group of hereditary genetic disorders that impair the body's ability to control blood clotting or coagulation, which is used to stop bleeding when a blood vessel is broken.

In the next section we give a mathematical model of the biological system corresponding to the hemophilia. The evolution of such system will be given by a nonlinear (quadratic) evolution operator which is called a gonosomal operator. Thus study of the biological system is reduced to the study of the nonlinear dynamical system generated by the gonosomal operator. In a general setting, this operator is considered as a mapping from $\mathbb{R}^n$, $n \geq 2$ to itself. In Section 4 we give some detailed properties of the dynamical system. In particular, for a gonosomal operator at $n = 4$ we explicitly give all (two) fixed points. Then limit points of the trajectories of the corresponding dynamical system are studied. In the last section we consider the normalized version of the gonosomal operator. In the case $n = 4$, for the normalized gonosomal operator we show uniqueness of fixed point and study limit points of the dynamical system.

2 Bisexual population: Gonosomal evolution operator

Type partition is called differentiation. The simplest example is sex differentiation. In bisexual population (BP) any kind of differentiation must agree with the sex differentiation, i.e. all the organisms of one type must belong to the same sex. Thus, it is possible to speak of male and female types.

In many cases, the sex determination is genetic, in particular, it is controlled by two chromosomes called gonosomes. Gonosomal inheritance is a mode of inheritance that is observed for traits related to a gene encoded on the sex chromosomes.

Let us discuss one example of sex-linked inheritance. Haemophilia is a lethal recessive X-linked disorder: a female carrying two alleles for hemophilia die. Therefore if we denote by X^h the gonosome X carrying the hemophilia, there are only two female genotypes: XX and XX^h (X^hX^h is lethal) and two male genotypes: XY and X^hY. We have four types of crosses:

$$\begin{aligned}
&XX \times XY \rightarrowtail \tfrac{1}{2}XX,\ \ \tfrac{1}{2}XY;\\
&XX \times X^hY \rightarrowtail \tfrac{1}{2}XX^h,\ \ \tfrac{1}{2}XY;\\
&XX^h \times XY \rightarrowtail \tfrac{1}{4}XX, \tfrac{1}{4}XX^h, \tfrac{1}{4}XY, \tfrac{1}{4}X^hY;\\
&XX^h \times X^hY \rightarrowtail \tfrac{1}{3}XX^h, \tfrac{1}{3}XY, \tfrac{1}{3}X^hY.
\end{aligned} \tag{1}$$

Let $F = \{XX, XX^h\}$ and $M = \{XY, X^hY\}$ be sets of genotypes. Assume state of the set F is given by a real vector (x, y) and state of M by a real vector (u, v). Then a state of $F \cup M$ is given by the vector $s = (x, y, u, v) \in \mathbb{R}^4$. If $s' = (x', y', u', v')$ is a state of the system $F \cup M$ in the next generation then by the rule (1) we get the evolution operator $W:\mathbb{R}^4\to\mathbb{R}^4$ defined by

$$W:\begin{cases} x'=\frac{1}{2}xu+\frac{1}{4}yu\\ y'=\frac{1}{2}xv+\frac{1}{4}yu+\frac{1}{3}yv\\ u'=\frac{1}{2}xu+\frac{1}{2}xv+\frac{1}{4}yu+\frac{1}{3}yv\\ v'=\frac{1}{4}yu+\frac{1}{3}yv. \end{cases} \tag{2}$$

This example can be generalized: suppose that the set of female types is $F=\{1,2,\dots,n\}$ and the set of male types is $M=\{1,2,\dots,\nu\}$. Let $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ be a state of F and $y=(y_1,\dots,y_\nu)\in\mathbb{R}^\nu$ be a state of M.

Consider $\gamma^{(f)}_{ik,j}$ and $\gamma^{(m)}_{ik,l}$ as some inheritance real coefficients (not necessary probabilities) with

$$\sum_{j=1}^{n}\gamma^{(f)}_{ik,j}+\sum_{l=1}^{\nu}\gamma^{(m)}_{ik,l}=1. \tag{3}$$

Consider an evolution operator $W:\mathbb{R}^{n+\nu}\to\mathbb{R}^{n+\nu}$ defined as

$$W:\begin{cases} x'_j=\sum_{i,k=1}^{n,\nu}\gamma^{(f)}_{ik,j}x_iy_k, \quad j=1,\dots,n\\ y'_l=\sum_{i,k=1}^{n,\nu}\gamma^{(m)}_{ik,l}x_iy_k, \quad l=1,\dots,\nu. \end{cases} \tag{4}$$

This operator is called gonosomal evolution operator. This means that the association $s=(x,y)\in\mathbb{R}^{n+\nu}\to s'=(x',y')\in\mathbb{R}^{n+\nu}$ defines a map W. The population evolves by starting from an arbitrary state s, then passing to the state $s'=W(s)$ (in the next 'generation'), then to the state $s''=W(W(s))$, and so on. Thus, states of the population described by the following discrete-time dynamical system

$$s^{(0)},\ s^{(1)}=W(s^{(0)}),\ s^{(2)}=W^2(s^{(0)}),\ s^{(3)}=W^3(s^{(0)}),\dots \tag{5}$$

where $s^{(0)}\in\mathbb{R}^{n+\nu}$ is a given initial point and $W^n(s)=\underbrace{W(W(...W}_{n}(s))...)$ denotes the n times iteration of W to s.

The main problem for a given dynamical system is to describe the limit points of the trajectory $\{s^{(n)}\}_{n=0}^\infty$ for arbitrary given $s^{(0)}$.

3 Dynamical system generated by the operator (2)

Note that operator (4) describes evolution of a hemophilia. The dynamical systems generated by gonosomal operator (4) is complicated. In this paper we study the dynamical system generated by gonosomal operator (2), which is a particular case of (4), obtained by $n=\nu=2$ and the following coefficients:

$$\begin{array}{llll} \gamma^{(f)}_{11,1}=\frac{1}{2} & \gamma^{(f)}_{11,2}=0 & \gamma^{(m)}_{11,1}=\frac{1}{2} & \gamma^{(m)}_{11,2}=0\\ \gamma^{(f)}_{12,1}=0 & \gamma^{(f)}_{12,2}=\frac{1}{2} & \gamma^{(m)}_{12,1}=\frac{1}{2} & \gamma^{(m)}_{12,2}=0\\ \gamma^{(f)}_{21,1}=\frac{1}{4} & \gamma^{(f)}_{21,2}=\frac{1}{4} & \gamma^{(m)}_{21,1}=\frac{1}{4} & \gamma^{(m)}_{21,2}=\frac{1}{4}\\ \gamma^{(f)}_{22,1}=0 & \gamma^{(f)}_{22,2}=\frac{1}{3} & \gamma^{(m)}_{22,1}=\frac{1}{3} & \gamma^{(m)}_{22,2}=\frac{1}{3} \end{array}$$

3.1 Fixed points

A point s is called fixed point if $W(s)=s$. Let us find all fixed points of W given by (2), i.e. we solve the following system of equations

$$W:\begin{cases} x=\frac{1}{2}xu+\frac{1}{4}yu \\ y=\frac{1}{2}xv+\frac{1}{4}yu+\frac{1}{3}yv \\ u=\frac{1}{2}xu+\frac{1}{2}xv+\frac{1}{4}yu+\frac{1}{3}yv \\ v=\frac{1}{4}yu+\frac{1}{3}yv. \end{cases} \tag{6}$$

First it is easy to see that $s_0=(0,0,0,0)$ is a solution to system (6).

To find another solution from the first equation of this system we get $(4-2u)x=yu$. Assuming $u=2$ from this equation we get $y=0$ then the last equation of (6) gives $v=0$ consequently from the third equation of the system we get $x=2$. Thus we obtained the solution $s_2=(2,0,2,0)$.

From the last equation we get $(12-4y)v=3yu$, assume first that $y=3$ then this equation gives $u=0$, consequently the first equation of the system (6) gives $x=0$. Then from the second equation we get $v=3$. But these values do not satisfy the third equation of the system. Assume now $u\neq 2$ and $y\neq 3$ then we have

$$x=\frac{yu}{4-2u},\quad v=\frac{3yu}{12-4y}. \tag{7}$$

Using (7) from the second and third equations of the system (6) we obtain

$$\begin{cases} 16(3-y)(2-u)=3u(8-4u+yu) \\ 16(3-y)(2-u)=y(24-yu). \end{cases}$$

From the first equation of the last system we find

$$y=\frac{12(u^2-6u+8)}{3u^2-16u+32}.$$

Substituting this to the second equation of the last system we obtain the following equation

$$(u-2)^2(u-8)\left(3u^2-14u+24\right)=0.$$

This equation gives $u=2$, $u=8$ and $3u^2-14u+24=0$. For case $u=2$ we have solution s_2 mentioned above. The case $u=8$ gives $y=3$ which does not give solution of the system (6) as was discussed above. Thus only remains $3u^2-14u+24=0$ which does not have real solutions.

We proved the following

Proposition 1. *The gonosomal operator (2) has exactly two fixed points:* $s_0=(0,0,0,0)$ *and* $s_2=(2,0,2,0)$.

3.2 The type of the fixed points

Now we shall examine the type of the fixed points.

Definition 1. (see [12]). A fixed point s of the operator W is called hyperbolic if its Jacobian J at s has no eigenvalues on the unit circle.

Definition 2. (see [12]). A hyperbolic fixed point s is called:

- attracting if all the eigenvalues of the Jacobi matrix $J(s)$ are less than 1 in absolute value;
- repelling if all the eigenvalues of the Jacobi matrix $J(s)$ are greater than 1 in absolute value;

- a saddle otherwise.

To find the type of a fixed point of the operator (2) we write the Jacobi matrix:

$$J(s) = J_W = \begin{pmatrix} \frac{1}{2}u & \frac{1}{4}u & \frac{1}{2}x+\frac{1}{4}y & 0 \\ \frac{1}{2}v & \frac{1}{4}u+\frac{1}{3}v & \frac{1}{4}y & \frac{1}{2}x+\frac{1}{3}y \\ \frac{1}{2}u+\frac{1}{2}v & \frac{1}{4}u+\frac{1}{3}v & \frac{1}{2}x+\frac{1}{4}y & \frac{1}{2}x+\frac{1}{3}y \\ 0 & \frac{1}{4}u+\frac{1}{3}v & \frac{1}{4}y & \frac{1}{3}y \end{pmatrix}.$$

It is easy to see that $J(s_0)$ has all eigenvalues equal to 0, therefore s_0 is an attracting point.

The Jacobian $J(s_2)$ has eigenvalues $-\frac{1}{2}, 0, 1, 2$, therefore the fixed point is not hyperbolic.

3.3 Dynamics on invariant sets

A set A is called invariant with respect to W if $W(A) \subset A$.

Denote

$$\begin{aligned} O &= \{(0,0,u,v) \in \mathbb{R}^4 : u,v \in \mathbb{R}\} \cup \{(x,y,0,0) \in \mathbb{R}^4 : x,y \in \mathbb{R}\}, \\ I &= \{s = (x,y,u,v) \in \mathbb{R}^4 : y = v = 0\}, \\ J &= \{s \in I : x = u\}, \\ P &= \{s = (x,y,u,v) \in \mathbb{R}^4 : x \geq 0, y \geq 0, u \geq 0, v \geq 0\}, \\ Q_a &= \{s = (x,y,u,v) \in P : x+y+u+v \leq a\}, \qquad a \in [0,4], \\ \mathscr{N} &= \{s = (x,y,u,v) \in \mathbb{R}^4 : x \leq 0, y \leq 0, u \leq 0, v \leq 0\}, \\ \mathscr{N}_0 &= \{s = (x,y,u,v) \in \mathbb{R}^4 : x \leq 0, y \leq 0, u \geq 0, v \geq 0\}, \\ \mathscr{N}_1 &= \{s = (x,y,u,v) \in \mathbb{R}^4 : x \geq 0, y \geq 0, u \leq 0, v \leq 0\}. \end{aligned}$$

Lemma 2. 1) *The sets I, J, P and Q_a ($a \in [0,4]$) are invariant with respect to W.*

2) $W(O) = \{(0,0,0,0)\}$.

3) $W(Q_a) \subset Q_{a^2/4}$.

4) $W(\mathscr{N}) \subset P$.

5) $W(\mathscr{N}_0) \subset \mathscr{N}, \quad W(\mathscr{N}_1) \subset \mathscr{N}$.

Proof. We give the proof for Q_a, for other sets it simply follows from (2). Take any $s = (x,y,u,v) \in Q_a$ then we have $0 \leq x+y \leq a$ and $0 \leq u+v \leq a$. From (2) we get $x' \geq 0$, $y' \geq 0$, $u' \geq 0$, $v' \geq 0$ and

$$x'+y'+u'+v' = (x+y)(u+v) \leq (\frac{x+y+u+v}{2})^2 \leq \frac{a^2}{4}.$$

Thus $s' = (x',y',u',v') \in Q_{\frac{a^2}{4}} \subset Q_a$.

Reduce W on J then we get the mapping $x' = f(x) = \frac{1}{2}x^2$. This function has two fixed points $x = 0$ and $x = 2$. Moreover, 0 is attractive ($f'(0) = 0 < 1$) and 2 is repeller ($f'(2) = 2 > 1$). Take an initial point $x_0 \in J$ and iterate the function f, then we get

$$x_n = f^n(x_0) = 2^{-(1+2+2^2\ldots+2^{n-1})}x_0^{2^n} = 2^{-2^n+1}x_0^{2^n} = 2(\frac{x_0}{2})^{2^n}.$$

Hence we have

$$\lim_{n\to\infty} x_n = \begin{cases} 0, & \text{if } |x_0| < 2 \\ 2, & \text{if } |x_0| = 2 \\ +\infty, & \text{if } |x_0| > 2. \end{cases}$$

Now reduce the operator W on I:

$$V:\begin{cases} x' = \frac{1}{2}xu \\ u' = \frac{1}{2}xu. \end{cases}$$

Thus for any $t_0 = (x_0, 0, u_0, 0) \in I$ we have $W(x_0, 0, u_0, 0) \in J$. Consequently, we have full characterization of the dynamical system on the invariant set J, i.e., we proved the following

Proposition 3. *For any initial point $t_0 = (x_0, 0, u_0, 0) \in I$ we have*

$$\lim_{n\to\infty} W^n(t_0) = \begin{cases} (0,0,0,0) & \text{if } |x_0u_0| < 4 \\ (2,0,2,0) & \text{if } |x_0u_0| = 4 \\ +\infty, & \text{if } |x_0u_0| > 4. \end{cases}$$

Let us now consider the dynamical system on the other sets (which may intersect with J).

Lemma 4. *Let $a \in [0,4)$. Then for any initial point $s = (x, y, u, v) \in Q_a$ we have*

$$\lim_{n\to\infty} W^n(s) = (0,0,0,0). \tag{8}$$

Proof. Let $f(a) = a^2/4$. By Lemma 2 we have

$$W^n(Q_a) \subset W^{n-1}(Q_{f(a)}) \subset W^{n-2}(Q_{f^2(a)}) \subset \cdots \subset Q_{f^n(a)}.$$

It is easy to see that $f(x)$ has two fixed points 0 and 4. Moreover, 0 is attracting point and 4 is repelling point. For any $a \in [0,4)$ we have $\lim_{n\to\infty} f^n(a) = 0$. Consequently, we get $\lim_{n\to\infty} W^n(Q_a) \subset Q_0 = \{(0,0,0,0)\}$.

Lemma 5. *For an initial point $s = (x, y, u, v) \in Q_4$ the following hold*

i. if there is $k \geq 0$ such that $y^{(k)}v^{(k)} \neq 0$ then (8) is satisfied,

ii. if $y^{(k)}v^{(k)} = 0$ for any $k \geq 0$ then

$$\lim_{n\to\infty} W^n(s) = (2,0,2,0), \tag{9}$$

where $y^{(k)}$ and $v^{(k)}$ are second and fourth coordinates of the vector $W^k(s)$.

Proof. In the case $a = 4$ we have $0 \leq x+y+u+v \leq 4$. From this inequality it follows that

$$0 \leq x+y \leq 2 \ \text{ or } \ 0 \leq u+v \leq 2,$$

if both $x+y$ and $u+v$ large than 2 then their sum is large than 4. Without loss of generality we assume that $t = x+y \leq 2$ then $u+v \leq 4-t$. Hence we have

$$x'+y'+u'+v' = (x+y)(u+v) \leq t(4-t) = \begin{cases} 4, & \text{if } t = 2 \\ < 4, & \text{if } t < 2 \end{cases}$$

hence $W(s) \in Q_{t(4-t)}$, i.e., for $t < 2$ the case is reduced to the case of $a < 4$. Consider now the case $t = 2$. Then if $u+v < 2$ we can reduce the case to the case $a < 4$. But if

$$t = x+y = 2 \ \text{ and } \ u+v = 2 \tag{10}$$

then from (2) we get

$$x'+y'+u'+v' = 4, \ \ x'+y' = 2 - \frac{yv}{6} \ \text{ and } \ u'+v' = 2 + \frac{yv}{6}. \tag{11}$$

Consequently, if $yv \neq 0$ then $W^2(s) \in Q_{4-(\frac{yv}{6})^2}$. By (10) we have $0 \leq y \leq 2$ and $0 \leq v \leq 2$, hence $0 < 4 - (\frac{yv}{6})^2 < 4$. Thus condition (10) together with $yv \neq 0$, by Lemma 4 gives (8).

Let now $yv = 0$ then (11) is reduced to the case (10). Repeating above argument we see that if $y'v' \neq 0$ then $W^3(s) \in Q_{4-(\frac{y'v'}{6})^2}$, otherwise we iterate the argument again. By this way one can show that if (10) is satisfied and there exists $k \geq 0$ such that $y^{(k)}v^{(k)} \neq 0$ then we have (8).

Suppose now (10) is satisfied and

$$y^{(k)}v^{(k)} = 0 \ \text{ for any } \ k \geq 0 \tag{12}$$

then similarly to (11) we get

$$x^{(n)} + y^{(n)} + u^{(n)} + v^{(n)} = 4, \ \ x^{(n)} + y^{(n)} = 2 \ \text{ and } \ u^{(n)} + v^{(n)} = 2, \ \text{ for any } \ n \geq 0. \tag{13}$$

To complete the proof we need to the following

Lemma 6. *If conditions* (10) *and* (12) *are satisfied then*

$$y^{(k)} = v^{(k)} = 0 \ \textit{ for any } \ k \geq 0.$$

Proof. From (2) we get

$$\begin{aligned} y^{(k+1)} &= \tfrac{1}{2}x^{(k)}v^{(k)} + v^{(k+1)} \\ v^{(k+1)} &= \left(\tfrac{1}{4}u^{(k)} + \tfrac{1}{3}v^{(k)}\right)y^{(k)}. \end{aligned} \tag{14}$$

Now using (13) and (12) from (14) we get

$$\begin{aligned} y^{(k+1)} &= \tfrac{1}{2}(2 - y^{(k)})v^{(k)} + v^{(k+1)} = v^{(k)} + v^{(k+1)} \\ v^{(k+1)} &= \left(\tfrac{1}{4}(2 - v^{(k)}) + \tfrac{1}{3}v^{(k)}\right)y^{(k)} = \tfrac{1}{2}y^{(k)}. \end{aligned} \tag{15}$$

If $v^{(0)} = 0$ then from the first equation of (15) we get $y^{(1)} = v^{(1)} = 0$. Consequently the second equation gives $v^{(2)} = 0$. Then using the first equation we get $y^{(2)} = 0$ and so on, we get $y^{(k)} = v^{(k)} = 0$ for any $k \geq 0$.

If $y^{(0)} = 0$ then the second equation gives $v^{(1)} = 0$. Assume $v^{(0)} = v \neq 0$ then from the first equation we get $y^{(1)} = v + v^{(1)} = v$. Then $v^{(2)} = \frac{1}{2}v$. Consequently, $y^{(2)} = \frac{1}{2}v$. Now condition $y^{(2)}v^{(2)} = 0$ gives $v = 0$. This completes the proof.

Now by Lemma 6 and property (13) we get

$$x^{(n)} = 2 \ \text{ and } \ u^{(n)} = 2, \ \text{ for any } \ n \geq 0.$$

This completes the proof Lemma 5.

Lemma 7. *If $s = (x, y, u, v) \in P$ is an initial point with $x + y + u + v > 4$, for which*

(a) if there exists $k \geq 0$ such that $(x^{(k)} + y^{(k)})(u^{(k)} + v^{(k)}) < 4$ then (8) is satisfied.

(b) if $\max\{\frac{xu}{4}, \frac{yu}{16}, \frac{yv}{9}\} > 1$ then

$$\lim_{n\to\infty} W^n(s) = \infty, \ \textit{ i.e. at least one coordinate of } \ W^n(s) \ \textit{ goes to } \ \infty.$$

Proof. (a) This simply follows from the equality

$$x^{(k+1)} + y^{(k+1)} + u^{(k+1)} + v^{(k+1)} = (x^{(k)} + y^{(k)})(u^{(k)} + v^{(k)}).$$

Indeed, from this equality it follows that $W(s^{(k+1)}) \in Q_4$. Since Q_4 is invariant the part (a) follows from Lemma 4.

(b) Let us prove it for the case $\max\{\frac{xu}{4}, \frac{yu}{16}, \frac{yv}{9}\} = \frac{xu}{4} > 1$. For other cases the proof is similar. From (2) for any $s = (x,y,u,v) \in P$ we get

$$x^{(k+1)} \geq \frac{1}{2}x^{(k)}u^{(k)}, \quad u^{(k+1)} \geq \frac{1}{2}x^{(k)}u^{(k)}, \ \ k \geq 0. \tag{16}$$

Iterating these inequalities we obtain

$$x^{(k+1)} \geq \frac{1}{2}x^{(k)}u^{(k)} \geq 2^{-(1+2+2^2+\dots+2^k)}(xu)^{2^k} = 2\left(\frac{xu}{4}\right)^{2^k}, \quad k \geq 0. \tag{17}$$

Similarly

$$u^{(k+1)} \geq 2\left(\frac{xu}{4}\right)^{2^k}, \quad k \geq 0.$$

This completes the proof.

Denote

$$P_0 = \{s = (x,y,u,v) \in P : \ (x+y)(u+v) < 4\},$$

$$F = \{s = (x,y,u,v) \in P : \ x+y+u+v > 4, \ \max\{\tfrac{xu}{4}, \tfrac{yu}{16}, \tfrac{yv}{9}\} > 1\}.$$

Summarizing above-mentioned results we get the following

Theorem 8. *If $s = (x,y,u,v) \in \mathbb{R}^4$ is such that*

(i) one of the following conditions is satisfied

1) $s \in P_0$;

2) $s \in Q_4$ and the condition of part i) of Lemma 5 is hold;

3) $s \in \mathcal{N}$, $W(s) \in P_0$;

4) $s \in \mathcal{N}_0$, $W^2(s) \in P_0$;

5) $s \in \mathcal{N}_1$, $W^2(s) \in P_0$

then

$$\lim_{n\to\infty} W^n(s) = (0,0,0,0).$$

(ii) one of the following conditions is satisfied

a) $s \in F$;

b) $s \in \mathcal{N}$, $W(s) \in F$;

c) $s \in \mathcal{N}_0$, $W^2(s) \in F$;

d) $s \in \mathcal{N}_1$, $W^2(s) \in F$

then

$$\lim_{n\to\infty} W^n(s) = +\infty.$$

Proof. (i) The case 1) follows from Lemma 4 and Lemma 7. The case 2) is result of Lemma 5. By Lemma 2 we have $W(\mathcal{N}) \subset P$, $W^2(\mathcal{N}_0) \subset P$ and $W^2(\mathcal{N}_1) \subset P$. Consequently, parts 3)-5) follow from 1).

Part (ii) is a result of Lemma 7 and Lemma 2.

Remark 1. The sum of sets for initial points considered in Theorem 8 is not equal to $\mathbb{R}^4$. But for each fixed point this theorem already gives a large set for the initial point, trajectory of which converges to the fixed point. Each point $s\in\mathbb{R}^n$ can be considered as a state of the system, which is a generalized measure (or charge) on the set $\{XX, XX^n, XY, X^hY\}$. We considered such measure, because for certain purposes, it is useful to have a "measure" whose values are not restricted to the non-negative reals or infinity. Moreover, the dynamical systems considered in this section are interesting because they are higher dimensional and such dynamical systems are important, but there are relatively few dynamical phenomena that are currently understood [12], [13]. In the next section we reduce our operators to the invariant set of vectors with non-negative coordinates (the usual measures which take non-negative values). Then to get a stochastic system of probability measures we use a normalization of the non-negative measures.

4 A normalized gonosomal opertor

We note that the gonosomal operator (4) does not map the simplex

$$S^{n+\nu-1}=\{s=(x_1,\dots,x_n,y_1,\dots,y_\nu)\in\mathbb{R}^{n+\nu}: x_i\geq 0, y_j\geq 0, \sum_{i=1}^n x_i+\sum_{j=1}^{\nu} y_j=1\}$$

to itself, since

$$\sum_{i=1}^n x_i'+\sum_{j=1}^{\nu} y_j'=(\sum_{i=1}^n x_i)(\sum_{j=1}^{\nu} y_j) \tag{18}$$

is not equal to 1 in general.

We denote

$$\mathscr{O}=\left\{s\in S^{n+\nu-1}:(x_1,\dots,x_n)=(0,\dots,0)\,\text{or}\,(y_1,\dots,y_\nu)=(0,\dots,0)\right\}.$$

$$\mathscr{S}^{n,\nu}=S^{n+\nu-1}\setminus\mathscr{O}.$$

It is easy to see that $W(\mathscr{O})=\{(0,\dots,0)\}$. So the points from $\mathscr{O}$ do not give any contribution to the dynamical system generated by W.

Therefore we introduce the normalized gonasomal operator as the following. Consider the coefficients of the operator (4) with the following properties

$$\begin{aligned}&\gamma^{(f)}_{ik,j}\geq 0,\ \ \gamma^{(m)}_{ik,l}\geq 0,\\ &\textstyle\sum_{j=1}^n\gamma^{(f)}_{ik,j}+\sum_{l=1}^{\nu}\gamma^{(m)}_{ik,l}=1,\ \ \text{for all}\ \ i,k,j,l.\end{aligned} \tag{19}$$

An normalized evolution operator V, with coefficients (19) is defined as

$$V:\begin{cases} x_j' &= \dfrac{\sum_{i,k=1}^{n,\nu}\gamma^{(f)}_{ik,j}x_iy_k}{(\sum_{i=1}^n x_i)(\sum_{j=1}^{\nu}y_j)},\ j=1,\dots,n\\ y_l' &= \dfrac{\sum_{i,k=1}^{n,\nu}\gamma^{(m)}_{ik,l}x_iy_k}{(\sum_{i=1}^n x_i)(\sum_{j=1}^{\nu}y_j)},\ l=1,\dots,\nu.\end{cases} \tag{20}$$

Proposition 9. *The operator V defined by (20) with coefficients (19) maps $\mathscr{S}^{n,\nu}$ to itself if and only if the following condition*

$$(\gamma^{(f)}_{ik,1},\dots,\gamma^{(f)}_{ik,n},\gamma^{(m)}_{ik,1},\dots,\gamma^{(m)}_{ik,\nu})\in\mathscr{S}^{n,\nu},\ \ \textit{for all}\ \ i,k. \tag{21}$$

is satisfied.

Proof. *Necessity.* Suppose for any $s \in \mathscr{S}^{n,\nu}$ we have $s' = V(s) \in \mathscr{S}^{n,\nu}$ then we shall show that (21) is satisfied. Assume that (21) is not true, then there is $i_0 \in \{1,\dots,n\}$ and $k_0 \in \{1,\dots,\nu\}$ such that

$$(\gamma^{(f)}_{i_0k_0,1},\dots,\gamma^{(f)}_{i_0k_0,n}) = (0,\dots,0), \tag{22}$$

or

$$(\gamma^{(m)}_{i_0k_0,1},\dots,\gamma^{(m)}_{i_0k_0,\nu}) = (0,\dots,0). \tag{23}$$

Consider the case (22) (the case (23) is similar). Take now some $s \in \mathscr{S}^{n,\nu}$ such that $x_{i_0} \neq 0$, $x_i = 0$ for $i \neq i_0$ and $y_{k_0} \neq 0$, $y_k = 0$ for $k \neq k_0$. Then for this s we have

$$x'_j = \frac{\sum_{i,k=1}^{n,\nu} \gamma^{(f)}_{ik,j} x_i y_k}{(\sum_{i=1}^n x_i)(\sum_{j=1}^\nu y_j)} = \frac{\gamma^{(f)}_{i_0k_0,j} x_{i_0} y_{k_0}}{(\sum_{i=1}^n x_i)(\sum_{j=1}^\nu y_j)} = 0, \text{ for all } j = 1,\dots,n,$$

i.e., $s' \in \mathscr{O}$. This is contradiction to the assumption that $s' \in \mathscr{S}^{n,\nu}$.

Sufficiency. Assume the conditions (19) and (21) are satisfied, we want to show that if $s \in \mathscr{S}^{n,\nu}$ then $s' = V(s) \in \mathscr{S}^{n,\nu}$. By the construction of the operator (20) it is easy to see that $s' \in S^{n+\nu+1}$ so it remains to show that $s' \notin \mathscr{O}$. Assume that $s' \in \mathscr{O}$, i.e, $(x'_1,\dots,x'_n) = (0,\dots,0)$ (the case $(y'_1,\dots,y'_\nu) = (0,\dots,0)$ is similar). Then by (20) we should have

$$\sum_{i,k=1}^{n,\nu} \gamma^{(f)}_{ik,j} x_i y_k = 0, \text{ for each } j = 1,\dots,n. \tag{24}$$

Since $s \in \mathscr{S}^{n,\nu}$ there is $i_0 \in \{1,\dots,n\}$ and $k_0 \in \{1,\dots,\nu\}$ such that $x_{i_0} > 0$ and $y_{k_0} > 0$. From our conditions it follows that $\gamma^{(f)}_{ik,j} x_i y_k \geq 0$, for all i,j,k. Hence from (24) we get

$$\gamma^{(f)}_{i_0k_0,j} x_{i_0} y_{k_0} = 0, \text{ for each } j = 1,\dots,n,$$

consequently,

$$\gamma^{(f)}_{i_0k_0,j} = 0 \text{ for each } j = 1,\dots,n.$$

This is contradiction to the condition (21).

A fixed point $s = (x_1,\dots,x_n,y_1,\dots,y_\nu)$ of the gonosomal operator (4) is called non-negative and normalizeable if all coordinates of this point are non-negative and $\sum_{i=1}^n x_i + \sum_{k=1}^\nu y_k > 0$.

Proposition 10. *There is one-to-one correspondence between non-negative and normalizeable fixed points of (4) and all fixed points of (20).*

Proof. Let $s = (x_1,\dots,x_n,y_1,\dots,y_\nu)$ be a non-negative and normalizeable fixed point of (4). Denote $Z = \sum_{i=1}^n x_i + \sum_{k=1}^\nu y_k$, and consider the point

$$\tilde{s} = (x_1/Z,\dots,x_n/Z,y_1/Z,\dots,y_\nu/Z).$$

By (18) for the fixed point we have

$$Z = (\sum_{i=1}^n x_i)(\sum_{k=1}^\nu y_k). \tag{25}$$

Using formula (25) one can see that $\tilde{s}$ is a fixed point of (20).

Now let $\tilde{s} = (\tilde{x}_1,\dots,\tilde{x}_n,\tilde{y}_1,\dots,\tilde{y}_\nu)$ be a fixed point of (20), i.e. it satisfies the following system

$$\begin{cases} \tilde{x}_j = \dfrac{\sum_{i,k=1}^{n,\nu} \gamma^{(f)}_{ik,j} \tilde{x}_i \tilde{y}_k}{(\sum_{i=1}^n \tilde{x}_i)(\sum_{j=1}^\nu \tilde{y}_j)}, \ j = 1,\dots,n \\[2ex] \tilde{y}_l = \dfrac{\sum_{i,k=1}^{n,\nu} \gamma^{(m)}_{ik,l} \tilde{x}_i \tilde{y}_k}{(\sum_{i=1}^n \tilde{x}_i)(\sum_{j=1}^\nu \tilde{y}_j)}, \ l = 1,\dots,\nu. \end{cases} \tag{26}$$

Denote

$$\tilde{Z} = (\sum_{i=1}^{n} \tilde{x}_i)(\sum_{k=1}^{\nu} \tilde{y}_k). \tag{27}$$

Dividing both side of (26) to $\tilde{Z}$ it is easy to see that the following point is a fixed point of (4) :

$$s = (\tilde{x}_1/\tilde{Z}, \dots, \tilde{x}_n/\tilde{Z}, \tilde{y}_1/\tilde{Z}, \dots, \tilde{y}_\nu/\tilde{Z}).$$

In this section we consider the normalized version of the evolution operator (2), i.e.,

$$V : \begin{cases} x' &= \dfrac{2xu + yu}{4(x+y)(u+v)} \\ y' &= \dfrac{6xv + 3yu + 4yv}{12(x+y)(u+v)} \\ u' &= \dfrac{6xu + 6xv + 3yu + 4yv}{12(x+y)(u+v)} \\ v' &= \dfrac{3yu + 4yv}{12(x+y)(u+v)}. \end{cases} \tag{28}$$

It is easy to see that the operator (28) satisfies the conditions of Proposition 9, hence $V : \mathscr{S}^{2,2} \to \mathscr{S}^{2,2}$.

The following lemmas give some useful estimates.

Lemma 11. *Let $s = (x,y,u,v) \in \mathscr{S}^{2,2}$ and $s^{(1)} = (x',y',u',v') = V(s)$ for the operator (28) then*

$$\begin{array}{rcccl} & \frac{u}{4(u+v)} \le & x' & \le \frac{u}{2(u+v)} & \le \frac{1}{2}, \\ & \frac{v}{3(u+v)} \le & y' & \le \frac{u+2v}{4(u+v)} & \le \frac{1}{2}, \\ \frac{1}{4} \le & \frac{2x+y}{4(x+y)} \le & u' & \le \frac{3x+2y}{6(x+y)} & \le \frac{1}{2}, \\ & \frac{y}{4(x+y)} \le & v' & \le \frac{y}{3(x+y)} & \le \frac{1}{3}, \end{array}$$

$$\frac{1}{3} + \frac{u}{6(u+v)} \le x' + y' \le \frac{1}{2},$$

$$\frac{1}{2} \le u' + v' \le \frac{1}{2} + \frac{yv}{6(x+y)(u+v)} \le \frac{2}{3},$$

$$v' \le y' \le u',$$

$$x' \le u'.$$

Proof. Straightforward.

Lemma 12. *Let $s = (x,y,u,v) \in \mathscr{S}^{2,2}$ and $s^{(n)} = (x^{(n)}, y^{(n)}, u^{(n)}, v^{(n)}) = V^n(s)$ for the operator (28) then*

1.

$$\frac{5}{12} \le x^{(2)} + y^{(2)} \le \frac{1}{2};$$

2. There exists $\alpha \in (0,1)$ such that

$$v^{(n+1)} \le \alpha y^{(n)}, \ \ n \ge 2.$$

Proof. 1. Using Lemma 11 we have

$$\frac{1}{3} + \frac{u'}{6(u'+v')} \le x^{(2)} + y^{(2)} \le \frac{1}{2}.$$

Consequently, since $u' \geq v'$ we get

$$\frac{1}{3}+\frac{u'}{6(u'+v')} \geq \frac{1}{3}+\frac{1}{12}=\frac{5}{12}.$$

2. Using $u=1-x-y-v$ we can rewrite the operator (28) as

$$V:\begin{cases} x'=\frac{(2x+y)(1-x-y-v)}{4(x+y)(1-x-y)} \\ y'=\frac{3y(1-x-y)+(6x+y)v}{12(x+y)(1-x-y)} \\ v'=y\cdot\frac{3(1-x-y)+v}{12(x+y)(1-x-y)}. \end{cases} \tag{29}$$

For any $n \geq 1$ we have

$$V^{n+1}\begin{cases} x^{(n+1)}=\frac{(2x^{(n)}+y^{(n)})(1-x^{(n)}-y^{(n)}-v^{(n)})}{4(x^{(n)}+y^{(n)})(1-x^{(n)}-y^{(n)})} \\ y^{(n+1)}=v^{(n+1)}+\varphi(x^{(n)},y^{n})v^{(n)} \\ v^{(n+1)}=\psi(x^{(n)},v^{(n)})y^{(n)}, \end{cases} \tag{30}$$

where

$$\varphi(x,y)=\frac{x}{2(x+y)(1-x-y)},\quad \psi(x,v)=\frac{3(1-x-y)+v}{12(x+y)(1-x-y)}=\frac{3u+4v}{12(u+v)(1-u-v)}.$$

Using above mentioned inequalities we get

$$\varphi(x,y)\leq 1,\quad \psi(x,v)=\frac{1}{3(1-u-v)}-\frac{u}{12(u+v)(1-u-v)}\leq\frac{2}{3}-\frac{1}{8}=\frac{13}{24}.$$

Consequently we get from (30) the following

$$\begin{cases} y^{(n+1)}\leq v^{(n+1)}+v^{(n)} \\ v^{(n+1)}\leq \frac{13}{24}y^{(n)}. \end{cases} \tag{31}$$

This completes the proof.

Theorem 13. *The operator (28) has a unique fixed point* $p=(1/2,0,1/2,0)$ *and there is an open neighborhood* $\mathscr{U}(p)\subset\mathscr{S}^{2,2}$ *of p such that for any initial point* $s\in\mathscr{U}(p)$ *we have*

$$\lim_{n\to\infty}V^{n}(s)=p.$$

Proof. The existence and uniqueness of p follow from Propositions 1 and 10. To check the Jacobi matrix of the operator (28) at the fixed point p, one has to replace v by $v=1-x-y-u$ in the operator then construct a 3×3 Jacobi matrix. It is easy to see that this matrix at the fixed point has eigenvalues $-0.5, 0, 1$, i.e the point is attractive, so [12, Theorem 6.3] completes the proof.

Using Lemma 11 and Lemma 12 one can see that the trajectory of any initial point after few iterations comes close to the fixed point p. Moreover, using Maple one can see that the limit point of the trajectory is always p. Thus the following should be true

Conjecture. For any initial point $s\in\mathscr{S}^{2,2}$ we have $\lim_{n\to\infty}V^{n}(s)=p$.

Remark 2. The results have the following biological interpretations: Let $s=(x,y,u,v)\in\mathscr{S}^{2,2}$ be an initial state (the probability distribution on the set $\{XX, XX^{h}; XY, X^{h}Y\}$ of genotypes). Theorem 13 says that, as a rule, the population tends to the equilibrium state $p=(1/2,0,1/2,0)$ with the passage of time, i.e. the future of the population is stable: genotypes XX and XY are survived always, but the genotypes XX^{h} and $X^{h}Y$ (therefore hemophilia) will disappear in the future. It follows that hemophilia is maintained in a population only if it occurs mutations on the genes coding for the coagulation factors.

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Exact Analytic Solutions of Pochammer-Chree and Boussinesq Equations by Invariant Painlevé Analysis and Generalized Hirota Techniques

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Abstract

Combinations of truncated Painlevé expansions, invariant Painlevé analysis, and generalized Hirota series are used to solve ('partially reduce to quadrature') the integrable Boussinesq and the cubic and quintic generalized Pochammer-Chree (GPC) equation families. Although the multisolitons of the Boussinesq equation are very well-known, the solutions obtained here for all the three NLPDEs are novel, and non-trivial. All of the solutions obtained via invariant Painlevé analysis are complicated rational functions, with arguments which themselves are trigonometric functions of various distinct traveling wave variables. This is reminiscent of doubly-periodic elliptic function solutions when nonlinear ODE systems are reduced to quadratures. The solutions obtained using recently-generalized Hirota-type expansions are closer in functional form to conventional hyperbolic secant solutions, although with non-trivial traveling-wave arguments which are distinct for the two GPC equations.

1 Introduction

Although not yet fully proven, the Painlevé tests [1] seem to provide extremely useful necessary conditions for identifying the completely integrable cases of a wide variety of families of nonlinear ordinary and partial differential equations, as well as integrodifferential equations. Originally, Ablowitz et al. [2] conjectured that a nonlinear partial differential equation is integrable if *all* its exact reductions to ordinary differential equations have the Painlevé property. This approach poses the obvious operational difficulty of finding *all* exact reductions. This difficulty was circumvented by Weiss et al. [3] by postulating that a partial differential equation has the Painlevé property if its solutions are single-valued about a movable singular manifold. In this paper, we follow this latter approach to perform the Painlevé analysis of several nonlinear evolution equations.

The usefulness of the Painlevé approach is not limited to integrability prediction, and use of the generalized Weiss algorithm [5, 6] yields auto-Bäcklund transformations and Lax pairs for the integrable cases. Painlevé analysis also yields a systematic procedure for obtaining special solutions when the equation possesses only the conditional Painlevé property [7]- [12], when the compatibility conditions of the Painlevé analysis result in constraint equations for the movable singular manifold which is no longer completely arbitrary.

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Weiss' original technique [3, 6] was extensively developed by others (see [13, 14] for instance). This approach, which will be briefly reviewed in Section 2, involves the Weiss strategy of truncating the Painlevé singularity expansion for the solution of the system of NLPDEs at the constant term, thereby imposing a specific choice of singular manifold function. The truncated (singular part) of the Painlevé expansion is then used to semi-algorithmically derive an auto-Bäcklund transformation between two different solutions of the NLPDE(s), and also to derive the associated linear scattering problem or Lax Pair. The latter step is not completely algorithmic since it involves linearizing the overdetermined system of PDEs connecting various derivatives of the singularity manifold by employing a 'Weiss substitution' which may often involve prior, extraneous knowledge about the NLPDE(s) under consideration. References [13] and [14] also discuss the connections between Painlevé analysis and other properties of, and approaches to, integrable systems such as Lie symmetries and Hirota's method. However, the original semi-algorithmic character of the Weiss SMM persists.

A second, more recent, recent approach, which has opened up a whole new sub-field, involves making the entire process of singularity analysis invariant under the homographic or Möbius transformation [16]. This significantly simplifies the testing for integrability [16], the derivation of Lax Pairs [17, 18], as well as the derivation of special families of analytic solutions (see [19]- [22] for instance). Some of these special families of analytic solutions have also been employed in tandem with Melnikov theory to analytically investigate the breakdown of coherent structure solutions and the onset of chaos in NLPDEs under forcing. Note that the invariant analysis yields a fully algorithmic procedure for finding Lax pairs, but none for auto-BTs, tau functions, and multisoliton solutions.

In the context of the above, we shall use three different techniques to derive analytic solutions of the Boussinesq and two generalized Pochammer-Chree (GPC) equations. These are briefly introduced in Section 2. In Section 3, we employ truncations of regular Painlevé expansions to derive auto-Bäcklund transformations for the Boussinesq and generalized Pochammer-Chree (GPC) equations. Section 4 begins with a quick overview of the less-familiar invariant Painlevé technique. Combining this with the auto-Bäcklund structure derived in Section 3 leads to complex analytic traveling wave solutions for both our NLPDEs. We obtain one family of traveling wave solutions for the Boussinesq equation, and two distinct families for the two distinct types of GPC equations. All of the above solutions are complicated rational functions, with arguments which themselves are trigonometric functions of various distinct traveling wave variables. They are thus quite non-trivial and reminiscent of doubly-periodic elliptic function solutions when nonlinear systems are reduced to quadratures. In the current case of course, the Painlevé procedure provides the algorithm for quadrature or partial integrability

To explore possible solutions further, we also employ a recent generalization or modification involving embedding Hirota's technique in the truncated Painlevé expansions in Section 5.Once again, complicated analytic solutions are obtained for both types of GPC equations. The resulting solutions are closer in functional form to conventional hyperbolic secant solutions, although with highly non-trivial traveling-wave arguments which are distinct for the two GPC equations.

2 The Boussinesq and Generalized Pochammer-Chree Equations

In this section, we list the NLPDEs we shall be working with in this paper.

2.1 Boussinesq Equation

The Boussinesq equation[3-5] is extremely well-known and widely investigated, we shall just take it in the form

$$u_{tt} - u_{xx} - u_{xxxx} + 3(u^2)_{xx} = 0. \tag{1}$$

2.2 Generalized Pochammer-Chree Equations

The propagation of longitudinal deformation waves [23]– [25] in elastic rods, under different assumptions, is governed by either of the following generalized Pochammer-Chree (GPC) equations, with cubic and quintic nonlinearities respectively:

$$(u-u_{xx})_{tt}-(a_1u+a_2u^2+a_3u^3)_{xx}= 0, \quad (2)$$
$$(u-u_{xx})_{tt}-(a_1u+a_3u^3+a_5u^5)_{xx}= 0, \quad (3)$$

Some solutions to these equations were given in special cases in [24]– [26].

3 Painlevé Analysis Method

Given a nonlinear partial differential equation in $(n+1)$-dimensions, without specifying initial or boundary conditions, we may find a solution about a movable singular manifold

$$\phi-\phi_0=0$$

, as an infinite Laurent series given by

$$u(x_1,\ldots,x_n,t)=\phi^{-\alpha}\sum_{m=0}^{\infty}u_m\phi^m. \quad (4)$$

Note that when $m\in(\mathbb{Q}-\mathbb{Z})$ (4) is more commonly known as a Puiseux series. One can avoid dealing with Puiseux series if proper substitutions are made, as we will see a little later on. Plugging this infinite series into the NLPDE yields a recurrence relation for the u_m's. As with most series-type solution methods for NLPDEs we will seek a solution to our NLPDE as (4) truncated at the constant term. Plugging this truncated series into our original NLPDE and collecting terms in decreasing order of ϕ will give us a set of determining equations for our unknown coefficients $u_0,\ldots,u_\alpha$ known as the Painlevé-Bäcklund equations. We now define new functions

$$C_0(x_0,\ldots,x_n,t)=\frac{\phi_t}{\phi_{x_0}} \quad (5)$$
$$C_1(x_0,\ldots,x_n,t)=\frac{\phi_{x_1}}{\phi_{x_0}} \quad (6)$$
$$\vdots \quad (7)$$
$$C_n(x_0,\ldots,x_n,t)=\frac{\phi_{x_n}}{\phi_{x_0}} \quad (8)$$
$$V(x_0,\ldots,x_n,t)=\frac{\phi_{x_0x_0}}{\phi_{x_0}} \quad (9)$$

which will allow us to eliminate all derivatives of ϕ other than ϕ_{x_0}. For simplicity it is common to allow $C_i(x_0,\ldots,x_n,t)$ and $V(x_0,\ldots,x_n,t)$ to be constants, thereby reducing a system of PDEs (more than likely nonlinear) in $\{C_i(x,t),V(x,t)\}$ to an algebraic system in $\{C_i,V\}$ for $(i=0,\ldots,n)$.

3.1 Boussinesq Equation

Plugging (4) into (1) with $\alpha=2$ gives us the following recurrence relation for the u_m's

$$\begin{aligned}0=\;&u_{m-4,tt}-u_{m-4,xxxx}-u_{m-4,xx}-4(m-5)u_{m-3,xxx}\phi_x-6(m-5)u_{m-3,xx}\phi_{xx}\\&-6(m-4)(m-5)u_{m-2,xx}\phi_x^2-4(m-5)u_{m-3,x}\phi_{xxx}-12(m-4)(m-5)u_{m-2,x}\phi_x\phi_{xx}\\&-4(m-3)(m-4)(m-5)u_{m-1,x}\phi_x^3+(m-5)u_{m-3}\phi_{xxxx}-4(m-4)(m-5)u_{m-2}\phi_x\phi_{xxx}\\&-3(m-4)(m-5)u_{m-2}\phi_{xx}^2-6(m-3)(m-4)(m-5)u_{m-1}\phi_x^2\phi_{xx}\end{aligned}$$

$$
\begin{aligned}
&-(m-2)(m-3)(m-4)(m-5)u_m\phi_x^4-2(m-5)u_{m-3,x}\phi_x-(m-5)u_{m-3}\phi_{xx}\\
&-(m-4)(m-5)u_{m-2}\phi_x^2+2(m-5)u_{m-3,t}\phi_t+(m-5)u_{m-3}\phi_{tt}+(m-4)(m-5)u_{m-2}\phi_t^2\\
&+6\sum_{k=0}^{m}(u_{k,x}u_{m-k-2,x}+(k-2)u_ku_{m-k-1,x}\phi_x+(m-k-3)u_{k,x}u_{m-k-1}\phi_x\\
&+(k-2)(m-k-2)u_ku_{m-k}\phi_x^2+u_ku_{m-k-2,xx}+2(m-k-1)u_k+(m-k-3)u_ku_{m-k-1}\phi_{xx}\\
&+(m-k-2)(m-k-3)u_ku_{m-k}\phi_x^2)
\end{aligned}
$$

where $u_m=0$ if $m<0$ and $u_{m,x}\equiv\frac{\partial}{\partial x}(u_m)$. Thus we have at the different orders of ϕ

$$
\begin{aligned}
m&=0: u_0=2\phi_x^2\\
m&=1: u_1=-\frac{18}{5}\phi_{xx}\\
m&=2: u_2=\frac{1}{150\phi_x^2}\left(-171\phi_{xx}^2+300\phi_x\phi_{xxx}+25\phi_x^2-25\phi_t^2\right)
\end{aligned}
$$

3.2 Generalized Pochammer-Chree Equations

We shall denote the unknown functions and all corresponding parts for (2) and (3) with superscripts 1 and 2, respectively. We find that balancing the highest nonlinear term with the highest order derivative in (3) we arrive at $\alpha=\frac{1}{2}$. As we mentioned earlier this can be avoided with a proper substitution, namely $u(x,t)=\sqrt{u(x,t)}$. Following the procedure and using the proper substitution for equation (3) we find that $\alpha^{(i)}=1$ $(i=1,2)$ and thus seek a solution of the form

$$
u^{(i)}(x,t)=\frac{u_0^{(i)}(x,t)}{\phi(x,t)}+u_1^{(i)}(x,t)
$$

Plugging (4) into (2) with $\alpha=1$ gives us the following recurrence relation for the u_m's

$$
\begin{aligned}
0=&-2(m-2)(m-3)(m-4)\phi_x\phi_t^2u_{m-1,x}-(m-3)(m-4)(2\phi_t\phi_{xxt}+\phi_{tt}\phi_{xx}+2\phi_x\phi_{xtt})u_{m-2}\\
&-2(m-3)(m-4)(\phi_t\phi_{xx}+2\phi_x\phi_{xt})u_{m-2,t}-4(m-3)(m-4)\phi_t\phi_xu_{m-2,xt}\\
&-2(m-3)(m-4)(\phi_x\phi_{tt}+2\phi_t\phi_{xt})u_{m-2,x}+u_{m-4,tt}-(m-1)(m-2)(m-3)(m-4)\phi_t^2\phi_x^2u_m\\
&-u_{m-4,xxtt}-2(m-2)(m-3)(m-4)\phi_t\phi_x^2u_{m-1,t}-2(m-3)(m-4)\phi_{xt}^2u_{m-2}\\
&-(m-3)(m-4)\phi_t^2u_{m-2,xx}-(m-2)(m-3)(m-4)\phi_t^2\phi_{xx}u_{m-1}+(m-3)(m-4)\phi_t^2u_{m-2}\\
&-(m-2)(m-3)(m-4)\phi_x^2\phi_{tt}u_{m-1}-(m-3)(m-4)\phi_x^2u_{m-2,tt}-a_1((m-3)(m-4)\phi_x^2u_{m-2}\\
&+2(m-4)\phi_xu_{m-3,x}+(m-4)\phi_{xx}u_{m-3}+u_{m-4,xx})-4(m-2)(m-3)(m-4)\phi_x\phi_t\phi_{xt}u_{m-1}\\
&+2(m-4)\phi_tu_{m-3,t}-2(m-4)\phi_tu_{m-3,xxt}-2(m-4)\phi_xu_{m-3,xtt}-(m-4)\phi_{xx}u_{m-3,tt}\\
&-2(m-4)\phi_{xtt}u_{m-3,x}-(m-4)\phi_{xxtt}u_{m-3}-2(m-4)\phi_{xxt}u_{m-3,t}-(m-4)\phi_{tt}u_{m-3,xx}\\
&-4(m-4)\phi_{xt}u_{m-3,xt}+(m-4)\phi_{tt}u_{m-3}-2a_2\sum_{j=0}^{m}[(m-j-2)(m-j-3)\phi_x^2u_ju_{m-j-1}\\
&+2(m-j-3)\phi_xu_ju_{m-j-2,x}+(m-j-3)\phi_{xx}u_ju_{m-j-2}+u_ju_{m-j-3,xx}]\\
&-3a_3\sum_{j=0}^{m}\sum_{k=0}^{m-j}[(m-k-j-1)(m-k-j-2)\phi_x^2u_ju_ku_{m-k-j}+u_ju_ku_{m-k-j-2,xx}\\
&+(m-k-j-2)\phi_{xx}u_ju_ku_{m-k-j-1}+2(m-k-j-2)\phi_xu_ju_ku_{m-k-j-1,x}]\\
&-6a_3\sum_{j=0}^{m}\sum_{k=0}^{m-j}[(m-k-j-1)(k-1)\phi_x^2u_ju_ku_{m-k-j}+(m-k-j-2)\phi_xu_ju_{k,x}u_{m-k-j-1}
\end{aligned}
$$

$$+u_j u_{k,x} u_{m-k-j-2,x} + (k-1)\phi_x u_j u_k u_{m-j-k-1,x}] - 2a_2 \sum_{j=0}^{m} [(j-1)(m-j-2)\phi_x^2 u_j u_{m-j-1}$$
$$+(j-1)\phi_x u_j u_{m-j-2,x} + (m-j-3)\phi_x u_{j,x} u_{m-j-2} + u_{j,x} u_{m-j-3,x}],$$

where $u_m = 0$ if $m < 0$ and $u_{m,x} \equiv \frac{\partial}{\partial x}(u_m)$. Consider the equation given as

$$(u - u_{xx})_{tt} - (\sum_{i=1}^{p} a_i u^i)_{xx} = 0, \tag{10}$$

where we take $p = 2m+1$ $(m > 1)$ and $a_p \neq 0$. We find that the leading order analysis leads to a noninteger value for α, namely $\alpha = \frac{2}{p-1}$. Therefore substituting $(u(x,t))^{\frac{p-1}{2}}$ for $u(x,t)$ changes α to -1 for all values of p greater than or equal to 3. After a bit of manipulation we arrive at the following rather complicated new NLPDE

$$\begin{aligned} 0 = &(m-1)(m-2)(m-3)u_t^2 u_x^2 - (m-1)(m-2)u(u_x^2 u_{tt} + 4u_x u_{xt} u_t + u_{xx} u_t^2) \\ &+(m-1)u^2(u_t^2 + 2u_x u_{xtt} + u_{xx} u_{tt} + 2u_{xxt} u_t + 2u_{xt}^2) + u^3(u_{tt} - u_{xxtt}) \\ &-u^2 \sum_{i=1}^{2m+1} i a_i u^{im}((im-1)u_x^2 + u u_{xx}). \end{aligned} \tag{11}$$

Plugging (4) with $\alpha = 1$ truncated at the constant term into (11) we obtain, by setting the coefficients of the different orders of ϕ to 0, a set of determining equations for the u_i and ϕ. We find that if $p \geq 5$, where we have assumed p is odd (since there are currently no interesting cases for other values of p), that the lowest order of ϕ can be found in the summation and is given as

$$O(\phi^{-2m^2-m}) : 0 = (2m+1)(2m^2+m+1)a_{2m+1} u_0^2 \phi_x^2, \tag{12}$$

from which we have that either $u_0 = 0$ or $\phi_x = 0$. This contradicts assumptions made on both u_0 and ϕ_0 and leads to only the trivial solution $u(x,t) = 0$. Therefore we have shown that for an equation of the form (10) with p odd and greater than 3 the standard Painlevé analysis method of solution fails and yields only the trivial solution.

Therefore we proceed with only the third-order for this section. Continuing on and utilizing the previous recurrence relation we have at the different orders of ϕ

$$m = 0 : u_0^{(1)} = -\frac{2\phi_t}{\sqrt{-2a_3}} \tag{13}$$

$$m = 1 : u_1^{(1)} = -\frac{1}{3}\frac{a_2\phi_t\sqrt{-2a_3} - 3a_3\phi_{tt}}{a_3\phi_t\sqrt{-2a_3}}. \tag{14}$$

Substituting in $C^{(1)}$ and $V^{(1)}$, letting both be constants we arrive at the following

$$u_0^{(1)}(x,t) = \frac{2C^{(1)}\phi_x^{(1)}}{\sqrt{-2a_3}} \tag{15}$$

$$u_1^{(1)}(x,t) = \frac{\sqrt{2}(3C^{(1)}V^{(1)}a_3 + a_2\sqrt{-2a_3})}{6(-a_3)^{3/2}} \tag{16}$$

$$C^{(1)} = -\text{csgn}\left(3a_1a_3 - a_2^2\right) \tag{17}$$

$$V^{(1)} = \frac{\sqrt{6a_3(3a_1a_3 - a_2^2 - 3a_3)}}{3a_3} \tag{18}$$

$$\phi^{(1)}(x,t) = C_1 + C_2 e^{V^{(1)}(x+C^{(1)}t)}, \tag{19}$$

where $\text{csgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$. This again lends itself to a rather nice expression for $u^{(1)}(x,t)$ as

$$u^{(1)}(x,t) = \frac{2C^{(1)}}{\sqrt{-2a_3}}\left(\log(\phi(x,t))\right)_x + \frac{\sqrt{2}(3C^{(1)}V^{(1)}a_3 + a_2\sqrt{-2a_3})}{6(-a_3)^{3/2}}. \tag{20}$$

4 Invariant Painlevé method

For an $(n+1)$-dimensional nonlinear partial differential equation algebraic in the unknown function $u(x_1,\ldots,x_n,t)$ and its derivatives about a movable singular manifold

$$\phi - \phi_0 = 0$$

following the invariant Painlevé method (IPM)[15-21] we seek a solution as an expansion of the form

$$u(x_1,\ldots,x_n,t) = \chi^{-\alpha}\sum_{m=0}^{\infty} u_m \chi^m \tag{21}$$

where the coefficients u_m are invariant under a group of homographic transformations on ϕ. χ is chosen to be

$$\chi \equiv \frac{\psi}{\psi_{x_1}} = \left(\frac{\phi_{x_1}}{\phi-\phi_0} - \frac{\phi_{x_1x_1}}{2\phi_{x_1}}\right)^{-1}, \tag{22}$$

$$\psi = (\phi-\phi_0)\phi_{x_1}^{-1/2}. \tag{23}$$

The expansion variable χ then satisfies the following Ricatti equations

$$\chi_{x_1} = 1 + \frac{1}{2}S\chi^2 \tag{24}$$

$$\chi_t = -C_1 + C_{1,x_1}\chi - \frac{1}{2}(C_1S + C_{1,x_1x_1})\chi^2 \tag{25}$$

$$\chi_{x_2} = -C_2 + C_{2,x_1}\chi - \frac{1}{2}(C_2S + C_{2,x_1x_1})\chi^2 \tag{26}$$

$$\vdots \tag{27}$$

$$\chi_{x_n} = -C_n + C_{n,x_1}\chi - \frac{1}{2}(C_nS + C_{n,x_1x_1})\chi^2, \tag{28}$$

and thus ψ satisfies the following linear equations

$$\psi_{x_1x_1} = -\frac{1}{2}S\psi \tag{29}$$

$$\psi_t = \frac{1}{2}C_{1,x_1}\psi - C_1\psi_{x_1} \tag{30}$$

$$\psi_{x_2} = \frac{1}{2}C_{1,x_1}\psi - C_1\psi_{x_1} \tag{31}$$

$$\vdots \tag{32}$$

$$\psi_{x_n} = \frac{1}{2}C_{n,x_1}\psi - C_n\psi_{x_1}, \tag{33}$$

where S (Schwarzian derivative) and C_i are defined by

$$S = \frac{\phi_{x_1x_1x_1}}{\phi_{x_1}} - \frac{3}{2}\left(\frac{\phi_{x_1x_1}}{\phi_{x_1}}\right)^2 \tag{34}$$

$$C_1 = -\frac{\phi_t}{\phi_x} \tag{35}$$

$$C_i = -\frac{\phi_{x_i}}{\phi_x} \quad (1 < i \leq n), \tag{36}$$

and are also invariant under the group of homographic transformations. It is important to note that the systems (4)/(5) and (6)/(7) are equivalent to each other. These homographic invariants are linked by the cross derivative conditions

$$\phi_{x_1x_1x_1t} = \phi_{tx_1x_1x_1} \tag{37}$$

$$\phi_{x_1x_1x_1x_2} = \phi_{x_2x_1x_1x_1} \tag{38}$$

$$\vdots \tag{39}$$

$$\phi_{x_1x_1x_1x_n} = \phi_{x_nx_1x_1x_1}, \tag{40}$$

which are equivalent to

$$S_t + C_{1,x_1x_1x_1} + 2C_{1,x_1}S + C_1S_{x_1} = 0 \tag{41}$$

$$S_{x_2} + C_{2,x_1x_1x_1} + 2C_{2,x_1}S + C_2S_{x_1} = 0 \tag{42}$$

$$\vdots \tag{43}$$

$$S_{x_n} + C_{n,x_1x_1x_1} + 2C_{n,x_1}S + C_nS_{x_1} = 0. \tag{44}$$

In most cases mandating that the C_i's be constant will greatly simplify computations. Therefore if we force C_i to be constant in the beginning we see that equations (2),(6),(7) and (10) reduce down to the following system

$$\chi = \frac{\psi}{\psi_{x_1}} \tag{45}$$

$$\psi_{x_1x_1} = -\frac{1}{2}S\psi \tag{46}$$

$$\psi_t = -C_1\psi_{x_1} \tag{47}$$

$$\psi_{x_i} = -C_i\psi_{x_1} \quad (1 < i \le n) \tag{48}$$

$$S_t = -C_1S_{x_1} \tag{49}$$

$$S_{x_i} = -C_iS_{x_1} \quad (1 < i \le n) \tag{50}$$

4.1 Boussinesq Equation

Following the algorithmic procedure of the IPM in (1), leading order analysis yields $\alpha = 2$. We therefore seek a solution of the form

$$u(x,t) = \frac{u_0(x,t)}{\chi(x,t)^2} + \frac{u_1(x,t)}{\chi(x,t)} + u_2(x,t). \tag{51}$$

Plugging this truncated expansion into (1) and eliminating all derivatives of χ yields the following Painlevé-Bäcklund equations order by order in χ

$$\begin{aligned}
O(\chi^{-6}) : 0 &= 60u_0^2 - 120u_0 \\
O(\chi^{-5}) : 0 &= -24u_1 - 24u_0(u_{0,x} - u_1) + 96u_{0,x} + 6u_0(2u_1 - 4u_{0,x}) + 36u_1u_0 \\
O(\chi^{-4}) : 0 &= -36u_{0,xx} - 120u_0S + 6u_0C^2 - 24u_0(-u_0S + u_{1,x}) + 6(u_{0,x} - u_1)^2 - 6u_0 \\
&\quad + 6u_0(4u_0S + u_{0,xx} - 2u_{1,x}) + 6u_1(2u_1 - 4u_{0,x}) + 36u_2u_0 + 24u_{1,x} \\
O(\chi^{-3}) : 0 &= 6u_0(u_{1,xx} - u_0S_x + (u_1 - 2u_{0,x})S) + 6u_1(4u_0S + u_{0,xx} - 2u_{1,xx}) + 6u_2(2u_1 - 4u_{0,x}) \\
&\quad + (80u_{0,x} - 20u_1)S + 8u_{0,xxx} + 30u_0S_x - 2u_0(-C_t + CC_x) - 12u_0(2u_{2,x} - u_1S) \\
&\quad + 12(u_{0,x} - u_1)(u_{1,x} - u_0S) + 4u_{0,x} + 4u_{0,t}C + 2u_1C^2 - 12u_0CC_x - 2u_1 - 12u_{1,xx}
\end{aligned}$$

$$
\begin{aligned}
O(\chi^{-2}) : 0 = & -u_{0,xx} + 4u_{1,xxx} + 2u_{1,x} + u_{0,tt} - u_{0,xxxx} - 4u_1 C C_x + 3u_0 \left(u_0 S^2 + 2u_{2,xx} - 2u_{1,x} S - u_1 S_x\right) \\
& +6u_1(u_{1,xx} - u_0 S_x + u_1 S - 2u_{0,x} S) + 6u_2(4u_0 S + u_{0,xx} - 2u_{1,x}) - 4u_0 S \\
& +6(u_{0,x} - u_1)(2u_{2,xx} - u_1 S) + 6(u_{1,x} - u_0 S)^2 + 5u_1 S_x + 6u_0(C^2 S + C C_{xx} + C_x^2) \\
& +2u_{1,t} C + u_1(C_t + C C_x) - 4u_{0,t} C_x + 6u_0 S^2 - 2u_0(C_{xt} + C_x^2 + C^2 S + C C_{xx}) - 24u_{0,xx} S \\
& -20u_{0,x} S_x + 2u_0(3S_{xx} + 4S^2) + 16u_{1,x} S - 12u_0(S_{xx} + S^4) \\
O(\chi^{-1}) : 0 = & -6u_0 S S_x - 6u_0 C_x(C S + C_{xx}) - u_1 S + u_0 S_x + 2u_{0,x} S + 6(u_{1,x} - u_0 S)(2u_{2,x} - u_1 S) \\
& +8u_1 S^2 - 2u_{1,t} C_x - 4u_{1,x} S_x - 4u_1(S_{xx} + 4S^2) + 3u_1(u_0 S^2 + 2u_{2,xx} - 2u_{1,x} S - u_1 S_x) \\
& +6u_2(u_{1,xx} - u_0 S_x + u_1 S - 2u_{0,x} S) + u_1(3S_{xx} + 4S^2) - u_1(C_{xt} + C_x^2 + C^2 S + C C_{xx}) \\
& +2u_{0,t}(2CS + C_{xx}) + u_0(3SCC_x + 3C_x C_{xx} + SC_t + CS_t + C_{xxt}) + 4u_{0,xxx} S + 6u_{0,xx} S_x \\
& +4u_{0,x}(S_{xx} + 4S^2) + u_0(S_{xxx} + S S_x) - 6u_{1,xx} S + 2u_1(C^2 S + C C_{xx} + C_x^2) - u_{1,xx} + u_{1,tt} - u_{1,xxxx} \\
O(\chi^0) : 0 = & (-6u_{0,x} - 8u_1) S S_x - u_0(C_{xx} + SC)^2 - 2u_1 C_x(C_{xx} + SC) - 6u_0 S(S_{xx} + 4S^2) + \frac{3}{2}(2u_{2,x} - u_1 S)^2 \\
& -u_{2,xx} - \frac{1}{2}u_0 S^2 + u_{1,x} S + \frac{1}{2}u_1 S_x + 30u_0 S^3 - \frac{9}{2}u_0(4S^3 + S_x^2) - 6u_{1,x} S^2 + 3u_2(u_0 S^2 \\
& +2u_{2,xx} - 2u_{1,x} S - u_1 S_x) + \frac{1}{2}u_1(S_{xxx} + 19 S S_x) + \frac{3}{2}u_0(C_{xx} + CS)^2 + u_{1,t}(C_{xx} + CS) \\
& +\frac{1}{2}u_1(2C_x C_{xx} + 2SCC_x + SC_t + CS_t + C_{xxt}) - 3u_{0,xx} S^2 + u_0(3S_x^2 + 4S S_{xx} + 7S^3) \\
& +2u_{1,xxx} S + 3u_{1,xx} S_x + 2u_{1,x}(S_{xx} + 4S^2) + u_{2,tt} - u_{2,xxxx}.
\end{aligned}
$$

We now have a system of equations with unknown functions u_0, u_1, u_2, C and S. It is often useful to impose certain conditions (such as C, S constant) to reduce computational complexity, however a major drawback as one may deduce is that our solutions will become more trivial. Solving this system with the aid of a CAS (computer algebra system) yields the following results

$$u_0 = 2 \tag{52}$$

$$u_1 = 0 \tag{53}$$

$$u_2 = \frac{1}{6} + \frac{2}{3}S - \frac{1}{6}C^2 \tag{54}$$

$$C = C \tag{55}$$

$$S = S, \tag{56}$$

where S and C are taken to be arbitrary constants. Using our equations for ψ and χ we have the following

$$\psi = c_1 \cos[\sqrt{\frac{S}{2}}(Ct - x)] + c_2 \sin[\sqrt{\frac{S}{2}}(Ct - x)] \tag{57}$$

$$\chi = \sqrt{\frac{2}{S}}\left(\frac{c_1 \cos[\sqrt{\frac{S}{2}}(Ct - x)] + c_2 \sin[\sqrt{\frac{S}{2}}(Ct - x)]}{c_1 \sin[\sqrt{\frac{S}{2}}(Ct - x)] - c_2 \cos[\sqrt{\frac{S}{2}}(Ct - x)]}\right), \tag{58}$$

which from our equation for $u(x,t)$ gives us the following solution

$$u(x,t) = S\left(\frac{c_1 \sin[\sqrt{\frac{S}{2}}(Ct - x)] - c_2 \cos[\sqrt{\frac{S}{2}}(Ct - x)]}{c_1 \cos[\sqrt{\frac{S}{2}}(Ct - x)] + c_2 \sin[\sqrt{\frac{S}{2}}(Ct - x)]}\right)^2 + \frac{1}{6} + \frac{2}{3}S - \frac{1}{6}C^2. \tag{59}$$

4.2 Generalized Pochammer-Chree equations

We begin as before with the leading order analysis of the IVM. Following this, we get $\alpha^{(1)} = 1$ and $\alpha^{(2)} = \frac{1}{2}$ for equations (25) and (26), respectively. Due to the non-integer value of $\alpha^{(2)}$ we need to use a proper substitution in (26), namely $u^{\frac{1}{2}}(x,t)$. Running through the leading order analysis again yields $\alpha^{(2)} = 1$. Therefore we seek solutions of the form

$$u^{(1)}(x,t) = \frac{u_0^{(1)}(x,t)}{\chi(x,t)} + u_1^{(1)}(x,t), \tag{60}$$

$$u^{(2)}(x,t) = \frac{u_0^{(2)}(x,t)}{\chi(x,t)} + u_1^{(2)}(x,t). \tag{61}$$

Plugging these truncated expansions into their respective equations and eliminating all derivatives of χ yields the PB equations order by order in χ for (2) and (3). Due to the complexity of these systems of equations we shall once again mandate that $C(x,t)$ be a constant. Solving the first two equations for both cases yields the following results

$$u_0^{(1)} = -iC\sqrt{\frac{2}{a_3}}, \tag{62}$$

$$u_1^{(1)} = -\frac{a_2}{3a_3}, \tag{63}$$

$$u_0^{(2)} = \frac{i\sqrt{3}C}{2\sqrt{a_5}}, \tag{64}$$

$$u_1^{(2)} = -\frac{3a_3}{8a_5}. \tag{65}$$

Using the remaining PB equations to solve for $S(x,t)$ and $C(x,t)$ in each case, once again using superscripts to differentiate between the two equations, gives us

$$C^{(1)} = C \tag{66}$$

$$S^{(1)} = \frac{1}{3}\frac{a_2^3 + 3C^2a_3 - 3a_1a_3}{C^2a_3} \tag{67}$$

$$C^{(2)} = -\frac{1}{4}\sqrt{16a_1 - 3\frac{a_3^2}{a_5}} \tag{68}$$

$$S^{(2)} = \frac{6a_3^2}{16a_1a_5 - 3a_3^2} \tag{69}$$

We will use $\psi^{(1)}(x,t)$ and $\psi^{(2)}(x,t)$ for the first and second equations, respectively. From our values for the $S^{(i)}$'s and $C^{(i)}$'s

$$\psi^{(1)}(x,t) = c_1\cos\left(\sqrt{\frac{a_2^3}{6a_3} + \frac{C^2}{2} - \frac{a_1}{2}}\left(\frac{x}{C} - t\right)\right) + c_2\sin\left(\sqrt{\frac{a_2^3}{6a_3} + \frac{C^2}{2} - \frac{a_1}{2}}\left(\frac{x}{C} - t\right)\right) \tag{70}$$

$$\psi^{(2)}(x,t) = -c_1\cos\left(\frac{\sqrt{3}a_3(4x\sqrt{a_5} + t\sqrt{16a_1a_5 - 3a_3^2})}{4\sqrt{a_5}\sqrt{16a_1a_5 - 3a_3^2}}\right) + c_2\sin\left(\frac{\sqrt{3}a_3(4x\sqrt{a_5} + t\sqrt{16a_1a_5 - 3a_3^2})}{4\sqrt{a_5}\sqrt{16a_1a_5 - 3a_3^2}}\right) \tag{71}$$

Therefore we have the following traveling wave solutions

$$u^{(1)}(x,t) = \frac{iC\sqrt{\frac{2}{a_3}}\kappa(c_1\sin(\kappa(\frac{x}{C}-t)) - c_2\cos(\kappa(\frac{x}{C}-t))))}{c_1\cos(\kappa(\frac{x}{C}-t))) + c_2\sin(\kappa(\frac{x}{C}-t))} - \frac{a_2}{3a_3} \tag{72}$$

$$u^{(2)}(x,t) = (\frac{3i(32a_1a_5 - 9a_3^2)a_3(c_1\sin(y(x,t)) + c_2\cos(y(x,t)))}{4\sqrt{a_5}(-c_1\cos(y(x,t)) + c_2\sin(y(x,t)))} - \frac{3a_3}{8a_5} - \frac{3a_3}{8a_5})^{1/2} \tag{73}$$

where $\kappa = \sqrt{\frac{a_2^3}{6a_3} + \frac{C^2}{2} - \frac{a_1}{2}}$ and $y(x,t) = \frac{\sqrt{3}a_3(4x\sqrt{a_5}+t\sqrt{16a_1a_5-3a_3^2})}{4\sqrt{a_5}\sqrt{16a_1a_5-3a_3^2}}$.

5 Series method

In this section we will use the series method [27] for finding exact solutions to our various nonlinear PDEs. Before we delve into this, we will give a brief informal outline of the procedure. In essence, it plugs a generalized Hirota series into the expressions and auto-BT obtained from the use of truncated Painlevé series earlier in this paper.

As with most truncated expansion methods we begin with the leading order analysis. Just as with standard Painlevé analysis, and in fact we will many similarities, we seek a solution about the singular manifold $\phi - \phi_0 = 0$ of the form

$$u(x_1,\ldots,x_n,t) = \phi(x_1,\ldots,x_n,t)^{-\alpha}\sum_{n=0}^{\alpha} u_j(x_1,\ldots,x_n,t)\phi(x_1,\ldots,x_n,t)^n$$

for $\alpha \in \mathbb{Z}^+$. If we plug this into our NLPDE and sort by increasing orders of ϕ we arrive at a set of determining equations known more commonly as the Painlevé-Bäcklund or Painlevé-Darboux equations. The first equation will determine u_0, the second u_1, etc.. We continue until we have found $u_0,\ldots,u_{\alpha-1}$ and keep the remaining u_α unknown. The final term will take the form

$$u_\alpha(x_1,\ldots,x_n,t) = \sum_{i=0}^{\infty} u_{\alpha,i}(x_2,\ldots,x_n,t)x_1^i \tag{74}$$

Since α is finite and in general we will have an NLPDE of finite order of nonlinearity we will have finitely many Painlevé-Bäcklund equations. Therefore plugging (41) into the remaining PB equations will yield a heavily underdetermined system. Thus we will find that for some $N \in \mathbb{N}$ we have $u_{\alpha,i}(x_2,\ldots,x_n,t) = 0$ for $i \geq N$. Thus $u_\alpha(x_1,\ldots,x_n,t)$ can be rewritten as a truncated series to N. To introduce soliton-like solutions we force $\phi(x_1,\ldots,x_n,t)$ to be of the form $\phi(x_1,\ldots,x_n,t) = 1 + \exp\{\Gamma(t) + \sum_{l=1}^{n} x_l\Omega_l(t)\}$. In this paper we let $\Gamma(t) = k_1 + k_2 t$ and $\Omega_l(t) = k_{l+2}$ where $k_l \in \mathbb{C}$ $(l = 1,\ldots,n+2)$. to reduce computational complexity. Plugging the new expansion with known u_n, $n = \in [0,\alpha-1]$, into our Painlevé-Bäcklund equations gives a new set of determining equations in each PB equation for our unknown $u_{\alpha,i}(t)$ and k_l. In theory, if we can solve for these terms we will have found the last term for our truncated series and as all of the PB will be satisfied we will have a solution to the original NLPDE.

We present the results only for the GPC equations so as to keep the discussion to a reasonable length. The analysis for the Boussinesq equation is similar.

5.1 Generalized Pochammer-Chree equations

In this section we apply the series method on the third-order and fifth-order Pochammer-Chree equations. Starting in a similar fashion to that of Painlevé and keeping the notation from chapter 2 we find that $\alpha^{(1)} = \alpha^{(2)} = 1$

from which we find that

$$u_0^{(1)}(x,t) = -\frac{2\phi_t}{\sqrt{-2a_3}}, \quad (75)$$

$$u_0^{(2)}(x,t) = \frac{3\phi_t}{2\sqrt{-3a_5}}, \quad (76)$$

and find that with our ansatz for $\phi(x,t)$ and $u_1^{(i)}(x,t)$ that we also have

$$u_1^{(1)}(x,t) = -\frac{\sqrt{2}(-2k_2a_3 + a_2\sqrt{-2a_3})}{6(-a_3)^{3/2}}, \quad (77)$$

$$u_1^{(2)}(x,t) = \frac{\sqrt{6}(9a_3^2 + 2a_2\sqrt{6a_3a_5})}{36a_3\sqrt{a_3a_5}}, \quad (78)$$

$$\phi^{(1)}(x,t) = 1 + \exp\{k_1 + k_2t - \frac{k_2\sqrt{6}\sqrt{a_3(-3k_2^2a_3 + 6a_1a_3 - 2a_2^2)}}{-3k_2^2a_3 + 6a_1a_3 - 2a_2^2}x\}, \quad (79)$$

$$\phi^{(2)}(x,t) = 1 + \exp\{k_1 - \frac{3a_3}{2\sqrt{-3a_5}}t - \frac{6\sqrt{2}a_3\sqrt{a_5(32a_1a_5 - 6a_3^2)}}{\sqrt{-3a_5}(32a_1a_5 - 6a_3^2)}x\}. \quad (80)$$

from which we have the following solutions

$$u^{(1)}(x,t) = -\frac{2k_2\exp\{y^{(1)}(x,t)\}}{\sqrt{-2a_3}(1 + \exp\{y^{(1)}(x,t)\})} - \frac{\sqrt{2}(-2k_2a_3 + a_2\sqrt{-2a_3})}{6(-a_3)^{3/2}} \quad (81)$$

$$u^{(2)}(x,t) = \left(\frac{3k_2\exp\{y^{(2)}(x,t)\}}{2\sqrt{-3a_5}(1 + \exp\{y^{(2)}(x,t)\})} + \frac{\sqrt{6}(9a_3^2 + 2a_2\sqrt{6a_3a_5})}{36a_3\sqrt{a_3a_5}}\right)^{1/2} \quad (82)$$

where we have the

$$y^{(1)}(x,t) = k_1 + k_2t - \frac{k_2\sqrt{6}\sqrt{a_3(-3k_2^2a_3 + 6a_1a_3 - 2a_2^2)}}{-3k_2^2a_3 + 6a_1a_3 - 2a_2^2}x \quad (83)$$

$$y^{(2)}(x,t) = k_1 - \frac{3a_3}{2\sqrt{-3a_5}}t - \frac{6\sqrt{2}a_3\sqrt{a_5(32a_1a_5 - 6a_3^2)}}{\sqrt{-3a_5}(32a_1a_5 - 6a_3^2)}x \quad (84)$$

6 Conclusions

In conclusion, three distinct techniques were employed to obtain analytic solutions ('partially reduce to quadrature') the integrable Boussinesq and the cubic and quintic GPC equation families. Of course, the multisolitons of the Boussinesq equation are very well-known. However, the solutions obtained here for all the three NLPDEs are novel, and non-trivial.

All of the solutions obtained via invariant Painlevé analysis are complicated rational functions, with arguments which themselves are trigonometric functions of various distinct traveling wave variables. This is reminiscent of doubly-periodic elliptic function solutions when nonlinear ODE systems are reduced to quadratures. In the current case of course, the Painlevé procedure provides the algorithm for quadrature or partial integrability of our NLPDEs. The solutions obtained by the recently-generalized Hirota-type expansions are closer in functional form to conventional hyperbolic secant solutions, although with highly non-trivial traveling wave arguments which are distinct for the two GPC equations.

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