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ORIGINAL PAPER

Chattering as a singular problem

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Abstract This paper presents the chattering through the singularity point of view for the first time. The main novelty of this article is that impact moments are considered as a singularity phenomenon. A bouncing ball, an inverted pendulum and a hydraulic relief valve models are considered for the study. Moreover, the behavior of solutions of a spring-mass system is studied for the small mass. Simulations are given to support the theoretical analysis.

Keywords Singularity in impact moments · Chattering · Singular perturbation

1 Introduction

Investigations and observations show that the impact chattering meets in operating almost every mechanism and machine of impact-oscillating type [1–4]. Chattering is an important feature of impact systems [5,6]. It is known as an infinite number of discontinuity moments occurring in a finite time period. It is asserted in [7] that chattering resembles with the inelastic collapse. The balls dissipate their energy through an infinite number of collisions in a finite time interval. Budd and Dux [5] showed that chattering can occur for a periodically forced, single degree of freedom impact oscillator with a restitution law. They demonstrated that chattering can form part of a periodic motion, and this relates to certain types of chaotic behavior. Nordmark and Piiroinen [8] considered simulation problems for chattering as well as analysis of stability of the limit cycle, which is chattering by solving the first variational equations. Moreover, they used the mappings, which are constructed with the help of a solution, in simulation schemes. Similar to the one in paper [5], it shows the existence of chattering for a linear system.

In paper [9], authors consider the mechanical models with Newton's Law of impacts. They provided sufficient conditions for the presence of chattering by examination of the right hand side of the impact models. The criteria for the sets of initial data which always lead to chattering were established. Moreover, they subject the Moon–Holmes model to regular impact perturbations for the chattering generation. Using the chattering solutions, they generated the continuous chattering and they applied Pyragas control to the system in order to depress the chattering.

Two different types of chattering, namely complete and incomplete chattering [3,5,8], exist in impact systems. Complete chattering is the phenomenon wherein a system an infinite number of discontinuities in a finite time occurs, where the velocity tends to zero uniformly. Incomplete chattering bears on a sequence of the impacts that initially has the same behavior

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as complete chattering, but it ends after a large but finite number of impacts [8]. It is important to note that, in paper [3], authors showed that in an electrically driven impact microactuator, as the excitation voltage is increased the complete chattering is observed.

We will study three important models in mechanics; an inverted pendulum [10,11], a bouncing ball [12–14] and a hydraulic relief valve model [15], all of which has chattering solutions. Moreover, we will study a spring-mass system for the small mass. This model will have a chattering solution with the singular properties as well.

In other respects, singular perturbation problems are common in many areas of the science since they give a high level overview of certain problems that appear in the modeling of real-world problems by differential equations [16–24]. These problems depend on a small positive parameter such that the solution varies rapidly in some regions and varies slowly in other regions. In the book [16], author, in order to show the difficulty that arises when a small parameter multiplies the highest derivative, studies a second-order differential equation, and one can understand the nature of the singular perturbation through that discussion. On the other hand, impulse effects exist in a wide diversity of evolutionary processes that exhibit abrupt changes in their states [25–27]. In many systems, in addition to singular perturbation, there also have impulse effects [28–32]. Chen et al. [30] derived a sufficient condition that guarantees robust exponential stability for sufficiently small singular perturbation parameter by applying the Lyapunov function method and using a two-time scale comparison principle. In [31,32], authors proposed Lyapunov function method to set up the exponential stability criteria for singularly perturbed impulsive systems. This method can be efficiently used to overcome the impulsive perturbation such that the stability of the original system can be ensured. In [28], Lyapunov function method was further extended to study the exponential stability of singularly perturbed stochastic time-delay systems with impulse effect. However, the stability criteria in [28,31,32] are all based on Lyapunov functions. There is no systematic procedure supplied therein for constructing the appropriate Lyapunov functions. The results in [28,31,32] only guarantee the systems under consideration to be exponentially stable for a sufficiently small positive parameter.

In this article, we introduce a new type of singularity. The systems under consideration have singularity which appears through moments of impacts. More precisely, we say that the impact moments are singular if they are infinite and there exist accumulation points for the moments. Since there exist an infinite number of discontinuity moments in a finite time, the possibility of the blow up of solutions occurs here. This is why, the phenomenon has to be accepted as a singular one. We will consider the system where the singularity presents only through the discontinuity moments as well as the case when it occurs not only in the moments but also in the differential equation.

To say about contribution of the paper to the mechanics, one should remark that singularly perturbed equations, which are in the focus of the investigation, are mechanical models with a positive restitution coefficient, $\mu > 0$. That is, there are *elastic* impacts. Meantime, the degenerate models, which are assumed to be with $\mu = 0$, are *plastic* impacts in mechanical sense. Thus, our research is mainly about *elastic* impact processes, and the plastic (degenerated) ones are mentioned only with auxiliary aims if one considers mechanical purposes. These all correspond to the paradigm of singular perturbation, which is approximation of a solution $z(t, \mu)$ of a problem $P(\mu)$ having a small positive parameter with a solution $\bar{z}(t) = z(t, 0)$ of the degenerate problem $P(0)$, wherein the parameter is zero.

In continuous dynamics, a parameter-dependent problem $P(\mu)$ is *singular* if the convergence of a solution $z(t, \mu)$ to a solution $\bar{z}(t)$ of degenerate equation $P(0)$ is not uniform [33,34]. In the present paper, we provide a rigorous argument that the problem under investigation is singular from this fundamental point of view. However, during the analysis, we have found that there are additional arguments to be a singular problem which are usually not mentioned in the literature. They are:

1. The solutions $z(t, \mu)$ and $\bar{z}(t)$ are from different functional spaces. In our case, they are functions with infinitely many discontinuity moments and those with a finite number of discontinuities.
2. The set of discontinuity moments is in an interval which shrinks to a point.

Possibly, in the future, the first and the second features can be considered as sufficient conditions for a problem with discontinuities to be a singular one.

2 Preliminaries: chattering in mechanical models

Consider the problem of impact interaction of a body falling in the uniform gravity force field with a fixed horizontal base. After colliding with the base the body bounces back with the velocity whose norm is equal to the norm of the pre-impact velocity multiplied by μ, where μ is the restitution coefficient, $0 < \mu < 1$. Then, after some time interval the body will fall on the base again and the norm of its velocity will be equal to the norm of bouncing velocity in the previous collision multiplied by μ. The process cannot end in a finite number of collisions. Thus, the considered phenomenon consists in following: after the initial collision a series of repeated collisions of attenuated to zero, which ends in a finite time with establishing a long contact between interacted bodies. Arising this contact results in decreasing number of degrees of freedom of the system by a unit or more.

In this section, we will demonstrate that some mechanical models with chattering solutions.

2.1 A bouncing ball

The most famous model in mechanics is a bouncing ball model [12–14]. Therefore, first of all, we start with a bouncing ball model. A ball is dropped from a height h_0 without initial velocity. The ball falls vertically onto a smooth horizontal surface. During the free fall, we assume that the ball is subjected only to gravity. Besides, during each bounce, the collision is assumed instantaneous, i.e., the duration of contact is zero, and inelastic, i.e., a part of the kinetic energy of the ball dissipated. Therefore, the ball's velocity after a collision is smaller than before the collision, and consequently the height of bounces decreases with time. Let μ be the ratio between the ball velocity after and before the impact. This ratio is between the ball and the surface, which is assumed constant for all impacts. Thus, if v_n is the ball velocity before the nth bounce, we have the following expression

$$v_{n+1} = \mu v_n, \quad n = 1, 2, 3, \ldots$$

Consider Fig. 1, it is easy to show that the ball first strikes the surface after a time $t_0 = \sqrt{2h_0/g}$, with a velocity $v_1 = \sqrt{2h_0 g}$ where g is the acceleration of gravity. Let t_n be the time of flight of the ball between the nth and $n+1$th bounces. Let us compute the time $t_n, n = 1, 2, 3, \ldots$:

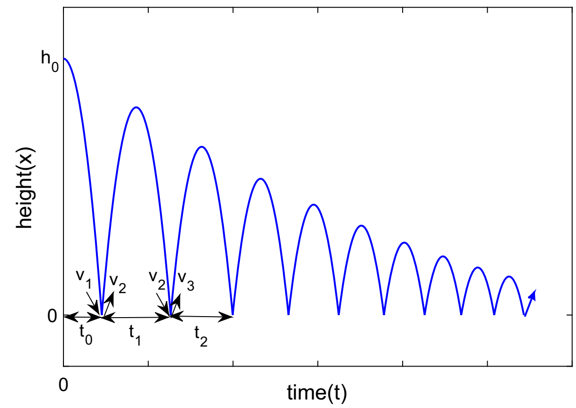

Fig. 1 Representation of a bouncing ball

$$t_n = \frac{2v_n}{g} = \mu^n \sqrt{\frac{8h_0}{g}}, \quad n = 1, 2, 3, \ldots$$

Now, we describe the system as follows:

$$\begin{aligned} &\ddot{x} = -g, \quad \Delta\dot{x}\ |_{x=0} = -(1+\mu)\dot{x}, \\ &x(0) = h_0, \quad \dot{x}(0) = 0, \end{aligned} \tag{1}$$

where $x \geq 0$. In Fig. 1 and from the above assumption, we can calculate the impact moments as: $\theta_0 = \sqrt{2h_0/g}$, $\theta_{i+1} = \theta_i + \mu^{i+1}\sqrt{\frac{8h_0}{g}}$, $i = 0, 1, 2, \ldots$. Hence,

$$\theta_\infty = \theta_0 + \sum_{i=1}^{\infty} t_n = \theta_0 + t_1 \sum_{i=1}^{\infty} \mu^{n-1} = \frac{1+\mu}{1-\mu}\sqrt{\frac{2h_0}{g}},$$

since $\mu < 1$. As a result, the ball admits infinitely many impacts and stays on the surface without bouncing for $t > \theta_\infty$.

It is easily seen that if one fix the moments θ_i and take $x = y$, $\dot{x} = z$ in (1), then the motion of the bouncing ball satisfies the following equations.

$$\begin{aligned} &\dot{z} = -g, \quad \Delta z|_{t=\theta_i} = -(1+\mu)z, \\ &\dot{y} = z, \\ &z(0,\mu) = 0, \quad y(0,\mu) = h_0, \end{aligned} \tag{2}$$

where $y \geq 0$. It can be seen in Fig. 2 that solutions of system (2) with initial values $z(0,\mu) = 1$, $y(0,\mu) = 0$ for different values of μ have many impact moments. It is obvious that as the parameter μ decreases to zero, the time of the ball to rest decreases and the impact

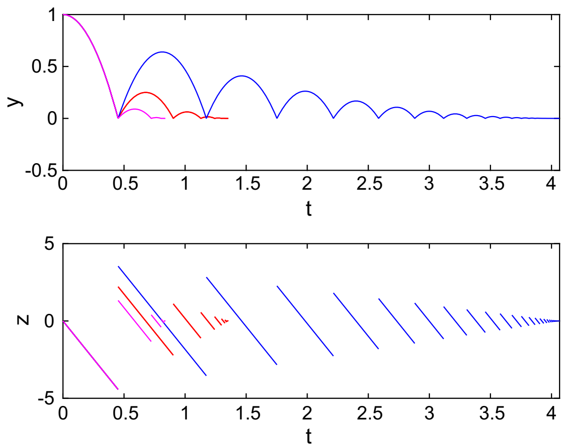

Fig. 2 Solution of system (2) with initial values $z(0, \mu) = 1$, $y(0, \mu) = 0$ for different values of μ (blue, red and magenta represent the coordinates of (2) for $\mu = 0.8$, $\mu = 0.5$ and $\mu = 0.3$, respectively). It is obviously seen that as the parameter μ decreases to zero, the time of the ball to rest also decreases. (Color figure online)

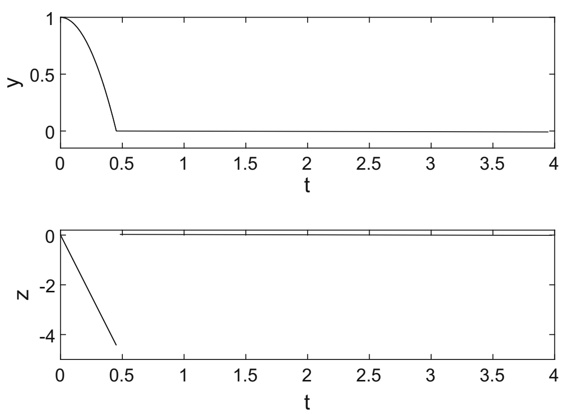

Fig. 3 Ultimate form of the solution of system (2) as $\mu \to 0$

moments tend to the first impact moment. That is, as $\mu \to 0$, the solution $(z(t, \mu), y(t, \mu))$ of (2) ultimately looks like in Fig. 3. There arise questions: (1) *are the functions in Fig. 3 are the limits of the solutions;* (2) *what is the type of the convergence;* (3) *is there a model such that the functions are their solutions?* In what follows, we will answer the questions, and moreover, we specify relations between the original model and the model with zero value of the parameter as *a singular perturbation*, such that one can approximate the solution of (2) by solutions of a degenerate equation. Moreover, we shall show that the interval $(\theta_0, \theta_\infty)$ is a boundary layer of the problem. Similar discussion can be made for the next two mechanical models.

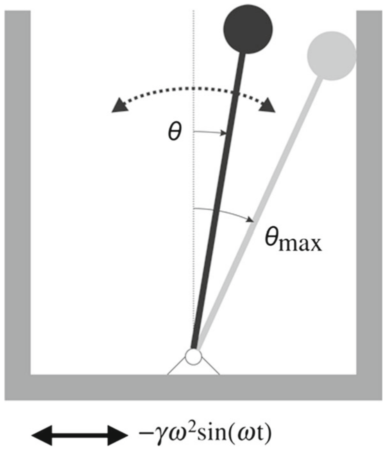

Fig. 4 The impacting inverted pendulum [11]

2.2 An inverted pendulum

Next, we will consider the inverted pendulum. It is used in the modeling of various engineering applications, such as rings, printers, machine tools, dynamics of rigid standing structures and rolling railway wheel set [10,11]. The model in [11] will be discussed which has a lateral obstacle for the chattering. The inverted pendulum has impact against the rigid flat wall with a constant restitution coefficient μ. The mechanical model can be observed in Fig. 4. The dynamics of the inverted pendulum between the lateral walls is described by the equations

$$\begin{aligned} &\ddot{x} + 2\delta\dot{x} - x = \gamma \sin(\omega t), \, |x| < 1, \\ &\Delta\dot{x}|_{|x|=1} = -(1+\mu)\dot{x}, \end{aligned} \tag{3}$$

where $x = \theta/\theta_{\max}$ is the normalized angle (Fig. 4), δ is the viscous damping $(0 < \delta < 1)$, $f(t) = \gamma \sin(\omega t)$ is the harmonic excitation representing the horizontal acceleration of the base. During the motion of the impacting pendulum, we will take the wall at the position $x = 1$ as an impacting surface, $0 \le x \le 1$, $\gamma = 0.001$, $\omega = 5$, $\delta = -0.005$ and $\mu = 0.9$. Denote $x = y$, $\dot{x} = z$. Then, system (3) will be

$$\begin{aligned} &\dot{z} = 0.01z + y + 0.001 \sin(5t), \\ &\Delta z|_{y=1} = -(1+\mu)z, \\ &\dot{y} = z, \end{aligned} \tag{4}$$

where $0 \le y \le 1$. In Fig. 5, one can observe that the pendulum performs many strikes in finite time if

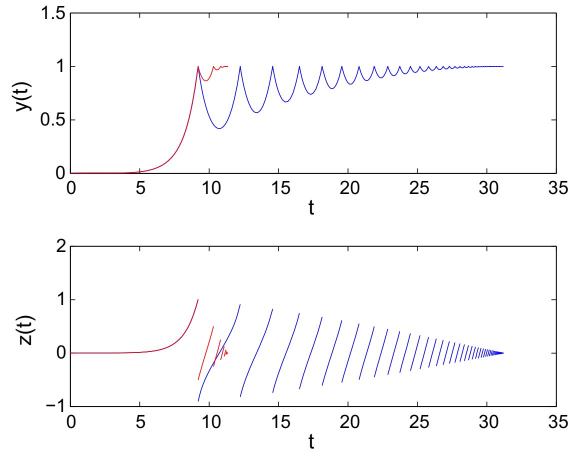

Fig. 5 Solutions of (4) with initial values $z(0, \mu) = 0$, $y(0, \mu) = 0$. Here, red and blue lines represent the solutions for $\mu = 0.5$ and $\mu = 0.8$, respectively. (Color figure online)

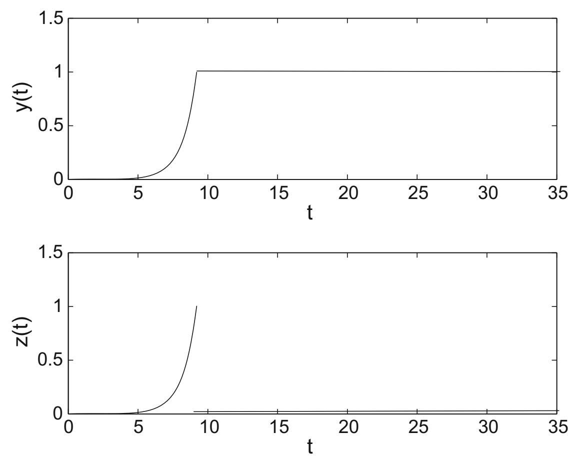

Fig. 6 Demonstration of the solution of (4) with initial values $z(0, \mu) = 0$, $y(0, \mu) = 0$ as $\mu \to 0$ in ultimate situation

the initial values are $z(0, \mu) = 0$, $y(0, \mu) = 0$. (The detailed mathematical investigations are presented in paper [11]. In particular, it was shown that there are infinitely many strikes.)

Moreover, Fig. 5 tells us that when μ decreases solutions of (4) get closer to functions demonstrated in Fig. 6.

2.3 A hydraulic pressure relief valve

In this subsection, a mathematical model describing the dynamics of a single stage relief valve embedded within a simple hydraulic circuit, which is derived in [15], will be discussed.

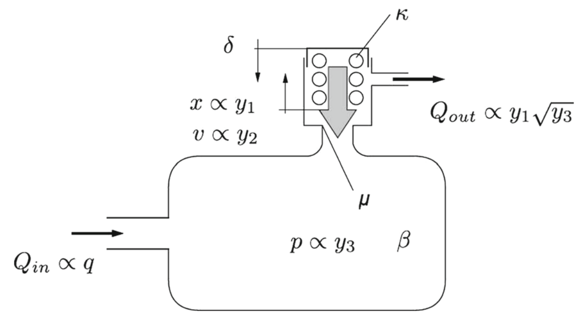

Fig. 7 Sketch of the physical system. Here $y_{1,2,3}$ stand for the dimensionless displacement, velocity and pressure, q is the dimensionless flow rate entering the system, δ and κ are the (dimensionless) spring precompression and damping coefficients, μ is the restitution coefficient between the seat and the valve body, β is a measure of the compressibility parameter of the fluid and the elastic hoses [15]

The equation of motion for the valve poppet system, which is described in Fig. 7, is of the form:

$$\begin{aligned}
\dot{y}_1 &= y_2,\\
\dot{y}_2 &= -\kappa y_2 - (y_1 + \delta) + y_3,\\
\dot{y}_3 &= \beta(q - y_1\sqrt{y_3}),\\
\Delta y_2|_{y_1=0} &= -(1+\mu)y_2,
\end{aligned} \tag{5}$$

with the initial values the initial values $y_1(0, \mu) = 10$, $y_2(0, \mu) = 0$, $y_3(0, \mu) = 10$, where y_1 is the position and y_2 is the velocity of the poppet, y_3 is the pressure in the chamber. $y_1 > 0$ if the valve is open and $y_1 = 0$ if it is closed. It is assumed that $y_3 > 0$, that is, the reservoir pressure is above the ambient pressure; hence, the flow direction is always outwards from the reservoir. δ and κ are the spring precompression and damping coefficients, μ is the restitution coefficient between the seat and the valve body, β is a measure of the compressibility parameter of the fluid and the elastic hoses.

In [15], it is shown that in the case of low flow rates, i.e., for small values of q, and $y_3 < \delta$ chattering motion exists. Therefore, it is reasonable to discuss this model as an example for the chattering. Figure 8 represents the solutions $y_1(t, \mu)$, $y_2(t, \mu)$, $y_3(t, \mu)$ for different values of μ. Clearly, we can see that as the restitution coefficient μ decays to zero, they approach to functions $y_1(t)$, $y_2(t)$, $y_3(t)$, which are represented in Fig. 9.

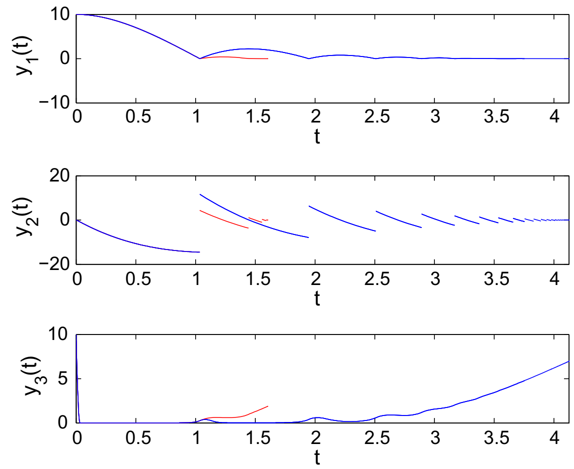

Fig. 8 Solutions of system (5) for $y_1(0,\mu)=10$, $y_2(0,\mu)=0$, $y_3(0,\mu)=10$, where $\beta=20, q=0.3, \kappa=1.25, \delta=20$. Here, blue and red represent the solutions for $\mu=0.8$ and $\mu=0.3$, respectively. (Color figure online)

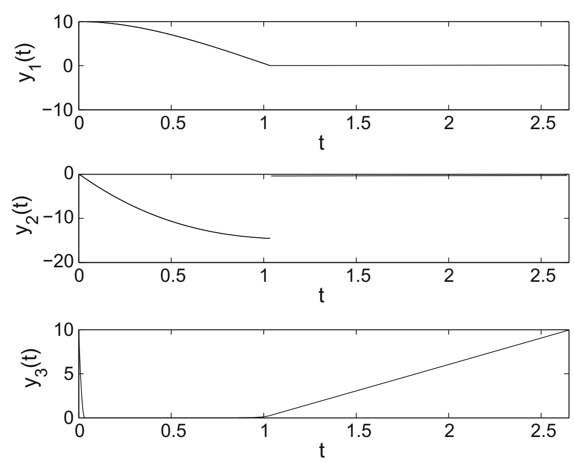

Fig. 9 Representation of solution of system (5) for $y_1(0,\mu)=10$, $y_2(0,\mu)=0$, $y_3(0,\mu)=10$, where $\beta=20, q=0.3, \kappa=1.25, \delta=20$ as $\mu\to 0$ in ultimate situation

3 Main results

3.1 Singularity in impact moments

We start to consider the following impulsive system

$$\frac{dz}{dt}=F(z,y,t), \quad \Delta z|_{t=\theta_i(\mu)}=I(z,y) \tag{6a}$$

$$\frac{dy}{dt}=f(z,y,t), \quad \Delta y|_{t=\xi_j}=J(z,y) \tag{6b}$$

where functions F, f, I and J are m-dimensional vector valued functions, $z,y\in\mathbb{R}^m$, $t\in[0,T]$, $\theta_1(\mu)=d_1(\mu), \theta_{i+1}(\mu)=\theta_i(\mu)+d_{i+1}(\mu), i\geq 1$, $\sum_{i=1}^{\infty}d_i(\mu)$ is uniformly convergent and $d_i(0)=0$, $0<\xi_1<\xi_2<\cdots<\xi_k<T$, and $\xi_i, 1\leq j\leq k$, is fixed. Consequently, limit $\lim_{i\to\infty}\theta_i(\mu)=\theta_\infty(\mu)$ exists and model (6) has infinitely many discontinuity moments in finite time.

Let us take $\mu=0$ in (6). Then we obtain

$$\frac{d\bar{z}}{dt}=F(\bar{z},\bar{y},t), \tag{7a}$$

$$\frac{d\bar{y}}{dt}=f(\bar{z},\bar{y},t), \quad \Delta\bar{y}|_{t=\xi_j}=J(\bar{z},\bar{y}). \tag{7b}$$

We will call system (7) as a degenerate equation for system (6).

Define the initial conditions (for simplicity, we set $t_0=0$ and it is not a jump moment.)

$$z(0,\mu)=z^0,\ y(0,\mu)=y^0, \tag{8}$$

where z^0 and y^0 are assumed to be independent of μ. Let us investigate the solution $z(t,\mu)$, $y(t,\mu)$ of (6) and (8) on segment $0\leq t\leq T$.

Define the domain $H=\{0\leq t\leq T, |y|<a, |z|<b\}$. Let $\tilde{J}(z,y)=z+I(z,y), z,y\in H$. Assume that $\tilde{J}(z,y)$ satisfies the Lipschitz condition:

(C1) $\|\tilde{J}(z_1,y)-\tilde{J}(z_2,y)\|<L\|z_1-z_2\|, 0<L<1$, $z_1,z_2,y\in H$.

We write $x=\tilde{J}_\infty(z,y), z,y\in H$, if the limit

$$\lim_{n\to\infty}\underbrace{\tilde{J}(\tilde{J}(\ldots\tilde{J}(z,y)\ldots,y),y)}_{n\text{-}times}=x$$

exists.

Fix $z^0, y^0\in H$ such that

(C2) $\tilde{J}_\infty(z^0,y^0)=\varphi$.

Consider the following initial conditions for (7)

$$\bar{z}(0)=\varphi,\ \bar{y}(0)=y^0. \tag{9}$$

We need the following conditions:

(C3) Functions $F(z,y,t), f(z,y,t), I(z,y)$ and $J(z,y)$ are continuous in each argument, and $F(z,y,t), f(y,z,t)$ are continuously differentiable with respect to z and y in the domain H.

(C4) Functions $F(z,y,t), f(z,y,t)$ are bounded on H, i.e., $\|F(z,y,t)\|\leq M<\infty$ and $\|f(z,y,t)\|\leq m<\infty$ for $(z,y,t)\in H$.

Theorem 1 *If conditions* (C1)–(C4) *are true, then solutions* $z(t,\mu)$ *and* $y(t,\mu)$ *of problem* (6) *with initial conditions* (8) *exist on* $0 \le t \le T$, *are unique, and satisfy*

$$\lim_{\mu\to 0} y(t,\mu) = \bar{y}(t) \text{ for } 0 \le t \le T \tag{10}$$

and

$$\lim_{\mu\to 0} z(t,\mu) = \bar{z}(t) \text{ for } 0 < t \le T, \tag{11}$$

where $\bar{z}(t)$, $\bar{y}(t)$ *are solutions of* (7) *and* (9).

In general, the initial condition z^0 is not equal to φ. This is why, the solution of (6) does not converge to the solution of (7) uniformly and the problem is singularly perturbed.

Proof Let $z^0, y^0 \in H$. Then the existence and uniqueness of solutions $z(t,\mu)$ and $y(t,\mu)$ of (6) with (8) follow from [25, Theorems 2.3.2 and 2.3.4] since condition (C3) holds.

Now, consider the following system

$$\frac{\mathrm{d}\tilde{z}}{\mathrm{d}t} = 0, \quad \Delta\tilde{z}|_{t=\theta_i(\mu)} = I(\tilde{z}, y^0) \tag{12a}$$

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}t} = f(\bar{z}, \bar{y}, t), \quad \Delta\bar{y}|_{t=\xi_j} = J(\bar{z}, \bar{y}). \tag{12b}$$

Solution $\tilde{z}(t)$ of (12a) with initial value $\tilde{z}(0) = z^0$, for each $\mu > 0$ is equal to φ if $t \ge \theta_\infty(\mu)$.

$\theta_\infty(\mu) \to 0$ as $\mu \to 0$. Therefore, there exists $\mu_0 > 0$ such that $\theta_\infty(\mu) < \xi_1$ for $0 \le \mu < \mu_0$. Define the recursive formula

$$T_n(\mu) = LT_{n-1}(\mu) + d_n(\mu), \quad n \ge 2,$$

where $T_1(\mu) = Md_1(\mu)$.

Then, for $0 \le t \le \theta_\infty(\mu)$ compare (6a) and (12a). If $t \in [0, \theta_1(\mu)]$, we obtain

$$\|z(t,\mu) - \tilde{z}(t)\| = \left\| z^0 + \int_0^t F(z,y,s)\mathrm{d}s - z^0 \right\| \le \int_0^t M\mathrm{d}s \le Mt \le Md_1(\mu),$$

and for $t \in (\theta_1(\mu), \theta_2(\mu)]$:

$$\begin{aligned}\|z(t,\mu) - \tilde{z}(t)\| = \Big\| z^0 &+ \int_0^{\theta_1} F(z,y,s)\mathrm{d}s + I(z^0 + \int_0^{\theta_1} F(z,y,s)\mathrm{d}s, y^0) \\ &+ \int_{\theta_1}^t F(z,y,s)\mathrm{d}s - (z^0 + I(z^0, y^0)) \Big\| \\ &\le LMd_1(\mu) + Md_2(\mu).\end{aligned}$$

For $t \in (\theta_2(\mu), \theta_3(\mu)]$:

$$\begin{aligned}\|z(t,\mu) - \tilde{z}(t)\| = \Big\| z^0 &+ \int_0^{\theta_1} F(z,y,s)\mathrm{d}s \\ &+ I(z^0 + \int_0^{\theta_1} F(z,y,s)\mathrm{d}s, y^0) \\ &+ \int_{\theta_1}^{\theta_2} F(z,y,s)\mathrm{d}s \\ &+ I\Big(z^0 + \int_0^{\theta_1} F(z,y,s)\mathrm{d}s + I(z^0 + \\ &+ \int_0^{\theta_1} F(z,y,s)\mathrm{d}s, y^0) \\ &+ \int_{\theta_1}^{\theta_2} F(z,y,s)\mathrm{d}s, y^0\Big) + \int_{\theta_2}^t F(z,y,s)\mathrm{d}s \\ &- (z^0 + I(z^0, y^0) + I(z^0 + I(z^0,y^0), y^0)) \Big\| \\ &\le L^2Md_1(\mu) + LMd_2(\mu) + Md_3(\mu).\end{aligned}$$

$\vdots$

By induction, one can show that for $t \in (\theta_{n-1}(\mu), \theta_n(\mu)]$, $\|z(t,\mu) - \tilde{z}(t)\| \le T_n(\mu)$,

$$T_n(\mu) = \sum_{i=1}^n L^{n-i} Md_i(\mu) < M \sum_{i=1}^n d_i(\mu) = M\theta_n(\mu).$$

$T_n(\mu) \to 0$ as $\mu \to 0$. Therefore, for $t \in (\theta_{n-1}(\mu), \theta_n(\mu)]$, $z(t,\mu)$ is in the neighborhood of $\tilde{z}(t)$. Moreover, as $n \to \infty$, we have $z(\theta_\infty(\mu), \mu) \to \tilde{z}(\theta_\infty(\mu)) = \varphi$. At time $t = \theta_\infty(\mu)$,

$$\bar{z}(\theta_\infty(\mu)) = \varphi + \int_0^{\theta_\infty(\mu)} F(\bar{z}, \bar{y}, s)\mathrm{d}s \text{ and } \tilde{z}(\theta_\infty(\mu)) = \varphi,$$

where $\bar{z}(t)$ is the solution of (7) and (9). Hence,

$$\|\bar{z}(\theta_\infty(\mu)) - \tilde{z}(\theta_\infty(\mu))\| \le M\theta_\infty(\mu).$$

Since $\theta_\infty(\mu) \to 0$ as $\mu \to 0$, $\tilde{z}(\theta_\infty(\mu)) \to \bar{z}(\theta_\infty(\mu))$, and so $z(\mu, \theta_\infty(\mu)) \to \bar{z}(\theta_\infty(\mu))$.

Now, if $\theta_\infty(\mu) < t \le T$, consider the systems (6a) and (7a). By continuous dependence $z(t,\mu)$ is the neighborhood of $\bar{z}(t)$.

Similarly, if $t \in [0, \theta_1(\mu)]$, we get

$$\begin{aligned}\|y(t,\mu) - \bar{y}(t)\| &= \left\| y^0 + \int_0^t f(z,y,s)\mathrm{d}s - y^0 \right.\\ &\left. - \int_0^t f(\bar{z},\bar{y},s)\mathrm{d}s \right\|\\ &\le 2mt \le 2md_1(\mu).\end{aligned}$$

For $t \in (\theta_1(\mu), \theta_2(\mu)]$:

$$\begin{aligned}\|y(t,\mu) - \bar{y}(t)\| &= \left\| y(\theta_1(\mu),\mu) \right.\\ &+ \int_{\theta_1(\mu)}^t f(z,y,s)\mathrm{d}s - \bar{y}(\theta_1(\mu))\\ &\left. - \int_{\theta_1(\mu)}^t f(\bar{z},\bar{y},s)\mathrm{d}s \right\|\\ &\le 2md_1(\mu) + 2m(t-\theta_1(\mu))\\ &\le 2md_1(\mu) + 2md_2(\mu).\\ &\vdots\end{aligned}$$

By induction, for $t \in (\theta_{n-1}(\mu), \theta_n(\mu)]$, we have $\|y(t,\mu) - \bar{y}(t)\| \le \sum_{i=1}^n 2md_i(\mu) = 2m\theta_n(\mu)$. Since $\theta_n(\mu) \to 0$ as $\mu \to 0$, $y(t,\mu) \to \bar{y}(t)$. Thus, $y(\theta_\infty(\mu),\mu) \to \bar{y}(\theta_\infty(\mu))$ as $\mu \to 0, n \to \infty$. Now, we examine the solution for $t > \theta_\infty(\mu)$. It is readily seen that by continuous dependence $y(t,\mu)$ is in the neighborhood of $\bar{y}(t)$ on $(\theta_\infty(\mu), T]$. Theorem is proved.

This is the time to explain, on the basis of the above theorem and proof, why the problem investigated in our paper is a singular perturbation problem. Indeed, a perturbation is singular if the convergence is not uniform [33,34]. In our case, solution $(z(t,\mu), y(t,\mu))$ converges to $(\bar{z}(t), \bar{y}(t))$ uniformly on each interval $[\varepsilon, T]$, $\varepsilon > 0$, but there is no convergence at the point $t = 0$ since $z^0 \neq \varphi$. This is why, the convergence is not uniform and it is the sufficient argument to say that in Theorem 1 a singular problem is considered. Another remarkable fact in our research is that the region in which the impact moments are placed shrinks to a single point when the parameter diminishes, i.e., $[0, \theta_\infty(\mu)] \to 0$ as $\mu \to 0$. From the above research, one can make a conclusion that $[0, \theta_\infty(\mu)]$ is a boundary layer. Finally, it should be emphasized that the singularity in our paper is not a consequence of the small parameter multiplied by the derivative, but it is caused by singularity in impact moments.

In the next section, we will combine the singular perturbation through the small parameter multiplying the highest derivative and singularity in discontinuity moments.

3.2 Singularity in impact moments and small parameter multiplying the derivative

In the previous subsection, we show the singularity emerging from impact moments. Now, in addition to this, we will demonstrate the singularity in both from impact moments and from the small parameter in front of the derivative which can be described as follows

$$\mu\frac{\mathrm{d}z}{\mathrm{d}t} = F(z,y,t), \quad \Delta z|_{t=\theta_i(\mu)} = I(z,y), \tag{13a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(z,y,t), \quad \Delta y|_{t=\xi_j} = J(z,y), \tag{13b}$$

where all functions, discontinuity moments, domain, initial conditions are defined in Sect. 3.1. This system is different from system (6) as follows: we have a small parameter multiplying the derivative and singularity in impact moments. Hence, additional condition is needed.

(C5) Suppose that $\lim\limits_{\mu\to 0^+} \dfrac{\theta_i(\mu)}{\mu} = 0, \quad i = 1, 2, \ldots.$

Let us investigate the solution $z(t,\mu)$, $y(t,\mu)$ of (13) and (8) on segment $0 \le t \le T$.

Take $\mu = 0$ in (13). Then we obtain

$$\begin{aligned}&0 = F(\bar{z},\bar{y},t),\\ &\frac{\mathrm{d}\bar{y}}{\mathrm{d}t} = f(\bar{z},\bar{y},t), \quad \Delta\bar{y}|_{t=\xi_j} = J(\bar{z},\bar{y}).\end{aligned} \tag{14}$$

Assume that $0 = F(\bar{z},\bar{y},t)$ has a root $\bar{z} = \varphi$ such that condition (C2) is true. Hence, we can write

$$\begin{aligned}&\frac{\mathrm{d}\bar{y}}{\mathrm{d}t} = f(\varphi,\bar{y},t), \quad \Delta\bar{y}|_{t=\xi_j} = J(\varphi,\bar{y}),\\ &\bar{y}(0) = y^0.\end{aligned} \tag{15}$$

Introduce the adjoint system

$$\frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tau} = F(\tilde{z},y,t), \tag{16}$$

where y and t are considered as parameters, $\tilde{z} = \varphi$ is an isolated stationary point of (16) for $y, t \in H$. We need the following condition, also, (C6) the stationary point $\tilde{z} = \varphi$ of (16) is uniformly asymptotically stable.

Theorem 2 *If conditions* (C1)–(C6) *are true, then solutions* $z(t,\mu)$ *and* $y(t,\mu)$ *of problem* (13) *with initial conditions* (8) *exist on* $0 \le t \le T$, *are unique, and satisfy*

$$\lim_{\mu\to 0} y(t,\mu) = \bar{y}(t) \text{ for } 0 \le t \le T \tag{17}$$

and

$$\lim_{\mu\to 0} z(t,\mu) = \varphi \text{ for } 0 < t \le T, \tag{18}$$

where $\bar{y}(t)$ *is the solution of* (15).

Proof Let $z^0, y^0 \in H$. Similarly, the existence and uniqueness of solutions $z(t,\mu)$ and $y(t,\mu)$ of (13) with (8) follow from [25, Theorems 2.3.2 and 2.3.4] since condition (C3) holds.

Now, consider the following system

$$\frac{d\hat{z}}{dt} = 0, \quad \Delta\hat{z}|_{t=\theta_i(\mu)} = I(\hat{z}, y^0) \tag{19a}$$

$$\frac{d\bar{y}}{dt} = f(\varphi, \bar{y}, t), \quad \Delta\bar{y}|_{t=\xi_j} = J(\varphi, \bar{y}) \tag{19b}$$

which has the same discontinuity moments, impulse function and initial condition as Eq. (13). Solution $\hat{z}(t)$ of (19) with initial value $\hat{z}(0) = z^0$, for each $\mu > 0$ is equal to φ if $t \ge \theta_\infty(\mu)$. Consequently, one can say

$$\lim_{\mu\to 0} \hat{z}(t) = \varphi, \quad 0 < t \le T.$$

$\theta_\infty(\mu) \to 0$ as $\mu \to 0$. Therefore, there exists $\mu_0 > 0$ such that $\theta_\infty(\mu) < \xi_1$ for $0 \le \mu < \mu_0$. Define the recursive formula

$$H_n(\mu) = LH_{n-1}(\mu) + \frac{d_n(\mu)}{\mu}, \quad n \ge 2,$$

where $H_1(\mu) = M\frac{d_1(\mu)}{\mu}$.

Similar to proof of Theorem 1, for $t \in (\theta_{n-1}(\mu), \theta_n(\mu)]$, one has $\|z(t,\mu) - \hat{z}(t)\| \le H_n(\mu)$, and

$$H_n(\mu) = \sum_{i=1}^{n} L^{n-i} M \frac{d_i(\mu)}{\mu} < M \sum_{i=1}^{n} \frac{d_i(\mu)}{\mu} = M\frac{\theta_n(\mu)}{\mu}.$$

It follows from condition (C5) that $H_n(\mu) \to 0$ as $\mu \to 0$. Therefore, for $t \in (\theta_{n-1}(\mu), \theta_n(\mu)]$, $z(t,\mu)$ is in the neighborhood of $\hat{z}(t)$. Moreover, as $n \to \infty$, we have $z(\theta_\infty(\mu),\mu) \to \hat{z}(\theta_\infty(\mu)) = \varphi$.

Now, if $\theta_\infty(\mu) < t \le T$, consider the systems (13a) and (19a). By condition (C6), $z(t,\mu)$ is the neighborhood of $\tilde{z}(t)$.

Limit (17) is follows from the proof of Theorem 1. Theorem is proved. □

4 Examples

The first example is a scalar one.

Example 1 Consider the initial value problem

$$\frac{dz}{dt} = z^2, \qquad \Delta z|_{t=\theta_i(\mu)} = -0.3z, \quad z(0) = z^0, \tag{20}$$

where $\theta_1(\mu) = h\mu, \theta_{i+1}(\mu) = \theta_i(\mu) + h\mu^{i+1} i \ge 1$, and $z \in \mathbb{R}$. Here, $J(x) = x + I(x) = 0.7x$. Let us check the conditions of Theorem 1.

$$\lim_{n\to\infty} \underbrace{J(J(\ldots J(z_0)))}_{n-times} = \lim_{n\to\infty} (0.7)^n z_0 = 0 = \varphi.$$

Also, $|J(x) - J(y)| \le 0.7|x - y|$. The system corresponding (7) and (9) is

$$\frac{d\bar{z}}{dt} = \bar{z}^2, \quad \bar{z}(0) = 0. \tag{21}$$

Therefore, for any $z^0 \in \mathbb{R}$, solution $z(t,\mu)$ of (20) approaches to solution $\bar{z} = \varphi = 0$ of (21) as $\mu \to 0$. It can be seen in Fig. 10 that solution $z(t,\mu)$ of (20) approaches $\bar{z} = 0$ as $\mu \to 0$ for the initial value $z^0 = 1$.

Example 2 Let us consider the second example as follows.

$$\frac{dz}{dt} = z^2 - 5z + y, \quad \Delta z|_{t=\theta_i(\mu)} = -0.8z + y^2, \qquad \frac{dy}{dt} = yz, \quad \Delta y|_{t=\xi_j} = z, \tag{22}$$

where $0 < \mu < 1$, $\theta_1(\mu) = \sin\mu^2, \theta_{i+1}(\mu) = \theta_i(\mu) + (\sin\mu)^{i+2}, i > 1, \xi_j = 0.3 + j/24, j = 12, 13, 14, 15, 16$, with initial conditions $z(0,\mu) = 1, y(0,\mu) = 1$, on the domain $H = \{0 \le t \le 1.5, \|z\| < 2, \|y\| < 10\}$. It is easily seen that the conditions of Theorem 1 are satisfied. That is, $J(z,y) = z + I(z,y) = 0.2z + y^2$ satisfies

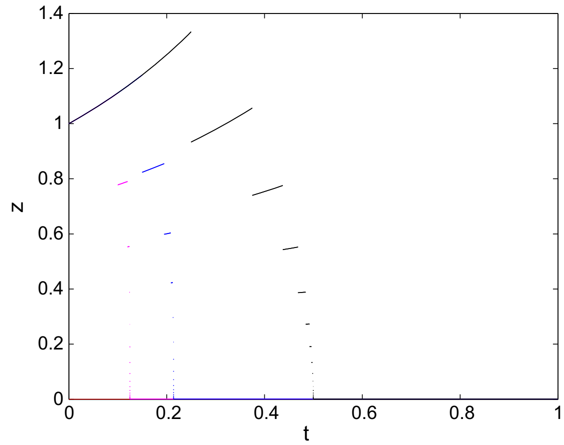

Fig. 10 Black, blue, magenta and red represent the solution $z(t,\mu)$ of (20) for values of $\mu = 0.5, 0.3, 0.2, 0$, respectively, with initial value $z^0 = 1$. (Color figure online)

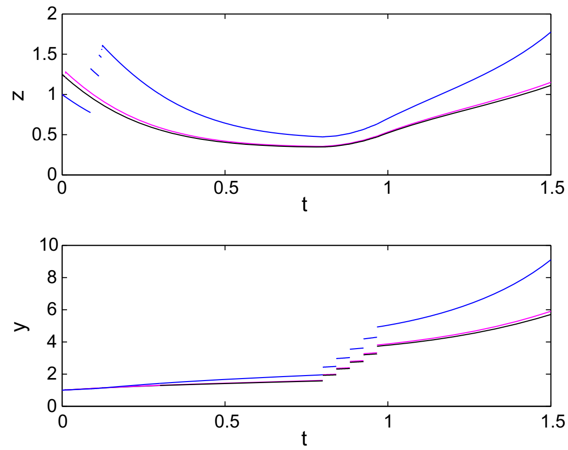

Fig. 11 Magenta and blue represent the coordinates $z(t,\mu), y(t,\mu)$ of (22) with initial conditions $z(0,\mu) = 1, y(0,\mu) = 1$, for $\mu = 0.1$, $\mu = 0.3$, respectively, and black represents the coordinates $\bar{z}(t), \bar{y}(t)$ of (23) with $\bar{z}(0) = 5/4, \bar{y}(0) = 1$. (Color figure online)

$$\|J(z_1, y) - J(z_2, y)\| \leq 0.2\|z_1 - z_2\|,$$
$$J_\infty(1,1) = \lim_{n\to\infty}\left((0.2)^n + \sum_{i=1}^{n-1}(0.2)^i\right) = 5/4.$$

Now, consider the following system which is obtained by taking $\mu = 0$ in (22) with new initial conditions:

$$\begin{aligned}\frac{d\bar{z}}{dt} &= \bar{z}^2 - 5\bar{z} + \bar{y},\\ \frac{d\bar{y}}{dt} &= \bar{y}\bar{z}, \quad \Delta\bar{y}|_{t=\xi_j} = \bar{z},\end{aligned} \tag{23}$$

with $\bar{z}(0) = 5/4, \bar{y}(0) = 1$. By Theorem 1, solutions $z(t,\mu), y(t,\mu)$ of (22) with $z(0,\mu) = 1, y(0,\mu) = 1$, tend to solutions $\bar{z}(t), \bar{y}(t)$ of (23) as $\mu \to 0$, respectively, which are illustrated in Fig. 11.

5 Asymptotic approximations

Solutions of the systems defined in this paper admit infinitely many jumps, and this makes, in general, impossible to find an exact solution or adequately to simulate it. So, in this section, we suggest a model with finitely many impacts to find the solution of the perturbed system approximately. In order to increase the precision of approximation, we follow the idea of asymptotic approximations. In systems, we will take finitely many discontinuity moments to find an asymptotic approximation. That is, the discussed systems are equipped with the same properties except infinite impact moments.

Let us discuss the asymptotic approximation for each proposed system. Through this section, [.] denotes the greatest integer function.

The solution of system (6) with (8) has the following asymptotic representation

$$z(t,\mu) = \begin{cases} z_m(t) & \text{if} \quad 0 \leq t \leq \theta_{m+1}(\mu), \\ z_m(t) + \bar{z}(t) \\ -\tilde{J}_m(z^0, y^0) + \varepsilon_1(t,\mu) & \text{if} \quad \theta_{m+1}(\mu) < t \leq T, \end{cases}$$

$$y(t,\mu) = \begin{cases} y_m(t) & \text{if} \quad 0 \leq t \leq \theta_{m+1}(\mu), \\ \bar{y}(t) + \tilde{\varepsilon}_1(t,\mu) & \text{if} \quad \theta_{m+1}(\mu) < t \leq T, \end{cases}$$

where $\tilde{J}_m(z,y)$ is defined in Sect. 3.1, $\varepsilon_1(t,\mu) \to 0$ and $\tilde{\varepsilon}_1(t,\mu) \to 0$ as $\mu \to 0$, $\bar{z}(t), \bar{y}(t)$ are the solutions of (7) and (9), $z_m(t), y_m(t)$ are the solution of

$$\frac{dz_m}{dt} = F(z_m, y_m, t), \quad \Delta z_m|_{t=\theta_i(\mu)} = I(z_m, y_m), \tag{24a}$$

$$\frac{dy_m}{dt} = f(z_m, y_m, t), \quad \Delta y_m|_{t=\xi_j} = J(z_m, y_m) \tag{24b}$$

with initial conditions $z_m(0) = z^0, y_m(0) = y^0, 1 \leq i \leq m$. Here, assume that (24) has the same properties as (6) except infinite impact moments $\theta_i(\mu)$s. Therefore, solutions $z(t,\mu), y(t,\mu)$ of (6) and $z_m(t), y_m(t)$

of (24) with the same initial values are equal on the interval $[0, \theta_{m+1}(\mu)]$. If $t \in (\theta_{m+1}(\mu), T]$,

$$\|\varepsilon_1(t,\mu)\| = \|z(t,\mu) - z_m(t) - \bar{z}(t) + \tilde{J}_m(z^0, y^0)\| \\ \leq \|z(t,\mu) - \bar{z}(t)\| + \|z_m(t) - \tilde{J}_m(z^0, y^0)\|$$

Hence, by Theorem 1, $\|z(t,\mu) - \bar{z}(t)\| < \frac{\varepsilon}{2}$ as $\mu \to 0$. Moreover, $\|z_m(t) - \tilde{J}_m(z^0, y^0)\| < \frac{\varepsilon}{2}$ as $\mu \to 0$ since $\theta_m(\mu) \to 0$ as $\mu \to 0$. Therefore, we have $\|\varepsilon_1(t,\mu)\| < \varepsilon$.

Consider, again, $t \in (\theta_{m+1}(\mu), T]$. In (6b), let us substitute the asymptotic approximation of $z(t,\mu)$. Then,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(z_m + \bar{z} - \tilde{J}_m(z^0, y^0) + \varepsilon_1(t,\mu), y, t), \\ \Delta y|_{t=\xi_j} = J(z_m + \bar{z} - \tilde{J}_m(z^0, y^0) + \varepsilon_1(t,\mu), y), \tag{25}$$

where $y(0,\mu) = y^0$. System (7b) with $\bar{y}(0) = y^0$ is the degenerated problem of (25). Thus, by regular perturbation theory for impulsive systems [25], $\|y(t,\mu) - \bar{y}(t)\| = \|\tilde{\varepsilon}_1(t,\mu)\| < \varepsilon$ for sufficiently small μ if $t \in (\theta_{m+1}(\mu), T]$. Consequently, we have proven the asymptotic representation for the solution of problem (6) with (8). Note that if we increase the impact moments, we obtain a better approximation.

The following is the final asymptotic approximation which is for solution of (13) and (8).

$$z(t,\mu) = \begin{cases} z_m(t) & \text{if} \quad 0 \leq t \leq \theta_{m+1}(\mu), \\ z_m(t) + \tilde{J}_{[\frac{1}{\mu}]}(z^0, y^0) & \\ -\tilde{J}_m(z^0, y^0) + \varepsilon_2(t,\mu) & \text{if} \quad \theta_{m+1}(\mu) < t \leq T, \end{cases}$$

$$y(t,\mu) = \begin{cases} y_m(t) & \text{if} \quad 0 \leq t \leq \theta_{m+1}(\mu), \\ \bar{y}(t) + \tilde{\varepsilon}_2(t,\mu) & \text{if} \quad \theta_{m+1}(\mu) < t \leq T, \end{cases}$$

where $\varepsilon_2(t,\mu) \to 0$ and $\tilde{\varepsilon}_2(t,\mu) \to 0$ as $\mu \to 0$, $\bar{y}(t)$ is the solution of (15), $z_m(t)$, $y_m(t)$ are the solution of

$$\mu\frac{\mathrm{d}z_m}{\mathrm{d}t} = F(z_m, y_m, t), \\ \Delta z_m|_{t=\theta_i(\mu)} = I(z_m, y_m), \tag{26a}$$

$$\frac{\mathrm{d}y_m}{\mathrm{d}t} = f(z_m, y_m, t), \\ \Delta y_m|_{t=\xi_j} = J(z_m, y_m) \tag{26b}$$

with initial conditions $z_m(0) = z^0, y_m(0) = y^0$, $1 \leq i \leq m$. Suppose that (26) has the same properties as (13) except infinite impact moments $\theta_i(\mu)$s. Therefore, solutions $z(t,\mu)$, $y(t,\mu)$ of (13) and $z_m(t)$, $y_m(t)$ of (26) with the same initial conditions are equal on the interval $[0, \theta_{m+1}]$. Now, we need to show for $\theta_{m+1}(\mu) < t \leq T$ the asymptotic representation is true. If $t \in (\theta_{m+1}(\mu), T]$,

$$\|\varepsilon_2(t,\mu)\| = \left\| z(t,\mu) - z_m(t) - \tilde{J}_{[\frac{1}{\mu}]}(z^0, y^0) + \tilde{J}_m(z^0, y^0) \right\| \\ \leq \left\| z(t,\mu) - \tilde{J}_{[\frac{1}{\mu}]}(z^0, y^0) \right\| + \|z_m(t) - \tilde{J}_m(z^0, y^0)\|$$

Here, since $\tilde{J}_{[\frac{1}{\mu}]}(z^0, y^0) \to \varphi$ as $\mu \to 0$, by Theorem 2, $\|z(t,\mu) - \tilde{J}_{[\frac{1}{\mu}]}(z^0, y^0)\| < \frac{\varepsilon}{2}$ as $\mu \to 0$. Also, $\|z_m(t) - \tilde{J}_m(z^0, y^0)\| < \frac{\varepsilon}{2}$ as $\mu \to 0$ since $\theta_m(\mu) \to 0$ as $\mu \to 0$. Thus, we have $\|\varepsilon_2(t,\mu)\| < \varepsilon$.

Consider $\theta_{m+1}(\mu) < t \leq T$ and substitute the asymptotic value of $z(t,\mu)$ into (13b) to obtain

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f\left(z_m + \tilde{J}_{[\frac{1}{\mu}]}(z^0, y^0) - \tilde{J}_m(z^0, y^0) + \varepsilon_2(t,\mu), y, t\right), \\ \Delta y|_{t=\xi_j} = J\left(z_m + \tilde{J}_{[\frac{1}{\mu}]}(z^0, y^0) - \tilde{J}_m(z^0, y^0) + \varepsilon_2(t,\mu), y\right), \tag{27}$$

where $y(0,\mu) = y^0$. System (27) with $y(0,\mu) = y^0$ is the regularly perturbed problem of (15). As a result, by regular perturbation theory for impulsive systems [25], $\|y(t,\mu) - \bar{y}(t)\| = \|\tilde{\varepsilon}_2(t,\mu)\| < \varepsilon$ for sufficiently small μ if $t \in (\theta_{m+1}(\mu), T]$. Finally, the asymptotic representation of the solution of problem (13) with (8) has been proven.

Remark 1 The precise asymptotic properties of ε_1, $\tilde{\varepsilon}_1$, ε_2 and $\tilde{\varepsilon}_2$ cannot be described in the section, since one needs concrete asymptotics for the sequence $d_i(\mu)$. Nevertheless, in the bouncing ball model in the next section the asymptotics for $d_i(\mu)$, ε_1 and for $\tilde{\varepsilon}_1$ will be found.

6 Chattering in the view of singularity

In this section, we will discuss the models in Sect. 2 as singular models defined in Sect. 3.

6.1 A bouncing ball

Consider again bouncing ball system (2) on $[\theta_0, T]$ since the parameter μ does not affect the solution on $[0, \theta_0]$. That is, we intend to apply Theorem 1 for a singular perturbation problem on the interval $[\theta_0, T]$, considering the moment $t = \theta_0$ instead of $t = 0$, discussed in the theorem. One can find that in the notations of the theorem $\tilde{J}(z, y) = z - (1+\mu)z = -\mu z$, $F(z, y, t) = -g$, $f(z, y, t) = z$, $t_0 = \theta_0 = \sqrt{2h_0/g}$, and impact moments $\theta_{i+1} = \theta_i + \mu^{i+1}\sqrt{\frac{8h_0}{g}}$, $i = 0, 1, 2, \ldots$, $\theta_\infty = \frac{1+\mu}{1-\mu}\sqrt{\frac{2h_0}{g}}$. Clearly, $\tilde{J}(z, y)$ has Lipschitz constant $\mu < 1$ and $\tilde{J}_\infty(z, y) = 0$ for any $z, y \in \mathbb{R}$. That is, $\varphi = 0$. Hence, conditions (C1) and (C2) are satisfied. Also, the functions $F(z, y, t) = -g$, $f(z, y, t) = z$, and $I(z, y) = -(1+\mu)z$ are continuously differentiable for any $z \in \mathbb{R}$, $F(z, y, t)$ and $f(z, y, t)$ are bounded in a finite domain H. It implies that conditions (C3) and (C4) hold as well. Take $\mu = 0$ in (2) and $\bar{y}(\theta_0) = 0$, Then

$$\begin{aligned} \dot{\bar{z}} &= -g, \\ \dot{\bar{y}} &= \bar{z}, \\ \bar{y}(\theta_0) &= 0, \quad \bar{z}(\theta_0) = 0. \end{aligned} \tag{28}$$

which is the degenerate system of (2). One should emphasize that the last system is considered as a model of the ball over the table, which is placed on the level $y = 0$. That is, the motion with zero initial values is an equilibrium, since the table is an obstacle for the ball to fall. As a result, the system has the solution $\bar{z}(t) = 0, \bar{y}(t) = 0$. Conditions of Theorem 1 are satisfied. Therefore, solutions of (2) with initial value $(z(\theta_0, \mu), y(\theta_0, \mu))$ tend to solutions of (28) as $\mu \to 0$ on the interval $[\theta_0, T]$. Moreover, the sequence $d_i(\mu) = \mu^i\sqrt{\frac{8h_0}{g}}$ is described and the interval $[\theta_0, \theta_\infty]$ shrinks to the single point θ_0 as $\mu \to 0$. This interval is the boundary layer.

Solution of (2) on the interval $[0.4515, 4]$ has the initial value $(z(0.4515, \mu), y(0.4515, \mu)) = (-4.429, 0)$. It is difficult to simulate the solution for small value of μ. Hence, we demonstrate the singularity by handmade picture and the coordinate $z(t, \mu)$ on the interval $[0.4515, 2]$ look likes in Fig. 12 and the limit position of the coordinate as $\mu \to 0$ is pictured in Fig. 13.

It is useful to consider degenerate model for small value of μ according to Theorem 1. One can see from Figs. 12 and 13 that the $z(t, \mu)$ coordinate of solution of (2) converge to the function in Fig. 13. However, the $\bar{z}(t)$ coordinate of the solution of (28) graphically is represented just by a line. As a result, the convergence is not uniform and it is a singular problem. However, approximation opportunity is approved by Theorem 1.

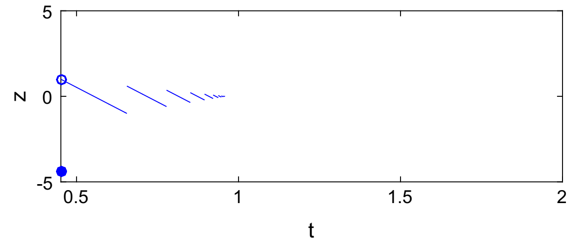

Fig. 12 A sketch of the coordinate $z(t, \mu)$ of the solution of (2) on $[\theta_0, 2]$, where $z(\theta_0, \mu) = -4.429$ and $\theta_0 = 0.4515$ for a small value of μ

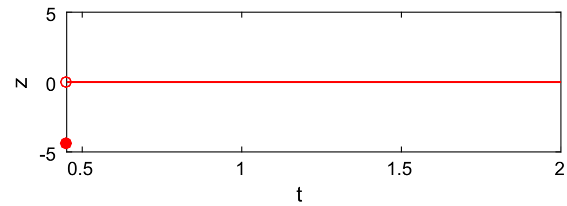

Fig. 13 The limit of the coordinate $z(t, \mu)$ of the solution of (2) on $[\theta_0, 4]$, where $z(\theta_0, \mu) = -4.429$ and $\theta_0 = 0.4515$. This figure shows that the uniform convergence of the coordinate z fails at θ_0

The chattering problem is difficult to solve explicitly. Theorem 1 says that if the parameter μ is sufficiently small, then we can accept the solution of non-perturbed system (28) as an approximate solution of system (2).

Now, we find an asymptotic approximation for the bouncing ball. In the notation of Sect. 5, the solution of system (2) on $[0.4515, 4]$ with $(z(0.4515, \mu), y(0.4515, \mu)) = (-4.429, 0)$ has the following asymptotic representation

$$z(t, \mu) = \begin{cases} z_m(t) & \text{if} \quad 0 \le t \le \theta_{m+1}(\mu), \\ -\tilde{J}_m(-4.429, 0) + & \\ +\varepsilon_1(t, \mu) & \text{if} \quad \theta_{m+1}(\mu) < t \le 4, \end{cases}$$

$$y(t, \mu) = \begin{cases} y_m(t) & \text{if} \quad 0 \le t \le \theta_{m+1}(\mu), \\ \tilde{\varepsilon}_1(t, \mu) & \text{if} \quad \theta_{m+1}(\mu) < t \le 4, \end{cases}$$

where $\varepsilon_1(t,\mu) \to 0$ and $\tilde{\varepsilon}_1(t,\mu) \to 0$ as $\mu \to 0$, since $(\bar{z}(t), \bar{y}(t)) = (0,0)$ is the solution of degenerate equation (28), $(z_m(t), y_m(t))$ is the solution of

$$\begin{aligned} &\frac{\mathrm{d}z_m}{\mathrm{d}t} = -g, \quad \Delta z_m|_{t=\theta_i} = -(1+\mu)z_m, \\ &\frac{\mathrm{d}y_m}{\mathrm{d}t} = z_m, \end{aligned} \tag{29}$$

with initial condition $(z_m(0), y_m(0)) = (-4.429, 0)$, $1 \le i \le m$. We know from Sect. 5 that on the interval $[0, \theta_{m+1}]$ the solution and the asymptotic approximation coincide. Moreover, $(z_m(t), y_m(t)) = (0,0)$ for $t \in (\theta_{m+1}, 4]$ since there is no impact moments in this interval, i.e., the ball stays on the surface. Therefore, the errors on the interval $(\theta_{m+1}, 4]$ are

$$\begin{aligned} \|\varepsilon_1(t,\mu)\| &= \|z(t,\mu) + \tilde{J}_m(-4.429, 0)\| \\ &= \|z(t,\mu) + 4.429(-1)^{m+1}\mu^m\| \\ &\le \|z(t,\mu)\| + 4.429\mu^m \\ &\le 8.878\mu^m \end{aligned}$$

and

$$\|\tilde{\varepsilon}_1(t,\mu)\| \le \mu^{2m}\frac{4.429^2}{2g}.$$

6.2 An inverted pendulum

Now, let us discuss the inverted pendulum model. Consider again system (4) on [9.205, 15] since the fist impact moment is 9.205 and the solution is not affected by the parameter μ on [0, 9.205]. Let $\tilde{J}(z,y) = z - (1+\mu)z = -\mu z$. Obviously, $\tilde{J}(z,y)$ has Lipschitz constant $\mu < 1$ and $\tilde{J}_\infty(z,y) = 0$ for any $z, y \in \mathbb{R}$, i.e., $\varphi = 0$. Let us consider the restitution coefficient as $\mu = 0$ in (4) with the initial condition $(\bar{z}(9.205), \bar{y}(9.205)) = (0,1)$.

$$\begin{aligned} &\dot{\bar{z}} = 0.01\bar{z} + \bar{y} + 0.001\sin(5t), \\ &\dot{\bar{y}} = \bar{z}, \\ &\bar{z}(9.205) = 0, \bar{y}(9.205) = 1. \end{aligned} \tag{30}$$

This system is the degenerate equations of (4). It means that at the position $y = 1$ the pendulum has zero velocity. Therefore, it admits the equilibrium solution $(\bar{z}(t), \bar{y}(t)) = (0,1)$. Obviously, conditions (C1)–(C4) of Theorem 1 are satisfied. Therefore, solutions $z(t,\mu)$, $y(t,\mu)$ of (4) with initial $(z(9.205,\mu), y(9.205,\mu)) = (1,1)$ tend to solutions $\bar{z}(t)$, $\bar{y}(t)$ of (30) with initial $(\bar{z}(9.205), \bar{y}(9.205)) = (0,1)$ as $\mu \to$ 0. Note that the convergence of $z(t,\mu) \to \bar{z}(t)$ is not uniform on [9.205, 15].

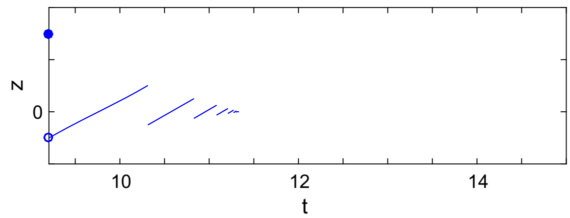

Fig. 14 A sketch of the coordinate $z(t,\mu)$ of the solution of (4) on [9.205, 15] for a small value of μ, where $z(9.205,\mu) = 1$ and 9.205 is the first impact moment of system (4)

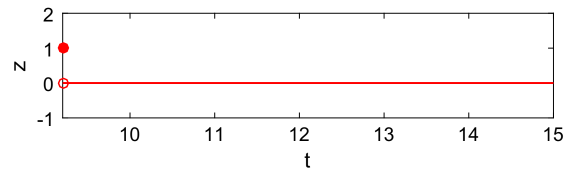

Fig. 15 The limit of the coordinate $z(t,\mu)$ of the solution of (4) with initial value $(z(9.205,\mu), y(9.205,\mu)) = (1,1)$ on [9.205, 15] as $\mu \to 0$

The solution of (4) on the interval [9.205, 15] has the initial value $(z(9.205,\mu), y(9.205,\mu)) = (1,1)$. We represent the $z(t,\mu)$ coordinate of the solution for small value of μ on the interval [9.205, 15] in Fig. 14. Moreover, the limit position of the coordinate $z(t,\mu)$ as $\mu \to 0$ is demonstrated in Fig. 15. It can be seen that the convergence is not uniform on the interval [9.205, 15]. So, it is a singularly perturbed problem.

6.3 A hydraulic relief valve

Our third model is the hydraulic relief valve. Consider the system (5) on the interval [1.038, 4] on which the parameter μ has an effect on the solutions. Take $\mu = 0$ in (5). Then, we obtain

$$\begin{aligned} &\dot{\bar{y}}_1 = \bar{y}_2, \\ &\dot{\bar{y}}_2 = -\kappa\bar{y}_2 - (\bar{y}_1 + \delta) + \bar{y}_3, \\ &\dot{\bar{y}}_3 = \beta(q - \bar{y}_1\sqrt{\bar{y}_3}). \end{aligned} \tag{31}$$

Define $y = (y_1, y_3)$ and $\tilde{J}(y_2, y) = y_2 + I(y_2, y) = -\mu y_2$, where $I(y_2, y) = -(1+\mu)y_2$. Clearly, $\tilde{J}_\infty(y_2, y) = 0$, for any $y_1, y_2, y_3 \in \mathbb{R}$, since $0 < \mu < 1$. Also, $\tilde{J}(y_2, y)$ has a Lipschitz constant $\mu < 1$, i.e., $\varphi = 0$. For this model take the initial values $\bar{y}_1(1.038) = 0$, $\bar{y}_2(1.038) = 0$, $\bar{y}_3(1.038) = 0$.

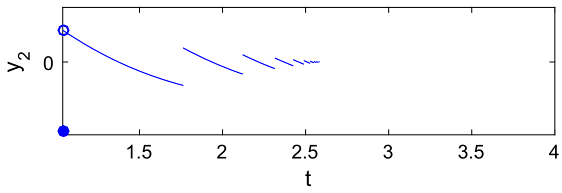

Fig. 16 A sketch of the coordinate $y_2(t,\mu)$ of the solution of (5) on [1.038, 4] for a small value of μ, where $z(1.038,\mu) = -14.54$, and 1.038 is the first impact moment of system (5)

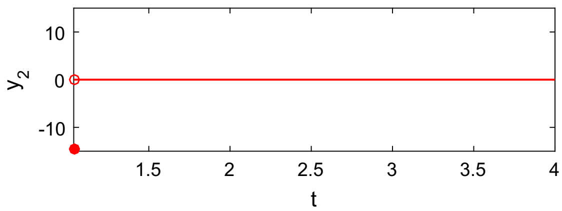

Fig. 17 The limit of the coordinate $y_2(t,\mu)$ of the solution of (5) with initial values $(y_1(1.038,\mu), y_2(1.038,\mu), y_3(1.038,\mu)) = (0, -14.54, 0)$ on the interval [1.038, 4] as $\mu \to 0$

Hence, system (31) with these initials is the degenerate equations of (5) with initials $y_1(1.038,\mu) = 0$, $y_2(1.038,\mu) = -14.54$, $y_3(1.038,\mu) = 0$. Conditions of Theorem 1 are satisfied. Therefore, in (5), if we choose $\beta = 20$, $q = 0.3$, $\kappa = 1.25$, $\delta = 20$, then solutions $y_1(t,\mu)$, $y_2(t,\mu)$, $y_3(t,\mu)$ of (5) with the given initial conditions tend to solutions $\bar{y}_1(t)$, $\bar{y}_2(t)$, $\bar{y}_3(t)$ of (31) as $\mu \to 0$ on the interval [1.038, 4].

Similar to the previous mechanical models, we represent the $z(t,\mu)$ coordinate of the solution for a small value of μ on the interval [1.038, 4] in Fig. 16. In addition, the limit position of the coordinate y_2 as $\mu \to 0$ is shown in Fig. 17. One can figure out that the convergence of the coordinate $y_2(t,\mu)$ to $\bar{y}_2(t)$ is not uniform on the interval [1.038, 4], which implies that this is a singularly perturbed problem as well.

6.4 A spring-mass system with a small mass

Now, we study a model for Theorem 2. Consider a small mass connected to a spring with a coefficient k. Assume that the surface on which mass is placed has no friction. It is released from a position x_0 without initial velocity and it moves onto a smooth vertical surface. During the process, we assume that the mass is subjected only to the spring's coefficient. The mathematical model of this problem is as follows (Fig. 18).

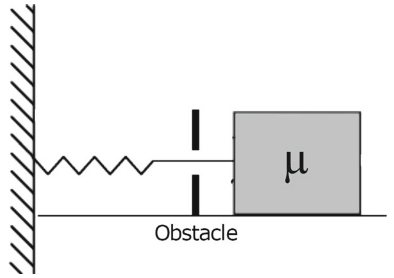

Fig. 18 A Spring-mass system with an obstacle

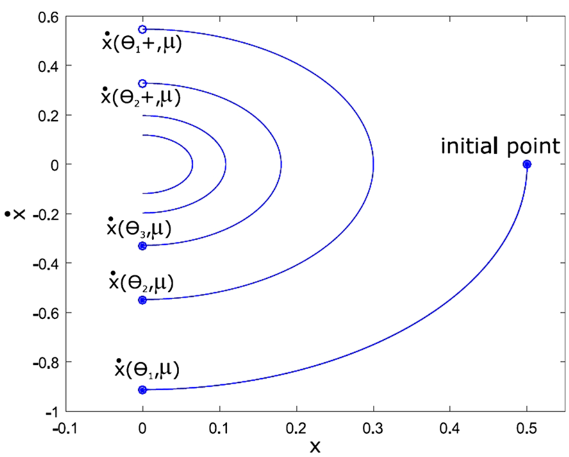

Fig. 19 The solution of (35) with 5 impacts and with the initial (0.5, 0), where $k = 2$, $\mu = 0.6$

$$\mu\ddot{x} + kx = 0, \tag{32}$$

$$\Delta\dot{x}|_{x=0} = -(1+\mu)\dot{x}, \tag{33}$$

where k is the spring coefficient, μ is the mass as well as the restitution coefficient. From the Eq. (33), one obtains

$$\dot{x}^+ = -\mu\dot{x}^- \text{ if } x = 0. \tag{34}$$

Fix a small $\mu > 0$. One can find from the simple calculation that any solution $x(t,\mu)$ of (32) and (34) with initial value $x(0,\mu) = x_0, \dot{x}(0,\mu) = 0$ is chattering and the moments of impacts depend on μ. A simulation of this solution with $x(0,\mu) = 0.5, \dot{x}(0,\mu) = 0$ is presented in Fig. 19. Consequently, one can obtain that the solution satisfies the system

$$\begin{aligned} &\mu\dot{z} = -ky, \quad \Delta z|_{t=\theta_i(\mu)} = -(1+\mu)z, \\ &\dot{y} = z, \end{aligned} \tag{35}$$

where $y = x$, $z = \dot{x}$. It is easy to verify that system (35) satisfies the conditions of Theorem 2, and the degenerate system (14) admits the form

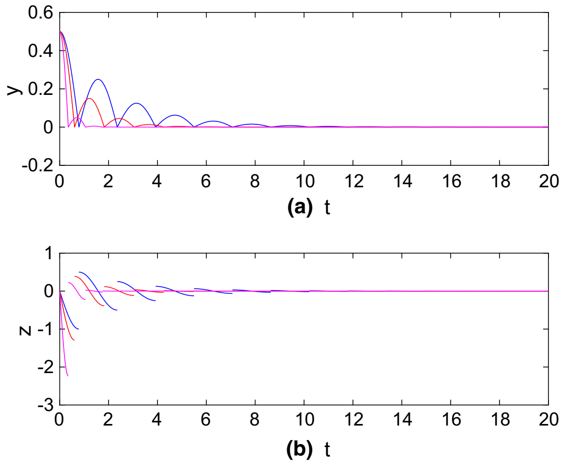

Fig. 20 The coordinates of (32) and (34) with the initial value (0.5, 0) where $k = 2$. Blue, red and magenta lines represent the coordinates for $\mu = 0.5$, $\mu = 0.3$ and $\mu = 0.1$, respectively. (Color figure online)

$$\begin{aligned} 0 &= -k\bar{y}, \\ \dot{\bar{y}} &= \bar{z}, \end{aligned} \tag{36}$$

such that (0, 0) is the solution of this problem. It is asymptotically stable as the mass is pressed to the wall motionless. Thus, one can observe the consequences of the singular perturbation through the simulations. They are presented in Fig. 20. It is important to say that, in Fig. 20a, the limit process is very similar to the that one can see in simulations of continuous dynamics with singular perturbation. This confirms one more time that the singular problem is under discussion.

7 Conclusion

This article has discussed the chattering through the singularity point of view. The chattering property is known as the appearance of an infinite number of impact moments occurring in a finite time. The singularity of impact moments has been introduced. The chattering problem has been discussed as a singularly perturbed problem. Three important mechanical models; a bouncing ball, an inverted pendulum and a hydraulic relief valve models have been studied for the chattering problem as a singular one. Additionally, the spring-mass system with the small mass and with the chattering solution has been discussed as a singularly perturbed problem. Models with chattering are sophisticated for analysis because of infinitely many impact moments accumulated in finite time. The result of this article which formulates the chattering as a singular problem will help researchers to consider degenerate systems without chattering to approximate initial models and reducing the complexity. We consider the system with singularity in impact moments as well as the system with both singularity in impact moments and small parameter multiplying the derivative.

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References

1. Nayfeh, A.H., Chin, C.-M., Pratt, J.: Perturbation methods in nonlinear dynamics—applications to machining dynamics. J. Manuf. Sci. Eng. **119**(4A), 485 (1997)
2. Zhao, X., Dankowicz, H., Reddy, C.K., Nayfeh, A.H.: Modeling and simulation methodology for impact microactuators. J. Micromech. Microeng. **14**(6), 775 (2004)
3. Zhao, X., Reddy, C.K., Nayfeh, A.H.: Nonlinear dynamics of an electrically driven impact microactuator. Nonlinear Dyn. **40**(3), 227–239 (2005)
4. Nagaev, R.F., Kremer, E.B.: Mechanical Processes with Repeated Attenuated Impacts. World Scientific, Singapore (1999)
5. Budd, C., Dux, F.: Chattering and related behaviour in impact oscillators. Philos. Trans. Phys. Sci. Eng. **347**(1683), 365–389 (1994)
6. Chillingworth, D.R.J.: Dynamics of an impact oscillator near a degenerate graze. Nonlinearity **23**(11), 2723 (2010)
7. Giusepponi, S., Marchesoni, F., Borromeo, M.: Randomness in the bouncing ball dynamics. Phys. A **351**(1), 142–158 (2005)
8. Nordmark, A.B., Piiroinen, P.: Simulation and stability analysis of impacting systems with complete chattering. Nonlinear Dyn. **58**(1–2), 85–106 (2009)
9. Akhmet, M., Çağ, S.: Analysis of impact chattering. Miskolc Math. Notes **17**(2), 707–721 (2016)
10. Arkhipova, I.M., Luongo, A., Seyranian, A.P.: Vibrational stabilization of the upright statically unstable position of a double pendulum. J. Sound Vib. **331**(2), 457–469 (2012)
11. Demeio, L., Lenci, S.: Asymptotic analysis of chattering oscillations for an impacting inverted pendulum. Q. J. Mech. Appl. Math. **59**(3), 419–434 (2006)
12. Falcon, E., Laroche, C., Fauve, S., Coste, C.: Behavior of one inelastic ball bouncing repeatedly off the ground. Eur. Phys. J. B **3**(1), 45–57 (1998)
13. Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Applied Mathematical Sciences. Springer, Berlin (1990)
14. McNamara, S., Young, W.: Inelastic collapse and clumping in a one-dimensional granular medium. Phys. Fluids A **4**, 496–504 (1992)

15. Hös, C., Champneys, A.R.: Grazing bifurcations and chatter in a pressure relief valve model. Phys. D **241**(22), 2068–2076 (2012)
16. Nayfeh, A.H.: Perturbation Methods. Wiley, Hoboken (2004)
17. Luongo, A., Casciati, S., Zulli, D.: Perturbation method for the dynamic analysis of a bistable oscillator under slow harmonic excitation. Smart Struct. Syst. **18**(1), 183–196 (2016)
18. Damiano, E.R., Rabbitt, R.D.: A singular perturbation model of fluid dynamics in the vestibular semicircular canal and ampulla. J. Fluid Mech. **307**, 333–372 (1996)
19. Gondal, I.: On the application of singular perturbation techniques to nuclear engineering control problems. IEEE Trans. Nucl. Sci. **35**, 1080–1085 (1988)
20. Hek, G.: Geometric singular perturbation theory in biological practice. J. Math. Biol. **60**, 347–386 (2010)
21. Kokotovic, P.V.: Applications of singular perturbation techniques to control problems. SIAM Rev. **26**, 501–550 (1984)
22. Michaelis, L., Menton, M.L.: Die kinetik der invertinwirkung. Biochem. Z. **49**, 333–69 (1913)
23. Owen, M.R., Lewis, M.A.: How predation can slow, stop, or reverse a prey invasion. Bull. Math. Biol. **63**, 655–684 (2001)
24. Segel, L.A., Slemrod, M.: The quasi-steady state assumption: a case study in perturbation. SIAM Rev. **31**, 446–477 (1989)
25. Akhmet, M.: Principles of Discontinuous Dynamical Systems. Springer, New York(2010)
26. Akhmet, M.: Nonlinear Hybrid Continuous/Discrete-Time Models. Atlantis Studies in Mathematics for Engineering and Science. Atlantis Press, Paris (2011)
27. Akhmet, M., Fen, M.: Replication of Chaos in Neural Networks, Economics and Physics. Nonlinear Physical Science. Springer, Berlin (2015)
28. Chen, W.-H., Chen, F., Lu, X.: Exponential stability of a class of singularly perturbed stochastic time-delay systems with impulse effect. Nonlinear Anal. Real World Appl. **11**(5), 3463–3478 (2010)
29. Chen, W.-H., Wei, D., Lu, X.: Exponential stability of a class of nonlinear singularly perturbed systems with delayed impulses. J. Frankl. Inst. **350**(9), 2678–2709 (2013)
30. Chen, W.-H., Yuan, G., Zheng, W.X.: Robust stability of singularly perturbed impulsive systems under nonlinear perturbation. IEEE Trans. Autom. Control **58**, 168–174 (2013)
31. Simeonov, P., Bainov, D.: Stability of the solutions of singularly perturbed systems with impulse effect. J. Math. Anal. Appl. **136**(2), 575–588 (1988)
32. Simeonov, P., Bainov, D.: Exponential stability of the solutions of singularly perturbed systems with impulse effect. J. Math. Anal. Appl. **151**(2), 462–487 (1990)
33. Vasil'eva, A., Butuzov, V., Kalachev, L.: *The Boundary Function Method for Singular Perturbed Problems. Studies in Applied Mathematics*. Society for industrial and applied mathematics. SIAM, Philadelphia (1995)
34. O'Malley, R.E.J.: Singular Perturbation Methods for Ordinary Differential Equations. Applied Mathematical Sciences. Springer, New York (1991)