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Stability of the Zero Solution of Impulsive Differential Equations by the Lyapunov Second Method

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The paper is concerned with the stability of the zero solution of the impulsive system

$$\frac{dx}{dt} = f(t, x), \qquad t \neq \theta_i(x)$$

$$\Delta x|_{t=\theta_i(x)} = J_i(x), \qquad i \in N = \{1, 2, \dots\},$$

where $\Delta x|_{t=\theta} := x(\theta+) - x(\theta)$, $x(\theta+) = \lim_{t\to\theta^+} x(t)$. The Lyapunov second method is used as a tool in obtaining the criteria for stability, asymptotic stability, and instability of the trivial solution. © 2000 Academic Press

Key Words: stability; instability; Lyapunov's second method; impulse effect; variable moments.

1. INTRODUCTION AND PRELIMINARIES

In recent years there have been intensive studies on the qualitative behavior of solutions of impulsive differential equations; see for instance [8, 14] and the references cited therein. The theory of stability for impulsive differential equations has also been well developed during the past several years. To the best of our knowledge, the first article devoted to the Lyapunov second method for impulsive differential equations is due to Milman and Myshkis [13], where the authors considered the stability of the zero solution of differential equations with fixed moments of impulse points. Later, the Lyapunov second method was used for differential equations with impulses at variable times [4, 5].

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Although there are several papers dealing with the stability of solutions of impulsive equations with fixed moments of impulse effects (see [11] and the references cited therein), much less is known in the case of variable impulse actions since these equations exhibit more difficulties. One should even modify the stability definition of solutions of such equations since it is not possible to reduce the problem of stability of a nontrivial solution to that of the zero solution [1, 2, 10, 15]. No modification is necessary if only the zero solution is concerned.

In this paper, by employing the Lyapunov second method, we investigate the stability, asymptotic stability, and instability of the zero solution of a system of impulsive differential equations with impulse actions on surfaces of the form

$$\frac{dx}{dt} = f(t,x), \qquad t \neq \theta_i(x),$$
$$\Delta x|_{t=\theta_i(x)} = J_i(x), \qquad i \in N = \{1,2,\dots\}, \tag{1}$$

where $\Delta x|_{t=\theta} := x(\theta+) - x(\theta)$, $x(\theta+) = \lim_{t \to \theta+} x(t)$.

For our purpose we first introduce the notation

$$G = \{(t,x) : t \geq 0,\ x \in S_\rho\}, \qquad S_\rho = \{x \in R^n : \|x\| < \rho\},$$

where $\rho > 0$ is a fixed real number and $\|x\|$ denotes the euclidean norm of $x \in R^n$. Next, for each $i \in N$ we define $\theta_i^0 = \theta_i(0)$ and let

$$G_i = \{(t,x) \in G : t \text{ is between } \theta_i^0 \text{ and } \theta_i(x)\}.$$

With regard to (1) we assume that the following conditions are satisfied.

(a) $f(t,x)$: $R_+ \times S_\rho \to R^n$ is piecewise continuous with discontinuities of the first kind at the boundary points of G_i, where it is left continuous with respect to t; $f(t,0) = 0$ for all $t \geq t_0$; $\sup_{(t,x)\in G} \|f(t,x)\| = M < \infty$.

(b) $\theta_i(x)$: $S_\rho \to R_+$ are continuous; $\theta_i(x + J_i(x)) \leq \theta_i(x)$ and $\theta_i(x) < \theta_{i+1}(x)$ for all $x \in S_\rho$ and $i \in N$; $\lim_{i\to\infty} \theta_i(0) = \infty$.

(c) $J_i(x)$: $S_\rho \to R^n$; $J_i(0) = 0$.

As is well known, the solutions of differential equations with variable moments of impulse effect may experience pulse phenomena, namely, they may hit a given surface of discontinuity a finite or infinite number of times, causing rhythmic beating [8, 14]. This results in additional complications in studying such systems and therefore in most cases it is necessary to find conditions that guarantee the absence of beating.

There are various sets of sufficient conditions for the absence of pulse phenomena [1, 2]. For instance, if there exists an $L > 0$ such that $|\theta_i(x) - \theta_i(y)| \le L\|x - y\|$ for all $x, y \in S_\rho$ and $i \in N$, and $\theta_i(x + J_i(x)) \le \theta_i(x)$ for all $x \in S_\rho$ and $i \in N$, then $ML < 1$ is sufficient for the absence of beating of solutions on a surface of discontinuity. These conditions are not necessary; see the remark after Example 3.1 at the end of this paper.

Therefore, it is essential to assume that

(d) the beating of solutions of (1) on each surface of discontinuity is absent.

It is also well known that the solution curve of a differential equation with discontinuous right-hand side may intersect a surface of discontinuity at a certain time and stay there for some period of time [3]. Therefore some care should be taken when such a system is under consideration. In our case, however, this phenomenon does not exist. We have two kinds of surfaces of discontinuities, namely, $t = \theta_i^0$ and $t = \theta_i(x)$. It can be shown that no solution curve of (1) can stay on the surface $t = \theta_i^0$, for a period of time. In the case $t = \theta_i(x)$, this behavior is still not possible since each $t = \theta_i(x)$ is also a surface of impulse points.

In this paper we will state and prove three theorems. The first two theorems are about the stability and asymptotic stability of the zero solution of (1). The third one is a Chetaev type instability theorem for the zero solution of (1). Examples are also inserted to illustrate the results of the paper.

We should point out that the arguments developed in [4, 5] were based on a comparison method. Specifically, the change of a Lyapunov function in the interval of continuity was compared with its changes at the moments of discontinuity. Our technique is also based on a comparison, but it is somewhat different. We compare the changes of a Lyapunov function in the vicinity of the moments where solutions meet a surface of discontinuity. Therefore, the results of this paper seem to be very useful for stabilization and controllability of impulsive systems [7].

The following definition is extracted from [3].

DEFINITION 1.1. A solution $x(t) = x(t, a, b)$ of

$$\dot{x} = f(t, x), \qquad x(a) = b \tag{E}$$

is called unique from the right if for each solution $y(t)$ of (E) there exists $a_1 > a$ such that $x(t) = y(t)$ for all $t \in [a, a_1]$.

We will assume that for a given $a \in [0, \infty)$, the solution $x(t, a, 0)$ of (E) is unique from the right. We do not assume the uniqueness for all solutions since the system (1) is not a *reduced* system. In fact if we try to reduce the problem of stability of a nontrivial solution to that of the zero solution, then because of the difference between the discontinuity points the resulting system becomes very complicated [10, 15].

Let us recall the following definitions.

DEFINITION 1.2. The zero solution of (1) is called stable if for any given $\epsilon > 0$ and $t_0 \in R_+$ there exists $\delta = \delta(\epsilon, t_0)$ such that $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0)\| < \epsilon$ for all $t \geq t_0$.

DEFINITION 1.3. The zero solution of (1) is called asymptotically stable if it is stable and there exists $\bar{\delta} > 0$ such that any solution $x(t, t_0, x_0)$ with $\|x_0\| < \bar{\delta}$ satisfies $\lim_{t \to \infty} x(t) = 0$.

DEFINITION 1.4. A continuous function a: $R_+ \to R_+$ is said to belong to class $\mathscr{K}$ if a is strictly increasing and $a(0) = 0$.

DEFINITION 1.5. A function $V(t, x)$ is called positive definite on G if there exists $a \in \mathscr{K}$ such that $V(t, x) \geq a(\|x\|)$ for all $(t, x) \in G$; it is called positive semidefinite on G if $V(t, x) \geq 0$ for all $(t, x) \in G$. The function $V(t, x)$ is called negative definite (negative semidefinite) on G if $-V(t, x)$ is positive definite (positive semidefinite) on G.

2. MAIN RESULTS

Let $V(t, x)$ be a continuous real-valued function defined on G with $V(t, 0) = 0$ for $t \geq t_0$. We assume that $V(t, x)$ is locally Lipschitz in x and denote

$$D^+V(t, x) = \limsup_{h \to 0^+} \frac{V(t+h, x+hf(t,x)) - V(t,x)}{h}.$$

Note that if V is differentiable then

$$D^+V(t, x) = \dot{V}(t, x) = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t, x).$$

The following basic comparison results are extracted from [9].

LEMMA 2.1. *Let $V(t, x)$ be as above, and*

$$D^+V(t, x) \leq g(t, V(t, x)), \qquad (t, x) \in G, \tag{I_1}$$

where $g \in C[R_+ \times R_+, R]$. Suppose that $u(t) = u(t, t_0, u_0)$ is the maximal solution of the scalar differential equation

$$u' = g(t, u), \qquad u(t_0) = u_0 \geq 0,$$

which exists to the right of t_0. *If* $x(t) = x(t, t_0, x_0)$ *is any solution of* (I_1) *such that* $V(t_0, x_0) \leq u_0$, *then* $V(t, x(t)) \leq u(t)$ *for* $t \geq t_0$.

LEMMA 2.2. *Let* $V(t, x)$ *be as above, and*

$$D^+V(t,x) \geq g(t, V(t,x)), \qquad (t,x) \in G, \qquad (I_2)$$

where $g \in C[R_+ \times R_+, R]$. *Suppose that* $u(t) = u(t, t_0, u_0)$ *is the minimal solution of the scalar differential equation*

$$u' = g(t,u), \qquad u(t_0) = u_0 \geq 0,$$

which exists to the right of t_0. *If* $x(t) = x(t, t_0, x_0)$ *is any solution of* (I_2) *such that* $V(t_0, x_0) \geq u_0$, *then* $V(t, x(t)) \geq u(t)$ *for* $t \geq t_0$.

Remark 2.1. By taking $g(t,u) \equiv 0$, one can easily verify that if $D^+V(t,x) \geq 0$ then $V(t, x(t))$ is nondecreasing, and if $D^+V(t,x) \leq 0$ then $V(t, x(t))$ is nonincreasing.

In what follows we denote by $\mathscr{A}$ the set of all continuous functions f: $R \to R$ such that $f(0) = 0$ and $f(s) > 0$ for $s > 0$.

THEOREM 2.1. *Assume that the following conditions are fulfilled.*

(i) $V(t,x)$ *is positive definite on* G.

(ii) $D^+V(t,x)$ *is negative semidefinite on* G.

(iii) $D^+V(t,x) \leq -\varphi(V(t,x))$ *for some* $\varphi \in \mathscr{A}$ *and for all* $(t,x) \in \bigcup_{i \in N} G_i$.

(iv) $V(\theta_i(x), x + J_i(x)) \leq \psi(V(\theta_i(x), x))$ *for some* $\psi \in \mathscr{A}$ *and for all* $x \in S_\rho$ *and* $i \in N$.

(v) *There exists an* $L > 0$ *such that* $|\theta_i(x) - \theta_i(y)| \leq L\|x - y\|$ *for all* $x, y \in S_\rho$ *and* $I \in N$.

(vi) *There exists an* $L_1 > 0$ *such that* $|\theta_i(x) - \theta_i^0| \geq L_1\|x\|$ *for all* $x \in S_\rho$ *and* $i \in N$.

(vii) *There exists a* $\gamma \geq 0$ *such that*

$$\int_{V(\theta_i(x),x)}^{\psi(V(\theta_i(x),x))} \frac{ds}{\varphi(s)} \leq (L_1 - \gamma)\|x\|$$

for all $x \in S_\rho$ *and* $i \in N$.

Then, the zero solution of (1) *is stable if* $\gamma = 0$ *and is asymptotically stable if* $\gamma > 0$.

Proof. Let $x(t) = x(t, t_0, x_0)$ be a solution of (1) having discontinuities at $t = \tau_i$ for $i \in N$. It follows from (i) and Remark 2.1 that $v(t) := V(t, x(t))$ is nonincreasing on each of the intervals $[t_0, \tau_1)$ and (τ_i, τ_{i+1}),

$i \in N$. To obtain more information on the behavior of $v(t)$ we need to investigate its change in G_i. For each fixed $i \in N$, there are two possible cases:

Case 1. $\tau_i \geq \theta_i^0$. We let $u(t)$ be the maximal solution of $u' = -\varphi(u)$ on $[\theta_i^0, \tau_i]$ such that $u(\theta_i^0) = v(\theta_i^0)$. In view of Lemma 2.1 and (vi), we have

$$L_i \| x(\tau_i) \| \leq \tau_i - \theta_i^0 = \int_{u(\tau_i)}^{u(\theta_i^0)} \frac{ds}{\varphi(s)} \leq \int_{v(\tau_i)}^{v(\theta_i^0)} \frac{ds}{\varphi(s)}. \tag{2}$$

Using (iv) and (vii) we also have

$$(\gamma - L_1) \| x(\tau_i) \| \leq -\int_{v(\tau_i)}^{\psi(v(\tau_i))} \frac{ds}{\varphi(s)} \leq -\int_{v(\tau_i)}^{v(\tau_i +)} \frac{ds}{\varphi(s)}. \tag{3}$$

Summing (2) and (3) leads to

$$\int_{v(\tau_i +)}^{v(\theta_i^0)} \frac{ds}{\varphi(s)} \geq \gamma \| x(\tau_i) \| \tag{4}$$

and hence we obtain that

$$v(\theta_i^0) \geq v(\tau_i +). \tag{5}$$

Case 2. $\tau_i < \theta_i^0$. We let $u(t)$ be the maximal solution of $u' = -\varphi(u)$ on $[\tau_i, \theta_i^0]$ such that $u(\tau_i) = v(\tau_i +)$. It follows that

$$L_1 \| x(\tau_i) \| \leq \theta_i^0 - \tau_i = \int_{u(\theta_i^0)}^{u(\tau_i)} \frac{ds}{\varphi(s)} \leq \int_{v(\theta_i^0)}^{v(\tau_i +)} \frac{ds}{\varphi(s)}. \tag{6}$$

From (3) and (6) we get

$$\int_{v(\theta_i^0)}^{v(\tau_i)} \frac{ds}{\varphi(s)} \geq \gamma \| x(\tau_i) \| \tag{7}$$

and so

$$v(\tau_i) \geq v(\theta_i^0). \tag{8}$$

Define $\Gamma := \bigcup_{i=1}^{\infty} (\xi_i, \zeta_i]$ and $\Lambda := [t_0, \infty) \setminus \Gamma$, where $\xi_i = \tau_i$ and $\zeta_i = \theta_i^0$ if $\tau_i \leq \theta_i^0$, $\xi_i = \theta_i^0$ and $\zeta_i = \tau_i$ if $\tau_i \geq \theta_i^0$. From (5) and (8) we may write $v(\zeta +) \leq v(\xi_i)$, and hence we conclude that $v(t)$ is nonincreasing on Λ.

Let $0 < \epsilon < \rho$ and t_0 be given. Without any loss of generality we may assume that $t_0 \in \Lambda$. It is now clear that $v(t) \leq v(t_0)$ for all $t \in \Lambda$. Set

$$\epsilon_1 = \frac{\epsilon}{2(1 + ML)} \qquad \text{and} \qquad \eta = \inf_{t \geq t_0, \|x\| \geq \epsilon_1} V(t, x).$$

Since $V(t, x)$ is continuous and $V(t, 0) = 0$, it is possible to find a positive real number δ such that $\delta < \epsilon_1$ and

$$\sup_{\|x\| \leq \delta} V(t_0, x) < \gamma < \eta.$$

We first claim that if $\|x_0\| < \delta$ then $\|x(t, t_0, x_0)\| < \epsilon_1$ for all $t \in \Lambda$. Suppose that this is not true. Then there would exist a $t^* \in \Lambda$ such that $\|x(t^*, t_0, x_0)\| \geq \epsilon_1$. But this leads us to the contradiction that $\eta \leq v(t^*) \leq v(t_0) < \eta$.

Next suppose that $t \in (\xi_i, \zeta_i]$ for some i. Clearly,

$$x(t) = x(\xi_i) + \int_{\xi_i}^{t} f(s, x(s))\, ds \qquad \text{if } \tau_i > \theta_i^0 \tag{9}$$

and

$$x(t) = x(\zeta_i) + \int_{\zeta_i}^{t} f(s, x(s))\, ds \qquad \text{if } \tau_i < \theta_i^0. \tag{10}$$

In view of (v), we easily obtain from both (9) and (10) that

$$x(t) \leq \epsilon_1(1 + ML) < \epsilon.$$

Therefore, the zero solution of (1) is stable.

We shall now show that if $\gamma > 0$ then $\lim_{t \to \infty} x(t) = 0$. We first observe that since $v(t)$ is positive and nonincreasing on Λ, there is a nonnegative real number μ such that

$$\lim_{t \to \infty} v(t) = \mu, \qquad t \in \Lambda. \tag{11}$$

We claim that $\mu = 0$. Suppose on the contrary that $\mu > 0$. Because of (11), there exists a positive real number μ_1 such that

$$\|x(t)\| \geq \mu_1 \qquad \text{for all } t \in \Lambda. \tag{12}$$

If $\theta_i^0 > \tau_i$ then, since $\tau_i \in \Lambda$, (12) implies that

$$\|x(\tau_i)\| \geq \mu_1. \tag{13}$$

Suppose that $\theta_i^0 < \tau_i$. In this case, $\theta_i^0 \in \Lambda$, and therefore by (12) we have

$$\left\| x\left(\theta_i^0\right) \right\| \geq \mu_1. \tag{14}$$

Using (v), we also have

$$\left| \theta_i(x(\tau_i)) - \theta_i^0 \right| \leq L \| x(\tau_i) \|. \tag{15}$$

In view of (14) and (15), we easily obtain from

$$x(\tau_i) = x\left(\theta_i^0\right) + \int_{\theta_i^0}^{\tau_i} f(s, x(s))\, ds$$

that

$$\| x(\tau_i) \| \geq \mu_1 - ML \| x(\tau_i) \|$$

and hence

$$\| x(\tau_i) \| \geq \mu_1/(1 + ML). \tag{16}$$

Thus, we see from (13) and (16) that

$$\| x(\tau_i) \| \geq \mu_2 \qquad \text{for all } i \in N, \tag{17}$$

where $\mu_2 = \mu_1/(1 + ML)$.

On the other hand, since $(\zeta_i, \xi_{i+1}] \in \Lambda$ for all $i \in N$, $\lim_{i \to \infty} v(\zeta_i +) = \lim_{i \to \infty} v(\xi_i) = \mu$ and $v(\xi_i) \geq v(\zeta_i +) \geq \mu$. Letting

$$m = \min_{\mu \leq s \leq v(t_0)} \varphi(s),$$

it follows from (4), (7) and (17) that

$$v(\xi_i) - v(\zeta_i +) \geq \gamma m \mu_2 \qquad \text{for all } i \in N. \tag{18}$$

Using $v(\xi_{i+1}) \leq v(\zeta_i +)$ in (18) and then summing the resulting inequality over i from 1 to k we get

$$v(\xi_1) - v(\xi_{k+1}) \geq (\gamma m \mu_2) k \qquad \text{for all } k \in N. \tag{19}$$

It is clear from (19) that if k is sufficiently large, then the function v takes on negative values. But this contradicts the fact that v is positive definite. Thus, we must have $\mu = 0$.

As in the classical case, it follows that $\lim_{t \to \infty} x(t) = 0$, and hence we may conclude that the zero solution is asymptotically stable.

COROLLARY 2.1. *Let all conditions of Theorem* 2.1 *except* (v) *be satisfied. In addition, suppose that the family* $\{\theta_i(x)\}$ *is equicontinuous at* $x = 0$ *and that* $\theta_i^0 \geq \theta_i(x)$ *for all* $x \in S_\rho$. *Then the conclusion of Theorem* 2.1 *remains valid.*

Proof. We proceed as in the proof of Theorem 2.1 until ϵ_1 is picked. Now since the family $\{\theta_i(x)\}$ is equicontinuous at $x = 0$ and $\theta_i^0 \geq \theta_i(x)$ for all $x \in S_\rho$, given any ϵ_2, $0 < \epsilon_2 < \epsilon/M$, we can find $\epsilon_3 > 0$ such that $\theta_i^0 - \theta_i(x) < \varepsilon_2$ for all $\|x\| < \epsilon_3$ and $i \in N$. Fix $\epsilon_1 > 0$ such that $\epsilon_1 < \min\{\epsilon_3, \epsilon - M\epsilon_2\}$. Then it follows from (9) and (10) that

$$\|x(t)\| \leq \epsilon_1 + M\epsilon_2 < \epsilon.$$

Clearly, (11), (12), and (13) hold and by our assumption the case $\theta_i^0 < \tau_i$ does not exist. Thus (17) is satisfied with $\mu_2 = \mu_1$. The rest of the proof is the same as that of Theorem 2.1.

In the next theorem we do not require that $D^+V(t,x)$ be negative semidefinite on $\bigcup_{i \in N} G_i$.

THEOREM 2.2. *Assume that the following conditions are fulfilled.*

(i) $V(t,x)$ *is positive definite on* G.

(ii) $D^+V(t,x)$ *is negative semidefinite on* $G \setminus \bigcup_{i \in N} G_i$.

(iii) $D^+V(t,x) \leq \varphi(V(t,x))$ *for some* $\varphi \in \mathscr{A}$ *and for all* $(t,x) \in \bigcup_{i \in N} G_i$.

(iv) $V(\theta_i(x), x + J_i(x)) \leq \psi(V(\theta_i(x), x)$ *for some* $\psi \in \mathscr{A}$ *and for all* $x \in S_\rho$ *and* $i \in N$.

(v) *There exists an* $L > 0$ *such that* $|\theta_i(x) - \theta_i(y)| \leq L\|x - y\|$ *for all* $x, y \in S_\rho$ *and* $i \in N$.

(vi) *There exists a* $\gamma \geq 0$ *such that*

$$\int_{V(\theta_i(x),x)}^{\psi(V(\theta_i(x),x))} \frac{ds}{\varphi(s)} \geq (L + \gamma)\|x\|$$

for all $x \in S_\rho$ *and* $i \in N$.

Then, the zero solution of (1) *is stable if* $\gamma = 0$ *and is asymptotically stable if* $\gamma > 0$.

Proof. Let $u(t)$ be the maximal solution of $u' = \varphi(u)$ on $[\xi_i, \zeta_i]$ such that $u(\xi_i) = v(\xi_i +)$, where ξ_i and ζ_i are as defined in the proof of Theorem 2.1. Proceeding as in the proof of Theorem 2.1 we easily obtain

$$\int_{v(\theta_i^0)}^{v(\tau_i)} \frac{ds}{\varphi(s)} \leq L\|x(\tau_i)\| \qquad \text{for } \tau_i \geq \theta_i^0 \tag{20}$$

and

$$\int_{v(\tau_i+)}^{v(\theta_i^0)} \frac{ds}{\varphi(s)} \leq L\|x(\tau_i)\| \qquad \text{for } \tau_i < \theta_i^0. \tag{21}$$

Using (iv) and (vi) we also have

$$\int_{v(\tau_i)}^{v(\tau_i+)} \frac{ds}{\varphi(s)} \leq -(L+\gamma)\|x(\tau_i)\|. \tag{22}$$

It follows from (20), (21), and (22) that $v(\zeta_i+) \leq v(\xi_i)$ for all $i \in N$. The remainder of the proof is similar to that of Theorem 2.1 and hence is omitted.

Our last result in this paper is a Chetaev-type instability theorem, see page 216 in [12], for the zero solution of (1).

THEOREM 2.3. *Assume that the following conditions are fulfilled.*

(i) *For every* $\epsilon > 0$ *and for every* $t \geq t_0$ *there exist points* $\bar{x} \in S_\epsilon$ *such that* $V(t, \bar{x}) > 0$. *The set B of all points* (t, x) *such that* $\bar{x} \in S_\rho$ *and such that* $v(t, \bar{x}) > 0$ *is called the "domain* $v > 0$." *The set B is bounded by the hypersurfaces* $\|x\| = \rho$ *and* $v(t, x) = 0$. *We assume that v is bounded from above in B and* $0 \in \partial B$ *for all* $t \geq t_0$.

(ii) $D^+V(t, x)$ *is positive semidefinite on* $B \setminus \bigcup_{i \in N}(G_i \cap B)$.

(iii) $D^+V(t, x) \geq -\varphi(V(t, x))$ *for some* $\varphi \in \mathscr{A}$ *and for all* $(t, x) \in \bigcup_{i \in N}(G_i \cap B)$.

(iv) $V(\theta_i(x), x + J_i(x)) \geq \psi(V(\theta_i(x), x)$ *for some* $\psi \in \mathscr{A}$ *and for all* $(t, x) \in \bigcup_{i \in N}(\Gamma_i \cap B)$, *where* $\Gamma_i = \{(t, x) : t = \theta_i(x)\}$.

(v) *There exists an* $L > 0$ *such that* $|\theta(x) - \theta_i^0| \leq L\|x\|$ *for* $x \in S_\rho$ *and* $i \in N$.

(vi) *There exists a* $\gamma > 0$ *such that*

$$\int_{V(\theta_i(x), x)}^{\psi(V(\theta_i(x), x))} \frac{ds}{\varphi(s)} \geq (L+\gamma)\|x\|$$

for all $x \in S_\rho$ *and* $i \in N$.

Then the zero solution of (1) *is unstable.*

Proof. Fix $\epsilon > 0$ and t_0, $(t_0, x_0) \in B$, and let $x(t) = x(t, t_0, x_0)$ be a solution of (1) having discontinuities at $t = \tau_i$ for $i \in N$. We shall show that $x(t)$ must leave the ball S_ϵ in finite time. In view of (ii) and Remark 2.1 we see that $v(t)$ is nondecreasing on each interval of its continuity in Λ. We need to prove that $v(t)$ is nondecreasing for all $t \in \Lambda$. So we let

$u(t)$ be the minimal solution of $u' = -\varphi(u)$ on $[\xi_i, \zeta_i]$ such that $u(\xi_i) = v(\xi_i +)$, where ξ_i and ζ_i are as defined in the proof of Theorem 2.1. By using Lemma 2 and (vi) we see that

$$\int_{v(\theta_i^0)}^{v(\tau_i+)} \frac{ds}{\varphi(s)} \geq \gamma \| x(\tau_i) \| \qquad \text{if } \tau_i > \theta_i^0 \tag{23}$$

and

$$\int_{v(\tau_i)}^{v(\theta_i^0)} \frac{ds}{\varphi(s)} \geq \gamma \| x(\tau_i) \| \qquad \text{if } \tau_i < \theta_i^0. \tag{24}$$

From (23) and (24) we may deduce that $v(\xi_i) \leq v(\zeta_i +)$. Therefore, $v(t) \geq v(t_0)$ for all $t \in \Lambda$, implying that $(t, x(t)) \in B \setminus \bigcup_{i \in N} G_i$ for all $t \in \Lambda$.

Let $M > 0$ be a real number such that $V(t, x) \leq M$ for all $(t, x) \in B$, which is possible by (i). Since $v(t) \geq v(t_0)$, there is a $\mu_1 > 0$ such that $\|x(t)\| \geq \mu_1$ for all $t \in \Lambda$. If we now define

$$m = \min_{v(t_0) \leq s \leq M} \varphi(s)$$

then it follows from (23) and (24) that

$$v(\zeta_i +) - v(\xi_i) \geq \gamma m \mu_1. \tag{25}$$

Since $v(\xi_{i+1}) \geq v(\zeta_i +)$ we get

$$v(\xi_{i+1}) - v(\xi_i) \geq \gamma m \mu_1 \qquad \text{for all } i \in N. \tag{26}$$

Summing (26) over i from 1 to k we see that

$$v(\xi_{k+1}) - v(\xi_1) \geq (\gamma m \mu_1) k \qquad \text{for all } k \in N. \tag{27}$$

But (27) leads to a contradiction that $v(t)$ is unbounded in B. This completes the proof.

3. EXAMPLES

EXAMPLE 3.1. Let $\theta_i(x) = i - \sqrt{x_1^2 + x_2^2}$ so that $G_i = \{(t, x) \in G : i - \sqrt{x_1^2 + x_2^2} < t \leq i\}$. We define $S = \bigcup_{i=1}^{\infty} G_i$ and consider the impulsive system

$$\dot{x}_1 = \begin{cases} -x_2, & (t, x) \notin S, \\ -x_1, & (t, x) \in S \end{cases}$$

$$\dot{x}_2 = \begin{cases} x_1, & (t, x) \notin S, \\ -x_2, & (t, x) \in S \end{cases}$$

$$\Delta x_1|_{t=\theta_i(x)} = -\alpha x_1 + \beta x_2,$$

$$\Delta x_2|_{t=\theta_i(x)} = \beta x_1 - \alpha x_2.$$

We choose $V(x) = x_1^2 + x_2^2$ and make the following observations:

(a) $\dot{V}(x) = 0$ if $(t, x) \notin S$ and $\dot{V}(x) = -2V(x)$ if $(t, x) \in S$. Since $V(x + \Delta x) = ((1 - \alpha)^2 + \beta^2)(x_1^2 + x_2^2) - 4\beta(\alpha - 1)x_1 x_2$ we have

$$V(x + \Delta x) \leq \ell(\alpha, \beta) V(x),$$

where $\ell(\alpha, \beta) = (|\beta| + |1 - \alpha|)^2$.

(b) $\|x + \Delta x\|^2 = ((1 - \alpha)^2 + \beta^2)(x_1^2 + x_2^2) - 4\beta(\alpha - 1)x_1 x_2$ and so $\|x + \Delta x\|^2 \geq (|1 - \alpha| - |\beta|)^2 \|x\|^2$. It follows that if $||1 - \alpha| - |\beta|| \geq 1$, then

$$\theta_i(x + \Delta x) \leq \theta_i(x).$$

(c) $|\theta_i(x) - \theta_i^0| = \sqrt{x_1^2 + x_2^2} = \|x\|$.

(d) Let $\varphi(s) = 2s$ and $\psi(s) = \ell(\alpha\beta)s$, and fix a positive number $\gamma < 1$. If $\ell(\alpha, \beta) \leq 1$, then $\ln \ell(\alpha, \beta) \leq 2(1 - \gamma)\|x\|$ and hence

$$\int_V^{\ell V} \frac{ds}{2s} \leq (1 - \gamma)\|x\|.$$

In view of Theorem 2.1 we deduce that the zero solution of (1) is asymptotically stable if

$$||1 - \alpha| - |\beta|| \geq 1 \qquad \text{and} \qquad |1 - \alpha| + |\beta| \leq 1.$$

If we take $\theta_i(x) = i - \sqrt[4]{x_1^2 + x_2^2}$, then it is easy to verify that condition (v) is not satisfied and therefore Theorem 2.1 does not apply. However, since the additional conditions stated in Corollary 2.1 are true, we may conclude that the above conclusion is valid. We note that the beating is still absent in this case, since $(d/dt)\|x(t)\| = 0$ for all $(t, x) \notin S$ and (b) holds.

EXAMPLE 3.2. Let $\theta_i(x) = i + x_1^2 + x_2^2$. Clearly $G_i = \{(t, x) : i < t \leq i + x_1^2 + x_2^2\}$. Define $S = \bigcup_{i=1}^{\infty} G_i$ and consider the impulsive system

$$\dot{x}_1 = \begin{cases} -x_2, & (t, x) \notin S, \\ -x_2 + x_1^3, & (t, x) \in S \end{cases}$$

$$\dot{x}_2 = \begin{cases} x_1, & (t, x) \notin S, \\ x_1 + x_2^3, & (t, x) \in S, \end{cases}$$

$$\Delta x_1|_{t = \theta_i(x)} = -\alpha x_1 + \beta x_2,$$

$$\Delta x_2|_{t = \theta_i(x)} = \beta x_1 - \alpha x_2.$$

We choose $V(x) = x_1^2 + x_2^2$ and make the following observations:

(a) $\dot{V}(x) = 0$ if $(t, x) \notin S$ and $\dot{V}(x) \leq 2V^2(x)$ if $(t, x) \in S$. Since $V(x + \Delta x) = ((1 - \alpha)^2 + \beta^2)(x_1^2 + x_2^2) - 4\beta(\alpha - 1)x_1 x_2$, we have

$$V(x + \Delta x) \leq \ell(\alpha, \beta) V(x),$$

where $\ell(\alpha, \beta) = (|\beta| + |1 - \alpha|)^2$.

(b) Let $x, y \in R^n$ such that $\|x\| \leq h$ and $\|y\| \leq h$, where $h > 0$ is some real number. It follows that

$$|\theta_i(x) - \theta_i(y)| \leq 2h\|x - y\|.$$

(c) If $\ell(\alpha, \beta) < 1$ then $\theta_i(x + \Delta x) \leq i + \ell(\alpha, \beta)\|x\| < \theta_i(x)$.

(d) Let $g = (-x_2 + x_1^3, x_1 + x_2^3)$ and $m(h) = \max_{\|x\| \leq h} \|g\|$. Clearly, $m(h) \to 0$ as $h \to 0$ and so there exists h_0 such that $2hm(h) < 1$ for all $h \leq h_0$.

(e) Let $\varphi(s) = 2s^2$ and $\psi(s) = \ell(\alpha, \beta)s$, and fix a positive real number γ. Choose $\|x\| \leq \min\{h_0, \sqrt[3]{(1 - l)/(2l(2h_0 + \gamma))}\}$. It follows that $1 - \ell \geq 2\ell(2h + \gamma)\|x\|^3$ and so

$$\int_{\ell V}^{V} \frac{ds}{2s^2} \geq (2h + \gamma)\|x\|.$$

By Theorem 2.2, the zero solution of (1) is asymptotically stable if

$$|\beta| + |\alpha - 1| < 1.$$

In this case, $2hM < 1$ is sufficient for the absence of beating.

We end this paper by pointing out that the results of this paper can be generalized easily if

$$G_i = \{(t, x) \in G : t \text{ is between } t = \omega_i(x) \text{ and } \theta_i(x)\},$$

where $t = \omega_i(x)$ is any given surface like $\theta_i(x)$ such that $\omega_i(0) = \theta_i(0)$. In this article, $\omega_i(x) = \theta_i(0)$.

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