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The controllability of boundary-value problems for quasilinear impulsive systems

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1. Introduction

The question of the control of linear and quasilinear systems of ordinary differential equations has attracted the attention of many authors for several years [3, 4, 7, 9]. Recently, there has been a lot of activity in the investigation of differential equations with impulse effect [1, 2, 6, 8, 10]. The controllability of boundary-value problems for linear impulsive systems was considered in [2].

In this paper, by the help of some results from [1, 2, 6, 8, 10], we will investigate the problem of the control of boundary-value problems for quasilinear impulsive systems. We will consider not only fixed but also variable moments of impulse control. A comparison method is used to investigate the systems with variable moments of impulse actions.

Let α and β be fixed real numbers such that $\alpha<\beta$, and r and p be fixed positive integers. Denote by $L_2^r[\alpha,\beta]$ the set of all square integrable and bounded functions $\eta:[\alpha,\beta]\to R^r$ and by $D^r[1,p]$ the set of all finite sequences $\{\xi_i\}$, $\xi_i\in R^r$, $i=1,\ldots,p$.

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We define a space $\Pi_p^r = L_2^r \times D^r$ and denote its elements by $\{\eta, \xi\}$, and let

$$\langle\{\eta,\xi\},\{w,v\}\rangle = \int_\alpha^\beta (\eta, w)\,\mathrm{d}t + \sum_{i=1}^{p} (\xi_i, v_i)$$

be an inner product in Π_p^r, where (,) is the euclidean scalar product in R^r.

The main aim of this paper is to consider the problem of the control of systems of differential equations with impulse actions on surfaces. The system under consideration is of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x(t) + C(t)u + f(t) + \mu g(t,x,u,\mu), \quad t \neq \theta_i + \mu\tau_i(x,\mu),$$
$$\Delta x|_{t=\theta_i+\mu\tau_i(x,\mu)} = B_i x + D_i v_i + J_i + \mu W_i(x, v_i, \mu) \tag{1}$$

with the boundary condition

$$x(\alpha) = a, \quad x(\beta) = b, \tag{2}$$

where $x \in R^n$, the symbol $\Delta x|_{t=\theta}$ means $x(\theta+) - x(\theta)$, A and C are matrix functions of the sizes $(n \times n)$ and $(n \times m)$, respectively, the elements of which belong to $L_2^1[\alpha,\beta]$, $m \leq n, \{\theta_i\}$, $i = 1,\ldots,p$, is a strictly increasing sequence of real numbers in (α,β), B_i and D_i are, respectively, $(n \times n)$ and $(n \times m)$ constant matrices with $\det(I + B_i) \neq 0$, $i = 1,\ldots,p$, $\{f,J\} \in \Pi_p^n[\alpha,\beta]$, μ is a small positive parameter, g, W_i, and τ_i are continuous and continuously differentiable functions in x, u, and v.

The control problem (1) and (2), which we shall denote by Σ_μ, is said to be solvable if given any bounded set $G \subset R^n$ there exists a positive $\mu_0 \in R$, $\mu_0 = \mu_0(G)$, such that for all arbitrary $a, b \in G$ and $\mu < \mu_0$ there is a control $\{u, v\} \in \Pi_p^m$ for which system (1) admits a solution $x(t)$ satisfying Eq. (2).

The process defined by Eq. (1) for fixed μ and $\{u, v\}$ operates as follows: the point $P_t\,(t, x(t))$, starting at (t_0, x_0), moves along the curve defined by the solution $x(t) = x(t, t_0, x_0)$ of the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x(t) + C(t)u + f(t) + \mu g(t,x,u,\mu). \tag{3}$$

The motion along this curve terminates at time $t = v_i$ when the point P_t arrives at one of the surfaces of discontinuity so that $v_i = \theta_i + \mu\tau_i(x(v_i),\mu)$. At that moment the point P_t performs a jump $\Delta x|_{t=v_i} = B_i x(v_i) + D_i v_i + J_i + \mu W_i(x(v_i), v_i, \mu)$ and proceeds to move along the curve described by the solution $x(t, v_i, x(v_i+))$ of system (3), until it meets another surface of discontinuity, and so on.

We should note that system (1) considered in this paper belongs to a class of systems with impulses at non-fixed moments. It is possible for the integral curve of this system to meet more than one time and even infinitely many times one and the same surface of discontinuity. This phenomenon is called beating [8, 10]. Therefore, the investigation of such a system needs conditions for the absence of beating. Below we shall first deal with this consideration.

Let s be a positive real number, and let Π_s be the subspace of elements (x,u,v) satisfying the inequality $|x|+|u|+|v| \le s$, where $|\cdot|$ is the euclidean norm in R^n, and let

$$G_s = \{(x,u,v,t,i,\mu) | (x,u,v) \in \Pi_s,\ \alpha \le t \le \beta,\ i=1,\dots,p,\ \mu \le \mu_1\},$$

where μ_1 is a fixed positive real number.

Fix a set $G \in R^n$, and a positive real number $h \in R$ such that for all $x \in G$, $|x| < h$. Let $H > h$ be a real number, and set

$$m_1 = \max\left\{\sup_t |A(t)|,\ \sup_t |C(t)|,\ \max_i |B_i|\right\},$$

$$m_2 = \max\left\{\sup_t |f(t)|,\ \max_i |J_i|\right\}$$

and

$$m_3 = \max\left\{\max_{G_H} |g|,\ \max_{G_H} |W|,\ \max_{G_H} |\tau|\right\}.$$

If it is assumed that

$$\mu_1 m_3 < \min\{\theta_1 - \alpha,\ \beta - \theta_p\}$$

and that for any x, μ, and i from G_H, the relation $\theta_{i+1} + \mu\tau_{i+1}(x,\mu) < \theta_i + \mu\tau_i(x,\mu)$ is true, then it follows that every solution of Eq. (1), which is in G_H and defined on $[\alpha,\beta]$, intersects each of the surface $t = \theta_i + \mu\tau_i(x,\mu)$, $i=1,\dots,p$.

In view of the differentiability of functions g, W_i, and τ_i, there exists a positive real number l such that uniformly in G_H,

$$|g(t,x_1,u_1,v^1,\mu) - g(t,x_2,u_2,v^2,\mu)| \le l\{|x_1-x_2| + |u_1-u_2| + |v^1-v^2|\},$$

$$|W_i(x_1,v^1,\mu) - W_i(x_2,v^2,\mu)| \le l\{|x_1-x_2| + |v^1-v^2|\},$$

$$|\tau_i(x_1,\mu) - \tau_i(x_2,\mu)| \le l\,|x_1-x_2|.$$

Now if we let $\mu_2 < \mu_1$ be a positive real number which satisfies

$$\mu_2 l(2m_1 H + m_2 + \mu_2 m_3) < 1,$$

then we can obtain the following lemma. The proof is similar to that of Lemma 5 on p. 22 in [10], and hence is omitted.

Lemma 1. *Let system* (1) *be considered in* G_H, *and let* $\mu < \mu_2$. *If the relation*

$$\tau_i(x,\mu) \ge \tau_i(x + B_i x + D_i v + J_i + \mu W_i(x,v,\mu),\mu)$$

is valid, then every solution $x(t)$ *of Eq.* (1) *meets any given surface* $t = \theta_i + \mu\tau_i(x,\mu)$ *at most once.*

To investigate system (1) we shall use a comparison method [6]. Let $x_0(t)$ be a solution of system (3) with initial condition $x_0(\theta_i) = x$ for a fixed i, $i=1,\dots,p$, and

$t=\zeta_i$ be the instant of meet of solution $x_0(t)$ with the surface $t=\theta_i+\mu\tau_i(x,\mu)$. Suppose that $x_1(t)$ is a solution of the Cauchy problem

$$x_1(\zeta_i)=(I+B_i)x_0(\zeta_i)+D_iv_i+J_i+\mu W_i(x_0(\zeta_i),v_i,\mu)$$

for system (3). We define

$$\begin{aligned}S_i(x,u,v,\mu)=(I+B_i)\int_{\theta_i}^{\zeta_i}[A(t)x_0(t)+C(t)u(t)+f(t)+\mu g(t,x_0(t),u(t),\mu)]\,\mathrm{d}t\\ +\mu W_i(x_0(\zeta_i),v_i,\mu)+\int_{\zeta_i}^{\theta_i}[A(t)x_1(t)+C(t)u(t)+f(t)\\ +\mu g(t,x_1(t),u(t),\mu)]\,\mathrm{d}t.\end{aligned} \tag{4}$$

As in [6] one can easily show that each S_i, $i=1,\dots,p$, is a continuously differentialable function in x and v_i. Using the definition of S_i, we see that system (1) and

$$\begin{aligned}&\frac{\mathrm{d}y}{\mathrm{d}t}=A(t)y+C(t)u+f(t)+\mu g(t,y,u,\mu), \quad t\neq\theta_i,\\ &\Delta y|_{t=\theta_i}=B_iy+D_iv_i+J_i+S_i(y,u,v_i,\mu)\end{aligned} \tag{5}$$

have the property Ω in G_H [6]. This means that if $\mu<\mu_2$, where μ_2 satisfies $\mu_2m_2(2m_1H+m_2+\mu_2m_3)<H-h$, then given any solution $x(t)$ of (1), $|x(t)|<h$, $t\in[\alpha,\beta]$, there is a solution $y(t)$ of (5), $|y(t)|<H$, such that $x(t)=y(t)$ for all $t\in[\alpha,\beta]$ except possibly at points $t\in[\theta_i,\zeta_i]$, $i=1,\dots,p$. Conversely, given any solution $y(t)$ of Eq. (5), $|y(t)|<h$, $t\in[\alpha,\beta]$, there is a solution $x(t)$ of Eq. (1), $|x(t)|<H$, such that $x(t)=y(t)$ for all $t\in[\alpha,\beta]$ except possibly at points $t\in[\theta_i,\zeta_i]$, $i=1,\dots,p$.

Now, we look at the dependence of S_i on the control function $u(t)$. So let $\hat{u}(t)$ be another control function, and let $y_0(t), y_0(\theta_i)=x$, be a solution of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t}=A(t)x(t)+C(t)\hat{u}(t)+f(t)+\mu g(t,x,\hat{u}(t),\mu). \tag{6}$$

Suppose that $t=v_i$ is the instant of meet of $y_0(t)$ with the surface $t=\theta_i+\mu\tau_i(x,\mu)$. Without loss of generality, we can assume that $\theta_i<v_i<\zeta_i$. Let $y_1(t)$ be also a solution of Eq. (6) satisfying the initial condition

$$y_1(v_i)=(I+B_i)y_0(v_i)+D_iv_i+J_i+\mu W_i(y_0(v_i),v_i,\mu).$$

Clearly,

$$\begin{aligned}S_i(x,\hat{u},v,\mu)=(I+B_i)\int_{\theta_i}^{v_i}[A(t)y_0(t)+C(t)\hat{u}(t)+f(t)+\mu g(t,y_0(t),\hat{u}(t),\mu)]\,\mathrm{d}t\\ +\mu W_i(y_0(v_i),v_i,\mu)+\int_{v_i}^{\theta_i}[A(t)y_1(t)+C(t)\hat{u}(t)+f(t)\\ +\mu g(t,y_1(t),\hat{u}(t),\mu)]\,\mathrm{d}t.\end{aligned} \tag{7}$$

Pick a positive real number $\mu_3\leq\mu_2$ such that

$$\mu_3l(2m_1H+\mu_3m_3+m_2)<1 \tag{8}$$

and define

$$\|u-\hat{u}\|_1 = \max_{\theta_i \le t \le \zeta_i} |u-\hat{u}|.$$

The following lemma holds.

Lemma 2. *Suppose that $(x, u(t), v)$ and $(x, \hat{u}(t), v)$ for all $t \in [\theta_i, \zeta_i]$ are in Π_h. Then*

$$|S_i(x,u,v,\mu) - S_i(x,\hat{u},v,\mu)| \le \mu k(H) \|u-\hat{u}\|_1 \tag{9}$$

for all $\mu < \mu_3$ and $t \in [\theta_i, \zeta_i]$, where $k(H)$ is a bounded function.

Proof. It is not difficult to verify that the solutions x_0, x_1 of Eq. (3) and the solutions y_0, y_1 of Eq. (6) which we have used in Eqs. (4) and (7) remain in G_H. Subtracting Eq. (4) from Eq. (7) we find that

$$\begin{aligned} S_i(x,\hat{u},v,\mu) - S_i(x,u,v,\mu) = &\int_{\theta_i}^{v_i} \{(I+B_i)A(t)(y_0(t)-x_0(t)) - A(t)(y_1(t)-x_1(t)) \\ &+\mu[(I+B_i)(g(t,y_0(t),\hat{u}(t),\mu) - g(t,x_0(t),u(t),\mu)) \\ &-(g(t,y_1(t),\hat{u}(t),\mu) - g(t,x_1(t),u(t),\mu))]\}\,\mathrm{d}t \\ &+B_i \int_{\theta_i}^{v_i} C(t)(\hat{u}(t)-u(t))\,\mathrm{d}t \\ &+\mu[W_i(y_0(v_i),v_i,\mu) - W_i(x_0(\zeta_i),v_i,\mu)] \\ &-\int_{v_i}^{\zeta_i} \{(I+B_i)A(t)x_0(t) - A(t)x_1(t) + B_iC(t)u(t) \\ &+\mu[(I+B_i)g(t,x_0(t),u(t),\mu) - g(t,x_1(t),u(t),\mu)]\}\,\mathrm{d}t. \end{aligned} \tag{10}$$

Since the right-hand side of Eq. (3) satisfies a Lipschitz condition in u, there exists a constant l_1, see [5], such that

$$|y_0(t) - x_0(t)| \le l_1 \|u-\hat{u}\|_1. \tag{11}$$

It follows that

$$|g(t,y_0(t),\hat{u}(t),\mu) - g(t,x_0(t),u(t),\mu)| \le l(1+l_1)\|u-\hat{u}\|_1. \tag{12}$$

On the other hand, since Eq. (8) is satisfied and

$$\begin{aligned} \zeta_i - v_i &= \mu\,\tau_i(x_0(\zeta_i),\mu) - \mu\tau_i(y_0(v_i),\mu) \\ &\le \mu l \left(\left| \int_{v_i}^{\zeta_i} [A(t)x_0(t) + C(t)u(t) + f(t) + \mu g(t,x_0(t),u(t),\mu)]\,\mathrm{d}t \right| \right. \\ &\left. \quad + l_1\|u-\hat{u}\|_1 \right) \le \mu l[(\zeta_i - v_i)(2m_1H + \mu m_3 + m_2) + l_1\|u-\hat{u}\|_1], \end{aligned}$$

we have

$$\zeta_i - v_i \le \frac{\mu l l_1}{1-\mu l h_1(H,\mu)}\|u-\hat{u}\|_1, \quad h_1(H,\mu) = 2m_1 H + \mu m_3 + m_2. \tag{13}$$

Using Eq. (13) and the expression

$$x_1(v_i) = x_1(\zeta_i) + \int_{\zeta_i}^{v_i} [A(t)x_1(t) + C(t)u(t) + f(t) + \mu g(t, x_1(t), u(t), \mu)]\,\mathrm{d}t,$$

we find that

$$|x_1(v_i) - x_1(\zeta_i)| \le h_2(H,\mu)\|u-\hat{u}\|_1, \quad h_2(H,\mu) = \frac{l l_1 h_1(H,\mu)}{1-\mu l h_1(H,\mu)}.$$

It follows that

$$|y_1(t) - x_1(t)| \le \mu l_2 h_2(H,\mu)\|u-\hat{u}\|_1, \tag{14}$$

where l_2 is a fixed positive real number.

Now, we shall estimate the integrals in Eq. (10). Denote them in order by I_1, I_2, and I_3. Using Eqs. (11), (12), and (14) we obtain

$$\begin{aligned}|I_1| \le \mu m_3\{&(1+m_1)m_1 l_1 + \mu m_1 h_2(H,\mu) + \mu[(1+m_1)l(1+l_1)\\ &+ l(1+\mu h_2(H,\mu))]\}\|u-\hat{u}\|_1 = \mu k^1(H)\|u-\hat{u}\|_1,\end{aligned} \tag{15}$$

$$|I_2| \le \mu m_1^2 m_3 \|u-\hat{u}\|_1 = \mu k^2(H)\|u-\hat{u}\|_1 \tag{16}$$

and

$$\begin{aligned}|I_3| \le \frac{\mu l l_1}{1-\mu l h_2(H,\mu)}&\{(1+m_1)m_2 H + (m_1+m_1^2)H\\ &+\mu[(1+m_1)m_3 + m_3]\} = \mu k^3(H)\|u-\hat{u}\|_1.\end{aligned} \tag{17}$$

Moreover,

$$\begin{aligned}\mu|W_i(y_0(v_i), v_i, \mu) - W_i(x_0(\zeta_i), v_i, \mu)| &\le \mu l(l_1 + h_2(H,\mu))\|u-\hat{u}\|_1\\ &= \mu k^4(H)\|u-\hat{u}\|_1.\end{aligned} \tag{18}$$

By summing relations (15)–(18) and noting that k^i, $i = 1, \ldots, 4$, are bounded functions of H, we see that Eq. (9) holds. Thus, the lemma is proved. □

In what follows, we denote by $X(t), X(\alpha) = I$, a fundamental matrix of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x, \quad t \ne \theta_i$$
$$\Delta x|_{t=\theta_i} = B_i x$$

and define

$$\Psi(t)=\int_\alpha^t Q(t)Q^{\mathrm{T}}(t)\,\mathrm{d}t+\sum_{\alpha<\theta_i<t} P_iP_i^{\mathrm{T}},$$

where $Q(t)=X^{-1}(t)C(t)$ and $P_i=X^{-1}(\theta_i)D_i$.

The problem Σ_μ when $\mu=0$ was considered in [2] and the following theorems were obtained.

Theorem 1. *The problem Σ_0 is solvable if and only if the matrix $\Psi(\beta)$ is non-singular.*

Theorem 2. *Let Σ_0 be solvable. Then the pair $\{u,v\}\in\Pi_p^m$ is a solving control for Σ_0 if and only if it has the form*

$$u=Q^{\mathrm{T}}(t)c+\hat{u}(t),\quad t\in[\alpha,\beta],\quad v_i=P_i^{\mathrm{T}}c+\hat{v}_i,\quad i=1,\dots,p,$$

where

$$c=\Psi^{-1}(\beta)\left[X^{-1}(\beta)b-X^{-1}(\alpha)a-\int_\alpha^\beta X^{-1}(t)f(t)\,\mathrm{d}t-\sum_{i=1}^{p}X^{-1}(\theta_i)J_i\right],$$

and $\{\hat{u},\hat{v}\}\in\Pi_p^m$ is orthogonal to all columns of $[Q^{\mathrm{T}},P_i^{\mathrm{T}}]$.

2. Control of systems with fixed moments of impulse actions

Consider the following system:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=A(t)x(t)+C(t)u+f(t)+\mu g(t,x,u,\mu),\quad t\neq\theta_i$$
$$\Delta x|_{t=\theta_i}=B_ix+D_iv_i+J_i+\mu W_i(x,v_i,\mu) \tag{19}$$

with boundary condition (2). All quantities here are as in Eq. (1) except that the moments of impulse actions are now fixed. We denote this problem by π_μ.

Theorem 3. *If the matrix $\Psi(\beta)$ is non-singular, then π_μ is solvable. The solving control of this problem is the limit of a uniformly convergent sequence obtained by the method of successive approximations.*

Proof. We will prove that π_μ is solvable with a control $\{u,v\}$ of the following form:

$$u=Q^{\mathrm{T}}(t)c+\hat{u}(t),\quad t\in[\alpha,\beta],\quad v_i=P_i^{\mathrm{T}}c+\hat{v}_i,\quad i=1,\dots,p, \tag{20}$$

where $c\in R^n$ is a constant vector and $\{\hat{u},\hat{v}\}\in\Pi_p^m$ is orthogonal to all columns of $[Q^{\mathrm{T}},P_i^{\mathrm{T}}]$.

The problem π_μ is equivalent to solving

$$x(t)=X(t)\left[a+\int_\alpha^t Q(s)(u(s)+C^{-1}(s)f(s)+\mu C^{-1}(s)g(s,x(s),u(s),\mu))\,ds\right.$$

$$\left.+\sum_{\alpha<\theta_i<t} P_i(v_i+D_i^{-1}J_i+\mu D_i^{-1}W_i(x(\theta_i),v_i,\mu))\right],\quad x(\beta)=b. \tag{21}$$

Substituting Eq. (20) into Eq. (21) we obtain the vector c as

$$c=\Psi^{-1}(\beta)\left[X^{-1}(\beta)b-X^{-1}(\alpha)a-\int_\alpha^\beta X^{-1}(t)(f(t)+\mu g(t,x(t),u(t),\mu))\,dt\right.$$

$$\left.-\sum_{i=1}^{p} X^{-1}(\theta_i)(J_i+\mu W_i(x(\theta_i),v_i,\mu))\right]. \tag{22}$$

If we let

$$u_0(t)=Q^{\mathrm{T}}(t)\Psi(\beta)^{-1}K+\hat{u}(t),\qquad v_i^0=P_i^{\mathrm{T}}\Psi(\beta)^{-1}K+\hat{v}_i,$$

$$x_0(t)=X(t)\left[a+\int_\alpha^t (Q(s)u_0(s)+X^{-1}(s)f(s))\,ds+\sum_{\alpha<\theta_i<t}(P_iv_i^0+X^{-1}(\theta_i)J_i)\right],$$

$$\phi_0=(x_0(t),u_0(t),v_i^0),\quad \phi=(x(t),u(t),v_i),$$

$$\kappa(t,\phi,\mu)=\int_\alpha^t X^{-1}(s)g(s,x(s),u(s),\mu)\,ds,$$

$$\psi(t,\phi,\mu)=\sum_{\alpha<\theta_i<t} X^{-1}(\theta_i)W_i(x(\theta_i),v_i,\mu),$$

where

$$K=X^{-1}(\beta)b-X^{-1}(\alpha)a-\int_\alpha^\beta X^{-1}(t)f(t)\,dt-\sum_{i=1}^{p}X^{-1}(\theta_i)J_i,$$

then, by substituting Eq. (22) into Eq. (21) we find that ϕ satisfies the following relation:

$$\phi=\phi_0+\mu\mathscr{P}(\phi,\mu), \tag{23}$$

where $\mathscr{P}=(\mathscr{P}_1,\mathscr{P}_2,\mathscr{P}^i)$,

$$\mathscr{P}_1(t,\phi)=X(t)[\kappa(t,\phi,\mu)+\psi(t,\phi,\mu)-\Psi(t)\Psi^{-1}(\beta)(\kappa(\beta,\phi,\mu)+\psi(\beta,\phi,\mu))],$$

$$\mathscr{P}_2(t,\phi)=Q(t)^{\mathrm{T}}\Psi^{-1}(\beta)[\kappa(\beta,\phi,\mu)+\psi(\beta,\phi,\mu)],$$

$$\mathscr{P}^i(\phi)=P_i^{\mathrm{T}}\Psi^{-1}(\beta)[\kappa(\beta,\phi,\mu)+\psi(\beta,\phi,\mu)].$$

We introduce the norm $\|\cdot\|$,

$$\|\phi\| = \max_t |x(t)| + \max_t |u(t)| + \max_i |v_i|,$$

in the space of all elements ϕ of the form $\phi = (x(t), u(t), v_i)$.

Suppose that the number h which is fixed earlier satisfies $h > \|\phi_0\|$, and that μ_4 is a positive real number such that $\mu_4 \le \mu_1$ and

$$\mu_4 \max_{\substack{\|\phi\| \le h \\ 0<\mu\le\mu_1}} \|\mathscr{P}(\phi,\mu)\| < h - \|\phi_0\|.$$

Let Π be the subspace consisting of elements $\phi = (x(t), u(t), v^i)$ such that $\|\phi\| \le h$, where $x(t)$ is piecewise absolutely continuous, continuous on the left, and has discontinuities of the first kind at moments θ_i, $i = 1, \dots, p$, and $\{u, v\} \in \Pi_p^m$. It follows that if $\mu \le \mu_4$ then the operator $\phi_0 + \mu\mathscr{P}(\phi,\mu)$ maps Π to itself.

We shall show that if μ is sufficiently small then $\mathscr{P}$ is a contraction mapping.

Let

$$m_4 = \max\left\{ \max_{t,s} |Q(t)^{\mathrm{T}}\Psi^{-1}(\beta)X^{-1}(s)|, \max_{i,s} |P_i^{\mathrm{T}}\Psi^{-1}(\beta)X^{-1}(s)|, \right.$$
$$\left. \max_{t,s} |X(t)X^{-1}(s)|, \max_{t,s} |X(t)\Psi(t)\Psi^{-1}(\beta)X^{-1}(s)| \right\}.$$

Then, we have

$$\|\mathscr{P}(\phi_1,\mu) - \mathscr{P}(\phi_2,\mu)\| \le 4\mu m_4 l(\beta - \alpha + p)\|\phi_1 - \phi_2\|$$

uniformly with respect to $t \in [\alpha, \beta]$, $\mu \in [0, \mu_4]$, and $i = 1, \dots, p$ for any $\phi_1, \phi_2 \in \Pi$.

Now if we let $\mu_0 < \mu_4$ be such that

$$4\mu_0 m_4 l(\beta - \alpha + p) < 1,$$

then for $\mu < \mu_0$, $\mathscr{P}$ becomes a contraction mapping.

Thus, we can obtain a sequence $\{\phi_i\}$, $i = 0, 1, 2, \dots$, $\phi_{i+1} = \phi_0 + \mu\mathscr{P}(\phi_i, \mu)$ converging to some element $\phi^0 \in \Pi$, $\phi^0 = (x^0, u^0, v^0)$.

Substituting ϕ^0 in Eq. (19), differentiating $x^0(t)$, and checking the boundary condition (2) and the condition of discontinuities, one can verify that ϕ^0 solves π_μ. Thus, the proof is complete. □

3. Control of systems with variable moments of impulse actions

Now, we return to considering problem Σ_μ. In view of the property Ω of systems (1) and (5), the problem Σ_μ is reduced to a problem of the kind π_μ for system (5). Let us denote it by γ_μ.

The following lemma is concerned with the solvability of γ_μ.

Lemma 3. *If $\Psi(\beta)$ is non-singular then γ_μ is solvable.*

Proof. Let $\mu<\mu_3$, where μ_3 is as in Eq. (8), and let $(x,u,v)\in\Pi_h$. Then using Eq. (4) we obtain

$$|S_i(x,u,v_i,\mu)|\leq \mu m_5,$$

where $m_5=m_3[(2+m_1)(2m_1h+m_2+\mu_3m_3)+\mu_3m_3]$.

Since S_i is differentiable in x and v_i, there is a positive real number μ_5 such that for $\mu<\mu_5$ it satisfies a Lipschitz condition in x and v. The corresponding Lipschitz constants have the form $\mu k_2(H)$ and $\mu k_3(H)$, where k_2 and k_3 are bounded functions.

Setting $k(H)=\max\{k_1,k_2,k_3\}$ and $\mu_6=\min\{\mu_3,\mu_5\}$, we see, in view of Lemma 2, that if $\mu<\mu_6$ then

$$|S_i(x_1,u_1,v^1,\mu)-S_i(x_2,u_2,v^2,\mu)|\leq \mu k(H)\{|x_1-x_2|+\|u_1-u_2\|_1+|v^1-v^2|\}$$

uniformly in Π_h.

Now, we let $\phi_0\in\Pi$ be as in Theorem 3 and consider the operator $\phi=\phi_0+\mu\mathscr{T}(\phi,\mu)$ in Π, where $\mathscr{T}=(\mathscr{T}_1,\mathscr{T}_2,\mathscr{T}^i)$,

$$\mathscr{T}_1(\phi,\mu)=X(t)[\kappa(t,\phi,\mu)+\tilde{\psi}(t,\phi,\mu)-\Psi(t)\Psi^{-1}(\beta)(\kappa(\beta,\phi,\mu)+\tilde{\psi}(\beta,\phi,\mu))],$$

$$\mathscr{T}_2(\phi,\mu)=Q(t)^{\mathrm{T}}\Psi^{-1}(\beta)[\kappa(\beta,\phi,\mu)+\tilde{\psi}(\beta,\phi,\mu)],$$

$$\mathscr{T}^i(\phi,\mu)=P_i^{\mathrm{T}}\Psi^{-1}(\beta)[\kappa(\beta,\phi,\mu)+\tilde{\psi}(\beta,\phi,\mu)]$$

with $\kappa(t,\psi,\mu)$ as defined in Eq. (23) and

$$\tilde{\psi}(t,\phi,\mu)=\frac{1}{\mu}\sum_{\alpha<\theta_i<t}X^{-1}(\theta_i)S_i(x(\theta_i),u,v_i,\mu).$$

Choosing μ_0 as

$$\mu_0=\min\left\{\mu_6,\frac{4(h-\|\phi_0\|)}{(\beta-\alpha)m_3m_4+pm_4m_5},\frac{4}{lm_4(\beta-\alpha)+k(H)m_4p}\right\},$$

we can see similarly as in the proof of Theorem 3 that if $\mu<\mu_0$, the operator $\phi_0+\mu\mathscr{T}(\phi,\mu)$ maps Π to itself and is contractive. Thus, the lemma is proved. □

Theorem 4. *If $\Psi(\mu)$ is non-singular then Σ_μ is solvable.*

Proof. Lemma 3 implies that there is a control $\{\hat{u},\hat{v}\}$ such that system (5) has a solution $y(t)$ which satisfies $y(\alpha)=a$ and $y(\beta)=b$.

Since Eqs. (1) and (5) have the property Ω, system (1) with the control $\{\hat{u},\hat{v}\}$ admits a solution $x(t)$ such that $x(t)=y(t)$ for all t except for $t\in[\theta_i,\zeta_i]$, $i=1,\dots,p$. Since α and β are not in $[\theta_i,\zeta_i]$, $i=1,\dots,p$, the solution $x(t)$ also satisfies the boundary condition (2). This completes the proof. □

References

[1] M.U. Akhmetov, N.A. Perestyuk, On comparison method for differential systems with pulse effect, Differential Equations 9 (26) (1990) 1079–1086.
[2] M.U. Akhmetov, N.A. Perestyuk, M.A. Tleubergenova, The control of linear impulsive systems, Ukrainian. Math. Zh. 3 (47) (1995) 307–314.
[3] S.A. Aysagaliev, K.O. Onaybaer, T.G. Mazakov, The controllability of nonlinear systems, Izv. Akad. Nauk. Kazakh.-SSR.-Ser. Fiz.-Mat (1) (1985) 83–87.
[4] S. Barnett, Introduction to Mathematical Control Theory, Clarendon Press, Oxford, 1975.
[5] P. Hartman, Ordinary Differential Equations, Wiley, New York, 1964.
[6] A. Halanay, D. Wexler, Qualitative Theory of Impulse Systems, Editura Academiei Republici Socialiste Romania, Bucuresti, 1968.
[7] N.N. Krasovsky, Theory of Motion Control, Linear Systems, Nauka, Moscow, 1973.
[8] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[9] E.B. Lee, L. Markus, Foundations of Optimal Control Theory, Wiley, New York, 1967.
[10] A.M. Samoilenko, N.A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.