



Controllability of two-point nonlinear boundary-value problems by the numerical-analytic method

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Abstract

By employing a numerical-analytic method, we establish sufficient conditions for the controllability of systems

$$\begin{aligned}\frac{dy}{dt} &= A(t)y + B(t)u + g(t) + f(t, y, z, u) \\ \frac{dz}{dt} &= F(t, y, z, u),\end{aligned}$$

with boundary conditions of the form

$$y(0) = a, \quad y(T) = c, \quad z(0) = b, \quad z(T) = d$$

or

$$y(0) = a, \quad y(T) = c, \quad Q_0z(0) + Q_1z(T) = b,$$

where all functions involved are continuous in their domain of definition, T is a fixed real number, Q_j , $j = 0, 1$, are constant matrices, $y \in R^n$, $z \in R^k$, and $u \in R^m$.

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1. Introduction

In this paper we are concerned with the controllability of nonlinear differential equations of the form

$$\begin{aligned} \frac{dy}{dt} &= A(t)y + B(t)u + g(t) + f(t, y, z, u) \\ \frac{dz}{dt} &= F(t, y, z, u), \end{aligned} \quad (1.1)$$

subject to boundary conditions

$$y(0) = a, \quad y(T) = c, \quad z(0) = b, \quad z(T) = d \quad (1.2)$$

or

$$y(0) = a, \quad y(T) = c, \quad Q_0z(0) + Q_1z(T) = b, \quad (1.3)$$

where T is a fixed real number, Q_j , $j = 0, 1$, are constant matrices, $y \in R^n$, $z \in R^k$, and $u \in R^m$. It is assumed that all functions involved are continuous in their domain of definition.

The problems (1.1) and (1.2) is to be referred to as problem γ_1 , and (1.1) and (1.3) as problem γ_2 . As usual, we say that γ_i is solvable if there is a control function u for which the problem admits a solution.

The controllability of boundary-value problems were investigated by many authors, see for instance [1–8] and the references cited therein. A summary and discussion of the methods employed for studying the controllability of nonlinear systems can be found in [1].

To the best of our knowledge the controllability of systems of the form (1.1) have been rarely considered in the literature due to a difficulty caused by the nonlinearity of the function F . In fact, even if F can be linearized, the combined system might only be investigated under some conditions not generally fulfilled. For instance, condition (2.1) may not be satisfied for the whole system. The advantage of our method lies in the fact that we need only one of the equations to be quasilinear, which surely suffices to construct a control function u necessary to deal with the problem. Our technique is based on a numerical-analytic method introduced in [9] and on their derivatives used in [10–13]. It is worth mentioning that the numerical-analytic method have certain similarities with many of the other methods available for investigating boundary-value problems, see for example [14–18]. It is therefore natural expect that the method can be employed to investigate a wide range of boundary-value problems of different nature such as the one in this work.

2. Preliminaries

Let G be a set defined by

$$G = \{(t, y, z, u) | 0 \leq t \leq T, (y, z, u) \in G_y \times G_z \times G_u\},$$

where $G_y \times G_z \times G_u$ is a compact subset of $R^n \times R^k \times R^m$. If f is a function defined on G , then by $\|f\|_0$ we shall mean $\|f\|_0 = \max_G \|f\|$, where $\|f\|$ denotes the Euclidean norm of f .

Let $Y(t)$, $Y(0) = I$, be the matrix solution of $(dy/dt) = A(t)y$, and $S(t)$ and $V(t)$ be matrices defined by

$$S(t) = Y^{-1}(t)B(t)$$

and

$$V(t) = \int_0^t S(\tau)S^T(\tau) d\tau.$$

Assuming that

$$\det V(T) \neq 0 \tag{2.1}$$

we set

$$\begin{aligned} M_1 &= \|f\|_0, \\ M_2 &= \|F\|_0, \\ M_3 &= \max\{M_{31}, M_{32}, M_{33}\}, \end{aligned}$$

where

$$\begin{aligned} M_{31} &= \max_{t,\tau} \|S^T(t)V^{-1}(T)Y^{-1}(\tau)\|, \\ M_{32} &= \max_{t,\tau} \|Y(t)Y^{-1}(\tau)\|, \\ M_{33} &= \max_{t,\tau} \|Y(t)V(t)V^{-1}(T)Y^{-1}(\tau)\|. \end{aligned}$$

We also assume that there exist positive real numbers ℓ_1 and ℓ_2 such that for all (t, y_1, z_1, u_1) and $(t, y_2, z_2, u_2) \in G$,

$$\|f(t, y_1, z_1, u_1) - f(t, y_2, z_2, u_2)\| \leq \ell_1(\|y_1 - y_2\| + \|z_1 - z_2\| + \|u_1 - u_2\|)$$

and

$$\|F(t, y_1, z_1, u_1) - F(t, y_2, z_2, u_2)\| \leq \ell_2(\|y_1 - y_2\| + \|z_1 - z_2\| + \|u_1 - u_2\|).$$

Moreover, we require that

$$3\ell_1 M_3 + \ell_2 \left[\frac{3\ell_1 M_3 T}{2} + \frac{\ell_2 T}{3} \right] < 1. \tag{2.2}$$

Finally, we give the following two lemmas which we rely on. Although these lemmas can be found in [9] we give their proofs here for convenience. The lemmas are concerned with the function $\alpha(t)$ defined by

$$\alpha(t) = 2t \left(1 - \frac{t}{T}\right), \quad t \in [0, T].$$

Lemma 2.1. *If $\varphi(t)$ is a continuous function defined on $[0, T]$, then*

$$\left\| \int_0^t \left[\varphi(s) - \frac{1}{T} \int_0^T \varphi(\xi) d\xi \right] ds \right\| \leq \alpha(t) \|\varphi\|_0$$

for all $t \in [0, T]$.

Proof. Since

$$\int_0^t \left[\varphi(s) - \frac{1}{T} \int_0^T \varphi(\xi) d\xi \right] ds = \left(1 - \frac{t}{T}\right) \int_0^t \varphi(s) ds - \frac{t}{T} \int_t^T \varphi(s) ds,$$

we see that

$$\begin{aligned} & \left\| \int_0^t \left[\varphi(s) - \frac{1}{T} \int_0^T \varphi(\xi) d\xi \right] ds \right\| \\ & \leq \left(1 - \frac{t}{T}\right) \int_0^t \|\varphi\|_0 ds + \frac{t}{T} \int_t^T \|\varphi\|_0 ds. \quad \square \end{aligned}$$

Lemma 2.2. *It is true that*

$$\left(1 - \frac{t}{T}\right) \int_0^t \alpha(s) ds + \frac{t}{T} \int_t^T \alpha(s) ds \leq \frac{T}{3} \alpha(t)$$

for all $t \in [0, T]$.

Proof. Evaluating the integrals, we have

$$\left(1 - \frac{t}{T}\right) \int_0^t \alpha(s) ds + \frac{t}{T} \int_t^T \alpha(s) ds = \alpha(t) \left[\frac{T}{6} + \frac{\alpha(t)}{3} \right].$$

Since $\alpha(t) \leq \frac{T}{2}$ for all $t \in [0, T]$, the proof is complete. \square

In what follows, we fix $u^0 \in L_2(0, T)$ such that

$$\int_0^T S(t)u^0(t) dt = 0. \tag{2.3}$$

3. Solvability of problem γ_1

In this section we investigate the solvability problem of solutions of (1.1) satisfying (1.2).

For our purpose we let

$$\begin{aligned} u_0(t) &= S^T(t)V^{-1}(T) \left[Y^{-1}(T)c - a - \int_0^T Y^{-1}(t)g(t) dt \right] + u^0(t), \\ y_0(t) &= Y(t) \left[a + \int_0^t S(\tau)u_0(\tau) d\tau + \int_0^t Y^{-1}(\tau)g(\tau) d\tau \right], \\ z_0(t) &= b + \frac{t}{T}(d - b) \end{aligned}$$

and denote by $G(b)$ the set of points (t, y, z, u) such that $t \in [0, T]$ and

$$\begin{aligned} \|y - y_0(t)\| &< 2M_3M_1T, \\ \|z - z_0(t)\| &< \frac{M_2T}{2}, \\ \|u - u_0(t)\| &< M_3M_1T. \end{aligned}$$

Our first theorem asserts that if γ_1 is solvable, then under some reasonable conditions its solution can be written as a uniform limit of a certain sequence.

Theorem 3.1. *Suppose that (2.1)–(2.3) hold that*

$$G_z^0 = \{z_0 \in G_z | G(z_0) \subset G\} \text{ is not empty.} \tag{3.1}$$

If γ_1 is solvable and $\psi(t) = (\varphi(t), w(t), v(t))$ denotes its solution, then $\psi(t)$ is the uniform limit of the sequence $\{\psi_n(t)\} = \{(y_n(t), z_n(t), u_n(t))\}$, where

$$\begin{aligned} y_{n+1}(t) &= y_0(t) + Y(t) \left[\int_0^t Y^{-1}(\tau)f(\tau, \psi_n(\tau)) d\tau \right. \\ &\quad \left. - V(t)V^{-1}(T) \int_0^T Y^{-1}(\tau)f(\tau, \psi_n(\tau)) d\tau \right], \\ z_{n+1}(t) &= z_0(t) + \int_0^t \left[F(\tau, \psi_n(\tau)) - \frac{1}{T} \int_0^T F(s, \psi_n(s)) ds \right] d\tau, \\ u_{n+1}(t) &= u_0(t) - S^T(t)V^{-1}(T) \int_0^T Y^{-1}(\tau)f(\tau, \psi_n(\tau)) d\tau. \end{aligned} \tag{3.2}$$

Proof. We shall first show that the sequence $\{\psi_n(t)\}$ converges uniformly for $t \in [0, T]$. It is easy to see directly that

$$\|u_n(t) - u_0(t)\| \leq M_3M_1T$$

and

$$\|y_n(t) - y_0(t)\| \leq 2M_3M_1T.$$

By using Lemma 2.1 we also have

$$\|z_{n+1}(t) - z_0(t)\| \leq M_2T/2.$$

Therefore the sequence $\{\psi_n(t)\}$ belongs to $G_y \times G_z \times G_u$ for t in $[0, T]$.

Define

$$\|\psi_n(t)\|_1 = \|y_n(t)\| + \|z_n(t)\| + \|u_n(t)\|.$$

In view of (3.2) and the definition of $\alpha(t)$, it follows that

$$\|\psi_1 - \psi_0\|_1 \leq 3M_3M_1T + M_2\alpha(t) = p_1\alpha(t) + q_1 \leq p_1T/2 + q_1,$$

where $p_1 = M_2$ and $q_1 = 3M_1M_3T$. Similarly,

$$\|\psi_2(t) - \psi_1(t)\|_1 \leq 3\ell_1M_3(p_1T/2 + q_1) + \ell_2q_1\alpha(t) + \ell_2p_1T/3 = p_2\alpha(t) + q_2,$$

where $p_2 = \ell_2q_1$ and $q_2 = (3\ell_1M_3T/2 + \ell_2T/3)p_1 + 3\ell_1M_3q_1$. In fact, by using mathematical induction we obtain that for $n \geq 1$,

$$\|\psi_{n+1}(t) - \psi_n(t)\| \leq p_{n+1}\alpha(t) + q_{n+1} \leq p_{n+1}T/2 + q_{n+1}, \quad (3.3)$$

where p_n and q_n satisfy the following equation

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = H \begin{bmatrix} p_n \\ q_n \end{bmatrix} \quad \text{for } n \geq 1$$

with

$$H = \begin{bmatrix} 0 & \ell_2 \\ \frac{3\ell_1M_3T}{2} + \frac{\ell_2T}{3} & 3\ell_1M_3 \end{bmatrix} \quad (3.4)$$

In view of (2.2), we see that the eigenvalues λ_1 and λ_2 of the matrix H are inside the unit circle, and moreover, $-1 < \lambda_2 < 0 < \lambda_1 < 1$, and $|\lambda_2| < \lambda_1$. If we denote $\lambda = \lambda_1$, then it follows from (3.4) that

$$\|(p_n, q_n)\| \leq \delta_0\lambda^{n-1} \quad (3.5)$$

for some positive δ_0 . We should note that δ_0 can be computed explicitly if desired.

In view of (3.5) we easily obtain from (3.3) that

$$\begin{aligned} \|\psi_{j+i}(t) - \psi_j\| &\leq \sum_{k=0}^{i-1} \|\psi_{j+k+1} - \psi_{j+k}\| \leq \sum_{k=0}^{i-1} (p_{j+k+1}T/2 + q_{j+k+1}) \\ &\leq \delta_0(1 + T^2/4)^{1/2} \sum_{k=0}^{i-1} \lambda^{j+k}, \end{aligned} \quad (3.6)$$

which surely implies that $\{\psi_n\}$ converges uniformly for $t \in [0, T]$ to $\psi_\infty = (y_\infty, z_\infty, u_\infty)$, say. By using (3.2) it is not difficult to verify that ψ_∞ satisfies the following system:

$$\begin{aligned}
 y(t) &= y_0(t) + Y(t) \left[\int_0^t Y^{-1}(\tau) f(\tau, \rho(\tau)) d\tau \right. \\
 &\quad \left. - V(t) V^{-1}(T) \int_0^T Y^{-1}(\tau) f(\tau, \rho(\tau)) d\tau \right], \\
 z(t) &= z_0(t) + \int_0^t \left[F(\tau, \rho(\tau)) - \frac{1}{T} \int_0^T F(s, \rho(s)) ds \right] d\tau, \\
 u(t) &= u_0(t) - S^T(t) V^{-1}(T) \int_0^T Y^{-1}(\tau) f(\tau, \rho(\tau)) d\tau,
 \end{aligned} \tag{3.7}$$

where $\rho = (y, z, u)$.

We claim that ψ_∞ also solves the BVP

$$\begin{aligned}
 \frac{dy}{dt} &= A(t)y + B(t)u + g(t) + f(t, \rho(t)), \\
 \frac{dz}{dt} &= F(t, \rho) + \frac{d-b}{T} - \frac{1}{T} \int_0^T F(t, \rho(t)) dt,
 \end{aligned} \tag{3.8}$$

$$y(0) = a, \quad y(T) = c, \quad z(0) = b, \quad z(T) = d \tag{3.9}$$

Indeed, by the Cauchy’s formula

$$y(t) = Y(t)a + Y(t) \int_0^t Y^{-1}(\tau) [B(\tau)u(\tau) + g(\tau) + f(\tau, \rho(\tau))] d\tau. \tag{3.10}$$

If we use $y(T) = c$ then we obtain

$$Y^{-1}(T)c - a = \int_0^T S(t)u(t) dt + \int_0^T Y^{-1}(t) [g(t) + f(t, \rho(t))] dt. \tag{3.11}$$

We may express the control u as

$$u(t) = S(t)e + u^0(t), \tag{3.12}$$

where $e \in R^m$ is a constant vector. Substituting $u(t)$ into (3.11) and solving for e results in

$$e = V^{-1}(T) \left\{ Y^{-1}(T)c - a - \int_0^T Y^{-1}(t)g(t) dt - \int_0^T Y^{-1}(t)f(t, \rho(t)) dt \right\}.$$

Therefore from (3.12) we obtain the last expression in (3.7). The first equation in (3.7) now easily follows from (3.10) by inserting $u(t)$. It is not also difficult to see that the second equation in (3.7) is satisfied.

Consider the function $\psi(t) = (\varphi(t), w(t), v(t))$ which, in view of the condition of theorem, is the solution of BVP (1.1), (1.2). Since

$$w(T) = w(0) + \int_0^T F(\tau, \psi(\tau)) d\tau,$$

we have

$$d - b = \int_0^T F(\tau, \psi(\tau)) d\tau \quad (3.13)$$

and consequently $\psi(t)$ is a solution of (3.8).

Finally, it is not difficult to see that

$$\|\psi_n - \psi\| \leq p_{n+1}\alpha(t) + q_{n+1} \leq p_{n+1}T/2 + q_{n+1},$$

and hence $\lim_{n \rightarrow \infty} \psi_n = \psi$. This completes the proof of the theorem. \square

Now let us consider system (1.1) with boundary condition

$$y(0) = a, \quad y(T) = c, \quad z(0) = z_0, \quad z(T) = d. \quad (3.14)$$

In view of (3.2) one can easily construct a sequence

$$\{\psi_n(t, z_0)\} = \{(y_n(t, z_0), z_n(t, z_0), u_n(t, z_0))\}$$

which converges a solution $\psi(t, z_0)$ of problem (3.8), (3.14).

This observation allows us to define an ϵ -approximate solution of γ_1 .

Definition 3.1. We shall say that $\psi(t, z_0)$ is an ϵ -approximate solution of γ_1 if for a given $\epsilon > 0$,

$$d - z_0 = \int_0^T F(s, \psi(s, z_0)) ds, \quad (3.15)$$

whenever

$$\|z_0 - b\| < \epsilon. \quad (3.16)$$

In view of Theorem 3.1 we may easily formulate the following theorem on the existence of an ϵ -approximate solution of γ_1 .

Theorem 3.2. Suppose that the conditions of Theorem 3.1 are all satisfied. Then there is an ϵ -approximate solution of problem γ_1 if and only if for some $z_0 \in B(b, \epsilon)$

$$A(z_0) \equiv d - z_0 - \int_0^T F(s, \psi(s, z_0)) ds = 0, \quad (3.17)$$

where $\psi(s, z_0)$ is the limit of approximations defined in (3.2) with $b = z_0$.

The following lemma shows that the mapping Δ defined by (3.17) is continuous.

Lemma 3.1. *Suppose that (2.1) and (2.3) hold. Then $\Delta(z_0)$ is continuous on G_z^0 as long as it exists.*

Proof. Let

$$\begin{aligned} \bar{M}_1 &= \max_t \{ \|S^T(t)V^{-1}(T)Y^{-1}(T)\| \}, \\ \bar{M}_2 &= \max_{t,\tau} \{ \|Y(t)S(\tau)S^T(\tau)V^{-1}(T)Y^{-1}(T)\| \}, \\ \bar{M} &= \max \{ \bar{M}_1, \bar{M}_2 \}, \\ d_0 &= \bar{M}(1 + T) + 1. \end{aligned}$$

In view of Lemmas 2.1 and 2.2 it follows that

$$\| \psi_0(t, z_0^1) - \psi_0(t, z_0^2) \| \leq d_0 \| z_0^1 - z_0^2 \| \quad \text{for } z_0^1, z_0^2 \in G_z^0.$$

It is not also difficult to show that for $n \geq 1$,

$$\| \psi_n(t, z_0^1) - \psi_n(t, z_0^2) \| \leq (d_0 + \bar{q}_n + \bar{p}_n T / 2) \| z_0^1 - z_0^2 \|, \tag{3.18}$$

where

$$\begin{bmatrix} \bar{p}_{n+1} \\ \bar{q}_{n+1} \end{bmatrix} = H \begin{bmatrix} \bar{p}_n \\ \bar{q}_n \end{bmatrix} + \begin{bmatrix} \bar{p}_1 \\ \bar{q}_1 \end{bmatrix} \quad \text{for } n \geq 1$$

with $\bar{p}_1 = d_0 \ell_2$, $\bar{q}_1 = 3\bar{M}T\ell_1 d_0$, and H as in (3.4).

On the other hand, it follows as in (3.5) that

$$\| (\bar{p}_{n+1}, \bar{q}_{n+1}) \| \leq \delta_0 \| (\bar{p}_1, \bar{q}_1) \| \sum_{i=1}^n \lambda^i.$$

Using this inequality, we see from (3.18) that

$$\| \psi_n(t, z_0^1) - \psi_n(t, z_0^2) \| \leq \left[d_0 + \delta_0 \| (\bar{p}_1, \bar{q}_1) \| (1 + T^2/4)^{1/2} \sum_{i=1}^n \lambda^i \right] \| z_0^1 - z_0^2 \|^2$$

and hence

$$\| \psi_\infty(t, z_0^1) - \psi_\infty(t, z_0^2) \| \leq \delta_1 \| z_0^1 - z_0^2 \|^2, \tag{3.19}$$

where

$$\delta_1 = d_0 + \frac{\delta_0 \| (\bar{p}_1, \bar{q}_1) \|}{1 - \lambda} (1 + T^2/4)^{1/2}.$$

In view of (3.17) and (3.19) we observe that

$$\|A(z_0^1) - A(z_0^2)\| \leq (1 + \ell_2 T \delta_1) \|z_0^1 - z_0^2\|.$$

This means that A is continuous. \square

Note that since the expression in (3.17) involves a limiting process it is impossible to find a zero of $A(z_0)$. To overcome this difficulty we will make use of a sequence of functions $\{A_j(z_0)\}$, defined by

$$A_j(z_0) = \int_0^T F(t, \psi_j(t, z_0)) dt + d - z_0. \quad (3.20)$$

The following definitions and lemmas, which are extracted from [19], are necessary for our purpose.

Definition 3.2. Let Ω be a bounded domain in R^k with boundary $\partial\Omega$. Let Px be a continuous vector field defined on $\partial\Omega$ without zeros. The degree of the continuous mapping $\|Px\|^{-1}Px$ from $\partial\Omega$ onto the unit sphere $\|x\| = 1$ in R^k , denoted by $\gamma(P; \partial\Omega)$, is the rotation of the field P on $\partial\Omega$.

Lemma 3.2. Let the continuous vector field P be defined on a closed domain $\Omega \cup \partial\Omega$. If $\gamma(P; \partial\Omega) \neq 0$, then the field P vanishes for at least one point in the domain Ω .

Definition 3.3. Two vector fields P_0 and P_1 are said to be homotopic on $\partial\Omega$ if there exists a continuous vector-function $P(t; x)$ defined for $0 \leq t \leq 1$ and $x \in \partial\Omega$ with values in R^k such that $P(0; x) \equiv P_0x$, $P(1; x) \equiv P_1x$, and that $P(t, x) \neq 0$, $(t, x) \in [0, 1] \times \partial\Omega$.

Lemma 3.3. Homotopic vector fields have the same rotation.

Definition 3.4. A vector field P , $Px = x - \Psi x$, defined on a Banach space E is said to be compact if Ψ is a compact operator in E .

Definition 3.5. Let Ω be a bounded domain in a Banach space E . A compact vector field defined on $\partial\Omega$ is said to be nondegenerate if it has no zeros in $\partial\Omega$.

If Ψ is a compact operator defined in a neighborhood of an isolated fixed point $x_0 \in E$, then it follows that the associated vector field P is nondegenerate on all spheres centered at x_0 and having sufficiently small radius r . It turns out that the field has the same rotation on all these spheres. This common rotation is known as the index of the fixed point x_0 and is denoted by $\gamma(x_0; \Psi)$.

Lemma 3.4. *The index $\gamma(x_0; \Psi)$ is equal to the rotation of the field P on the boundary of any bounded domain in which x_0 is the only fixed point of Ψ .*

Theorem 3.3. *In addition to the conditions of Theorem 3.1, suppose that for a given $\epsilon > 0$ there is a convex closed set $\Omega_\epsilon \subset B(b, \epsilon) \cap G_z^0$ such that for some $j \geq 1$ the map $\Delta_j : \Omega_\epsilon \rightarrow R^k$ has a unique singular point $z = z_j^0$ of nonzero index in Ω_ϵ , and*

$$\inf_{z \in \partial\Omega_\epsilon} \|\Delta_j(z)\| > \delta_0 \ell_2 (1 + T^2/4)^{1/2} \lambda^j (1 - \lambda)^{-1}. \tag{3.21}$$

Then problem γ_1 has an ϵ -approximate solution.

Proof. Let us first show that the vector fields Δ_j and Δ are homotopic on $\partial\Omega_\epsilon$. To see this we construct a vector function

$$V(s, z) = \Delta_j(z) + s(\Delta(z) - \Delta_j(z))$$

defined for $(s, z) \in [0, 1] \times \partial\Omega_\epsilon$. In view of Lemma 3.1 we see that V is continuous in s and z . On the other hand, we see from (3.6) that

$$\|\Delta(z) - \Delta_j(z)\| \leq \frac{\ell_2}{T} \int_0^T \|\psi_\infty - \psi_j\| ds \leq \delta_0 \ell_2 (1 + T^2/4)^{1/2} \lambda^j (1 - \lambda)^{-1} \tag{3.22}$$

and, hence, in view (3.21),

$$\|V(s, z)\| \geq \|\Delta_j(z)\| - \|\Delta_j(z) - \Delta(z)\| > 0.$$

Thus V does not vanish for any value of the parameter $s \in [0, 1]$.

Since the point $z_j^0 \in \Omega_\epsilon$ is the unique singular point of $\Delta_j(z)$, by Lemma 3.4 we may deduce that the rotation of the vector fields Δ_j on $\partial\Omega_\epsilon$ is not zero. It follows from Lemma 3.3 that the rotation of Δ on $\partial\Omega_\epsilon$ is not zero as well, and so by using Lemma 3.2 we see that the equation $\Delta(z) = 0$ has at least one solution $z = z^0$ in Ω_ϵ . This, in view of Theorem 3.2, completes the proof. \square

Remark 3.1. It should be noted that the verification of the assumption that for a given $\epsilon > 0$ there is a convex closed set $\Omega_\epsilon \subset B(b, \epsilon) \cap G_z^0$ such that for some $j \geq 1$ the map $\Delta_j : \Omega_\epsilon \rightarrow R^k$ has a unique singular point $z = z_j^0$ of nonzero index in Ω_ϵ requires defining an index of an isolated singular point. In case $k = 2$ this is no problem at all, see [19]. The case $k > 2$ is not so easy to handle. However, it is well known that if Δ_j is a topological map of a neighborhood of the singular point then the index is either -1 or $+1$, see [20], and if Δ_j is continuously differentiable with a nonzero Jacobian at z_j^0 then the index of z_j^0 cannot be zero.

Theorem 3.4. *In addition to the conditions of Theorem 3.1, suppose that for a given sequence $\{\epsilon_n\}$ such that $\epsilon_n > 0$ and $\lim_{n \rightarrow \infty} \epsilon_n = 0$, there exists a sequence $\{\Omega_n\}$ of closed and convex sets such that $\Omega_n \subset B(b, \epsilon_n) \cap G_z^0$ and for some $j_n \geq 1$ the map $\Delta_{j_n} : \Omega_n \rightarrow R^k$ has a unique singular point $z = z_n$ of nonzero index in Ω_n , and*

$$\inf_{z \in \partial\Omega_n} \|\Delta_{j_n}(z)\| > \delta_0 \ell_2 (1 + T^2/4)^{1/2} \lambda^{j_n} (1 - \lambda)^{-1}.$$

Then, problem γ_1 is solvable.

Proof. By Theorem 3.3 there is a sequence $\{z_n^0\}$, $n \geq 1$, such that $\Delta(z_n^0) = 0$. Letting $\epsilon_n \rightarrow 0$ we see that $z_n^0 \rightarrow b$ as $n \rightarrow \infty$. In view of Lemma 3.1 we may conclude that $\Delta(b) = 0$. \square

Theorem 3.5. *If problem γ_1 is solvable, then*

$$\|\Delta_j(z)\| \leq \delta_0 \ell_2 (1 + T^2/4)^{1/2} \lambda^j (1 - \lambda)^{-1} + (1 + \ell_2 \delta_1) \|z - b\|$$

for every $j \geq 1$ and $z \in G_z^0$.

Proof. By using (3.21), (3.22), and $\Delta(b) = 0$, we see that

$$\begin{aligned} \|\Delta_j(z)\| &= \|\Delta(z) + (\Delta_j(z) - \Delta(z))\| \leq \|\Delta_j(z) - \Delta(z)\| + \|\Delta(z) - \Delta(b)\| \\ &\leq \delta_0 \ell_2 (1 + T^2/4)^{1/2} \lambda^j (1 - \lambda)^{-1} + (1 + \ell_2 \delta_1) \|z - b\| \end{aligned}$$

for every $j \geq 1$ and $z \in G_z^0$. This completes the proof of the theorem. \square

The following contrapositive form of Theorem 3.5 is obvious.

Corollary 3.1. *If there exist $j \geq 1$ and $z \in G_z^0$ such that*

$$\|\Delta_j(z)\| > \delta_0 \ell_2 (1 + T^2/4)^{1/2} \lambda^j (1 - \lambda)^{-1} + (1 + \ell_2 \delta_1) \|z - b\|$$

then problem γ_1 is not solvable.

Remark 3.2. It is possible to define an ϵ -approximate solution of γ_1 with respect to $z(T) = d$ as was done with respect to $z(0) = b$. In that case, one can easily prove theorems similar to Theorems 3.2–3.5.

4. Solvability of problem γ_2

In the previous section the boundary conditions (1.2) did not allow us to obtain the exact solution of problem γ_1 . To overcome this deficiency we consider a mixed type boundary condition (1.3) with respect to the variable z . The key is to introduce a parameterization with parameter $z_0 = z(0)$. In that case, we are able to prove that problem γ_2 is solvable and the solution can be represented as a limit of a uniformly convergent sequence of functions.

We assume that

$$\det(k_0Q_0 + k_1Q_1) \neq 0 \quad \text{for some real numbers } k_0 \text{ and } k_1, \tag{4.1}$$

and let

$$\begin{aligned} u_0(t) &= S^T(t)V^{-1}(T) \left[Y^{-1}(T)c - a - \int_0^T Y^{-1}(t)g(t) dt \right] + u^0(t), \\ y_0(t) &= Y(t) \left[a + \int_0^t S(\tau)u_0(\tau)d\tau + \int_0^t Y^{-1}(\tau)g(\tau) d\tau \right], \\ z_0(t) &= z_0 + \left[k_0 + \frac{t}{T}(k_1 - k_0) \right] [Q_0k_0 + Q_1k_1]^{-1} [b - (Q_0 + Q_1)z_0]. \end{aligned}$$

Now we define a sequence $\{\psi_n\} = \{(y_n, z_n, u_n)\}$ as follows:

$$\begin{aligned} y_{n+1}(t) &= y_0(t) + Y(t) \left[\int_0^t Y^{-1}(\tau)f(\tau, \psi_n(\tau)) d\tau \right. \\ &\quad \left. - V(t)V^{-1}(T) \int_0^T Y^{-1}(\tau)f(\tau, \psi_n(\tau)) d\tau \right], \\ z_{n+1}(t) &= z_0(t) + \int_0^t \left[F(\tau, \psi_n(\tau)) - \frac{1}{T} \int_0^T F(s, \psi_n(s)) ds \right] d\tau, \\ u_{n+1}(t) &= u_0(t) - S^T(t)V^{-1}(T) \int_0^T Y^{-1}(\tau)f(\tau, \psi_n(\tau)) d\tau. \end{aligned}$$

It is easy to verify that the sequence $\{\psi_n\}$ satisfies the boundary conditions (1.3). We denote by $G(a, z_0)$ the set of points (y, z, u) which satisfy for each fixed $t \in [0, T]$ the following inequalities:

$$\begin{aligned} \|y - y_0(t)\| &\leq 2M_1M_3, \\ \|z - z_0 - k_0(k_0Q_0 + k_1Q_1)[b - (Q_0 + Q_1)z_0]\| \\ &\leq \frac{M_2T}{2} + \|(k_0 - k_1)(k_0Q_0 + k_1Q_1)^{-1}[b - (Q_0 + Q_1)z_0]\|, \\ \|u - u_0(t)\| &\leq M_1M_3T. \end{aligned}$$

Theorem 4.1. *Suppose that (2.1)–(2.3), and (4.1) are satisfied, and that*

$$G_y^0 \times G_z^0 = \{(a, z_0) \in G_y \times G_z | G(a, z_0) \subset G\} \quad \text{is not empty.} \tag{4.2}$$

Then for any point $(a, z_0) \in G_y^0 \times G_z^0$ the sequence $\{\psi_n\}$ converges uniformly to the function ψ_∞ satisfying

$$\begin{aligned}
 y(t) &= y_0(t) + Y(t) \left[\int_0^t Y^{-1}(\tau) f(\tau, \psi(\tau)) \, d\tau \right. \\
 &\quad \left. - V(t) V^{-1}(T) \int_0^T Y^{-1}(\tau) f(\tau, \psi_n(\tau)) \, d\tau \right], \\
 z(t) &= z_0(t) + \int_0^t \left[F(\tau, \psi_n(\tau)) - \frac{1}{T} \int_0^T F(s, \psi_n(s)) \, ds \right] \, d\tau, \\
 u(t) &= u_0(t) - S^T(t) V^{-1}(T) \int_0^T Y^{-1}(\tau) f(\tau, \psi_n(\tau)) \, d\tau.
 \end{aligned}$$

and (1.3), where $\psi = (y, z, u)$.

Moreover, ψ_∞ is a solution of the system

$$\begin{aligned}
 \frac{dy}{dt} &= A(t)y + B(t)u + g(t) + f(t, \psi(t)), \\
 \frac{dz}{dt} &= F(t, \psi(t)) + \tilde{\Delta}(z_0)
 \end{aligned}$$

where

$$\tilde{\Delta}(z_0) = \frac{1}{T} (k_1 - k_0) (k_0 Q_0 + k_1 Q_1)^{-1} [b - (Q_0 + Q_1) z_0] - \frac{1}{T} \int_0^T F(s, \psi(s)) \, ds.$$

Proof. Using Lemma 2.1 we see that

$$\begin{aligned}
 \|u_n(t) - u_0(t)\| &\leq M_1 M_3 T, \\
 \|y_n(t) - y_0(t)\| &\leq 2M_3 M_1 T(T + 1), \\
 \|z_n(t) - z_0 - k_0(k_0 Q_0 + k_1 Q_1)^{-1} [b - (Q_0 + Q_1) z_0]\| \\
 &\leq \frac{M_2 T}{2} + \|(k_0 - k_1)(k_0 Q_0 + k_1 Q_1)^{-1} [b - (Q_0 + Q_1) z_0]\|
 \end{aligned}$$

and therefore ψ_n , $t \in [0, T]$, is well defined.

It is also easy to see that

$$\|\psi_1 - \psi_0\| \leq 3M_3 M_1 T + M_2 \alpha(t) \leq p_1 \alpha(t) + q_1 \leq p_1 T/2 + q_1,$$

where $p_1 = M_2$, $q_1 = 3M_1 M_3 T$. In fact, by using mathematical induction one can show that

$$\|\psi_{n+1} - \psi_n\| \leq p_{n+1} \alpha(t) + q_{n+1} \leq p_{n+1} T/2 + q_{n+1}, \quad n \geq 1,$$

where $p_{n+1} = \ell_2 q_n$ and $q_{n+1} = (3\ell_1 M_3 T/2 + \ell_2 T/3) p_n + 3\ell_1 M_3 q_n$. The rest of the proof is similar to that of Theorem 3.1, and hence it is omitted. \square

Let λ be the positive eigenvalue of the matrix H in (3.4) and

$$\tilde{\Delta}_j(z) = \frac{1}{T} (k_1 - k_0) (k_0 Q_0 + k_1 Q_1)^{-1} [b - (Q_0 + Q_1) z_0] - \frac{1}{T} \int_0^T F(s, \psi_j(s)) \, ds.$$

Theorem 4.2. *In addition to the conditions of Theorem 3.1, suppose that there is a convex closed domain $\hat{G}_z \subseteq G_z^0$ such that for a fixed $j \in N$ the map $\tilde{\Delta}_j : \hat{G}_z \rightarrow R^k$ has a unique singular point $z = z_j^0$ in \hat{G}_z with nonzero index, and that*

$$\inf_{z \in \partial \hat{G}_z} \|\tilde{\Delta}_j(z)\| > \lambda^{-j}(1 - \lambda)^{-1} \ell_2(1 + T^2/4)^{1/2}.$$

Then problem γ_2 is solvable in $G_y^0 \times G_z^0$.

Proof. The proof is similar to that of Theorem 3.3 and hence is omitted. \square

In this paper we have considered two nonlinear boundary-value problems, namely problem γ_1 and problem γ_2 . The first problem contains separated boundary conditions and the second has a mixed type one. As we have indicated in Section 1, the system under investigation is not easy to handle because of the lack of a linear part. By using the numerical-analytic method we have been able to overcome this difficulty in the case when one of the equations has a linear part. In particular, we have proved that γ_1 has an ϵ -approximate solution and γ_2 is solvable. In both cases, the solutions are expressible as uniform limits of certain sequences. It is worth mentioning that each sequence is defined by successive approximations and therefore one can obtain approximate solutions if desired. We note that this is not a trivial problem and therefore it should be addressed under a separate work.

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