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Stability of Hopfield neural networks with delay and piecewise constant argument

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Abstract

In this paper, by using the concept of differential equations with piecewise constant argument, the model of Hopfield neural networks with constant delay is developed. Sufficient conditions for the existence of an equilibrium as well as its global exponential stability by means of Lyapunov functionals and a linear matrix inequality (LMI) are obtained. An example is given to illustrate our results.

1 Introduction

Hopfield neural network models, which constitute a class of recurrent neural networks were first proposed by J.J. Hopfield [1]. These neural network models have been widely studied by many authors due to the extensive applications on image and signal processing, biology, pattern recognition and optimization problems [2]- [7]. The original Hopfield neural network model can be described by the following differential equations:

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)), \tag{1}$$

for $i = 1, \cdots, n$, where $a_i \geq 0$, x_j, b_{ij} and f_j denotes the state variable, interconnection strengths from neuron j to neuron i, and activation functions, respectively. In the literature, there are many papers where delays have been introduced to Hopfield neural networks [2]- [7]. Time delays are present due to finite switching speed of the amplifiers and communication time [8]- [11]. It essentially changes the characteristic properties of the neural network systems such that stability, convergence and divergence and encountered in the biological systems and computer sciences [11]- [14]. Up to now, various kinds of delays were introduced to the Hopfield neural network

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systems such that constant delays, single time delays, time varying delays and distributed delays. The following system with constant delays was introduced by Marcus and Westervelt [15]

$$x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j)), \tag{2}$$

for $i = 1, \cdots, n$. Neural networks with time varying delays were studied deeply in recent years. Exponential stability, asymptotic stability, existence and uniqueness of solutions of them have been analyzed by many authors. Further studies were taken about the following model with time variable delay [16]- [21]

$$x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j(t))), \tag{3}$$

for $i = 1, \cdots, n$.

Recently the differential equations with piecewise constant argument has been studied in many papers [22]- [34]. The main idea of differential equation with piecewise constant argument is combining the continuous and discrete dynamical systems. With this view, it is important for the modeling the biological and computer sciences problems. This type of differential equations has been under investigation since 1980s. Busenberg and Cooke firstly introduced the piecewise constant argument in 1982. Cooke and Wiener, Wiener, Shah and Wiener have studied the type of differential equations [21]- [23]. Neural networks with piecewise constant argument have been introduced into the following form [25]- [35]

$$x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(\beta(t))), \tag{4}$$

for $i = 1, \cdots, n$. Qualitative properties of this neural network system, such that existence and uniqueness of solutions, stability of equilibrium, existence and stability of periodic solutions are investigated.

In implementation of neural network models to real world problems, stability of them has a primary importance. So, the stability analysis of neural network systems is crucial. The linear matrix inequalities (LMIs) have been frequently used for the stability analysis of the neural networks as well as they have been used for dynamical systems. Many stability criteria based on LMI have been derived in the literature for different Hopfield neural network models because of the efficiency of this method [35]- [53]. Also this technique has been used in control theory [54].

In this paper we are concerned about a model including both delays and piecewise constant argument. It is the first time that global exponential stability of equilibrium of Hopfield neural networks model with both delays and piecewise constant argument is considered.

2 Model description and preliminaries

Let N and R^+ be the sets of natural and nonnegative real numbers, respectively, i.e., $N = \{0, 1, 2, ...\}$, $R^+ = [0, \infty)$, R^n denotes the n dimensional real space. The notation $X > 0$ (or $X < 0$) denotes that X is a symmetric and positive definite (or negative definite) matrix. For real symmetric matrices X and Y, the notation $X \neq Y$ (respectively, $X > Y$) means that the matrix X is positive semi-definite (respectively, positive definite). The notations X^T and X^{-1} refer, respectively, the transpose and the inverse of a square matrix X. $\lambda_{max}(X)$ and $\lambda_{min}(X)$ represent the maximal eigenvalue and minimal eigenvalue of X, respectively. The norm $\| \cdot \|$ means either one-norm: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$, $x \in R^n$ or the induced matrix 2-norm: $\|X\|_2 = \sqrt{\lambda_{max}(X^T X)}$. Let θ_i, and ζ_i, denote two fixed real-valued sequences such that $\theta_i < \theta_{i+1}$, $\theta_i \leq \zeta_i \leq \theta_{i+1}$ for all $i \in N$, with $\theta_i \to \infty$ as $i \to \infty$. Throughout the paper, we assume that there exists a positive constant $\bar{\theta}$ such that $\theta_{i+1} - \theta_i \leq \bar{\theta}$, $i \in N$.

In this section, we will consider the description of the following neural network with piecewise argument and constant delay:

$$x'(t) = -Ax(t) + Bg(x(t)) + Cg(x(\beta(t))) + Dg(x(t-\tau)) + E, \tag{5}$$

where $\beta(t) = \theta_k$ if $t \in [\theta_k, \theta_{k+1})$, $k \in N$, $t \in R^+$, $x = [x_1, \cdots, x_n]^T \in R^n$ is the neuron state vector, $g(x(t)) = [g_1(x_1(t)), \cdots, g_n(x_n(t))]^T \in R^n$ is the activation function of neurons, $E = [E_1, \cdots, E_n]^T$ is an external input vector.

Additionally, we have $A = diag(a_1, \cdots, a_n)$ where $a_i > 0$, $B = (b_{ij})_{n\times n}$, $C = (c_{ij})_{n\times n}$, $D = (d_{ij})_{n\times n}$, denote the connection weight matrices.

(A1) The activation function g satisfies $g(0) = 0$;

(A2) There exists Lipschitz constant

$$L = diag(L_1, \cdots, L_n) > 0,$$

such that

$$|g_i(u) - g_i(v)| \le L_i|u - v|,$$

for all $u, v \in R^n,\ i = 1, 2, \ldots, n$;

(A3) The activation function g is bounded, i.e. for some constant $M > 0$, $|g(x(t))| < M$, for all $t \in R$ and $x \in R$;

(A4) $\bar{\theta} < \tau$.

Consider the equilibrium point, $x^* = (x_1^*, \cdots, x_n^*)^T$, of the system (5).

Theorem 1. *Suppose that the assumptions* $(A1), (A2)$ *and* $(A3)$ *are satisfied. If*

$$a_i > L_i \sum_{j=1}^{n} (|b_{ij}| + |c_{ij}| + |d_{ij}|),\ \ i = 1, 2, \ldots, n, \tag{6}$$

then system (5) has a unique equilibrium point.

Proof. Step 1: Existence:vIf $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is an equilibrium point of the system (5), then each x_i^* satisfies the following equation:

$$\begin{aligned} x_i^*(t) &= \frac{1}{a_i}\left[\sum_{j=1}^{n} b_{ij} g_j(x_j^*) + \sum_{j=1}^{n} c_{ij} g_j(x_j^*) + \sum_{j=1}^{n} d_{ij} g_j(x_j^*)\right] + \frac{E_i}{a_i} \\ &= \sum_{j=1}^{n}\left[\frac{1}{a_i}(b_{ij} + c_{ij} + d_{ij})\right] g_j(x_j^*) + \frac{E_i}{a_i},\ \ i = 1, 2, \cdots, n. \end{aligned}$$

Denote

$$H(x_j^*) = \sum_{j=1}^{n}\left[\frac{1}{a_i}(b_{ij} + c_{ij} + d_{ij})\right] g_j(x_j^*) + \frac{E_i}{a_i}.$$

Thus, x^* is a fixed point of the map $H : R^n \to R^n$. The *i-th* component of the function $H(x)$ satisfies the following equation

$$\begin{aligned}
|H(x_j^*)| &= |\sum_{j=1}^{n}\left[\frac{1}{a_i}(b_{ij}+c_{ij}+d_{ij})\right]g_j(x_j^*)+\frac{E_i}{a_i}| \\
&\le \sum_{j=1}^{n}|\left[\frac{1}{a_i}(b_{ij}+c_{ij}+d_{ij})\right]||g_j(x_j^*)|+|\frac{E_i}{a_i}| \\
&\le \sum_{j=1}^{n}|\left[\frac{1}{a_i}(b_{ij}+c_{ij}+d_{ij})\right]|M+\frac{|E_i|}{a_i},
\end{aligned}$$

where $x = (x_1, x_2, \cdots, x_n)^T$. Then we have

$$|H(x_j^*)| \le \max_{1\le i\le n}\sum_{j=1}^{n}|\left[\frac{1}{a_i}(b_{ij}+c_{ij}+d_{ij})|\right]M+\frac{|E_i|}{a_i}, for\ \ i = 1,2,\cdots,n.$$

$H : R^n \to R^n$ is bounded for all $x \in R^n$. Also we can easily say that H is continuous. From Brouwer's Fixed Point Theorem, H has at least one fixed point.
Step 2: Uniqueness: Consider a mapping

$$f(x_1,x_2,\cdots,x_n) = \begin{pmatrix} f_1(x_1,x_2,\cdots,x_n) \\ \vdots \\ f_n(x_1,x_2,\cdots,x_n)\end{pmatrix},$$

x^* is a fixed point of the map $f : R^n \to R^n$.

$$x_i^* = f(x_1^*,x_2^*,\cdots,x_n^*) = \frac{1}{a_i}\{\sum_{j=1}^{n}(b_{ij}+c_{ij}+d_{ij})g_j(x_j^*)+E_i\}.$$

Suppose that there exists another fixed point denoted y^*. Then

$$a_i(x_i^*-y_i^*) = \sum_{j=1}^{n}(b_{ij}+c_{ij}+d_{ij})(g_j(x_j^*)-g_j(y_j^*)).$$

From conditions $(A1)$-$(A3)$ and $a > 0$,

$$a_i|x_i^*-y_i^*| - \sum_{j=1}^{n}(|b_{ij}|+|c_{ij}|+|d_{ij}|)L_{g_j}|x_j^*-y_j^*| \le 0,\ i \in I. \tag{7}$$

Consequently from (6) we obtain $x_i^* = y_i^*$. So there exists a unique equilibrium. The theorem is proved.

Now, we will consider the following initial value problem

$$x'(t) = -Ax(t)+BG(x(t))+CG(x(\beta(t)))+DG(x(t-\tau))+E, \tag{8}$$

$$x(t) = \varphi(t),\ \sigma-\tau \le t \le \sigma, \tag{9}$$

where $\beta(t) = \theta_k$ if $t \in [\theta_k, \theta_{k+1})$, $k \in N$, $t \in R^+$ and $\varphi(t)$ is a continuous function.

Theorem 2. *Suppose that the conditions* $(A1)-(A4)$ *are hold. Then for every* $(\sigma,\varphi)\in R^+\times R^n$*, there exists a unique solution* $x(t)=x(t,\sigma,\varphi)$ *of (8)-(9), such that* $x(\sigma)=\varphi(t)$ *on* R^+.

Proof. Existence: Equation (8)-(9) can be investigated step by step on intervals $[\theta_i,\theta_{i+1})$, $i\in Z$. We assume without loss of generality that $\theta_i\leq\sigma\leq\theta_{i+1}$ and $i=0$. We are looking for the solution $x(t)$, which satisfies the equation $x(t)=\varphi(t)$ for $[\sigma-\tau,\sigma]$. Consider the following cases:

(a) Assume that there exists an integer j such that $\theta_j\leq\sigma+\tau<\theta_{j+1}$, $j>1$. We will show that there exists a unique solution on the interval $[\sigma,\sigma+\tau)$. For $t\in[\sigma,\theta_1)$, $x(t)$ satisfies the following equation

$$x'(t)=-Ax(t)+BG(x(t))+CG(\varphi(\sigma))+DG(\varphi(t-\tau))+E. \tag{10}$$

Since the equation is quasilinear, with Lipschitzian nonlinear part, the solution exists, unique and is continuable to θ_1. For each $i<j$, $x(t)$ satisfies the following equation

$$x'(t)=-Ax(t)+BG(x(t))+CG(x(\theta_i))+DG(x(t-\tau))+E.$$

on the interval $[\theta_i,\theta_{i+1}]$. Consequently, repeating the discussion for the first interval, one can continue the solution till θ_j. Now, consider $t\in[\theta_j,\sigma+\tau)$. Again, similarly to the previous intervals one can show that the solution exists on $[\theta_j,\sigma+\tau)$.

(b) Now, assume that $\sigma+\tau<\theta_1<\sigma+2\tau$. Consider the interval $[\sigma,\sigma+\tau)$, then $x(t)$ satisfies the following quasilinear differential equation

$$x'(t)=-Ax(t)+BG(x(t))+CG(\varphi(\sigma))+DG(\varphi(t-\tau))+E.$$

It is obvious that the solution exists and is unique on the interval $[\sigma,\sigma+\tau)$.

Now for $t\in[\sigma+\tau,\theta_1)$, $x(t)$ satisfies the following equation;

$$x'(t)=-Ax(t)+BG(x(t))+CG(x(\sigma+\tau))+DG(x(t-\tau))+E.$$

The above equation is a quasilinear ordinary differential equation, since $x(\sigma+\tau)$ and $x(t-\tau)$ are known from the previous step. So, there exists a solution on $[\sigma+\tau,\theta_1)$.

One can see that by combination of the two cases, (a) and (b) the solution is continuable uniquely on the interval $[\sigma,\infty)$.

The theorem is proved.

Definition 1. The equilibrium $x=x^*$ of (5) is said to be globally exponentially stable if there exist positive constants α_1 and α_2 such that

$$||x(t)||\leq\alpha_1 e^{-\alpha_2 t}\sup_{-\tau\leq\xi\leq0}||x(\xi)||.$$

By means of the transformation $u(t)=x(t)-x^*$, system (5) can be simplified as

$$u'(t)=-Au(t)+BG(u(t))+CG(u(\beta(t)))+DG(u(t-\tau)), \tag{11}$$

where $G_j(u_j(t))=g_j(u_j(t)+x_i^*)-g_j(x_j^*)$, with $g_j(0)=0$.

It is obvious that the stability of the zero solution of (11) is equivalent to that of the equilibrium x^* of (5). Therefore, in what follows, we discuss the stability of the zero solution of (11).

Lemma 3. *Given any real matrices* U, W, Z *of appropriate dimensions and a scalar* $\varepsilon>0$ *such that* $0<W=W^T$*, then the following matrix inequality holds:*

$$U^TZ+Z^TU\leq\varepsilon U^TWU+\frac{1}{\varepsilon}Z^TW^{-1}Z.$$

3 Main Results

Theorem 4. *Suppose that (A1)-(A4) hold true. The equilibrium x^* of (11) is globally exponentially stable, if there exist matrices $P > 0$, $Q > 0$ and two diagonal matrices $R > 0$, $S > 0$ such that the following LMI holds;*

$$\begin{pmatrix} AP+PA-PBRB^TP-L(R^{-1}+Q+S)L & -PC & -PD \\ -C^TP & S & 0 \\ -D^TP & 0 & Q \end{pmatrix} > 0. \tag{12}$$

Proof. Firstly we choose a functional candidate for system (11) as below

$$V(u_t) = u^T(t)Pu(t) + \int_{t-\tau}^{t} G^T(u(\xi))QG(u(\xi))d\xi + \int_{\beta(t)}^{t} G^T(u(\xi))QG(u(\xi))d\xi.$$

Then we will find the time derivative of $V(u_t)$ along the trajectories of system (11)

$$\begin{aligned} \dot{V}(u_t) &= \dot{u}^T(t)Pu(t) + u^T(t)P\dot{u}(t) + G^T(u(t))QG(u(t)) - \\ &\quad -G^T(u(t-\tau))QG(u(t-\tau)) + G^T(u(t))SG(u(t)) - \\ &\quad -G^T(u(\beta(t)))QG(u(\beta(t))) \\ &= [-Au(t)+BG(u(t))+CG(u(\beta(t)))+DG(u(t-\tau))]^T Pu(t) + \\ &\quad +u^T(t)P[-Au(t)+BG(u(t))+CG(u(\beta(t)))+DG(u(t-\tau))] + \\ &\quad +G^T(u(t))QG(u(t)) - G^T(u(t-\tau))QG(u(t-\tau)) + G^T(u(t))SG(u(t)) - \\ &\quad -G^T(u(\beta(t)))QG(u(\beta(t))) \\ &= -u^T(t)(A^TP+PA)u(t) + G^T(u(t))B^TPu(t) + G^T(u(\beta(t)))C^TPu(t) + \\ &\quad +G(u(t-\tau))D^TPu(t) + u^T(t)PBG(u(t)) + u^T(t)PCG(u(\beta(t))) + \\ &\quad +u^T(t)PDG(u(t-\tau)) + G^T(u(t))QG(u(t)) - G^T(u(t-\tau))QG(u(t-\tau)) + \\ &\quad +G^T(u(t))SG(u(t)) - G^T(u(\beta(t)))QG(u(\beta(t))). \end{aligned} \tag{13}$$

It follows from Lemma (3),

$$u^T(t)PBG(u(t)) + G^T(u(t))B^TPu(t) \le u^T(t)PBRB^TPu(t) + G^T(u(t))R^{-1}G(u(t)). \tag{14}$$

Substituting (14) into (13), we have

$$\begin{aligned} \dot{V}(u(t),G(u(\beta(t))),G(u(t-\tau))) \le\ & u^T(t)(-AP-PA+PBRB^TP+ \\ & +L(R^{-1}+Q+S)L)u(t) + \\ & +G^T(u(\beta(t)))C^TPu(t) + u^T(t)PCG(u(\beta(t))) + \\ & +G^T(u(t-\tau))D^TPu(t) + u^T(t)PDG(u(t-\tau)) + \\ & +G^T(u(\beta(t)))SG(u(\beta(t))) \\ & +G^T(u(t-\tau))QG(u(t-\tau)). \end{aligned}$$

Then we obtain

$$\dot{V}(u_t) \le -\eta(t)\Sigma\eta^T(t), \tag{15}$$

where $\eta(t) = \left[u^T(t)\ G^T(u(\beta(t)))\ G^T(u(t-\tau))\right]$, and

$$\Sigma = \begin{pmatrix} AP+PA-PBRB^TP-L(R^{-1}+Q+S)L & -PC & -PD \\ -C^TP & S & 0 \\ -D^TP & 0 & Q \end{pmatrix}.$$

Now we will prove the global exponential stability of the solution. Note that $\ell = \max\limits_{1\le i\le n}\{L_i\}$ for $i = 1,\cdots,n$ and

$$V(u_t) \le \lambda_{max}(P)||u(t)||^2 + \lambda_{max}(Q)\ell^2 \int_{t-\tau}^{t} ||u(\xi)||^2 d\xi + \lambda_{max}(Q)\ell^2 \int_{\beta(t)}^{t} ||u(\xi)||^2 d\xi. \tag{16}$$

From (12) and (15), one can see that there exists a scalar $m > 0$ such that

$$\begin{pmatrix} AP+PA-PBRB^TP-L(R^{-1}+Q+S)L-mI & -PC & -PD \\ -C^TP & S & 0 \\ -D^TP & 0 & Q \end{pmatrix} > 0.$$

Then we can obtain easily the following equation for any scalar $c > 0$,

$$\begin{aligned} \frac{d}{dt}(e^{ct}V(u_t)) &= e^{ct}[b(V(u_t)) + \dot{V}(u_t)] \\ &\le e^{ct}\Big[(c\lambda_{max}(P)-m)||u(t)||^2 + c\lambda_{max}(Q)\ell^2\int_{t-\tau}^{t}||u(\xi)||^2d\xi + \\ &\quad +\lambda_{max}(Q)\ell^2\int_{\beta(t)}^{t}||u(\xi)||^2d\xi\Big] \\ &\le e^{ct}\left[(c\lambda_{max}(P)-m)||u(t)||^2 + 2c\lambda_{max}(Q)\ell^2\int_{t-\tau}^{t}||u(\xi)||^2d\xi\right]. \end{aligned}$$

By intergating two sides from 0 to $T > 0$, we obtain

$$\begin{aligned} e^{cT}V(u_T) - V(u_0) \le (c\lambda_{max}(P)-m)\int_0^T e^{ct}||u(t)||^2dt + \\ +2c\lambda_{max}(Q)\int_0^T\int_{t-\tau}^{t} e^{ct}||u(\xi)||^2d\xi dt. \end{aligned}$$

One can see that easily

$$\begin{aligned} \int_0^T\int_{t-\tau}^{t} e^{ct}||u(\xi)||^2d\xi dt &\le \tau\int_{-\tau}^{T} e^{c(t+\tau)}||u(t)||^2dt \\ &\le \tau e^{c\tau}\int_{-\tau}^{0}||u(t)||^2dt + \\ &\quad +\tau e^{c\tau}\int_0^T e^{ct}||u(t)||^2dt. \end{aligned}$$

Then we obtain

$$e^{cT}V(u_T) \leq (c\lambda_{max}(P) - m + 2c\lambda_{max}(Q)\ell^2\tau e^{c\tau})\int_0^T e^{ct}||u(t)||^2 dt +$$
$$+2c\lambda_{max}(Q)\ell^2\tau e^{c\tau}\int_{-\tau}^0 ||u(t)||^2 dt + V(u_0)$$

By choosing a scalar $c > 0$ such that $m = c\lambda_{max}(P) + 2c\lambda_{max}(Q)\ell^2\tau e^{c\tau}$, we have

$$e^{cT}V(u_T) \leq 2c\lambda_{max}(Q)\ell^2\tau e^{c\tau}\int_{-\tau}^0 ||u(t)||^2 dt + V(u_0) \tag{17}$$

We know from definition of the $V(u_t)$, $V(u_0)$ satisfies the following inequality

$$V(u_0) \leq \lambda_{max}(P)||u_0||^2 + \lambda_{max}(Q)\ell^2\int_{-\tau}^0 ||u(\xi)||^2 d\xi \tag{18}$$

Substituting (18) into (17), we have

$$e^{cT}V(u_T) \leq (2c\lambda_{max}(Q)\ell^2\tau^2 e^{c\tau} + \lambda_{max}(P) + \lambda_{max}(Q)\ell^2\tau)\sup_{-\tau\leq\xi\leq 0}||u(\xi)||^2.$$

Also we know from (16), $\lambda_{min}(P)||u(T)||^2 \leq V(u_T)$.
Consequently, we have

$$||u_T)|| \leq \sqrt{\frac{2c\lambda_{max}(Q)\ell^2\tau^2 e^{c\tau} + \lambda_{max}(P) + \lambda_{max}(Q)\ell^2\tau}{\lambda_{min}(P)}}e^{-cT/2}\sup_{-\tau\leq\xi\leq 0}||u(\xi)||^2.$$

The theorem is proved.

4 An Illustrative Example

Consider the following Hopfield neural network system with piecewise constant argument.

$$\begin{aligned} x'(t) = &-\begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0.01 & 0.02 \\ 0.03 & 0.01 \end{pmatrix}\begin{pmatrix} tanh(x_1(t)) \\ tanh(x_2(t)) \end{pmatrix} \\ &+ \begin{pmatrix} 0.01 & 0.02 \\ 0.02 & 0.03 \end{pmatrix}\begin{pmatrix} tanh(x_1(\beta(t))) \\ tanh(x_2(\beta(t))) \end{pmatrix}. \end{aligned} \tag{19}$$

Here the coefficients of the delay terms in the main model has been chosen zero. If $L_1 = 0.1$ and $L_2 = 0.1$, it can be shown easily that (19) satisfies the condition of Theorem 2.1. So there exists a unique equilibrium of (19) such that $x^* = [0.4372, 0.6623]^T$. For

$$P = \begin{pmatrix} 1.5 & 1 \\ 1 & 1.5 \end{pmatrix}, Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, R = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, S = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

the condition of the Theorem 4.1 is satisfied. So, the equilibrium of the system (19) is globally exponentially stable.

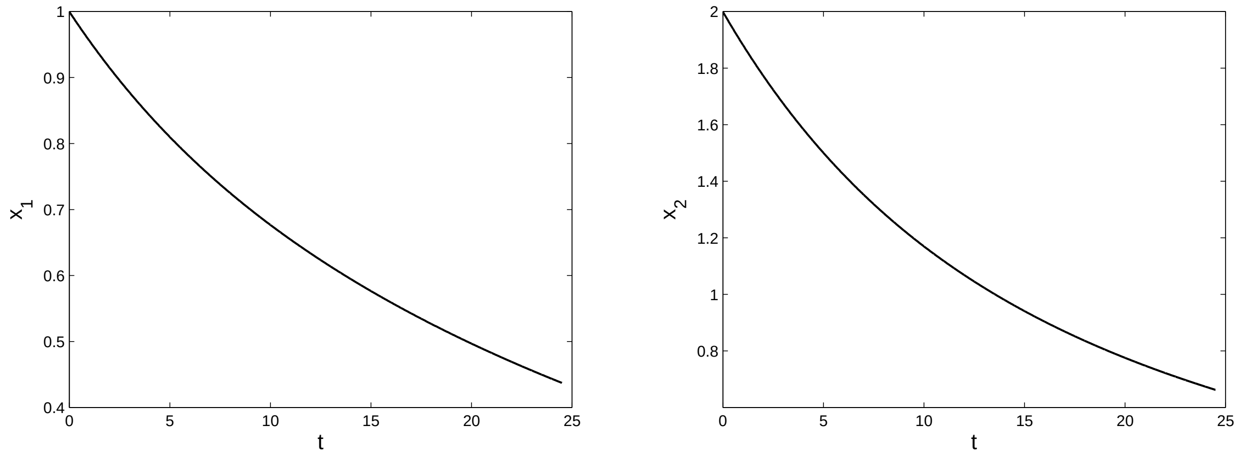

Fig. 1 Time response of $x_1(t)$ and $x_2(t)$ with piecewise constant arguments

5 Conclusion

In this paper, the Hopfield neural network with piecewise constant argument and constant delay has been studied. Up to now, various kinds of delays were introduced to the Hopfield neural network systems such that constant delays, single time delays, time varying delays and distributed delays. But it is the first time that Hopfield neural networks model with both piecewise constant argument and constant delay is considered. This combination provided a more realistic approximation to the real life problems. An LMI method has been used to obtain the global exponential stability of equilibrium point of the system. An example is given to illustrate our results.

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